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POSITIVE SOLUTIONS FOR A CLASS OF ϕ-LAPLACIAN DIFFERENTIAL SYSTEMS WITH MULTIPLE PARAMETERS

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Abstract. In this article, we consider the double eigenvalue problem for a ϕ-Laplacian differential system. We prove the existence of positive solutions under the ϕ-super-linear condition by means of the Guo-Krasnosel'skii fixed point theorem and the topological degree. It is shown that there exists a continuous curve splitting $\mathbb{R}^2_+\backslash\{(0,0)\}$ into disjoint subsets such that systems has at least two, at least one, or no positive solutions according to parameters in different subsets.

1. Introduction

Eigenvalue problems have been studied for many years. There are lots of important results on positive solutions of nonlinear problems. Among them, we refer to [3, 11, 18, 19, 22] for ordinary differential systems, to [4, 5, 7, 8, 9, 13, 14] for elliptic equations of partial differential systems, and to [1, 2, 6, 10, 15, 16, 17, 20, 21] for Laplacian systems. The technical methods are mainly the fixed point theory, Leray-Schauder degree theory, upper and lower solution method, and variational methods.

Dunninger and Wang [8] considered the Dirichlet boundary value problem of the problem

$$\begin{gathered}\Delta u+\lambda k_1(|x|)f(u,v)=0,\\ \Delta v+\mu k_2(|x|)g(u,v)=0,\end{gathered}$$

where $\Omega=\{x\in\mathbb{R}^N: R_1<|x|<R_2, R_1,R_2>0\}$ is an Annulus. They used the cone expansion and compression theorem to obtain sufficient conditions for the existence of the existence and multiplicity results. Later in [9], they extended the existence results to a more generalized boundary condition with $f(0,0)>0$, $g(0,0)>0$,

$$\begin{gathered}\alpha_1 u+\beta_1\frac{\partial u}{\partial n}=0,\quad \alpha_2 u+\beta_2\frac{\partial v}{\partial n}=0,\quad |x|=R_1,\\ \gamma_1 u+\delta_1\frac{\partial u}{\partial n}=0,\quad \gamma_2 u+\delta_2\frac{\partial v}{\partial n}=0,\quad |x|=R_2.\end{gathered}$$

They proved that superlinearity or sublinearity at zero or infinity of the nonlinearity can guarantee the existence results of differential systems.

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Lee [14] considered the the radial solutions for semilinear elliptic systems on an annulus. By applying consecutive change of variables, the author transformed the partial differential equations into the ordinary differential equations

$$\begin{aligned} u''(t) + \lambda h_1(t) f(u(t), v(t)) = 0, \\ v''(t) + \mu h_2(t) g(u(t), v(t)) = 0, \end{aligned}$$

with homogeneous and semi homogeneous Dirichlet boundary conditions. The author proved the existence of one positive and multiple positive solutions based on the upper and lower solution technique, cone expansion and compression theorem and degree theory.

For the multiple parameters of ϕ-Laplace systems, Wang [20, 21] considered the single parameter of the n-dimensional system

$$(\Phi(\mathbf{u}'))' + \lambda \mathbf{h}(t) \mathbf{f}(\mathbf{u}) = 0, \quad 0 < t < 1,$$

with one of the following conditions

$$\begin{gathered} \mathbf{u}(0) = \mathbf{u}(1) = 0, \\ \mathbf{u}'(0) = \mathbf{u}(1) = 0, \\ \mathbf{u}(0) = \mathbf{u}'(1) = 0, \end{gathered}$$

where λ is a single parameter and $\Phi = (\phi_1, \cdots, \phi_n)$. Under suitable sufficient conditions, the author proved that when the nonlinearity is ϕ-suplinearity or sublinearity at zero or infinity, there exists a λ_0 such that the above n-dimensional system has at least two, at least one, or no positive radial solutions according to λ large enough or not.

Lee et al [15] discussed a generalized Gelfand type Laplacian system with a vector parameter

$$\begin{gathered} (\Phi(\mathbf{u}'))' + \lambda \mathbf{h}(t) \mathbf{f}(\mathbf{u}) = 0, \\ \mathbf{u}(0) = \mathbf{u}(1) = 0, \end{gathered}$$

where $\lambda = diag(\lambda_1, \cdots, \lambda_n)$. By using the upper and lower solution method and fixed point index theory, they obtained a global multiplicity result with respect to the parameter. Recently, Lee and Xu [16] extended the existence result to a more general singular (p_1, p_2)-Laplacian system with multi-parameters.

Inspired by the work listed above, we aim to study the double eigenvalue problem of ϕ-Laplace differential equations with mixed boundary condition

$$\begin{gathered} (\phi(u'(t)))' + \lambda h_1(t) f(u(t), v(t)) = 0, \quad 0 < t < 1, \\ (\phi(v'(t)))' + \mu h_2(t) g(u(t), v(t)) = 0, \quad 0 < t < 1, \\ u'(0) = v'(0) = u(1) = v(1) = 0, \end{gathered} \tag{1.1}$$

where ϕ is an increasing and odd homemorphism. Throughout this article, we assume that the multiparameter $\lambda, \mu \in \mathbb{R}_+^2 \setminus \{(0,0)\}$, $\mathbb{R}_+ = [0, +\infty)$, $h_i \in C([0,1], \mathbb{R}_+)$ is not always zero in any subinterval of $[0,1]$, $i = 1, 2$. $f, g \in C(\mathbb{R}_+^2, \mathbb{R}_+)$.

We define

$$f_\infty := \lim_{|u|+|v| \to \infty} \frac{f(u,v)}{\phi(|u|+|v|)}, \quad g_\infty := \lim_{|u|+|v| \to \infty} \frac{g(u,v)}{\phi(|u|+|v|)}.$$

and use the assumptions

(H1) $f(u,v)$ and $g(u,v)$ are quasi-nondecreasing on u, (H2) $f_\infty = g_\infty = \infty$.

We summarize our main results as follows.

Theorem 1.1. *Suppose that* (A1) *and* (A2) *hold. Then there exists a continuous curve* Γ *separating* $\mathbb{R}^2_+ \setminus \{(0,0)\}$ *into two disjoint subsets* θ_1 *and* θ_2 *such that the eigenvalue problem* (1.1) *has at least one positive solution for* $(\lambda,\mu) \in \theta_1 \cup \Gamma$ *and has no solution for* $(\lambda,\mu) \in \theta_2$.

The article is organized as follows: In Section 2, we present some lemmas and a theorem of upper and lower solutions method for our systems. In Section 3, we show that the parameters lying in different area make a difference for the existence and nonexistence results. In Section 4, we demonstrate the multiplicity results.

2. Preliminaries

A cone and partial order are important definitions when the positive solution is discussed. After introducing the fixed point theorems on cones, we establish the upper and lower solution theory for our systems.

Definition 2.1. Let X be a Banach space and K be a closed and convex subset of X. We call that K is a cone of X if and only if

(1) If $x \in K$ and $\lambda \geq 0$, then $\lambda x \in K$.
(2) If $x \in K$ and $-x \in K$, then $x = \theta$, where θ is the zero element of X.

Definition 2.2. Let Ω be a subset of $\mathbb{R}^2$. We call that P is a binary partial order of Ω if the following conditions hold.

(1) For all $(\lambda_1,\mu_1) \in \Omega$, $\{(\lambda_1,\mu_1),(\lambda_1,\mu_1)\} \in P$.
(2) If $\{(\lambda_1,\mu_1),(\lambda_2,\mu_2)\} \in P$, and $\{(\lambda_2,\mu_2),(\lambda_1,\mu_1)\} \in P$, then $(\lambda_1,\mu_1) = (\lambda_2,\mu_2)$.
(3) If $\{(\lambda_1,\mu_1),(\lambda_2,\mu_2)\} \in P$ and $\{(\lambda_2,\mu_2),(\lambda_3,\mu_3)\} \in P$, then $\{(\lambda_1,\mu_1),(\lambda_3,\mu_3)\} \in P$.

If $\{(\lambda_1,\mu_1),(\lambda_2,\mu_2)\} \in P$, we denote $(\lambda_1,\mu_1) \leq (\lambda_2,\mu_2)$. For convenience, we also use $\lambda_1 \leq \lambda_2, \mu_1 \leq \mu_2$ to mean $(\lambda_1,\mu_1) \leq (\lambda_2,\mu_2)$.

The following lemmas are fixed point index theorems. We refer to Guo and Lakshmikantham [12] for proofs and further discussion of fixed point index.

Lemma 2.3. *Let X be a Banach space, and K be a cone in X. Ω is bounded open in X. Let $0 \in \Omega$ and $T : K \cap \overline{\Omega} \to K$ be condensing. Suppose that $Tx \neq \nu x$ for all $x \in k \cap \partial\Omega$ and $\nu \geq 1$. Then $i(T, K \cap \Omega, K) = 1$.*

Lemma 2.4. *Let X be a Banach space and K be a cone in X. For $r > 0$, define $K_r = \{x \in K : \|x\| < r\}$. Suppose that $T : \overline{K_r} \to K$ is a compact operator such that $Tx \neq x$ for $x \in \partial K_r$. If $\|x\| \leq \|Tx\|$ for $x \in \partial K_r$, then $i(T, K_r, K) = 0$.*

Consider the auxiliary boundary value problem

$$\begin{gathered}
(\phi(u'(t)))' + F_1(t,u(t),v(t)) = 0, \quad 0 < t < 1, \\
(\phi(v'(t)))' + F_2(t,u(t),v(t)) = 0, \quad 0 < t < 1, \\
u'(0) = v'(0) = a \geq 0, \\
u(1) = v(1) = b,
\end{gathered} \tag{2.1}$$

where $F_i : [0,1] \times \mathbb{R}^2 \to R$ are continuous functions for $i = 1, 2$.

Definition 2.5. We say a function $F(t,u,v)$ is quasi-monotone nondecreasing with respect to (u,v), if for every fixed $t\in[0,1]$,
$$F(t,u_1,v_1)\le F(t,u_2,v_2),\quad \text{whenever } u_1\le u_2,\ v_1\le v_2.$$
We say $F(t,u,v)$ is quasi-monotone nondecreasing with respect to v(or u) if
$$\begin{gathered} F(t,u,v_1)\le F(t,u,v_2)\quad \text{whenever } v_1\le v_2\,. \\ (\text{or } F(t,u_1,v)\le F(t,u_2,v)\quad \text{whenever } u_1\le u_2\,.) \end{gathered}$$

Definition 2.6. The functions $\alpha_1,\alpha_2\in C([0,1],\mathbb{R})$ are lower solutions of (2.1) if for all $t\in(0,1)$, it holds
$$\begin{gathered} (\phi(\alpha_1'(t)))'+F_1(t,\alpha_1(t),\alpha_2(t))\ge 0,\quad 0<t<1,\\ (\phi(\alpha_2'(t)))'+F_2(t,\alpha_1(t),\alpha_2(t))\ge 0,\quad 0<t<1,\\ \alpha_1'(0)\le a,\quad \alpha_1(1)\le b,\\ \alpha_2'(0)\le a,\quad \alpha_2(1)\le b. \end{gathered}$$
Similarly, we can define the upper solution $(\beta_1,\beta_2)\in C([0,1],\mathbb{R})\times C([0,1],\mathbb{R})$ of (2.1) if it satisfies the reverse inequality.

Consider the Banach space $X=C([0,1],\mathbb{R})\times C([0,1],\mathbb{R})$. For $u\in X$, the norm is defined by $\|u\|=\max_{t\in[0,1]}|u(t)|$, and
$$\|(u,v)\|=\|u\|+\|v\|.$$
Here, we recall that the norm in $\mathbb{R}^2$ can be defined by $|(u,v)|=|u|+|v|$. Let $\boldsymbol{\alpha}(t)=(\alpha_1(t),\alpha_2(t))$, $\boldsymbol{\beta}(t)=(\beta_1(t),\beta_2(t))$ and
$$\mathcal{D}_\alpha^\beta=\{(t,u,v)|t\in[0,1],\ \alpha_1(t)\le u(t)\le\beta_1(t),\ \alpha_2(t)\le v(t)\le\beta_2(t)\}.$$
For the next lemma we use the assumptions:

- (H3) $(\alpha_1(t),\alpha_2(t))\le(\beta_1(t),\beta_2(t))$ for $t\in[0,1]$;
- (H4) $|F_i(t,u,v)|\le w_i(t)$ for all $(t,u,v)\in\mathcal{D}_\alpha^\beta$, where $w_i\in C([0,1],\mathbb{R}_+)$ satisfy
$$\int_0^1\phi^{-1}\Big(\int_0^s w_i(r)dr\Big)ds<\infty,\quad i=1,2;$$
- (H5) $F_1(t,u,v)$ is quasi-monotone nondecreasing on v and $F_2(t,u,v)$ is quasi-monotone nondecreasing on u.

Lemma 2.7. *Let $(\alpha_1,\alpha_2),(\beta_1,\beta_2)$ be a pair of lower and upper solution of* (2.1). *Assume that* (H3)–(H5) *hold. Then problem* (2.1) *has at least one solution (u,v) in X satisfying*
$$(\alpha_1,\alpha_2)\le(u,v)\le(\beta_1,\beta_2).$$

To prove the above lemma, we need to prove the following result first.

Lemma 2.8. *If there exist $w_1,w_2\in C((0,1),\mathbb{R}_+)$ such that* (H4) *holds, then* (2.1) *is solvable.*

Proof. We define the operators q_1 and q_2 by
$$\begin{gathered} q_1(u,v)=b-\int_t^1\phi^{-1}\Big(\phi(a)-\int_0^s F_1\big(r,u(r),v(r)\big)dr\Big)ds,\\ q_2(u,v)=b-\int_t^1\phi^{-1}\Big(\phi(a)-\int_0^s F_2\big(r,u(r),v(r)\big)dr\Big)ds \end{gathered}$$

and let

$$T(u,v)(t) = \big(q_1(u,v)(t), q_2(u,v)(t)\big).$$

Clearly, T is well defined on X and the solution of (2.1) corresponds to the fixed point of T. From (H4), we can see that T is continuous. Furthermore, for any bounded set $B \subset X$, TB is uniformly bounded and if $(u,v) \in B$, $t_1, t_2 \in [0,1]$, it has

$$\begin{aligned}
&|q_i(u,v)(t_1) - q_i(u,v)(t_2)| \\
&\le \Big| \int_{t_1}^{t_2} \phi^{-1}\Big(\int_0^s \phi^{-1}\big(\phi(a) - F_i(\tau, u(\tau), v(\tau))\big) d\tau\Big) ds \Big| \\
&\le \Big| \int_{t_1}^{t_2} \phi^{-1}\Big(\int_0^s \phi^{-1}\big(\phi(a) + \eta_k w_1(\tau)\big) d\tau\Big) ds \Big| \\
&\to 0, \quad \text{uniformly as } t_1 \to t_2.
\end{aligned}$$

Which implies that TB is equi-continuous. By Arezla-Ascoli Theorem, T is completely continuous operator. Schauder fixed point theorem guarantees that T has a fixed point (u,v) in X which is the solution of (2.1). $\square$

Let

$$X_i(t) = b - \int_t^1 \phi^{-1}\Big(\phi(a) - \int_0^s w_i(r)dr\Big) ds, \quad i = 1, 2.$$

Then $\mathbf{X}(t) = (X_1(t), X_2(t))$ is the solution of the problem

$$\begin{gathered}
(\Phi(\mathbf{u}'))' + \mathbf{w}(t) = 0, \quad 0 < t < 1, \\
\mathbf{u}'(0) = a, \quad \mathbf{u}(1) = b,
\end{gathered}$$

where $\mathbf{w} = (w_1, w_2), w_1, w_2$ defined in Lemma 2.7. Similarly, we define

$$x_i(t) = b - \int_t^1 \phi^{-1}\Big(\phi(a) + \int_0^s w_i(r)dr\Big) ds, \quad i = 1, 2\,.$$

Then $\mathbf{x}(t) = (x_1(t), x_2(t))$ is the solution of the problem

$$\begin{gathered}
(\Phi(\mathbf{u}'))' - \mathbf{w}(t) = 0, \quad t \in (0,1), \\
\mathbf{u}'(0) = a, \quad \mathbf{u}(1) = b.
\end{gathered}$$

It is easy to prove that the solution of (2.1) satisfies $\mathbf{x} \le (u,v) \le \mathbf{X}$.

The proof of Lemma 2.7. We consider the auxiliary boundary value problem

$$\begin{gathered}
(\Phi(\mathbf{u}'))' + F^*(t, \mathbf{u}) = 0, \quad 0 < t < 1, \\
\mathbf{u}'(0) = a, \quad \mathbf{u}(1) = b,
\end{gathered} \tag{2.2}$$

where $\mathbf{u} = (u_1, u_2)$, $\Phi(\mathbf{u}) = (\phi(u_1), \phi(u_2))$, $F^* = (F_1^*, F_2^*)$ and $F_i^* : (0,1) \times \mathbb{R}^2 \to R$ is defined as

$$F_i^*(t, u_1, u_2) = \begin{cases} F_i(t, y_1(t), y_2(t)) - w_i(t)\frac{u_i - \beta_i(t)}{1 + u_i - \beta_i(t)}, & u_i > \beta_i(t), \\ F_i(t, y_1(t), y_2(t)), & \alpha_i(t) \le u_i(t) \le \beta_i(t), \\ F_i(t, y_1(t), y_2(t)) + w_i(t)\frac{\alpha_i(t) - u_i}{1 + \alpha_i(t) - u_i}, & u_i < \alpha_i(t), \end{cases}$$

where

$$y_i(t) = \begin{cases} \beta_i(t), & u_i > \beta_i(t), \\ u_i(t), & \alpha_i(t) \le u_i(t) \le \beta_i(t), \\ \alpha_i(t), & u_i < \alpha_i(t), \end{cases}$$

for $i = 1, 2$. Obviously, F_i^* is continuous and $|F_i^*(t, u_1, u_2)| \le w_i(t)$, $i = 1, 2$. From Lemma 2.8, we can see that (2.2) is solvable, its solution is denoted by $\mathbf{u} = (u_1, u_2)$. Let $\boldsymbol{\alpha}(t) = (\alpha_1(t), \alpha_2(t))$, $\boldsymbol{\beta}(t) = (\beta_1(t), \beta_2(t))$ and

$$\boldsymbol{\alpha}'(0) \le \mathbf{u}'(0) \le \boldsymbol{\beta}'(0), \quad \boldsymbol{\alpha}(1) \le \mathbf{u}(1) \le \boldsymbol{\beta}(1).$$

We will show that $\boldsymbol{\alpha}(t) \le \mathbf{u}(t) \le \boldsymbol{\beta}(t), t \in [0, 1)$. On the contrary, suppose the right inequality does not hold. Then there exists $j = 1$ or 2 such that $u_j(t) \not\le \beta_j(t)$. Suppose there is $t_0 \in [0, 1)$ such that

$$u_j(t_0) - \beta_j(t_0) = \max_{t \in [0,1]} (u_j - \beta_j)(t) > 0, \quad u_j'(t_0) - \beta_j'(t_0) = 0,$$

and for $t \in (t_0, t_0 + \delta]$,

$$u_j'(t) - \beta_j'(t) < 0. \tag{2.3}$$

Note that $y_j(t_0) = \beta_j(t_0)$ and $y_i(t_0) \le \beta_i(t_0)$ for $i \ne j$. From (H5) it follows that

$$\begin{aligned} F_j^*(t_0, u_1(t_0), u_2(t_0)) &= F_j(t_0, y_1(t_0), y_2(t_0)) - w_j(t_0) \frac{u_j(t_0) - \beta_j(t_0)}{1 + u_j(t_0) - \beta_j(t_0)} \\ &< F_j(t_0, y_1(t_0), y_2(t_0)) \le F_j(t_0, \beta_1(t_0), \beta_2(t_0)). \end{aligned}$$

Because F_j^* and F_j are continuous, we can choose $\delta_0 \in (0, \delta)$ such that $F_j^*(t, \mathbf{u}(t)) < F_j(t, \boldsymbol{\beta}(t))$ holds for $t \in (t_0, t_0 + \delta_0)$. Notice that β is the upper solution of (2.1), when $t \in (t_0, t_0 + \delta_0)$, it holds

$$\begin{aligned} \phi(u_j'(t)) - \phi(u_j'(t_0)) &= -\int_{t_0}^t F_j^*(s, \mathbf{u}(s)) ds \\ &> -\int_{t_0}^t F_j(s, \boldsymbol{\beta}(s)) ds \\ &> \phi(\beta_j'(t)) - \phi(\beta_j'(t_0)). \end{aligned}$$

Furthermore, because $u_j'(t_0) = \beta_j'(t_0)$ and ϕ is increasing, it has $u_j'(t) > \beta_j'(t)$, $t \in (t_0, t_0 + \delta_0) \subset (t_0, t_0 + \delta)$, which is an contradiction with (2.3). So $\mathbf{u}(t) \le \boldsymbol{\beta}(t)$ holds for $t \in [0, 1]$. The left inequality can be similarly proved. $\square$

3. Existence of positive solutions

In this section, we prove Theorem 1.1 based on the upper and lower solution technique. We first prove some lemmas. We define three operators on X by

$$\begin{gathered} T_\lambda(u, v)(t) = \int_t^1 \phi^{-1}\Big(\int_0^s \lambda h_1(\tau) f\big(u(\tau), v(\tau)\big) d\tau\Big) ds, \\ T_\mu(u, v)(t) = \int_t^1 \phi^{-1}\Big(\int_0^s \mu h_2(\tau) g\big(u(\tau), v(\tau)\big) d\tau\Big) ds, \\ T(u, v)(t) = \big(T_\lambda(u, v)(t), T_\mu(u, v)(t)\big) \end{gathered}$$

for $t \in [0, 1]$. Clearly, $T : X \to X$ is continuous and the fixed points of T corresponds to the solution of (1.1). Let $\mathcal{P}$ and $\mathcal{K}$ be defined by

$$\begin{gathered} \mathcal{P} = \{(u, v) \in X : u, v \ge 0\}, \\ \mathcal{K} = \{(u, v) \in \mathcal{P} : u'(0) = v'(0) = 0 = u(1) = v(1), v \text{ are convex on } (0, 1)\}. \end{gathered}$$

Then $\mathcal{P}$ and $\mathcal{K}$ are both cones on X. Obviously, $T(\mathcal{P}) \subset \mathcal{K}$.

Lemma 3.1. *If $u \in C^1([0,1],\mathbb{R}), \phi(u') \in C^1((0,1),\mathbb{R})$ and $(\phi(u'))' \le 0$, then*
$$u(t) \ge \min\{t, 1-t\} \max_{t\in[0,1]} u(t), \quad t \in [0,1].$$

Proof. Because $(\phi(u'))' \le 0$ and ϕ is increasing, u' is increasing. For $0 \le t_0 < t < t_1 \le 1$, we have
$$u(t) - u(t_0) = \int_{t_0}^t u'(s)ds \ge (t-t_0)u'(t),$$
that is,
$$u'(t) \le \frac{u(t)-u(t_0)}{t-t_0}.$$
Similarly, we have
$$u'(t) \ge \frac{u(t_1)-u(t)}{t_1-t}.$$
Therefore,
$$\frac{u(t_1)-u(t)}{t_1-t} \le \frac{u(t)-u(t_0)}{t-t_0}. \tag{3.1}$$
Which is equivalent to
$$u(t) \ge \frac{(t-t_0)u(t_1) + (t_1-t)u(t_0)}{t-t_0} \ge (t-t_0)u(t_1) + (t_1-t)u(t_0).$$
Let $\sigma = \{t^* | u(t^*) = \max_{t\in[0,1]} u(t)\}$, $t_0 = 0, t_1 = \sigma$ (or $t_0 = \sigma, t_1 = 1$). Then
$$u(t) \ge t \max_{t\in[0,1]} u(t), \quad t \in [0,\sigma],$$
$$\big(\text{or } u(t) \ge (1-t) \max_{t\in[0,1]} u(t), \quad t \in [\sigma,1]\big).$$
So the proof is complete. ☐

Remark 3.2. If ϕ and u satisfy the conditions in Lemma 3.1, then
$$\min_{1/4<t\le 3/4} u(t) \ge \frac{1}{4} \sup_{t\in[0,1]} u(t).$$

Lemma 3.3. *Suppose that* (H2) *holds. Let $\mathbb{R}^*$ be a compact subset of $\mathbb{R}^2_+ \setminus \{(0,0)\}$. Then there exists a constant $b_{\mathbb{R}^*} > 0$ such that for all $(\lambda,\mu) \in \mathbb{R}^*$ and all positive solutions (u,v) of* (1.1) *at (λ,μ), we have*
$$\|(u,v)\| < b_{\mathbb{R}^*}.$$

Proof. Suppose by contradiction that there is a sequence (u_n, v_n), being positive solutions of (1.1) at $(\lambda_n,\mu_n) \in \mathbb{R}^*(n = 1,2,\cdots)$, such that $\|(u_n,v_n)\| \to \infty$ as $n \to \infty$. From Remark 3.2,
$$\min_{1/4\le t\le 3/4}(u_n(t)+v_n(t)) \ge \frac{1}{4}(\|u_n\| + \|v_n\|). \tag{3.2}$$
Noticing that $(\lambda_n,\mu_n) \in \mathbb{R}^2_+ \setminus \{(0,0)\}$, without loss of generality, we suppose $\lambda_n > 0$. From condition (H2), we can choose $R_f > 0$ such that
$$f(u_n,v_n) \ge \eta\phi(u_n+v_n), \quad \|u_n\| + \|v_n\| \ge R_f, \tag{3.3}$$
where η satisfies
$$\frac{\phi^{-1}(\lambda_n\eta)}{4} \int_t^{3/4} \phi^{-1}\Big(\int_{1/4}^s h_1(\tau)d\tau\Big)ds > 1, \quad \text{for } t \in [1/4, 3/4). \tag{3.4}$$

By inequalities (3.2)–(3.4), we have

$$\begin{aligned}
\|u_n\| \geq u_n(t) &= \int_t^1 \phi^{-1}\Big(\int_0^s \lambda_n h_1(\tau) f\big(u_n(\tau), v_n(\tau)\big)d\tau\Big)ds \\
&\geq \int_t^{3/4} \phi^{-1}\Big(\int_{1/4}^s \lambda_n h_1(\tau) f\big(u_n(\tau), v_n(\tau)\big)d\tau\Big)ds \\
&\geq \int_t^{3/4} \phi^{-1}\Big(\int_{1/4}^s \lambda_n h_1(\tau)\eta\phi\big(u_n(\tau) + v_n(\tau)\big)d\tau\Big)ds \\
&\geq \int_t^{3/4} \phi^{-1}\Big(\int_{1/4}^s h_1(\tau)d\tau\Big)\phi^{-1}\Big(\lambda_n\eta\cdot\phi\big(\frac{1}{4}(\|u_n\| + \|v_n\|)\big)\Big)ds \\
&= \|u_n\| + \|v_n\|)\frac{\phi^{-1}(\underline{\lambda}\cdot\eta)}{4}\int_t^{3/4} \phi^{-1}\Big(\int_{1/4}^s h_1(\tau)d\tau\Big)ds \\
&> \|u_n\| + \|v_n\| > \|u_n\|,
\end{aligned}$$

for n sufficiently large and $\|u_n\| + \|v_n\| \geq R_f$, which is a contradiction. So u is bounded. For the case $\lambda_n = 0$, $\mu_n > 0$, it can be proved similarly by using $g_\infty = \infty$. □

Lemma 3.4. *Suppose that* (H1) *holds. If* (1.1) *has a positive solution at* $(\bar{\lambda}, \bar{\mu})$, *then* (1.1) *has a positive solution at* (λ, μ) *for all* $(\lambda, \mu) \leq (\bar{\lambda}, \bar{\mu})$.

Proof. Let $(\bar{u}, \bar{v})$ be a positive solution of (1.1) at $(\bar{\lambda}, \bar{\mu})$ and $(\lambda, \mu) \in \mathbb{R}_0^2 \setminus \{(0,0)\}$ satisfying $(\lambda, \mu) \leq (\bar{\lambda}, \bar{\mu})$. Then

$$\begin{gathered}
(\phi(\bar{u}'(t)))' = -\bar{\lambda}h_1(t)f(\bar{u}(t), \bar{v}(t)) \leq \lambda h_1(t)f(\bar{u}(t), \bar{v}(t)), \\
(\phi(\bar{v}'(t)))' = -\bar{\mu}h_2(t)g(\bar{u}(t), \bar{v}(t)) \leq \mu h_2(t)g(\bar{u}(t), \bar{v}(t)), \\
\bar{u}'(0) = \bar{v}'(0) = 0 = \bar{u}(1) = \bar{v}(1).
\end{gathered}$$

So $(\bar{u}, \bar{v})$ is an upper solution of (1.1) at (λ, μ). Similarly, $(0,0)$ is a lower solution of (1.1) at (λ, μ). Since $(\bar{u}, \bar{v}) \neq (0,0)$, $(\bar{u}, \bar{v}) \geq (0,0)$ and $(0,0)$ is not a solution of (1.1) at (λ, μ), Lemma 2.7 implies that (1.1) has a positive solution at (λ, μ). □

Lemma 3.5. *Suppose that* (H1) *holds. Then there exists* $(\lambda_0, \mu_0) > (0,0)$ *such that* (1.1) *has a positive solution for all* $(\lambda, \mu) \leq (\lambda_0, \mu_0)$.

Proof. Let

$$\beta_i(t) = \int_t^1 \phi^{-1}\Big(\int_0^s h_i(\tau)d\tau\Big)ds$$

for $i = 1, 2$. Then it is the unique solution of

$$\begin{gathered}
(\phi(u'(t)))' + h_i(t) = 0, \quad 0 < t < 1, \\
u'(0) = 0 = u(1).
\end{gathered}$$

Let $M_f = \max_{t\in[0,1]} f(\beta_1(t), \beta_2(t))$ and $M_g = \max_{t\in[0,1]} g(\beta_1(t), \beta_2(t))$. Then $M_f > 0, M_g > 0$, and when $(\lambda_0, \mu_0) = (1/M_f, 1/M_g)$, we have

$$\begin{aligned}
(\phi(\beta_1'(t)))' + \lambda_0 h_1(t)f(\beta_1(t), \beta_2(t)) = h_1(t)[\lambda_0 f(\beta_1(t), \beta_2(t)) - 1] \leq 0, \\
(\phi(\beta_2'(t)))' + \mu_0 h_2(t)g(\beta_1(t), \beta_2(t)) = h_2(t)[\mu_0 g(\beta_1(t), \beta_2(t)) - 1] \leq 0.
\end{aligned}$$

This indicates that (β_1, β_2) is an upper solution of (1.1) at (λ_0, μ_0). Meanwhile, $(0,0)$ is a lower solution of (1.1) at (λ_0, μ_0) and $(0,0) < (\beta_1, \beta_2)$. Lemma 2.7 implies that (1.1) has a positive solution at (λ_0, μ_0). $\square$

We define

$$\psi = \{(\lambda, \mu) \in \mathbb{R}_+^2 \setminus \{(0,0)\} : (1.1) \text{ has a positive solution at}(\lambda, \mu)\}.$$

From Lemma 3.5, we can see that $\psi \neq \emptyset$. $(\psi, \leq)$ is a partial ordered set.

Lemma 3.6. *Suppose that* (H1) *and* (H2) *hold. Then* $(\psi, \leq)$ *is bounded above.*

Proof. On the contrary, if $(\psi, \leq)$ is not bounded above, then there exists (λ_n, μ_n) such that (1.1) has solutions at (λ_n, μ_n) with $\lambda_n \to \infty$ or $\mu_n \to \infty$ as $n \to \infty$. Without loss of generality, we suppose $\lambda_n \to \infty$ as $n \to \infty$). Let (u_n, v_n) are the positive solutions of (H3) at (λ_n, μ_n). From $(H2)$, similarly to the proof of Lemma 3.3, when n is sufficient large and $\lambda_n > \delta$, we can obtain $\|u_n\| > \|u_n\|$, which is a contraction. So $(\psi, \leq)$ has an upper bound. $\square$

Lemma 3.7. *Suppose that* (H1, (H2) *hold. Then every chain in* ψ *has a unique supremum in* ψ.

Proof. Let $\bar{\psi}$ be a chain in ψ. Because ψ is bounded above, $\bar{\psi}$ has an upper bound. Without loss of generality, we choose a distinct sequence $\{(\lambda_n, \mu_n)\} \subset \bar{\psi}$ with $(\lambda_n, \mu_n) \leq (\lambda_{n+1}, \mu_{n+1})$, $n = 1, 2, \dots$. From Lemma 3.6, two sequences $\{\lambda_n\}$ and $\{\mu_n\}$ are convergent to, say, $\lambda_{\bar{\psi}}$ and $\mu_{\bar{\psi}}$, respectively. Next, we show that $\lambda_{\bar{\psi}}, \mu_{\bar{\psi}} \in \psi$. Because $\{(\lambda_n, \mu_n)\}_{n=1}^{\infty}$ has an upper bound, we can suppose that the sequence $\{(\lambda_n, \mu_n)\}_{n=1}^{\infty}$ belongs to a compact rectangle of $\mathbb{R}_+^2 \subset \{(0,0)\}$. Let (u_n, v_n) be the positive solutions of (H3) at (λ_n, μ_n). Lemma 3.3 implies that $\{(u_n, v_n)\}$ are uniformly bounded on X. So the sequence $\{(u_n, v_n)\}$ has a convergent subsequence, denoted by $\{(u_n, v_n)\} \to (u_{\bar{\psi}}, v_{\bar{\psi}}) \in X$. By using the Lebesgue convergence theorem, we can prove that $(u_{\bar{\psi}}, v_{\bar{\psi}})$ is a solution of (1.1) at $(\lambda_{\bar{\psi}}, \mu_{\bar{\psi}})$, that is, $(\lambda_{\bar{\psi}}, \mu_{\bar{\psi}}) \in \psi$. Hence, the proof is complete. $\square$

Lemma 3.8. *Suppose that* (H1) *and* (H2) *hold. Then there exists* $\lambda_*, \mu_* > 0$ *such that* $\{(\lambda, 0) | \lambda \in [0, \lambda_*]\} \cup \{(0, \mu) | \mu \in [0, \mu_*]\} \subset \partial(\operatorname{int}\psi)$ *and* $\overline{\operatorname{int}\psi} \subset [0, \lambda_*] \times [0, \mu_*]$.

Proof. Lemma 3.5 implies that $\operatorname{int}\psi$ is nonempty and there exists $(\lambda_0, \mu_0) \in \operatorname{int}\psi \subset \mathbb{R}_+^2$. Lemma 3.4 shows that $(0, \lambda_0] \times (0, \mu_0] \subset \operatorname{int}\psi$ and $\{(\lambda_0, 0), (0, \mu_0)\} \subset \partial(\operatorname{int}\psi)$. So $\{\lambda > 0 | (\lambda, 0) \in \partial(\operatorname{int}\psi)\}$ and $\{\mu > 0 | (0, \mu) \in \partial(\operatorname{int}\psi)\}$ are nonempty and bounded above. Let

$$\begin{aligned} \lambda_* &= \sup\{\lambda > 0 | (\lambda, 0) \in \partial(\operatorname{int}\psi)\}, \\ \mu_* &= \sup\{\mu > 0 | (0, \mu) \in \partial(\operatorname{int}\psi)\}, \end{aligned} \tag{3.5}$$

then $\{(\lambda_*, 0), (0, \mu_*)\} \subset \partial(\operatorname{int}\psi)$, and

$$\{(\lambda, 0) | \lambda \in [0, \lambda_*]\} \cup \{(0, \mu) | \mu \in [0, \mu_*]\} \subset \partial(\operatorname{int}\psi). \tag{3.6}$$

In fact, for all $\lambda_0 \in [0, \lambda_*)$, from Lemma 3.5, there exists

$$(\bar{\lambda}, \bar{\mu}) \in \operatorname{int}\psi \cap B((\lambda_*, 0), \lambda_* - \lambda_0)$$

such that $(0, \bar{\lambda}) \times (0, \bar{\mu}) \subset \operatorname{int}\psi$, where $B((\lambda_*, 0), \lambda_* - \lambda_0)$ is a circular area with $(\lambda_*, 0)$ as its center and $\lambda_* - \lambda_0$ as its radius. In view of $\lambda_0 < \bar{\lambda}$, so $(\lambda_0, 0) \in \partial(\operatorname{int}\psi)$. From the selection of λ_0, it holds $\{(\lambda, 0) | \lambda \in [0, \lambda_*)\} \subset \partial(\operatorname{int}\psi)$. Similarly, $\{(0, \mu) | \mu \in [0, \mu_*)\} \subset \partial(\operatorname{int}\psi)$.

Next, we prove $\overline{\operatorname{int}\psi} \subset [0,\lambda_*]\times[0,\mu_*]$. Suppose, on the contrary that

$$(\bar\lambda,\bar\mu)\in \operatorname{int}\psi \quad\text{and}\quad (\bar\lambda,\bar\mu)\notin[0,\lambda_*]\times[0,\mu_*]\,.$$

Then $\bar\lambda>\lambda_*$ or $\bar\mu>\mu_*$. Without loss of generality, we suppose $\bar\lambda>\lambda_*$. From Lemma 3.4, $[0,\bar\lambda]\times[0,\bar\mu]\subset\psi$. So $(\bar\lambda,0)\in\partial(\operatorname{int}\psi)$. While, (3.5) implies $\bar\lambda\le\lambda_*$, which is a contradiction. □

We define the line cluster

$$L(t)=\{(\lambda,\mu)\in\mathbb{R}^2_+\setminus\{(0,0)\}:\mu=\lambda-t\},$$

where $t\in[-\mu_*,\lambda_*]$. We define

$$\lambda^*(t)=\sup\{\lambda\mid(\lambda,\mu)\in L(t)\cap\overline{\operatorname{int}\psi}\},\quad \mu^*(t)=\lambda^*(t)-t,$$

and $\Gamma(t)=(\lambda^*(t),\mu^*(t))$.

Lemma 3.9. *Suppose that* (H1) *and* (H2) *hold. For any given* $t\in[-\mu_*,\lambda_*]$, $\Gamma(t)\in L(t)\cap\partial(\operatorname{int}\psi)$.

Proof. For any $t\in[-\mu_*,\lambda_*]$ and $\lambda^*(t)$, there exists $\{(\lambda_n,\mu_n)\}_{n=1}^\infty\subset L(t)\cap\overline{\operatorname{int}\psi}$ with λ_n converging to $\lambda^*(t)$. Notice $\mu^*(t)=\lambda^*(t)-t$ and $(\lambda^*(t),\mu^*(t))\in L(t)$. So $(\lambda^*(t),\mu^*(t))=\lim_{n\to\infty}(\lambda_n,\mu_n)$ and $(\lambda^*(t),\mu^*(t))\in L(t)\cap\overline{\operatorname{int}\psi}$. □

Lemma 3.10. *Suppose that* (H1) *and* (H2) *hold. Then*

(I) $\{\Gamma(t)\mid t\in[-\mu_*,\lambda_*]\}$ *is continuous;*
(II) $\{\Gamma(t)\mid t\in[-\mu_*,\lambda_*]\}\cap\{(\lambda,\mu)|\lambda\mu=0\}=\{(\lambda_*,0),(0,\mu_*)\}$.

Proof. (I) Obviously, $\lambda^*(t)$ is increasing and $\mu^*(t)$ is decreasing. Let $\lambda^*(t_n)=\lambda^*_n,\ \mu^*(t_n)=\mu^*_n$ for any $t_n\in[-\mu_*,\lambda_*]$. For any given $t_1,t_2\in[-\mu_*,\lambda_*],t_1<t_2$, we have

$$\begin{aligned}
|\Gamma(t_1)-\Gamma(t_2)| &= |(\lambda_1^*,\mu_1^*)-(\lambda_2^*,\mu_2^*)| = |(\lambda_1^*-\lambda_2^*,\mu_1^*-\mu_2^*)|\\
&= |\lambda_1^*-\lambda_2^*|+|\mu_1^*-\mu_2^*| = (\lambda_2^*-\lambda_1^*)+(\mu_1^*-\mu_2^*)\\
&= (\lambda_2^*-\mu_2^*)-(\lambda_2^*-\mu_2^*)\\
&= t_2-t_1=|t_2-t_1|,
\end{aligned}$$

where $|\cdot|$ denotes the norm of $\mathbb{R}^2$. Therefore, Γ is equi-continuous.

(II) It suffices to prove that $\{t\mid t\in[-\mu_*,\lambda_*],\lambda^*(t)=0\}=\{-\mu_*\}$ and $\{t\mid t\in[-\mu_*,\lambda_*],\mu^*(t)=0\}=\{\lambda_*\}$. Let $t\in(-\mu_*,0]$. By Lemma 3.8, we have $(0,\mu_*)\in\partial(\operatorname{int}\psi)$. And there exists $(\lambda_0,\mu_0)\in\operatorname{int}\psi\cap B((0,\mu_*),\epsilon/3)\subset\mathbb{R}_+\times\mathbb{R}_+$, where $\epsilon=t+\mu_*$. It is easy to see that $\lambda_0-t<\mu_0$. Furthermore, by Lemma 3.5,

$$(\lambda_0,\lambda_0-t)\in L(t)\cap\overline{\operatorname{int}\psi},\quad t\in(-\mu_*,0].$$

Therefore, $\lambda^*(t)\ge\lambda_0>0$. Meanwhile, because $\lambda^*(t)$ is increasing on $[-\mu_*,\lambda_*]$, it holds $\lambda^*(t)\ge\lambda^*(0)>0$ for all $t\in[0,\lambda_*]$. So $\{t\in[-\mu_*,\lambda_*],\lambda^*(t)=0\}=\{-\mu_*\}$. Similarly, we can prove that when $\{t\in[-\mu_*,\lambda_*]$, it holds $\mu_*(t)=0\}=\{\lambda_*\}$. □

Proof of Theorem 1.1. From Lemmas 3.4–3.10, we can see that Theorem 1.1 holds. The curve Γ divides the set $\mathbb{R}^2_+\setminus\{(0,0)\}$ into two disjoint subsets θ_1 and θ_2, where θ_1 is bounded and θ_2 is unbounded. Lemma 3.5 shows that (1.1) has a positive solution at $\Gamma(t),t\in[-\mu_*,\lambda_*]$. If $(\lambda,\mu)\in\theta_1$, then $(\lambda,\mu)\in L(t),t\in[-\mu_*,\lambda_*]$, and $(\lambda,\mu)<\Gamma(t)$, Lemma 3.8 shows that (1.1) has a solution at (λ,μ). If $(\lambda,\mu)\in\theta_2$, then either $t\notin[-\mu_*,\lambda_*]$ or $(\lambda,\mu)>\Gamma(t),t\in[-\mu_*,\lambda_*]$. In both cases, (1.1) has no solution at (λ,μ). □

4. Multiplicity of positive solutions

In this section, we prove that (1.1) has another positive solution when $(\lambda,\mu)\in\theta_1$. Choose $t_0\in[-\mu_*,\lambda_*]$ with $(\lambda,\mu)\in L(t_0)$. From Lemma 3.5, it is known that (1.1) has a positive solution at $\Gamma(t_0)$, denoted by (u_0^*,v_0^*), and set $\Gamma(t_0)=(\lambda_0^*,\mu_0^*)$. Obviously, $(\lambda,\mu)<(\lambda_0^*,\mu_0^*)$. For the convenience, set $u_\epsilon^*(t)=u_0^*(t)+\epsilon$, $v_\epsilon^*(t)=v_0^*(t)+\epsilon$ for $\epsilon>0$.

Lemma 4.1. *If $(\lambda,\mu)\in\theta_1$, then there exists $\epsilon_0>0$ such that for any $\epsilon\in(0,\epsilon_0]$, $(u_\epsilon^*,v_\epsilon^*)$ is an upper solution of (1.1) at (λ,μ).*

Proof. Choose a constant $H>0$ such that

$$f(u_0^*(t),v_0^*(t))\ge H,\quad g(u_0^*(t),v_0^*(t))\ge H,\quad 0<t<1.$$

Because f and g are continuous, there exists $\epsilon_0>0$ such that

$$\begin{gathered}\big|f\big(u_\epsilon^*(t),v_\epsilon^*(t)\big)-f\big(u_0^*(t),v_0^*(t)\big)\big|<\frac{H(\lambda_0^*-\lambda)}{\lambda},\\ \big|g\big(u_\epsilon^*(t),v_\epsilon^*(t)\big)-g\big(u_0^*(t),v_0^*(t)\big)\big|<\frac{H(\mu_0^*-\mu)}{\mu}\end{gathered}$$

hold for all $t\in[0,1]$ and $0<\epsilon\le\epsilon_0$. Obviously, $u_\epsilon'^*(0)=v_\epsilon'^*(0)=0$ and $u_\epsilon^*(1)=v_\epsilon^*(1)=\epsilon>0$. For $0<\lambda<\lambda_0^*$, we have

$$\begin{aligned}&\big(\phi\big(u_\epsilon^{*\prime}(t)\big)\big)'+\lambda h_1(t)f(u_\epsilon^*(t),v_\epsilon^*(t))\\ &=-\lambda_0^*h_1(t)f(u_0^*(t),v_0^*(t))+\lambda h_1(t)f(u_\epsilon^*(t),v_\epsilon^*(t))\\ &=\lambda h_1(t)[f(u_\epsilon^*(t),v_\epsilon^*(t))-f(u_0^*(t),v_0^*(t))]-(\lambda_0^*-\lambda)h_1(t)f(u_0^*(t),v_0^*(t))\\ &<H(\lambda_0^*-\lambda)h_1(t)-(\lambda_0^*-\lambda)h_1(t)f(u_0^*(t),v_0^*(t))\\ &=(\lambda_0^*-\lambda)h_1(t)(H-f(u_0^*(t),v_0^*(t))\\ &\le 0,\quad 0<t<1.\end{aligned}$$

Similarly,

$$\big(\phi\big(v_\epsilon^{*\prime}(t)\big)\big)'+\mu h_2(t)g(u_\epsilon^*(t),v_\epsilon^*(t))\le 0,\quad 0<t<1.$$

So $(u_\epsilon^*,v_\epsilon^*)$ is an upper solution of (1.1) at (λ,μ). □

In addition, for $0<\epsilon\le\epsilon_0$ and $(u_\epsilon^*,v_\epsilon^*)$, we have

$$u_\epsilon^*(t)>\int_t^1\phi^{-1}\Big(\int_0^s\lambda h_1(\tau)f(u_\epsilon^*(\tau),v_\epsilon^*(\tau))d\tau\Big)ds,\tag{4.1}$$

$$v_\epsilon^*(t)>\int_t^1\phi^{-1}\Big(\int_0^s\mu h_2(\tau)g(u_\epsilon^*(\tau),v_\epsilon^*(\tau))d\tau\Big)ds.\tag{4.2}$$

Theorem 4.2. *Suppose that* (H1) *and* (H2) *hold. Then there exists a continuous curve Γ splitting $\mathbb{R}_0^2\setminus\{(0,0)\}$ into two disjoint subsets θ_1 and θ_2 such that the eigenvalue problem (1.1) has at least two positive solutions on θ_1, at least one solution on Γ, and no solution on θ_2.*

Proof. From Theorem 1.1, suffices to prove that (1.1) has a second positive solution for $(\lambda,\mu)\in\theta_1$. Let $(\lambda,\mu)\in\theta_1$ and $(\lambda,\mu)\in L(t_0)$. Let

$$\Omega=\{(u,v)\in X:-\epsilon<u(t)<u_\epsilon^*(t),-\epsilon<v(t)<v_\epsilon^*(t),t\in[0,1]\}.$$

Clearly, $0\in\Omega$, $\Omega\subset X$ and Ω is bounded open. $T:\mathcal{K}\cap\bar\Omega\to\mathcal{K}$ is completely continuous. Suppose $(u,v)\in\mathcal{K}\cap\partial\Omega$. There exists $t_0\in[0,1]$ such that either

$u(t_0) = u^*_\epsilon(t_0)$ or $v(t_0) = v^*_\epsilon(t_0)$. Without loss of generality, we suppose $u(t_0) = u^*_\epsilon(t_0)$. Then by (H1) and (4.1), we arrive at

$$\begin{aligned} T_\lambda(u,v)(t_0) &= \int_{t_0}^1 \phi^{-1}\Big(\int_0^s \lambda h_1(\tau) f(u(\tau), v(\tau))d\tau\Big)ds \\ &\le \int_{t_0}^1 \phi^{-1}\Big(\int_0^s \lambda h_1(\tau) f(u^*_\epsilon(\tau), v^*_\epsilon(\tau))d\tau\Big)ds \\ &< u^*_\epsilon(t_0) = u(t_0) \le \nu u(t_0) \end{aligned}$$

for all $\nu \ge 1$. Thus for all $(u,v) \in \mathcal{K} \cap \partial\Omega$ and $\nu \ge 1$, $T(u,v) \ne \nu(u,v)$. If $v(t_0) = v^*_\epsilon(t_0)$, by (H1) and (4.2), the prior inequality also holds. From Lemma 2.3,

$$i(T, \mathcal{K} \cap \Omega, \mathcal{K}) = 1.$$

From (H2), there exists $R_f > 0$ such that

$$f(u,v) \ge \eta\phi(u+v), \quad \|u\| + \|v\| \ge R_f, \tag{4.3}$$

with η satisfies (3.4) with λ_n replacing with λ. Let $R = \max\{b_{\mathbb{R}^*}, 4R_f, \|(u^*_\epsilon, v^*_\epsilon)\|\}$, where $b_{\mathbb{R}^*}$ is given in Lemma 3.3 with $\mathbb{R}^*$ a rectangle in $\mathbb{R}^2_+ \setminus \{(0,0)\}$ containing (λ,μ). Let $\mathcal{K}_R = \{(u,v) \in \mathcal{K} : \|(u,v)\| < R\}$. Then $\|T(u,v)\| \ge \|T_\lambda(u,v)\| > \|(u,v)\|$. From Lemma 2.4, we have

$$i(T, \mathcal{K}_R, \mathcal{K}) = 0.$$

By the additivity of the fixed point index, we have

$$i(T, \mathcal{K}_R, \mathcal{K}) = i(T, \mathcal{K} \cap \Omega, \mathcal{K}) + i(T, \mathcal{K}_R \setminus \overline{\mathcal{K} \cap \Omega}, \mathcal{K})$$

So $i(T, \mathcal{K}_R \setminus \overline{\mathcal{K} \cap \Omega}, \mathcal{K}) = -1$, which implies that T has a fixed point in $\mathcal{K} \cap \Omega$ and another one in $\mathcal{K}_R \setminus \overline{\mathcal{K} \cap \Omega}$. These two fixed points of T are two positive solutions of (1.1). $\square$

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