

## DECAY RATES FOR TWO CAUCHY THERMOELASTIC LAMINATED TIMOSHENKO PROBLEMS OF TYPE III WITH INTERFACIAL SLIP

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ABSTRACT. In this article we study the decay of solutions for two systems of laminated Timoshenko beams with interfacial slip, in the whole space  $\mathbb{R}$  subject to a thermal effect of type III acting only on one component. When the thermal effect acts via the second or third component of the laminated Timoshenko beam (rotation angle displacement or dynamic of the slip), we prove that both systems are polynomially stable. Also we obtain stability estimates in the  $L^2(\mathbb{R})$ -norm of solutions and their higher order derivatives with respect of the space variable. The decay rates, and the absence or presence of the regularity-loss type property, depend on the regularity of the initial data and the speeds of wave propagations. However, when the thermal effect acts via the first component (transversal displacement), we introduce a new stability number  $\chi$  and prove that the stability of the system is equivalent to  $\chi \neq 0$ . An application to a case of lower order coupling terms is also given. To prove our results, we use the energy method in the Fourier space combined with well chosen weight functions to build appropriate Lyapunov functionals.

### 1. INTRODUCTION

A typical model of laminated Timoshenko beams of length  $L$  and with interfacial slip based on the Timoshenko theory can be formulated by the system (see [16, 17, 23] for more details)

$$\begin{aligned}\rho_1 \varphi_{tt} + k(u - \varphi_x)_x + F_1 &= 0, \\ \rho_2 (3v - u)_{tt} - b(3v - u)_{xx} - k(u - \varphi_x) + F_2 &= 0, \\ \tilde{\rho}_3 v_{tt} - \tilde{k}_0 v_{xx} + 3k(u - \varphi_x) + 4\tilde{\beta} v_t + \tilde{F}_3 &= 0,\end{aligned}\tag{1.1}$$

where the subscripts  $x$  and  $t$  denote the derivative with respect to space and time variables  $x$  and  $t$ , respectively,  $x \in ]0, L[$  and  $t > 0$ , combining some initial data and boundary conditions at  $x = 0$  and  $x = L$ . All the coefficients are positive constants and denote some physical properties of beams. The terms  $F_1 = F_1(x, t)$ ,  $F_2 = F_2(x, t)$  and  $\tilde{F}_3 = \tilde{F}_3(x, t)$  are external forces and play the role of controls. The functions  $\varphi = \varphi(x, t)$  and  $u = u(x, t)$  represent, respectively, the transverse and rotation angle displacements, and the function  $v = v(x, t)$  is proportional to

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the amount of slip along the interface, so the third equation in (1.1) describes the dynamics of the slip.

Using the change of variables

$$\begin{aligned} \rho_3 &= \frac{1}{9}\tilde{\rho}_3, & k_1 &= k, & k_2 &= b, & k_3 &= \frac{1}{9}\tilde{k}_0, & \beta &= \frac{4}{9}\tilde{\beta}, \\ w &= -3v, & \psi &= 3v - u, & F_3 &= \frac{1}{9}\tilde{F}_3, \end{aligned}$$

the system (1.1) can be rewritten as

$$\begin{aligned} \rho_1\varphi_{tt} - k_1(\varphi_x + \psi + w)_x + F_1 &= 0, \\ \rho_2\psi_{tt} - k_2\psi_{xx} + k_1(\varphi_x + \psi + w) + F_2 &= 0, \\ \rho_3w_{tt} - k_3w_{xx} + k_1(\varphi_x + \psi + w) + \beta w_t + F_3 &= 0. \end{aligned} \quad (1.2)$$

This system is mathematically a particular case of the following more general one of Bresse-type

$$\begin{aligned} \rho_1\varphi_{tt} - k_1(\varphi_x + \psi + lw)_x - \tilde{l}k_3(w_x - \tilde{l}\varphi) + F_1 &= 0, \\ \rho_2\psi_{tt} - k_2\psi_{xx} + k_1(\varphi_x + \psi + lw) + F_2 &= 0, \\ \rho_3w_{tt} - k_3(w_x - \tilde{l}\varphi)_x + lk_1(\varphi_x + \psi + lw) + \beta w_t + F_3 &= 0, \end{aligned} \quad (1.3)$$

where  $l$  and  $\tilde{l}$  are positive constants. System (1.3) coincides with (1.2) when  $l = 1$  and  $\tilde{l} = 0$ . When  $w = F_3 = l = \tilde{l} = 0$ , system (1.3) is reduced to the Timoshenko-type system

$$\begin{aligned} \rho_1\varphi_{tt} - k_1(\varphi_x + \psi)_x + F_1 &= 0, \\ \rho_2\psi_{tt} - k_2\psi_{xx} + k_1(\varphi_x + \psi) + F_2 &= 0. \end{aligned} \quad (1.4)$$

Systems (1.2), (1.3), and (1.4) were the subject of various studies in the literature during the previous thirty years, tackling well-posedness and stability questions by considering different types of controls  $F_j$  (dampings, memories, heat conduction effects, etc.). Let us mention here some of these studies related to our objectives in this paper.

For the well-posedness and stability questions in the case of bounded domains, we refer the readers to the non exhaustive list of references [1, 2, 3, 4, 5, 6, 7, 10, 12, 13, 14, 15, 21, 22, 23, 24, 25, 26, 27, 28, 34, 36].

We notice here that (1.2) was generally considered in the literature under the following restrictions: (1.2) is already damped via the control  $\beta w_t$  and the speeds of the wave propagations of the last two equations in (1.2) are the same; that is,

$$\beta > 0 \quad \text{and} \quad \frac{k_2}{\rho_2} = \frac{k_3}{\rho_3}. \quad (1.5)$$

For unbounded domains, the stability of (1.3) and (1.4) has been also treated in the literature for the previous few years. In this direction, we mention the papers [8, 11, 19, 20, 29, 31] (see also the references therein), where some polynomial stability estimates for  $L^2(\mathbb{R})$ -norm of solutions were proved using frictional damping, heat conduction effects or memory controls.

In this paper, we investigated the decay properties of two laminated Timoshenko beam with interfacial slip in the whole space  $\mathbb{R}$  and without the restrictions (1.5). In addition, only one external force  $F_j$  is considered and it is generated by a thermal

effect of type III. Without loss of generality, the coefficients  $\rho_j$  in (1.2) are taken equal to 1. The first system we consider is the following

$$\begin{aligned} \varphi_{tt} - k_1(\varphi_x + \psi + w)_x + \tau_1\gamma q_{xt} &= 0, \\ \psi_{tt} - k_2\psi_{xx} + k_1(\varphi_x + \psi + w) + \tau_2\gamma q_{xt} &= 0, \\ w_{tt} - k_3w_{xx} + k_1(\varphi_x + \psi + w) + \tau_3\gamma q_{xt} &= 0, \\ q_{tt} - k_4q_{xx} - k_5q_{xxt} + \gamma(\tau_1\varphi_{xt} + \tau_2\psi_{xt} + \tau_3w_{xt}) &= 0, \end{aligned} \tag{1.6}$$

where  $x \in \mathbb{R}$ ,  $t > 0$ ,  $k_j > 0$ ,  $\gamma \in \mathbb{R}^*$ ,  $q = q(x, t)$  denotes the temperature and

$$(\tau_1, \tau_2, \tau_3) \in \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}. \tag{1.7}$$

The thermal dissipation in (1.6) is generated by the term  $-k_5q_{xxt}$  (see (2.9) in Section 2). In the second system of interest, the thermal dissipation is generated by the term of lower order  $k_5q_t$ ; more precisely, we consider the system

$$\begin{aligned} \varphi_{tt} - k_1(\varphi_x + \psi + w)_x + \tau_1\gamma q_{xt} &= 0, \\ \psi_{tt} - k_2\psi_{xx} + k_1(\varphi_x + \psi + w) + \tau_2\gamma q_{xt} &= 0, \\ w_{tt} - k_3w_{xx} + k_1(\varphi_x + \psi + w) + \tau_3\gamma q_{xt} &= 0, \\ q_{tt} - k_4q_{xx} + k_5q_t + \gamma(\tau_1\varphi_{xt} + \tau_2\psi_{xt} + \tau_3w_{xt}) &= 0. \end{aligned} \tag{1.8}$$

Systems (1.6) and (1.8) are subject to the initial conditions

$$\begin{aligned} (\varphi, \psi, w, q)(x, 0) &= (\varphi_0, \psi_0, w_0, q_0)(x), \\ (\varphi_t, \psi_t, w_t, q_t)(x, 0) &= (\varphi_1, \psi_1, w_1, q_1)(x). \end{aligned} \tag{1.9}$$

The main objective of this article is to study the stability of (1.6) and (1.8) and to obtain some polynomial estimates in the  $L^2(\mathbb{R})$ -norm of solutions and their higher order derivatives with respect to  $x$ . We will show that, when  $(\tau_1, \tau_2, \tau_3) = (1, 0, 0)$ , both (1.6) and (1.8) are stable if and only if  $\chi \neq 0$ , where

$$\chi := k_3 - k_2. \tag{1.10}$$

However, when

$$(\tau_1, \tau_2, \tau_3) \in \{(0, 1, 0), (0, 0, 1)\}, \tag{1.11}$$

we prove that systems (1.6) and (1.8) are always stable, where the decay rate in the case

$$k_1 = k_2 = k_3 \tag{1.12}$$

is better than in the opposite one. Moreover, in the case (1.6), (1.12) allows to avoid the regularity restriction on the initial data known as the regularity-loss property (see [9, 18, 19, 30, 32, 33]). At the end of this article, we give an application to the case where the coupling terms between the laminated Timoshenko system and the equation of heat conduction in (1.6) and (1.8)

$$\tau_j\gamma q_{xt} \quad \text{and} \quad \gamma(\tau_1\varphi_{xt} + \tau_2\psi_{xt} + \tau_3w_{xt}) \tag{1.13}$$

are, respectively, replaced by the following ones of lower order:

$$\tau_j\gamma q_t \quad \text{and} \quad -\gamma(\tau_1\varphi_t + \tau_2\psi_t + \tau_3w_t). \tag{1.14}$$

Our stability results show that the effect of the heat conduction is better propagated to the whole system from the second or third equation of the laminated Timoshenko system than from the first one. The proof is based on the energy method combined with the Fourier analysis (by using the transformation in the Fourier space) and well chosen weight functions.

This article is organized as follows: in Section 2, we formulate (1.6) and (1.8) as a first order Cauchy system and give some preliminaries. In Section 3 we prove some differential identities. In Section 4, we prove our stability results. We end our paper by an application to the case (1.14) in Section 5.

## 2. FORMULATION OF THE PROBLEMS

We start by formulating (1.6) and (1.8) in an abstract first order system. To do so, we introduce the new variables

$$\begin{aligned} u &= \varphi_t, & y &= \psi_t, & \theta &= w_t, & \eta &= q_t, \\ v &= \varphi_x + \psi + w, & z &= \psi_x, & \phi &= w_x & \text{and} & \sigma &= q_x. \end{aligned} \quad (2.1)$$

Then systems (1.6) and (1.8) can be presented in the form

$$\begin{aligned} v_t - u_x - y - \theta &= 0, \\ u_t - k_1 v_x + \tau_1 \gamma \eta_x &= 0, \\ z_t - y_x &= 0, \\ y_t - k_2 z_x + k_1 v + \tau_2 \gamma \eta_x &= 0, \\ \phi_t - \theta_x &= 0, \\ \theta_t - k_3 \phi_x + k_1 v + \tau_3 \gamma \eta_x &= 0, \\ \sigma_t - \eta_x &= 0, \\ \eta_t - k_4 \sigma_x + (1 - k_0) k_5 \partial_x^{k_0} \eta + \gamma(\tau_1 u_x + \tau_2 y_x + \tau_3 \theta_x) &= 0, \end{aligned} \quad (2.2)$$

where  $k_0 = 2$  in case (1.6), and  $k_0 = 0$  in case (1.8). Let  $U$  and its initial data  $U_0$  be given by

$$U = (v, u, z, y, \phi, \theta, \sigma, \eta)^T \quad \text{and} \quad U_0 = (v, u, z, y, \phi, \theta, \sigma, \eta)^T(\cdot, 0).$$

System (2.2) and the initial conditions (1.9) are reduced to

$$\begin{aligned} U_t(x, t) + A_2 U_{xx}(x, t) + A_1 U_x(x, t) + A_0 U(x, t) &= 0, \\ U(x, 0) &= U_0(x), \end{aligned} \quad (2.3)$$

where

$$\begin{aligned} A_2 U_{xx} &= \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -\epsilon_0 k_5 \eta_{xx} \end{pmatrix}, & A_1 U_x &= \begin{pmatrix} -u_x \\ -k_1 v_x + \tau_1 \gamma \eta_x \\ -y_x \\ -k_2 z_x + \tau_2 \gamma \eta_x \\ -\theta_x \\ -k_3 \phi_x + \tau_3 \gamma \eta_x \\ -\eta_x \\ -k_4 \sigma_x + \gamma(\tau_1 u_x + \tau_2 y_x + \tau_3 \theta_x) \end{pmatrix}, \\ A_0 U &= \begin{pmatrix} -y - \theta \\ 0 \\ 0 \\ k_1 v \\ 0 \\ k_1 v \\ 0 \\ (1 - \epsilon_0) k_5 \eta \end{pmatrix}, \end{aligned} \quad (2.4)$$

and

$$\epsilon_0 = \begin{cases} 1 & \text{in case (1.6),} \\ 0 & \text{in case (1.8).} \end{cases} \tag{2.5}$$

For a function  $h : \mathbb{R} \rightarrow \mathbb{C}$ ,  $\text{Re } h$ ,  $\text{Im } h$ ,  $\bar{h}$  and  $\widehat{h}$  denote the real part, the imaginary part, the conjugate, and the Fourier transformation of  $h$ , respectively. Using the Fourier transformation (with respect to the space variable  $x$ ), (2.3) can be written in the Fourier space as the following first order Cauchy system

$$\begin{aligned} \widehat{U}_t(\xi, t) - \xi^2 A_2 \widehat{U}(\xi, t) + i\xi A_1 \widehat{U}(\xi, t) + A_0 \widehat{U}(\xi, t) &= 0, \quad \xi \in \mathbb{R}, t > 0, \\ \widehat{U}(\xi, 0) &= \widehat{U}_0(\xi), \quad \xi \in \mathbb{R}. \end{aligned} \tag{2.6}$$

The solution of (2.6) is

$$\widehat{U}(\xi, t) = e^{-(\xi^2 A_2 + i\xi A_1 + A_0)t} \widehat{U}_0(\xi). \tag{2.7}$$

The energy  $\widehat{E}$  associated with (2.6) is

$$\widehat{E}(\xi, t) = \frac{1}{2} [k_1 |\widehat{v}|^2 + |\widehat{u}|^2 + k_2 |\widehat{z}|^2 + |\widehat{y}|^2 + k_3 |\widehat{\phi}|^2 + |\widehat{\theta}|^2 + k_4 |\widehat{\sigma}|^2 + |\widehat{\eta}|^2]. \tag{2.8}$$

System (2.6) is dissipative because

$$\frac{d}{dt} \widehat{E}(\xi, t) = -k_5 \xi^{2\epsilon_0} |\widehat{\eta}|^2. \tag{2.9}$$

Indeed, the first equation in (2.6) is equivalent to

$$\begin{aligned} \widehat{v}_t - i\xi \widehat{u} - \widehat{y} - \widehat{\theta} &= 0, \\ \widehat{u}_t - ik_1 \xi \widehat{v} + i\tau_1 \gamma \xi \widehat{\eta} &= 0, \\ \widehat{z}_t - i\xi \widehat{y} &= 0, \\ \widehat{y}_t - ik_2 \xi \widehat{z} + k_1 \widehat{v} + i\tau_2 \gamma \xi \widehat{\eta} &= 0, \\ \widehat{\phi}_t - i\xi \widehat{\theta} &= 0, \\ \widehat{\theta}_t - ik_3 \xi \widehat{\phi} + k_1 \widehat{v} + i\tau_3 \gamma \xi \widehat{\eta} &= 0, \\ \widehat{\sigma}_t - i\xi \widehat{\eta} &= 0, \\ \widehat{\eta}_t - ik_4 \xi \widehat{\sigma} + k_5 \xi^{2\epsilon_0} \widehat{\eta} + i\gamma \xi (\tau_1 \widehat{u} + \tau_2 \widehat{y} + \tau_3 \widehat{\theta}). \end{aligned} \tag{2.10}$$

To obtain (2.9), we multiply the equations in (2.10) by  $k_1 \bar{\widehat{v}}$ ,  $\bar{\widehat{u}}$ ,  $k_2 \bar{\widehat{z}}$ ,  $\bar{\widehat{y}}$ ,  $k_3 \bar{\widehat{\phi}}$ ,  $\bar{\widehat{\theta}}$ ,  $k_4 \bar{\widehat{\sigma}}$ , and  $\bar{\widehat{\eta}}$ , respectively. Then adding the obtained equations, taking the real part of the resulting expression and using the following classical relation, for two differentiable functions  $h, d : \mathbb{R} \rightarrow \mathbb{C}$ :

$$\frac{d}{dt} \text{Re}(h\bar{d}) = \text{Re}(h_t \bar{d} + d_t \bar{h}). \tag{2.11}$$

We observe that the energy  $\widehat{E}$  is equivalent to  $|\widehat{U}|^2$  defined by

$$|\widehat{U}(\xi, t)|^2 = |\widehat{v}|^2 + |\widehat{u}|^2 + |\widehat{z}|^2 + |\widehat{y}|^2 + |\widehat{\phi}|^2 + |\widehat{\theta}|^2 + |\widehat{\sigma}|^2 + |\widehat{\eta}|^2$$

because, for  $\alpha_1 = \frac{1}{2} \min\{k_1, k_2, k_3, k_4, 1\}$  and  $\alpha_2 = \frac{1}{2} \max\{k_1, k_2, k_3, k_4, 1\}$ , we have

$$\alpha_1 |\widehat{U}(\xi, t)|^2 \leq \widehat{E}(\xi, t) \leq \alpha_2 |\widehat{U}(\xi, t)|^2, \quad \forall \xi \in \mathbb{R}, \forall t \in \mathbb{R}_+. \tag{2.12}$$

Before presenting and proving our stability results in the next three sections, we prove these two lemmas that will be used in the proofs.

**Lemma 2.1.** *Let  $r_1, r_2$  and  $r_3$  be real numbers such that  $r_1 > -1$  and  $r_2, r_3 > 0$ . Then there exists  $C_{r_1, r_2, r_3} > 0$  such that*

$$\int_0^1 \xi^{r_1} e^{-r_3 t \xi^{r_2}} d\xi \leq C_{r_1, r_2, r_3} (1+t)^{-(r_1+1)/r_2}, \quad \forall t \in \mathbb{R}_+. \quad (2.13)$$

*Proof.* For  $0 \leq t \leq 1$ , (2.13) is evident, for any  $C_{r_1, r_2, r_3} \geq \frac{2^{(r_1+1)/r_2}}{r_1+1}$ . For  $t > 1$ , we have

$$\begin{aligned} \int_0^1 \xi^{r_1} e^{-r_3 t \xi^{r_2}} d\xi &= \int_0^1 \xi^{r_1+1-r_2} e^{-r_3 t \xi^{r_2}} \xi^{r_2-1} d\xi \\ &= \int_0^1 (\xi^{r_2})^{(r_1+1-r_2)/r_2} e^{-r_3 t \xi^{r_2}} \xi^{r_2-1} d\xi. \end{aligned}$$

Taking  $\tau = r_3 t \xi^{r_2}$ , we have

$$\xi^{r_2} = \frac{\tau}{r_3 t} \quad \text{and} \quad \xi^{r_2-1} d\xi = \frac{1}{r_2 r_3 t} d\tau.$$

Substituting in the above integral, we find

$$\begin{aligned} \int_0^1 (\xi^{r_2})^{(r_1+1-r_2)/r_2} e^{-r_3 t \xi^{r_2}} \xi^{r_2-1} d\xi &= \int_0^{r_3 t} \left(\frac{\tau}{r_3 t}\right)^{(r_1+1-r_2)/r_2} e^{-\tau} \frac{1}{r_2 r_3 t} d\tau \\ &\leq \frac{1}{r_2 (r_3 t)^{(r_1+1)/r_2}} \int_0^{+\infty} \tau^{(r_1+1-r_2)/r_2} e^{-\tau} d\tau \\ &\leq \frac{2^{(r_1+1)/r_2}}{r_2 (r_1+1)/r_2} C_{r_1, r_2} (t+1)^{-(r_1+1)/r_2}, \end{aligned}$$

where

$$C_{r_1, r_2} = \int_0^{+\infty} \tau^{(r_1+1-r_2)/r_2} e^{-\tau} d\tau,$$

which is a convergent integral, for any  $r_1 > -1$  and  $r_2 > 0$ . This completes the proof of (2.13) with

$$C_{r_1, r_2, r_3} = \max \left\{ \frac{2^{(r_1+1)/r_2}}{r_1+1}, \frac{2^{(r_1+1)/r_2}}{r_2 (r_1+1)/r_2} C_{r_1, r_2} \right\}.$$

□

**Lemma 2.2.** *For any positive real numbers  $\sigma_1, \sigma_2$ , and  $\sigma_3$ , we have*

$$\sup_{|\xi| \geq 1} |\xi|^{-\sigma_1} e^{-\sigma_2 t |\xi|^{-\sigma_3}} \leq (1 + \sigma_1 / (\sigma_2 \sigma_3))^{\sigma_1 / \sigma_3} (1+t)^{-\sigma_1 / \sigma_3}, \quad \forall t \in \mathbb{R}_+. \quad (2.14)$$

*Proof.* Clearly (2.14) is satisfied for  $t = 0$ . Let  $t > 0$  and  $h(x) = x^{-\sigma_1} e^{-\sigma_2 t x^{-\sigma_3}}$ , for  $x \geq 1$ . Simple computations show that

$$h'(x) = (\sigma_2 \sigma_3 t x^{-\sigma_3} - \sigma_1) x^{-\sigma_1-1} e^{-\sigma_2 t x^{-\sigma_3}}.$$

If  $t \geq \sigma_1 / (\sigma_2 \sigma_3)$ , then

$$\begin{aligned} h(x) &\leq h(((\sigma_2 \sigma_3 t) / \sigma_1)^{1/\sigma_3}) \\ &= ((\sigma_2 \sigma_3) / \sigma_1)^{-\sigma_1 / \sigma_3} e^{-\sigma_1 / \sigma_3} (1 + 1/t)^{\sigma_1 / \sigma_3} (1+t)^{-\sigma_1 / \sigma_3} \\ &\leq ((\sigma_2 \sigma_3) / \sigma_1)^{-\sigma_1 / \sigma_3} (1 + (\sigma_2 \sigma_3) / \sigma_1)^{\sigma_1 / \sigma_3} (1+t)^{-\sigma_1 / \sigma_3} \\ &= (1 + \sigma_1 / (\sigma_2 \sigma_3))^{\sigma_1 / \sigma_3} (1+t)^{-\sigma_1 / \sigma_3}, \end{aligned}$$

which gives (2.14) by taking  $x = |\xi|$ . If  $0 < t < \sigma_1/(\sigma_2\sigma_3)$ , then

$$h(x) \leq h(1) = e^{-\sigma_2 t}(1+t)^{\sigma_1/\sigma_3}(1+t)^{-\sigma_1/\sigma_3} \leq (1 + \sigma_1/(\sigma_2\sigma_3))^{\sigma_1/\sigma_3}(1+t)^{-\sigma_1/\sigma_3},$$

which implies (2.14), for  $x = |\xi|$ . □

### 3. PRELIMINARY DIFFERENTIAL IDENTITIES

This section is dedicated to the proof of several identities, which will play a crucial role in the proofs. In the rest of the paper,  $C$  and  $\tilde{C}$  denote generic positive constants, and  $C_\varepsilon$  denotes a generic positive constant depending on some positive constant  $\varepsilon$ . These generic constants can be different from line to line.

Multiplying (2.10)<sub>4</sub> and (2.10)<sub>3</sub> by  $i\xi\bar{z}$  and  $-i\xi\bar{y}$ , respectively, and then adding the resulting equations, taking the real part and using (2.11), we obtain

$$\frac{d}{dt} \operatorname{Re}(i\xi\bar{y}\bar{z}) = \xi^2(|\hat{y}|^2 - k_2|\hat{z}|^2) - k_1 \operatorname{Re}(i\xi\bar{v}\bar{z}) + \tau_2\gamma\xi^2 \operatorname{Re}(\hat{\eta}\bar{z}). \quad (3.1)$$

Multiplying (2.10)<sub>2</sub> and (2.10)<sub>1</sub> by  $i\xi\bar{v}$ , and  $-i\xi\bar{u}$ , respectively, and then adding the resulting equations, taking the real part and using (2.11), we find

$$\frac{d}{dt} \operatorname{Re}(i\xi\bar{u}\bar{v}) = \xi^2(|\hat{u}|^2 - k_1|\hat{v}|^2) - \operatorname{Re}(i\xi\bar{y}\bar{u}) - \operatorname{Re}(i\xi\bar{\theta}\bar{u}) + \tau_1\gamma\xi^2 \operatorname{Re}(\hat{\eta}\bar{v}). \quad (3.2)$$

After, multiplying (2.10)<sub>6</sub> and (2.10)<sub>5</sub> by  $i\xi\bar{\phi}$  and  $-i\xi\bar{\theta}$ , respectively, adding the resulting equations, taking the real part and using (2.11), we obtain

$$\frac{d}{dt} \operatorname{Re}(i\xi\bar{\theta}\bar{\phi}) = \xi^2(|\hat{\theta}|^2 - k_3|\hat{\phi}|^2) - k_1 \operatorname{Re}(i\xi\bar{v}\bar{\phi}) + \tau_3\gamma\xi^2 \operatorname{Re}(\hat{\eta}\bar{\phi}). \quad (3.3)$$

Multiplying (2.10)<sub>6</sub> and (2.10)<sub>1</sub> by  $-\xi^2\bar{v}$  and  $-\xi^2\bar{\theta}$ , respectively, then adding the resulting equations, taking the real part and using (2.11), we have

$$\begin{aligned} \frac{d}{dt} \operatorname{Re}(-\xi^2\bar{\theta}\bar{v}) &= \xi^2(k_1|\hat{v}|^2 - |\hat{\theta}|^2) - \xi^2 \operatorname{Re}(i\xi\bar{u}\bar{\theta}) - k_3\xi^2 \operatorname{Re}(i\xi\bar{\phi}\bar{v}) \\ &\quad - \xi^2 \operatorname{Re}(\hat{y}\bar{\theta}) + \tau_3\gamma\xi^2 \operatorname{Re}t(i\xi\bar{\eta}\bar{v}). \end{aligned} \quad (3.4)$$

Also, multiplying (2.10)<sub>4</sub> and (2.10)<sub>1</sub> by  $-\xi^2\bar{v}$  and  $-\xi^2\bar{y}$ , respectively, then adding the resulting equations, taking the real part and using (2.11), we infer that

$$\begin{aligned} \frac{d}{dt} \operatorname{Re}(-\xi^2\bar{y}\bar{v}) &= \xi^2(k_1|\hat{v}|^2 - |\hat{y}|^2) - \xi^2 \operatorname{Re}(i\xi\bar{u}\bar{y}) - k_2\xi^2 \operatorname{Re}(i\xi\bar{z}\bar{v}) \\ &\quad - \xi^2 \operatorname{Re}(\hat{\theta}\bar{y}) + \tau_2\gamma\xi^2 \operatorname{Re}(i\xi\bar{\eta}\bar{v}). \end{aligned} \quad (3.5)$$

Multiplying (2.10)<sub>8</sub> and (2.10)<sub>7</sub> by  $i\xi\bar{\sigma}$  and  $-i\xi\bar{\eta}$ , respectively, adding the resulting equations, taking the real part and using (2.11), we obtain

$$\begin{aligned} \frac{d}{dt} \operatorname{Re}(i\xi\bar{\eta}\bar{\sigma}) &= \xi^2(|\hat{\eta}|^2 - k_4|\hat{\sigma}|^2) - k_5\xi^{2\varepsilon_0} \operatorname{Re}(i\xi\bar{\eta}\bar{\sigma}) + \gamma\xi^2 \operatorname{Re}(\bar{\sigma}(\tau_1\hat{u} + \tau_2\hat{y} + \tau_3\hat{\theta})). \end{aligned} \quad (3.6)$$

Similarly, multiplying (2.10)<sub>3</sub> and (2.10)<sub>6</sub> by  $i\xi\bar{\theta}$  and  $-i\xi\bar{z}$ , respectively, then adding the resulting equations, taking the real part and using (2.11), we have

$$\frac{d}{dt} \operatorname{Re}(i\xi\bar{z}\bar{\theta}) = -\xi^2 \operatorname{Re}(\hat{y}\bar{\theta}) + k_3\xi^2 \operatorname{Re}(\hat{\phi}\bar{z}) + k_1 \operatorname{Re}(i\xi\bar{v}\bar{z}) - \tau_3\gamma\xi^2 \operatorname{Re}(\hat{\eta}\bar{z}). \quad (3.7)$$

Multiplying (2.10)<sub>5</sub> and (2.10)<sub>4</sub> by  $i\xi\bar{\widehat{y}}$  and  $-i\xi\bar{\widehat{\phi}}$ , respectively, then adding the resulting equations, taking the real part and using (2.11), we arrive at

$$\frac{d}{dt} \operatorname{Re}(i\xi\widehat{\phi}\widehat{y}) = -\xi^2 \operatorname{Re}(\widehat{\theta}\widehat{y}) + k_2\xi^2 \operatorname{Re}(\widehat{z}\bar{\widehat{\phi}}) + k_1 \operatorname{Re}(i\xi\widehat{v}\bar{\widehat{\phi}}) - \tau_2\gamma\xi^2 \operatorname{Re}(\widehat{\eta}\bar{\widehat{\phi}}). \quad (3.8)$$

Multiplying (2.10)<sub>2</sub> and (2.10)<sub>3</sub> by  $-\bar{\widehat{z}}$  and  $-\bar{\widehat{u}}$ , respectively, then adding the resulting equations, taking the real part and using (2.11), it follows that

$$\frac{d}{dt} \operatorname{Re}(-\widehat{u}\bar{\widehat{z}}) = -k_1 \operatorname{Re}(i\xi\widehat{v}\bar{\widehat{z}}) - \operatorname{Re}(i\xi\widehat{y}\bar{\widehat{u}}) + \tau_1\gamma \operatorname{Re}(i\xi\widehat{\eta}\bar{\widehat{z}}). \quad (3.9)$$

Finally, multiplying (2.10)<sub>2</sub> and (2.10)<sub>5</sub> by  $-\bar{\widehat{\phi}}$  and  $-\bar{\widehat{u}}$ , respectively, then adding the resulting equations, taking the real part and using (2.11), it follows that

$$\frac{d}{dt} \operatorname{Re}(-\widehat{u}\bar{\widehat{\phi}}) = -k_1 \operatorname{Re}(i\xi\widehat{v}\bar{\widehat{\phi}}) - \operatorname{Re}(i\xi\widehat{\theta}\bar{\widehat{u}}) + \tau_1\gamma \operatorname{Re}(i\xi\widehat{\eta}\bar{\widehat{\phi}}). \quad (3.10)$$

#### 4. STABILITY

In this section, we investigate the asymptotic behavior, when time  $t$  goes to infinity, of the solution  $U$  of (2.3). First, we will show that  $|\widehat{U}|^2$  converges exponentially to zero (with respect to time  $t$ ) in case (1.11), and in case  $(\tau_1, \tau_2, \tau_3) = (1, 0, 0)$  with  $\chi \neq 0$ . In case  $(\tau_1, \tau_2, \tau_3) = (1, 0, 0)$  with  $\chi = 0$ , we prove that  $|\widehat{U}|^2$  does not converge to zero when  $t$  goes to infinity. Let us distinguish the three cases (1.7).

**Case 1.1:**  $(\tau_1, \tau_2, \tau_3) = (1, 0, 0)$  and  $\chi \neq 0$ . We start by presenting the exponential stability result for (2.6) in the next lemma.

**Lemma 4.1.** *Assume that  $\chi \neq 0$ ; that is  $k_2 \neq k_3$ . Let  $\widehat{U}$  be a solution of (2.6). Then there exist  $c, \tilde{c} > 0$  such that*

$$|\widehat{U}(\xi, t)|^2 \leq \tilde{c}e^{-cf(\xi)t} |\widehat{U}_0(\xi)|^2, \quad \forall \xi \in \mathbb{R}, \forall t \in \mathbb{R}_+, \quad (4.1)$$

where

$$f(\xi) = \frac{\xi^{4+2\epsilon_0}}{\tilde{f}(\xi)} \quad \text{and} \quad \tilde{f}(\xi) = \begin{cases} 1 + \xi^8 & \text{in case (1.6),} \\ 1 + \xi^6 & \text{in case (1.8).} \end{cases} \quad (4.2)$$

*Proof.* Multiplying (2.10)<sub>2</sub> and (2.10)<sub>8</sub> by  $i\frac{|\gamma|}{\gamma}\xi\bar{\widehat{\eta}}$  and  $-i\frac{|\gamma|}{\gamma}\xi\bar{\widehat{u}}$ , respectively, then adding the resulting equations, taking the real part and using (2.11), we obtain

$$\begin{aligned} \frac{d}{dt} \operatorname{Re}\left(i\frac{|\gamma|}{\gamma}\xi\widehat{u}\bar{\widehat{\eta}}\right) &= |\gamma|\xi^2(|\widehat{\eta}|^2 - |\widehat{u}|^2) + \frac{|\gamma|}{\gamma}k_4\xi^2 \operatorname{Re}(\widehat{\sigma}\bar{\widehat{u}}) \\ &\quad - \frac{|\gamma|}{\gamma}k_1\xi^2 \operatorname{Re}(\widehat{v}\bar{\widehat{\eta}}) + \frac{|\gamma|}{\gamma}k_5\xi^{2\epsilon_0} \operatorname{Re}(i\xi\widehat{\eta}\bar{\widehat{u}}). \end{aligned} \quad (4.3)$$

Multiplying (2.10)<sub>6</sub> and (2.10)<sub>8</sub> by  $\bar{\widehat{\eta}}$  and  $\bar{\widehat{\theta}}$ , respectively, adding the resulting equations, taking the real part and using (2.11), we find that

$$\begin{aligned} \frac{d}{dt} \operatorname{Re}(\widehat{\eta}\bar{\widehat{\theta}}) &= \gamma \operatorname{Re}(i\xi\widehat{\theta}\bar{\widehat{u}}) + k_4 \operatorname{Re}(i\xi\widehat{\sigma}\bar{\widehat{\theta}}) - k_5\xi^{2\epsilon_0} \operatorname{Re}(\widehat{\eta}\bar{\widehat{\theta}}) \\ &\quad + k_3 \operatorname{Re}(i\xi\widehat{\phi}\bar{\widehat{\eta}}) - k_1 \operatorname{Re}(\widehat{v}\bar{\widehat{\eta}}). \end{aligned} \quad (4.4)$$

Also, multiplying (2.10)<sub>4</sub> and (2.10)<sub>8</sub> by  $\bar{\eta}$  and  $\bar{y}$ , respectively, adding the resulting equations, taking the real part and using (2.11), we obtain

$$\begin{aligned} \frac{d}{dt} \operatorname{Re}(\bar{\eta}\bar{y}) &= -\gamma \operatorname{Re}(i\xi\bar{u}\bar{y}) + k_4 \operatorname{Re}(i\xi\bar{\sigma}\bar{y}) - k_5\xi^{2\epsilon_0} \operatorname{Re}(\bar{\eta}\bar{y}) \\ &\quad + k_2 \operatorname{Re}(i\xi\bar{z}\bar{\eta}) - k_1 \operatorname{Re}(\bar{v}\bar{\eta}). \end{aligned} \tag{4.5}$$

Multiplying (2.10)<sub>1</sub> and (2.10)<sub>7</sub> by  $i\xi\bar{\sigma}$  and  $-i\xi\bar{v}$ , respectively, then adding the resulting equations, taking the real part and using (2.11), we infer that

$$\frac{d}{dt} \operatorname{Re}(i\xi\bar{v}\bar{\sigma}) = -\xi^2 \operatorname{Re}(\bar{\sigma}\bar{u}) + \xi^2 \operatorname{Re}(\bar{v}\bar{\eta}) + \operatorname{Re}(i\xi\bar{y}\bar{\sigma}) + \operatorname{Re}(i\xi\bar{\theta}\bar{\sigma}). \tag{4.6}$$

Similarly, multiplying (2.10)<sub>3</sub> and (2.10)<sub>7</sub> by  $-\bar{\sigma}$  and  $-\bar{z}$ , respectively, then adding the resulting equations, taking the real part and using (2.11), we arrive at

$$\frac{d}{dt} \operatorname{Re}(-\bar{\sigma}\bar{z}) = \operatorname{Re}(i\xi\bar{\sigma}\bar{y}) + \operatorname{Re}(i\xi\bar{z}\bar{\eta}). \tag{4.7}$$

Multiplying (2.10)<sub>5</sub> and (2.10)<sub>7</sub> by  $-\bar{\sigma}$  and  $-\bar{\phi}$ , respectively, then adding the resulting equations, taking the real part and using (2.11), we entail

$$\frac{d}{dt} \operatorname{Re}(-\bar{\sigma}\bar{\phi}) = \operatorname{Re}(i\xi\bar{\sigma}\bar{\theta}) + \operatorname{Re}(i\xi\bar{\phi}\bar{\eta}). \tag{4.8}$$

Let  $\lambda_0, \dots, \lambda_5$  be positive constants to be defined later, and let (observe that  $\chi \neq 0$  by assumption)

$$\begin{aligned} \lambda_6 &= \frac{k_2}{\chi}(\lambda_4 + \lambda_5), & \lambda_7 &= -\frac{k_3}{\chi}(\lambda_4 + \lambda_5), & \lambda_8 &= \frac{k_2}{k_1}\lambda_5\xi^2 - \lambda_1 + \frac{k_2}{\chi}(\lambda_4 + \lambda_5), \\ \lambda_9 &= \frac{k_3}{k_1}\lambda_4\xi^2 - \lambda_3 - \frac{k_3}{\chi}(\lambda_4 + \lambda_5). \end{aligned}$$

We define the functional

$$\begin{aligned} F_0(\xi, t) &= \operatorname{Re} [i\xi (\lambda_1\bar{y}\bar{z} + \lambda_2\bar{u}\bar{v} + \lambda_3\bar{\theta}\bar{\phi} + \bar{\eta}\bar{\sigma} + \lambda_6\bar{z}\bar{\theta} + \lambda_7\bar{\phi}\bar{y})] \\ &\quad + \operatorname{Re} (-\lambda_4\xi^2\bar{\theta}\bar{v} - \lambda_5\xi^2\bar{y}\bar{v} - \lambda_8\bar{u}\bar{z} - \lambda_9\bar{u}\bar{\phi}). \end{aligned} \tag{4.9}$$

Multiplying (3.1)-(3.10) by  $\lambda_1, \dots, \lambda_5, 1, \lambda_6, \dots, \lambda_9$ , respectively, and then adding the obtained equations, we see that, thanks to the choices of  $\lambda_6, \dots, \lambda_9$ , the expression of  $\frac{d}{dt}F_0$  does not contain the terms  $\operatorname{Re}(i\xi\bar{v}\bar{z})$ ,  $\operatorname{Re}(i\xi\bar{v}\bar{\phi})$ ,  $\operatorname{Re}(\bar{y}\bar{\theta})$ , and  $\operatorname{Re}(\bar{\phi}\bar{z})$  because their coefficients vanish. So, we find that

$$\begin{aligned} \frac{d}{dt}F_0(\xi, t) &= -\xi^2 \left( k_3\lambda_3|\bar{\phi}|^2 + (\lambda_5 - \lambda_1)|\bar{y}|^2 + (\lambda_4 - \lambda_3)|\bar{\theta}|^2 \right. \\ &\quad \left. + (k_1\lambda_2 - k_1\lambda_4 - k_1\lambda_5)|\bar{v}|^2 \right) - \xi^2(k_2\lambda_1|\bar{z}|^2 + k_4|\bar{\sigma}|^2) \\ &\quad + I_1 \operatorname{Re}(i\xi\bar{\theta}\bar{u}) + I_2 \operatorname{Re}(i\xi\bar{y}\bar{u}) + \gamma\xi^2 \operatorname{Re}(\bar{\sigma}\bar{u}) + \xi^2(\lambda_2|\bar{u}|^2 + |\bar{\eta}|^2) \\ &\quad + \operatorname{Re} \left( i\gamma\lambda_8\xi\bar{\eta}\bar{z} + i\gamma\lambda_9\xi\bar{\eta}\bar{\phi} - ik_5\xi^{2\epsilon_0+1}\bar{\eta}\bar{\sigma} + \gamma\lambda_2\xi^2\bar{\eta}\bar{v} \right), \end{aligned} \tag{4.10}$$

where

$$I_1 = \lambda_4\xi^2 - \lambda_2 - \lambda_9 \quad \text{and} \quad I_2 = \lambda_5\xi^2 - \lambda_2 - \lambda_8. \tag{4.11}$$

To eliminate the terms  $\text{Re}(i\xi\widehat{\theta}\widehat{u})$ ,  $\text{Re}(i\xi\widehat{y}\widehat{u})$ , and  $\text{Re}(\widehat{\sigma}\widehat{u})$  from the right hand side of (4.10), we put

$$I_3 = \gamma + \frac{|\gamma|}{\gamma}k_4\lambda_0 + \frac{k_4}{\gamma}I_1, \quad I_4 = \gamma + \frac{|\gamma|}{\gamma}k_4\lambda_0 + \frac{k_4}{\gamma}I_2, \quad I_5 = \gamma + \frac{|\gamma|}{\gamma}k_4\lambda_0,$$

and introduce the functional

$$F_1(\xi, t) = F_0(\xi, t) + \frac{|\gamma|}{\gamma}\lambda_0 \text{Re}(i\xi\widehat{u}\widehat{\eta}) - \frac{1}{\gamma} \text{Re}(I_1\widehat{\eta}\widehat{\theta} + I_2\widehat{\eta}\widehat{y}) + I_5 \text{Re}(i\xi\widehat{v}\widehat{\sigma}) - I_3 \text{Re}(\widehat{\sigma}\widehat{\phi}) - I_4 \text{Re}(\widehat{\sigma}\widehat{z}). \tag{4.12}$$

Multiplying (4.3)-(4.8) by  $\lambda_0$ ,  $-\frac{1}{\gamma}I_1$ ,  $-\frac{1}{\gamma}I_2$ ,  $I_5$ ,  $I_4$  and  $I_3$ , respectively, and then adding the obtained equations and (4.10), we arrive at

$$\begin{aligned} \frac{d}{dt}F_1(\xi, t) &= -\xi^2\left(k_2\lambda_1|\widehat{z}|^2 + k_3\lambda_3|\widehat{\phi}|^2 + (\lambda_5 - \lambda_1)|\widehat{y}|^2 + (\lambda_4 - \lambda_3)|\widehat{\theta}|^2 \right. \\ &\quad \left. + (k_1\lambda_2 - k_1\lambda_4 - k_1\lambda_5)|\widehat{v}|^2\right) - \xi^2\left((|\gamma|\lambda_0 - \lambda_2)|\widehat{u}|^2 + k_4|\widehat{\sigma}|^2\right) \\ &\quad + (|\gamma|\lambda_0 + 1)\xi^2|\widehat{\eta}|^2 + \text{Re}\left(i\frac{|\gamma|}{\gamma}k_5\lambda_0\xi^{2\epsilon_0+1}\widehat{\eta}\widehat{u} - ik_5\xi^{2\epsilon_0+1}\widehat{\eta}\widehat{\sigma}\right) \\ &\quad + \text{Re}\left[\left(\frac{k_1}{\gamma}(I_1 + I_2) + (\gamma\lambda_2 + \gamma + \frac{|\gamma|}{\gamma}(k_4 - k_1)\lambda_0)\xi^2\right)\widehat{\eta}\widehat{v}\right. \\ &\quad \left. + \frac{k_5}{\gamma}I_1\xi^{2\epsilon_0}\widehat{\eta}\widehat{\theta} + \frac{k_5}{\gamma}I_2\xi^{2\epsilon_0}\widehat{\eta}\widehat{y}\right] \\ &\quad + \text{Re}\left[i\left(\gamma\lambda_8 + \frac{k_2}{\gamma}I_2 - I_4\right)\xi\widehat{\eta}\widehat{z} + i\left(\gamma\lambda_9 + \frac{k_3}{\gamma}I_1 - I_3\right)\xi\widehat{\eta}\widehat{\phi}\right]. \end{aligned} \tag{4.13}$$

Let  $\lambda$  be a positive constant. We introduce the functionals ( $\tilde{f}$  is defined in (4.2))

$$F(\xi, t) = \xi^{2+2\epsilon_0}F_1(\xi, t) \quad \text{and} \quad L(\xi, t) = \lambda\widehat{E}(\xi, t) + \frac{1}{\tilde{f}(\xi)}F(\xi, t). \tag{4.14}$$

For the rest of proofs, we will frequently use the inequality

$$|\xi|^{m_2} \leq |\xi|^{m_1} + |\xi|^{m_3}, \quad \forall \xi \in \mathbb{R}, \quad \forall 0 \leq m_1 \leq m_2 \leq m_3. \tag{4.15}$$

According to (4.15), we observe that

$$|I_j| \leq C(\xi^2 + 1), \quad j = 1, 2, 3, 4.$$

Then, applying Young's inequality for the terms depending on  $\widehat{\eta}$  in (4.13), it follows, for any  $\varepsilon > 0$ , that

$$\begin{aligned} \frac{d}{dt}F(\xi, t) &\leq -\xi^{4+2\epsilon_0}\left((k_2\lambda_1 - \varepsilon)|\widehat{z}|^2 + (k_3\lambda_3 - \varepsilon)|\widehat{\phi}|^2 + (\lambda_5 - \lambda_1 - \varepsilon)|\widehat{y}|^2 \right. \\ &\quad \left. + (\lambda_4 - \lambda_3 - \varepsilon)|\widehat{\theta}|^2\right) - \xi^{4+2\epsilon_0}\left((k_1\lambda_2 - k_1\lambda_4 - k_1\lambda_5 - \varepsilon)|\widehat{v}|^2 \right. \\ &\quad \left. + (|\gamma|\lambda_0 - \lambda_2 - \varepsilon)|\widehat{u}|^2 + (k_4 - \varepsilon)|\widehat{\sigma}|^2\right) + C_{\varepsilon, \lambda_0, \dots, \lambda_9}\tilde{f}(\xi)\xi^{2\epsilon_0}|\widehat{\eta}|^2. \end{aligned} \tag{4.16}$$

We choose  $\lambda_1, \lambda_3 > 0$ , then we select  $\lambda_0$  such that  $\lambda_0 > \frac{1}{|\gamma|}(\lambda_1 + \lambda_3)$ . After, we pick  $\lambda_4$  and  $\lambda_2$  such that

$$\lambda_3 < \lambda_4 < |\gamma|\lambda_0 - \lambda_1 \quad \text{and} \quad \lambda_1 + \lambda_4 < \lambda_2 < |\gamma|\lambda_0.$$

Finally, we take  $\lambda_5$  and  $\varepsilon$  such that  $\lambda_1 < \lambda_5 < \lambda_2 - \lambda_4$  and

$$0 < \varepsilon < \min\{\lambda_5 - \lambda_1, k_1(\lambda_2 - \lambda_4 - \lambda_5), \lambda_4 - \lambda_3, |\gamma|\lambda_0 - \lambda_2, k_2\lambda_1, k_3\lambda_3, k_4\}.$$

Hence, using the definition (2.8) of  $\widehat{E}$ , (4.16) leads to, for some positive constant  $c_1$ ,

$$\frac{d}{dt}F(\xi, t) \leq -c_1\xi^{4+2\epsilon_0}\widehat{E}(\xi, t) + C\tilde{f}(\xi)\xi^{2\epsilon_0}|\widehat{\eta}|^2. \tag{4.17}$$

Thus, from (2.9), (4.14), and (4.17), we have

$$\frac{d}{dt}L(\xi, t) \leq -c_1f(\xi)\widehat{E}(\xi, t) - (k_5\lambda - C)\xi^{2\epsilon_0}|\widehat{\eta}|^2, \tag{4.18}$$

where  $f$  is defined in (4.2). Moreover, using the definitions of  $\widehat{E}$ ,  $F$ ,  $L$ , and  $\tilde{f}$ , we obtain, for some  $c_2 > 0$  (independent of  $\lambda$ ),

$$|L(\xi, t) - \lambda\widehat{E}(\xi, t)| = \frac{1}{\tilde{f}(\xi)}|F(\xi, t)| \leq C\frac{(1 + \xi^2)\xi^{2+2\epsilon_0}}{\tilde{f}(\xi)} \leq c_2\widehat{E}(\xi, t). \tag{4.19}$$

Therefore, for  $\lambda$  large enough so that  $\lambda > \max\{\frac{C}{k_5}, c_2\}$ , we deduce from (4.18) and (4.19) that

$$\frac{d}{dt}L(\xi, t) + c_1f(\xi)\widehat{E}(\xi, t) \leq 0, \tag{4.20}$$

$$c_3\widehat{E}(\xi, t) \leq L(\xi, t) \leq c_4\widehat{E}(\xi, t), \tag{4.21}$$

where  $c_3 = \lambda - c_2 > 0$  and  $c_4 = \lambda + c_2 > 0$ . Consequently, a combination of (4.20) and the second inequality in (4.21) lead to, for  $c = \frac{c_1}{c_4}$ ,

$$\frac{d}{dt}L(\xi, t) + cf(\xi)L(\xi, t) \leq 0. \tag{4.22}$$

Finally, by integration (4.22) with respect to time  $t$  and using (2.12) and (4.21), (4.1) follows with  $\tilde{c} = \frac{c_4\alpha_2}{c_3\alpha_1}$ .  $\square$

**Theorem 4.2.** *Assume that  $\chi \neq 0$ ; that is  $k_2 \neq k_3$ . Let  $N, \ell \in \mathbb{N}$  such that  $\ell \leq N$ ,  $U_0 \in H^N(\mathbb{R}) \cap L^1(\mathbb{R})$  and  $U$  be the solution of (2.3). Then for any  $j \in \{0, \dots, N - \ell\}$ , there exists  $c_0 > 0$  such that*

$$\|\partial_x^j U\|_{L^2(\mathbb{R})} \leq c_0(1 + t)^{-1/12-j/6}\|U_0\|_{L^1(\mathbb{R})} + c_0(1 + t)^{-\ell/2}\|\partial_x^{j+\ell} U_0\|_{L^2(\mathbb{R})}, \tag{4.23}$$

for all  $t \in \mathbb{R}_+$  in case (1.6), and

$$\|\partial_x^j U\|_{L^2(\mathbb{R})} \leq c_0(1 + t)^{-1/8-j/4}\|U_0\|_{L^1(\mathbb{R})} + c_0(1 + t)^{-\ell/2}\|\partial_x^{j+\ell} U_0\|_{L^2(\mathbb{R})}, \tag{4.24}$$

for all  $t \in \mathbb{R}_+$  in case (1.8).

*Proof.* From (4.2) we have in case (1.6) (low and high frequencies)

$$f(\xi) \geq \begin{cases} \xi^6/5 & \text{if } |\xi| \leq 1, \\ \xi^{-2}/5 & \text{if } |\xi| > 1. \end{cases} \tag{4.25}$$

Applying Plancherel’s theorem and (4.1), we have

$$\begin{aligned}
 & \|\partial_x^j U\|_{L^2(\mathbb{R})}^2 \\
 &= \|\widehat{\partial_x^j U}(x, t)\|_{L^2(\mathbb{R})}^2 = \int_{\mathbb{R}} \xi^{2j} |\widehat{U}(\xi, t)|^2 d\xi \\
 &\leq \tilde{c} \int_{\mathbb{R}} \xi^{2j} e^{-cf(\xi)t} |\widehat{U}_0(\xi)|^2 d\xi \\
 &\leq \tilde{c} \int_{|\xi| \leq 1} \xi^{2j} e^{-cf(\xi)t} |\widehat{U}_0(\xi)|^2 d\xi + \tilde{c} \int_{|\xi| > 1} \xi^{2j} e^{-cf(\xi)t} |\widehat{U}_0(\xi)|^2 d\xi \\
 &:= J_1 + J_2.
 \end{aligned} \tag{4.26}$$

Using (2.13) (with  $r_1 = 2j$ ,  $r_3 = \frac{c}{5}$  and  $r_2 = 6$ ) and (4.25), it follows, for the low frequency region,

$$J_1 \leq C \|\widehat{U}_0\|_{L^\infty(\mathbb{R})}^2 \int_{|\xi| \leq 1} \xi^{2j} e^{-\frac{c}{5}t\xi^6} d\xi \leq C(1+t)^{-\frac{1}{6}(1+2j)} \|U_0\|_{L^1(\mathbb{R})}^2. \tag{4.27}$$

For the high frequency region, using (4.25), we observe that

$$\begin{aligned}
 J_2 &\leq C \int_{|\xi| > 1} |\xi|^{2j} e^{-\frac{c}{5}t\xi^{-2}} |\widehat{U}(\xi, 0)|^2 d\xi \\
 &\leq C \sup_{|\xi| > 1} \{|\xi|^{-2\ell} e^{-\frac{c}{5}t|\xi|^{-2}}\} \int_{\mathbb{R}} |\xi|^{2(j+\ell)} |\widehat{U}(\xi, 0)|^2 d\xi,
 \end{aligned}$$

then, using (2.14) (with  $\sigma_1 = 2\ell$ ,  $\sigma_2 = \frac{c}{5}$  and  $\sigma_3 = 2$ ),

$$J_2 \leq C(1+t)^{-\ell} \|\partial_x^{j+\ell} U_0\|_{L^2(\mathbb{R})}^2, \tag{4.28}$$

and so, by combining (4.26)–(4.28), we obtain (4.23).

The proof of (4.24) is very similar; we notice only, in case (1.8), that

$$f(\xi) \geq \begin{cases} \xi^4/4 & \text{if } |\xi| \leq 1, \\ \xi^{-2}/4 & \text{if } |\xi| > 1. \end{cases} \quad \square$$

**Remark 4.3.** It is well known that the behavior of the Fourier transform of  $U$  in the low frequency region determines the rate of decay of  $U$ , while its behavior in the high frequency region imposes a regularity restriction on the initial data known as the regularity-loss property; see [9, 18, 19, 30, 32, 33]. The fact that  $f$  tends to 0 when  $\xi$  goes to infinity leads to the regularity-loss property in the estimates on  $\|\partial_x^j U\|_{L^2(\mathbb{R})}$  because (4.23) and (4.24) with  $j = \ell = 0$  imply only the boundedness of  $\|U\|_{L^2(\mathbb{R})}$ . This remark is valid also in case (1.11) for (1.8), and in case (1.11) for (1.6) if (1.12) is not satisfied (see Theorem 4.6 and Theorem 4.9 below).

**Case 1.2:**  $(\tau_1, \tau_2, \tau_3) = (1, 0, 0)$  and  $\chi = 0$ . In this subsection, we prove that (2.6) is not stable if  $(\tau_1, \tau_2, \tau_3) = (1, 0, 0)$  and  $\chi = 0$ .

**Theorem 4.4.** Assume that  $\chi = 0$ ; that is  $k_2 = k_3$ . Then  $|\widehat{U}(\xi, t)|$  does not converge to zero when time  $t$  goes to infinity.

*Proof.* We show that, for any  $\xi \in \mathbb{R}$ , the matrix

$$A := -(-\xi^2 A_2 + i\xi A_1 + A_0) \tag{4.29}$$

has at least a pure imaginary eigenvalue; that is

$$\forall \xi \in \mathbb{R}, \exists \lambda \in \mathbb{C} : \operatorname{Re}(\lambda) = 0, \quad \operatorname{Im}(\lambda) \neq 0 \quad \text{and} \quad \det(\lambda I - A) = 0, \tag{4.30}$$

where  $I$  denotes the identity matrix. From (2.4) with  $(\tau_1, \tau_2, \tau_3) = (1, 0, 0)$  and  $k_2 = k_3$ , we have

$$\lambda I - A = \begin{pmatrix} \lambda & -i\xi & 0 & -1 & 0 & -1 & 0 & 0 \\ -ik_1\xi & \lambda & 0 & 0 & 0 & 0 & 0 & i\gamma\xi \\ 0 & 0 & \lambda & -i\xi & 0 & 0 & 0 & 0 \\ k_1 & 0 & -ik_2\xi & \lambda & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \lambda & -i\xi & 0 & 0 \\ k_1 & 0 & 0 & 0 & -ik_2\xi & \lambda & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \lambda & -i\xi \\ 0 & i\gamma\xi & 0 & 0 & 0 & 0 & -ik_4\xi & k_5\xi^{2\epsilon_0} + \lambda \end{pmatrix}.$$

A direct computation shows that

$$\begin{aligned} \det(\lambda I - A) &= 2k_1\lambda^2(\lambda^2 + k_2\xi^2) [\lambda(\lambda + k_5\xi^{2\epsilon_0}) + (k_4 + \gamma^2)\xi^2] + k_4\xi^2(\lambda^2 + k_1\xi^2)(\lambda^2 + k_2\xi^2)^2 \\ &\quad + \lambda(\lambda^2 + k_2\xi^2)^2 [\lambda^2(\lambda + k_5\xi^{2\epsilon_0}) + \gamma^2\lambda\xi^2 + k_1\xi^2(\lambda + k_5\xi^{2\epsilon_0})]. \end{aligned}$$

It is clear that, if  $\xi \neq 0$ , then  $\lambda = i\sqrt{k_2}\xi$  is a pure imaginary eigenvalue of  $A$ . If  $\xi = 0$ , then  $\lambda = i\sqrt{2}k_1$  is a pure imaginary eigenvalue of  $A$ . Consequently, according to (2.7) and (4.29) (see [35]), the solution of (2.6) does not converge to zero when time  $t$  goes to infinity.  $\square$

**Case 2:**  $(\tau_1, \tau_2, \tau_3) = (0, 1, 0)$ . We present, first, our exponential stability result for (2.6), where the proof is similar to the one of Lemma 4.1.

**Lemma 4.5.** *Let  $\widehat{U}$  be a solution of (2.6). Then there exist  $c, \tilde{c} > 0$  such that (4.1) is satisfied with*

$$f(\xi) = \frac{\xi^{4+2\epsilon_0}}{\tilde{f}(\xi)}, \quad \tilde{f}(\xi) = \begin{cases} 1 + \xi^6 & \text{for (1.6) and (1.8) under (1.12),} \\ 1 + \xi^{10} & \text{for (1.6) without (1.12),} \\ 1 + \xi^8 & \text{for (1.8) without (1.12).} \end{cases} \quad (4.31)$$

*Proof.* Multiplying (2.10)<sub>4</sub> and (2.10)<sub>8</sub> by  $i\frac{|\gamma|}{\gamma}\xi\bar{\eta}$  and  $-i\frac{|\gamma|}{\gamma}\xi\bar{y}$ , respectively, then adding the resulting equations, taking the real part and using (2.11), we obtain

$$\begin{aligned} \frac{d}{dt} \operatorname{Re}(i\frac{|\gamma|}{\gamma}\xi\bar{\eta}) &= |\gamma|\xi^2(|\bar{\eta}|^2 - |\bar{y}|^2) + \frac{|\gamma|}{\gamma}k_4\xi^2 \operatorname{Re}(\bar{\sigma}\bar{y}) - \frac{|\gamma|}{\gamma}k_1 \operatorname{Re}(i\xi\bar{v}\bar{\eta}) \\ &\quad - \frac{|\gamma|}{\gamma}k_2\xi^2 \operatorname{Re}(\bar{\eta}\bar{z}) + \frac{|\gamma|}{\gamma}k_5\xi^{2\epsilon_0} \operatorname{Re}(i\xi\bar{\eta}\bar{y}). \end{aligned} \quad (4.32)$$

Multiplying (2.10)<sub>1</sub> and (2.10)<sub>7</sub> by  $\xi^2\bar{\sigma}$  and  $\xi^2\bar{v}$ , respectively, then adding the resulting equations, taking the real part and using (2.11), we find that

$$\frac{d}{dt} \operatorname{Re}(\xi^2\bar{v}\bar{\sigma}) = \xi^2 \operatorname{Re}(\bar{\sigma}\bar{y}) + \xi^2 \operatorname{Re}(\bar{\sigma}\bar{\theta}) + \xi^2 \operatorname{Re}(i\xi\bar{u}\bar{\sigma}) + \xi^2 \operatorname{Re}(i\xi\bar{\eta}\bar{v}). \quad (4.33)$$

Also, multiplying (2.10)<sub>3</sub> and (2.10)<sub>7</sub> by  $i\xi\bar{\sigma}$  and  $-i\xi\bar{z}$ , respectively, adding the resulting equations, taking the real part and using (2.11), we obtain

$$\frac{d}{dt} \operatorname{Re}(i\xi\bar{z}\bar{\sigma}) = -\xi^2 \operatorname{Re}(\bar{\sigma}\bar{y}) + \xi^2 \operatorname{Re}(\bar{\eta}\bar{z}). \quad (4.34)$$

Multiplying (2.10)<sub>2</sub> and (2.10)<sub>8</sub> by  $\bar{\eta}$  and  $\bar{u}$ , respectively, then adding the resulting equations, taking the real part and using (2.11), we infer that

$$\frac{d}{dt} \operatorname{Re}(\widehat{u\eta}) = -\gamma \operatorname{Re}(i\xi\widehat{y\bar{u}}) + k_1 \operatorname{Re}(i\xi\widehat{v\bar{\eta}}) + k_4 \operatorname{Re}(i\xi\widehat{\sigma\bar{u}}) - k_5\xi^{2\epsilon_0} \operatorname{Re}(\widehat{\eta\bar{u}}). \quad (4.35)$$

Multiplying (2.10)<sub>5</sub> and (2.10)<sub>7</sub> by  $i\xi\bar{\sigma}$  and  $-i\xi\bar{\phi}$ , respectively, then adding the resulting equations, taking the real part and using (2.11), we see that

$$\frac{d}{dt} \operatorname{Re}(i\xi\widehat{\phi\bar{\sigma}}) = -\xi^2 \operatorname{Re}(\widehat{\sigma\bar{\theta}}) + \xi^2 \operatorname{Re}(\widehat{\eta\bar{\phi}}). \quad (4.36)$$

Finally, multiplying (2.10)<sub>6</sub> and (2.10)<sub>8</sub> by  $-i\xi\bar{\eta}$  and  $i\xi\bar{\theta}$ , respectively, adding the resulting equations, taking the real part and using (2.11), it follows that

$$\begin{aligned} & \frac{d}{dt} \operatorname{Re}(i\xi\widehat{\eta\bar{\theta}}) \\ &= \gamma\xi^2 \operatorname{Re}(\widehat{y\bar{\theta}}) - k_4\xi^2 \operatorname{Re}(\widehat{\sigma\bar{\theta}}) + k_3\xi^2 \operatorname{Re}(\widehat{\eta\bar{\phi}}) - k_5\xi^{2\epsilon_0} \operatorname{Re}(i\xi\widehat{\eta\bar{\theta}}) + k_1 \operatorname{Re}(i\xi\widehat{v\bar{\eta}}). \end{aligned} \quad (4.37)$$

Let  $\lambda_0, \dots, \lambda_5$  be positive constants, and let

$$\begin{aligned} \lambda_6 &= \frac{k_2}{k_3} \left[ \left( \frac{k_3}{k_1} - 1 \right) \lambda_4 \xi^2 - \lambda_2 - \lambda_3 \right], & \lambda_7 &= -\frac{k_3}{k_2} \lambda_6, \\ \lambda_8 &= -\frac{k_2}{k_1} \lambda_5 \xi^2 + \lambda_6 - \lambda_1, & \lambda_9 &= \lambda_4 \xi^2 + \lambda_2. \end{aligned}$$

We define the functional

$$\begin{aligned} F_0(\xi, t) &= \operatorname{Re} \left[ i\xi (\lambda_1 \widehat{y\bar{z}} - \lambda_2 \widehat{u\bar{v}} + \lambda_3 \widehat{\theta\bar{\phi}} + \widehat{\eta\bar{\sigma}} + \lambda_6 \widehat{z\bar{\theta}} + \lambda_7 \widehat{\phi\bar{y}}) \right] \\ &+ \operatorname{Re} \left( -\lambda_4 \xi^2 \widehat{\theta\bar{v}} + \lambda_5 \xi^2 \widehat{y\bar{v}} - \lambda_8 \widehat{u\bar{z}} - \lambda_9 \widehat{u\bar{\phi}} \right). \end{aligned} \quad (4.38)$$

Multiplying (3.1)-(3.10) by  $\lambda_1, -\lambda_2, \lambda_3, \lambda_4, -\lambda_5, 1, \lambda_6, \dots, \lambda_9$ , respectively, and adding the resulting equations, we find that

$$\begin{aligned} & \frac{d}{dt} F_0(\xi, t) \\ &= -\xi^2 \left( k_3 \lambda_3 |\widehat{\phi}|^2 + \lambda_2 |\widehat{u}|^2 + (\lambda_4 - \lambda_3) |\widehat{\theta}|^2 + (k_1 \lambda_5 - k_1 \lambda_4 - k_1 \lambda_2) |\widehat{v}|^2 \right) \\ &- \xi^2 (k_2 \lambda_1 |\widehat{z}|^2 + k_4 |\widehat{\sigma}|^2) + I_1 \operatorname{Re}(i\xi\widehat{y\bar{u}}) + I_2 \xi^2 \operatorname{Re}(\widehat{y\bar{\theta}}) + \gamma \xi^2 \operatorname{Re}(\widehat{\sigma\bar{y}}) \\ &+ \xi^2 ((\lambda_1 + \lambda_5) |\widehat{y}|^2 + |\widehat{\eta}|^2) \\ &+ \operatorname{Re} \left( \gamma \lambda_1 \xi^2 \widehat{\eta\bar{z}} - \gamma \lambda_7 \xi^2 \widehat{\eta\bar{\phi}} - ik_5 \xi^{2\epsilon_0+1} \widehat{\eta\bar{\sigma}} - i\gamma \lambda_5 \xi^3 \widehat{\eta\bar{v}} \right) \end{aligned} \quad (4.39)$$

(thanks to the choices of  $\lambda_6, \dots, \lambda_9$ ,  $\operatorname{Re}(i\xi(\widehat{v\bar{z}} + \widehat{\theta\bar{u}} + \widehat{v\bar{\phi}}))$  and  $\operatorname{Re}(\widehat{z\bar{\phi}})$  disappear), where

$$I_1 = -\lambda_5 \xi^2 + \lambda_2 - \lambda_8 \quad \text{and} \quad I_2 = \lambda_5 - \lambda_4 - \lambda_6 - \lambda_7.$$

We put

$$I_3 = \left( \frac{|\gamma|}{\gamma} k_4 \lambda_0 + \gamma \right) \xi^2 + \frac{k_4}{\gamma} I_1 \quad \text{and} \quad I_4 = \frac{k_4}{\gamma} (I_2 \xi^2 + I_1),$$

and introduce the functional

$$\begin{aligned}
 F_1(\xi, t) = & \xi^2 F_0(\xi, t) + \frac{|\gamma|}{\gamma} \lambda_0 \xi^2 \operatorname{Re}(i\xi \widehat{y} \overline{\widehat{\eta}}) + \frac{k_4}{\gamma} I_1 \xi^2 \operatorname{Re}(\widehat{v} \overline{\widehat{\sigma}}) + I_3 \operatorname{Re}(i\xi \widehat{z} \overline{\widehat{\sigma}}) \\
 & + \frac{1}{\gamma} I_1 \xi^2 \operatorname{Re}(\widehat{u} \overline{\widehat{\eta}}) + I_4 \operatorname{Re}(i\xi \widehat{\phi} \overline{\widehat{\sigma}}) - \frac{1}{\gamma} I_2 \xi^2 \operatorname{Re}(i\xi \widehat{\eta} \overline{\widehat{\theta}}).
 \end{aligned} \tag{4.40}$$

Multiplying (4.32)-(4.37) and (4.39) by  $\lambda_0 \xi^2$ ,  $\frac{k_4}{\gamma} I_1$ ,  $I_3$ ,  $\frac{1}{\gamma} I_1 \xi^2$ ,  $I_4$ ,  $-\frac{1}{\gamma} I_2 \xi^2$  and  $\xi^2$ , respectively, then adding the obtained expressions, we arrive at  $(\operatorname{Re}(i\xi \widehat{u} \overline{\widehat{\sigma}})$  and  $\operatorname{Re}(\widehat{\sigma} \overline{\widehat{y}} + \widehat{\sigma} \overline{\widehat{\theta}})$  disappear according to the definition of  $I_3$  and  $I_4$ )

$$\begin{aligned}
 & \frac{d}{dt} F_1(\xi, t) \\
 = & -\xi^4 (k_2 \lambda_1 |\widehat{z}|^2 + k_3 \lambda_3 |\widehat{\phi}|^2 + \lambda_2 |\widehat{u}|^2 + (\lambda_4 - \lambda_3) |\widehat{\theta}|^2 \\
 & + (k_1 \lambda_5 - k_1 \lambda_4 - k_1 \lambda_2) |\widehat{v}|^2) - \xi^4 (|\gamma| \lambda_0 - \lambda_1 - \lambda_5) |\widehat{y}|^2 + k_4 |\widehat{\sigma}|^2 \\
 & + (|\gamma| \lambda_0 + 1) \xi^4 |\widehat{\eta}|^2 + \xi^2 \operatorname{Re} \left[ (i I_5 \widehat{v} + I_6 \widehat{z} + I_7 \widehat{\phi} - i k_5 \xi^{2\epsilon_0 + 1} \overline{\widehat{\sigma}} \right. \\
 & \left. + i \frac{|\gamma|}{\gamma} k_5 \lambda_0 \xi^{2\epsilon_0 + 1} \overline{\widehat{y}} + i \frac{k_5}{\gamma} \xi^{2\epsilon_0 + 1} I_2 \overline{\widehat{\theta}} - \frac{k_5}{\gamma} \xi^{2\epsilon_0} I_1 \overline{\widehat{u}}) \widehat{\eta} \right],
 \end{aligned} \tag{4.41}$$

where

$$\begin{aligned}
 I_5 = & -\gamma \lambda_5 \xi^3 + \left( \frac{|\gamma|}{\gamma} k_1 \lambda_0 + \frac{k_4 - k_1}{\gamma} I_1 + \frac{k_1}{\gamma} I_2 \right) \xi, \\
 I_6 = & \left( -\frac{|\gamma|}{\gamma} k_2 \lambda_0 + \gamma \lambda_1 \right) \xi^2 + I_3, \quad I_7 = -\left( \frac{k_3}{\gamma} I_2 + \gamma \lambda_7 \right) \xi^2 + I_4.
 \end{aligned}$$

Observe that, by definition,

$$|I_1| \leq \begin{cases} C & \text{if (1.12) holds,} \\ C(1 + \xi^2) & \text{if not,} \end{cases} \quad |I_2| \leq \begin{cases} C & \text{if (1.12) holds,} \\ C(1 + \xi^2) & \text{if not,} \end{cases} \tag{4.42}$$

$$|I_5| \leq C(|\xi| + |\xi|^3), \quad |I_6| \leq C(1 + \xi^2), \quad |I_7| \leq \begin{cases} C(1 + \xi^2) & \text{if (1.12) holds,} \\ C(1 + \xi^4) & \text{if not.} \end{cases} \tag{4.43}$$

Then, applying Young's inequality, it follows, for any  $\varepsilon > 0$ , that

$$\begin{aligned}
 & \xi^2 \operatorname{Re} \left[ \left( i I_5 \widehat{v} + I_6 \widehat{z} + I_7 \widehat{\phi} - i k_5 \xi^{2\epsilon_0 + 1} \overline{\widehat{\sigma}} + i \frac{|\gamma|}{\gamma} k_5 \lambda_0 \xi^{2\epsilon_0 + 1} \overline{\widehat{y}} + i \frac{k_5}{\gamma} \xi^{2\epsilon_0 + 1} I_2 \overline{\widehat{\theta}} \right. \right. \\
 & \left. \left. - \frac{k_5}{\gamma} \xi^{2\epsilon_0} I_1 \overline{\widehat{u}} \right) \widehat{\eta} \right] \\
 \leq & \varepsilon \xi^4 (|\widehat{z}|^2 + |\widehat{\phi}|^2 + |\widehat{u}|^2 + |\widehat{\theta}|^2 + |\widehat{v}|^2 + |\widehat{\sigma}|^2 + |\widehat{y}|^2) \\
 & + C_\varepsilon (\xi^{4\epsilon_0} |I_1|^2 + \xi^{4\epsilon_0 + 2} |I_2|^2 + |I_5|^2 + |I_6|^2 + |I_7|^2 + \xi^{4\epsilon_0 + 2}) |\widehat{\eta}|^2 \\
 \leq & \varepsilon \xi^4 (|\widehat{z}|^2 + |\widehat{\phi}|^2 + |\widehat{u}|^2 + |\widehat{\theta}|^2 + |\widehat{v}|^2 + |\widehat{\sigma}|^2 + |\widehat{y}|^2) + C_{\varepsilon, \lambda_0, \dots, \lambda_9} \tilde{f}(\xi) |\widehat{\eta}|^2.
 \end{aligned} \tag{4.44}$$

By combining (4.41) and (4.44), we find that

$$\begin{aligned} \frac{d}{dt}F_1(\xi, t) &\leq -\xi^4 \left( (k_2\lambda_1 - \varepsilon)|\widehat{z}|^2 + (k_3\lambda_3 - \varepsilon)|\widehat{\phi}|^2 + (\lambda_2 - \varepsilon)|\widehat{u}|^2 \right. \\ &\quad \left. + (\lambda_4 - \lambda_3 - \varepsilon)|\widehat{\theta}|^2 \right) - \xi^4 \left( (k_1\lambda_5 - k_1\lambda_4 - k_1\lambda_2 - \varepsilon)|\widehat{v}|^2 \right. \\ &\quad \left. + (|\gamma|\lambda_0 - \lambda_1 - \lambda_5 - \varepsilon)|\widehat{y}|^2 + (k_4 - \varepsilon)|\widehat{\sigma}|^2 \right) \\ &\quad + C_{\varepsilon, \lambda_0, \dots, \lambda_9} \widetilde{f}(\xi) |\widehat{\eta}|^2. \end{aligned} \tag{4.45}$$

Let  $\lambda$  be a positive constant. We introduce the functionals

$$F(\xi, t) = \xi^{2\epsilon_0} F_1(\xi, t) \quad \text{and} \quad L(\xi, t) = \lambda \widehat{E}(\xi, t) + \frac{1}{\widetilde{f}(\xi)} F(\xi, t). \tag{4.46}$$

We choose  $0 < \lambda_1, 0 < \lambda_3 < \lambda_4 < \lambda_5, 0 < \lambda_2 < \lambda_5 - \lambda_4, \lambda_0 > \frac{1}{|\gamma|}(\lambda_1 + \lambda_5)$  and

$$0 < \varepsilon < \min \{ k_2\lambda_1, k_3\lambda_3, \lambda_2, \lambda_4 - \lambda_3, k_1\lambda_5 - k_1\lambda_4 - k_1\lambda_2, |\gamma|\lambda_0 - \lambda_1 - \lambda_5, k_4 \},$$

and use the definition of  $\widehat{E}$ , we deduce from (4.45) and (4.46), for some positive constant  $c_1$ , that

$$\frac{d}{dt}F(\xi, t) \leq -c_1 \xi^{4+2\epsilon_0} \widehat{E}(\xi, t) + C \widetilde{f}(\xi) \xi^{2\epsilon_0} |\widehat{\eta}|^2. \tag{4.47}$$

Then, from (2.9), (4.46) and (4.47), we infer that

$$\frac{d}{dt}L(\xi, t) \leq -c_1 f(\xi) \widehat{E}(\xi, t) - (k_5\lambda - C) \xi^{2\epsilon_0} |\widehat{\eta}|^2. \tag{4.48}$$

On the other hand, the definitions of  $\widehat{E}$ ,  $F$  and  $L$  imply that there exists  $c_2 > 0$  (independent of  $\lambda$ ) such that, for  $d_0 = 0$  if (1.12) holds, and  $d_0 = 5$  if not,

$$|L(\xi, t) - \lambda \widehat{E}(\xi, t)| \leq c_2 \frac{\xi^{2\epsilon_0} (1 + \xi^4 + |\xi|^{d_0})}{\widetilde{f}(\xi)} \widehat{E}(\xi, t) \leq 6c_2 \widehat{E}(\xi, t).$$

So, we choose  $\lambda > \max \{ \frac{C}{k_5}, 6c_2 \}$ , we obtain (4.20) and (4.21) with  $c_3 = \lambda - 6c_2 > 0$  and  $c_4 = \lambda + 6c_2 > 0$ . The proof can be ended as for Lemma 4.1.  $\square$

**Theorem 4.6.** *Let  $N, \ell \in \mathbb{N}$  such that  $\ell \leq N, U_0 \in H^N(\mathbb{R}) \cap L^1(\mathbb{R})$  and  $U$  be the solution of (2.3). Then for any  $j \in \{0, \dots, N - \ell\}$ , there exist  $c_0, \tilde{c}_0 > 0$  such that, for any  $t \in \mathbb{R}_+$ ,*

(i) **Case (1.6):**

$$\|\partial_x^j U\|_{L^2(\mathbb{R})} \leq c_0 (1+t)^{-1/12-j/6} \|U_0\|_{L^1(\mathbb{R})} + c_0 e^{-\tilde{c}_0 t} \|\partial_x^j U_0\|_{L^2(\mathbb{R})} \tag{4.49}$$

if  $k_1 = k_2 = k_3$ , and

$$\|\partial_x^j U\|_{L^2(\mathbb{R})} \leq c_0 (1+t)^{-1/12-j/6} \|U_0\|_{L^1(\mathbb{R})} + c_0 (1+t)^{-\ell/4} \|\partial_x^{j+\ell} U_0\|_{L^2(\mathbb{R})} \tag{4.50}$$

if not.

(ii) **Case (1.8):**

$$\|\partial_x^j U\|_{L^2(\mathbb{R})} \leq c_0 (1+t)^{-1/8-j/4} \|U_0\|_{L^1(\mathbb{R})} + c_0 (1+t)^{-\ell/2} \|\partial_x^{j+\ell} U_0\|_{L^2(\mathbb{R})} \tag{4.51}$$

if  $k_1 = k_2 = k_3$ , and

$$\|\partial_x^j U\|_{L^2(\mathbb{R})} \leq c_0 (1+t)^{-1/8-j/4} \|U_0\|_{L^1(\mathbb{R})} + c_0 (1+t)^{-\ell/4} \|\partial_x^{j+\ell} U_0\|_{L^2(\mathbb{R})} \tag{4.52}$$

if not.

*Proof.* For (1.6), from (4.31) (low and high frequencies) we have: if  $k_1 = k_2 = k_3$ , then

$$f(\xi) \geq \begin{cases} \xi^6/4 & \text{if } |\xi| \leq 1, \\ 1/4 & \text{if } |\xi| > 1; \end{cases} \tag{4.53}$$

otherwise

$$f(\xi) \geq \begin{cases} \xi^6/6 & \text{if } |\xi| \leq 1, \\ \xi^{-4}/6 & \text{if } |\xi| > 1. \end{cases} \tag{4.54}$$

The proof of (4.50) is identical to the one of Theorem 4.2 by using (4.54) and applying (2.13) (with  $r_1 = 2j$ ,  $r_2 = \frac{c}{6}$  and  $r_3 = 6$ ) and (2.14) (with  $\sigma_1 = 2l$ ,  $\sigma_2 = \frac{c}{6}$  and  $\sigma_3 = 4$ ). To obtain (4.49), noticing that the low frequencies can be treated as for (4.50). For the high frequencies, we observe that (4.53) implies that

$$\begin{aligned} \int_{|\xi|>1} |\xi|^{2j} e^{-cf(\xi)t} |\widehat{U}(\xi, 0)|^2 d\xi &\leq \int_{|\xi|>1} |\xi|^{2j} e^{-ct/4} |\widehat{U}(\xi, 0)|^2 d\xi \\ &\leq e^{-ct/4} \int_{\mathbb{R}} |\xi|^{2j} |\widehat{U}(\xi, 0)|^2 d\xi \\ &\leq e^{-ct/4} \|\partial_x^j U_0\|_{L^2(\mathbb{R})}^2, \end{aligned}$$

so (4.49) holds with  $\tilde{c}_0 = \frac{c}{8}$ . The proof of (4.51) and (4.52) is identical to the one of (4.50) by remarking, for (1.8), that: if  $k_1 = k_2 = k_3$ , then

$$f(\xi) \geq \begin{cases} \xi^4/4 & \text{if } |\xi| \leq 1, \\ \xi^{-2}/4 & \text{if } |\xi| > 1; \end{cases}$$

otherwise

$$f(\xi) \geq \begin{cases} \xi^4/5 & \text{if } |\xi| \leq 1, \\ \xi^{-4}/5 & \text{if } |\xi| > 1. \end{cases}$$

□

**Remark 4.7.** In case (1.6) under (1.12), the fact that  $f$  tends to 1 when  $\xi$  goes to infinity allows to avoid the regularity-loss property in the estimate (4.49) on  $\|\partial_x^j U\|_{L^2(\mathbb{R})}$  because one can take  $j = \ell = 0$ , and the stability of (2.3) is still satisfied with a decay estimate depending only on  $\|U_0\|_{L^1(\mathbb{R})}$  and  $\|U_0\|_{L^2(\mathbb{R})}$ . This remark is valid also for (1.6) in case  $(\tau_1, \tau_2, \tau_3) = (0, 0, 1)$  under (1.12) (see Theorem 4.9 below).

**Case 3:**  $(\tau_1, \tau_2, \tau_3) = (0, 0, 1)$ . In this case, we prove the same stability results for (2.6) and (2.3) that given in the previous subsection, and moreover, the proofs are very similar.

**Lemma 4.8.** *The result of Lemma 4.5 holds when  $(\tau_1, \tau_2, \tau_3) = (0, 0, 1)$ .*

*Proof.* Multiplying (2.10)<sub>6</sub> and (2.10)<sub>8</sub> by  $i\frac{|\gamma|}{\gamma}\xi\bar{\widehat{\eta}}$  and  $-i\frac{|\gamma|}{\gamma}\xi\bar{\widehat{\theta}}$ , respectively, then adding the resulting equations, taking the real part and using (2.11), we obtain

$$\begin{aligned} \frac{d}{dt} \operatorname{Re}\left(i\frac{|\gamma|}{\gamma}\xi\widehat{\theta\widehat{\eta}}\right) &= |\gamma|\xi^2(|\widehat{\eta}|^2 - |\widehat{\theta}|^2) + \frac{|\gamma|}{\gamma}k_4\xi^2 \operatorname{Re}(\widehat{\sigma\theta}) - \frac{|\gamma|}{\gamma}k_1 \operatorname{Re}(i\xi\widehat{v\widehat{\eta}}) \\ &\quad - \frac{|\gamma|}{\gamma}k_3\xi^2 \operatorname{Re}(\widehat{\eta\phi}) + \frac{|\gamma|}{\gamma}k_5\xi^{2\epsilon_0} \operatorname{Re}(i\xi\widehat{\eta\theta}). \end{aligned} \tag{4.55}$$

Also, multiplying (2.10)<sub>2</sub> and (2.10)<sub>8</sub> by  $\bar{\eta}$  and  $\bar{u}$ , respectively, adding the resulting equations, taking the real part and using (2.11), we infer that

$$\frac{d}{dt} \operatorname{Re}(\widehat{u\eta}) = -\gamma \operatorname{Re}(i\xi\widehat{\theta u}) + k_1 \operatorname{Re}(i\xi\widehat{v\eta}) + k_4 \operatorname{Re}(i\xi\widehat{\sigma u}) - k_5 \xi^{2e_0} \operatorname{Re}(\widehat{\eta u}). \quad (4.56)$$

Finally, multiplying (2.10)<sub>4</sub> and (2.10)<sub>8</sub> by  $-i\xi\bar{\eta}$  and  $i\xi\bar{y}$ , respectively, then adding the resulting equations, taking the real part and using (2.11), it follows that

$$\begin{aligned} \frac{d}{dt} \operatorname{Re}(i\xi\widehat{\eta y}) &= \gamma \xi^2 \operatorname{Re}(\widehat{y\theta}) - k_4 \xi^2 \operatorname{Re}(\widehat{\sigma y}) + k_2 \xi^2 \operatorname{Re}(\widehat{\eta z}) \\ &\quad - k_5 \xi^{2e_0} \operatorname{Re}(i\xi\widehat{\eta y}) + k_1 \operatorname{Re}(i\xi\widehat{v\eta}). \end{aligned} \quad (4.57)$$

Let  $\lambda_0, \dots, \lambda_5$  be positive constants, and let

$$\begin{aligned} \lambda_6 &= \frac{k_2}{k_3} \left[ \left(1 - \frac{k_3}{k_1}\right) \lambda_4 \xi^2 - \lambda_2 - \lambda_3 \right], \quad \lambda_7 = -\frac{k_3}{k_2} \lambda_6, \\ \lambda_8 &= \frac{k_2}{k_1} \lambda_5 \xi^2 + \lambda_6 - \lambda_1, \quad \lambda_9 = -\lambda_4 \xi^2 + \lambda_2. \end{aligned}$$

We define the functional

$$\begin{aligned} F_0(\xi, t) &= \operatorname{Re} \left[ \lambda_1 \widehat{y z} - \lambda_2 \widehat{u v} + \lambda_3 \widehat{\theta \phi} + \widehat{\eta \sigma} + \lambda_6 \widehat{z \theta} + \lambda_7 \widehat{\phi y} \right] \\ &\quad + \operatorname{Re}(\lambda_4 \xi^2 \widehat{\theta v} - \lambda_5 \xi^2 \widehat{y v} - \lambda_8 \widehat{u z} - \lambda_9 \widehat{u \phi}). \end{aligned} \quad (4.58)$$

Multiplying (3.1)-(3.10) by  $\lambda_1, -\lambda_2, \lambda_3, -\lambda_4, \lambda_5, 1, \lambda_6, \dots, \lambda_9$ , respectively, and then adding the resulting equations, we find that

$$\begin{aligned} \frac{d}{dt} F_0(\xi, t) &= -\xi^2 (k_3 \lambda_3 |\widehat{\phi}|^2 + \lambda_2 |\widehat{u}|^2 + (\lambda_5 - \lambda_1) |\widehat{y}|^2 + (k_1 \lambda_4 - k_1 \lambda_5 - k_1 \lambda_2) |\widehat{v}|^2) \\ &\quad - \xi^2 (k_2 \lambda_1 |\widehat{z}|^2 + k_4 |\widehat{\sigma}|^2) + I_1 \operatorname{Re}(i\xi\widehat{\theta u}) + I_2 \xi^2 \operatorname{Re}(\widehat{y\theta}) + \gamma \xi^2 \operatorname{Re}(\widehat{\sigma\theta}) \\ &\quad + \xi^2 ((\lambda_3 + \lambda_4) |\widehat{\theta}|^2 + |\widehat{\eta}|^2) \\ &\quad + \operatorname{Re} \left( \gamma \lambda_3 \xi^2 \widehat{\eta \phi} - \gamma \lambda_6 \xi^2 \widehat{\eta z} - i k_5 \xi^{2e_0+1} \widehat{\eta \sigma} - i \gamma \lambda_4 \xi^3 \widehat{\eta v} \right), \end{aligned} \quad (4.59)$$

where

$$I_1 = \lambda_5 \xi^2 + \lambda_2 - \lambda_8 \quad \text{and} \quad I_2 = \lambda_4 - \lambda_5 - \lambda_6 - \lambda_7.$$

We put

$$I_3 = \left( \frac{|\gamma|}{\gamma} k_4 \lambda_0 + \gamma \right) \xi^2 + \frac{k_4}{\gamma} I_1 \quad \text{and} \quad I_4 = \frac{k_4}{\gamma} (I_2 \xi^2 + I_1),$$

and introduce the functional

$$\begin{aligned} F_1(\xi, t) &= \xi^2 F_0(\xi, t) + \frac{|\gamma|}{\gamma} \lambda_0 \xi^2 \operatorname{Re}(i\xi\widehat{\theta \eta}) + \frac{k_4}{\gamma} I_1 \xi^2 \operatorname{Re}(\widehat{v\sigma}) + I_3 \operatorname{Re}(i\xi\widehat{z\sigma}) \\ &\quad + \frac{1}{\gamma} I_1 \xi^2 \operatorname{Re}(\widehat{u\eta}) + I_4 \operatorname{Re}(i\xi\widehat{\phi\sigma}) - \frac{1}{\gamma} I_2 \xi^2 \operatorname{Re}(i\xi\widehat{\eta y}). \end{aligned} \quad (4.60)$$

Multiplying (4.55), (4.33), (4.34), (4.56), (4.36), (4.57), and (4.59) by  $\lambda_0 \xi^2, \frac{k_4}{\gamma} I_1, I_4, \frac{1}{\gamma} I_1 \xi^2, I_3, -\frac{1}{\gamma} I_2 \xi^2$  and  $\xi^2$ , respectively, and then adding the obtained expressions,

we arrive at (observe that (4.33), (4.34) and (4.36) are valid also in case  $(\tau_1, \tau_2, \tau_3) = (0, 0, 1)$ )

$$\begin{aligned} & \frac{d}{dt} F_1(\xi, t) \\ &= -\xi^4 \left( k_2 \lambda_1 |\widehat{z}|^2 + k_3 \lambda_3 |\widehat{\phi}|^2 + \lambda_2 |\widehat{u}|^2 + (\lambda_5 - \lambda_1) |\widehat{y}|^2 + (k_1 \lambda_5 - k_1 \lambda_4 \right. \\ & \quad \left. - k_1 \lambda_2) |\widehat{v}|^2 \right) - \xi^4 \left( (|\gamma| \lambda_0 - \lambda_3 - \lambda_4) |\widehat{\theta}|^2 + k_4 |\widehat{\sigma}|^2 \right) + (|\gamma| \lambda_0 + 1) \xi^4 |\widehat{\eta}|^2 \quad (4.61) \\ & \quad + \xi^2 \operatorname{Re} \left[ (i I_5 \widehat{v} + I_6 \widehat{\phi} + I_7 \widehat{z} - i k_5 \xi^{2\epsilon_0+1} \widehat{\sigma} + i \frac{|\gamma|}{\gamma} k_5 \lambda_0 \xi^{2\epsilon_0+1} \widehat{\theta} \right. \\ & \quad \left. + i \frac{k_5}{\gamma} \xi^{2\epsilon_0+1} I_2 \widehat{y} - \frac{k_5}{\gamma} \xi^{2\epsilon_0} I_1 \widehat{u}) \widehat{\eta} \right], \end{aligned}$$

where

$$\begin{aligned} I_5 &= -\gamma \lambda_4 \xi^3 + \left( \frac{|\gamma|}{\gamma} k_1 \lambda_0 + \frac{k_4 - k_1}{\gamma} I_1 + \frac{k_1}{\gamma} I_2 \right) \xi, \\ I_6 &= \left( -\frac{|\gamma|}{\gamma} k_3 \lambda_0 + \gamma \lambda_3 \right) \xi^2 + I_3, \quad I_7 = -\left( \frac{k_2}{\gamma} I_2 + \gamma \lambda_6 \right) \xi^2 + I_4. \end{aligned}$$

We see that (4.42) and (4.43) are still valid. Then, applying Young’s inequality, we obtain (4.44). Therefore, we define  $F$  and  $L$  by (4.46) and choose  $0 < \lambda_3$ ,  $0 < \lambda_1 < \lambda_4 < \lambda_5$ ,  $0 < \lambda_2 < \lambda_5 - \lambda_4$ ,  $\lambda_0 > \frac{1}{|\gamma|} (\lambda_3 + \lambda_4)$  and

$$0 < \varepsilon < \min\{k_2 \lambda_1, k_3 \lambda_3, \lambda_2, \lambda_5 - \lambda_1, k_1 \lambda_5 - k_1 \lambda_4 - k_1 \lambda_2, |\gamma| \lambda_0 - \lambda_3 - \lambda_4, k_4\},$$

we obtain (4.47) and (4.48). Consequently, the proof can be ended as for Lemma 4.5.  $\square$

**Theorem 4.9.** *The stability result in Theorem 4.6 is satisfied when  $(\tau_1, \tau_2, \tau_3) = (0, 0, 1)$ .*

The proof of the above theorem is identical to the one of Theorem 4.6; therefore we omit it.

### 5. APPLICATION: LOWER ORDER COUPLING TERMS (1.14)

This section concerns the stability of (2.3) in case where the coupling terms (1.13) are replaced by the ones (1.14); more precisely, we study the stability of

$$\begin{aligned} & \varphi_{tt} - k_1(\varphi_x + \psi + w)_x + \tau_1 \gamma q_t = 0, \\ & \psi_{tt} - k_2 \psi_{xx} + k_1(\varphi_x + \psi + w) + \tau_2 \gamma q_t = 0, \\ & w_{tt} - k_3 w_{xx} + k_1(\varphi_x + \psi + w) + \tau_3 \gamma q_t = 0, \\ & q_{tt} - k_4 q_{xx} - k_5 q_{xxt} - \gamma(\tau_1 \varphi_t + \tau_2 \psi_t + \tau_3 w_t) = 0 \end{aligned} \quad (5.1)$$

and

$$\begin{aligned} & \varphi_{tt} - k_1(\varphi_x + \psi + w)_x + \tau_1 \gamma q_t = 0, \\ & \psi_{tt} - k_2 \psi_{xx} + k_1(\varphi_x + \psi + w) + \tau_2 \gamma q_t = 0, \\ & w_{tt} - k_3 w_{xx} + k_1(\varphi_x + \psi + w) + \tau_3 \gamma q_t = 0, \\ & q_{tt} - k_4 q_{xx} + k_5 q_t - \gamma(\tau_1 \varphi_t + \tau_2 \psi_t + \tau_3 w_t) = 0 \end{aligned} \quad (5.2)$$

with the initial conditions (1.9). We define  $U$ , its initial data  $U_0$  and the energy  $\widehat{E}$  as in Section 2. It is clear that (2.3), (2.6), (2.7), and (2.9) are valid with  $A_2$  as in (2.4),

$$A_1 U_x = \begin{pmatrix} -u_x \\ -k_1 v_x \\ -y_x \\ -k_2 z_x \\ -\theta_x \\ -k_3 \phi_x \\ -\eta_x \\ -k_4 \sigma_x \end{pmatrix}, \quad A_0 U = \begin{pmatrix} -y - \theta \\ \tau_1 \gamma \eta \\ 0 \\ k_1 v + \tau_2 \gamma \eta \\ 0 \\ k_1 v + \tau_3 \gamma \eta \\ 0 \\ (1 - \epsilon_0)k_5 \eta - \gamma(\tau_1 u + \tau_2 y + \tau_3 \theta) \end{pmatrix}. \quad (5.3)$$

So, instead of (2.10), we have

$$\begin{aligned} \widehat{v}_t - i\xi \widehat{u} - \widehat{y} - \widehat{\theta} &= 0, \\ \widehat{u}_t - ik_1 \xi \widehat{v} + \tau_1 \gamma \widehat{\eta} &= 0, \\ \widehat{z}_t - i\xi \widehat{y} &= 0, \\ \widehat{y}_t - ik_2 \xi \widehat{z} + k_1 \widehat{v} + \tau_2 \gamma \widehat{\eta} &= 0, \\ \widehat{\phi}_t - i\xi \widehat{\theta} &= 0, \\ \widehat{\theta}_t - ik_3 \xi \widehat{\phi} + k_1 \widehat{v} + \tau_3 \gamma \widehat{\eta} &= 0, \\ \widehat{\sigma}_t - i\xi \widehat{\eta} &= 0, \\ \widehat{\eta}_t - ik_4 \xi \widehat{\sigma} + k_5 \xi^{2\epsilon_0} \widehat{\eta} - \gamma(\tau_1 \widehat{u} + \tau_2 \widehat{y} + \tau_3 \widehat{\theta}) &= 0. \end{aligned} \quad (5.4)$$

**Lemma 5.1.** *Let  $\widehat{U}$  be a solution of (2.6). Then*

(i) *If  $(\tau_1, \tau_2, \tau_3) = (1, 0, 0)$  and  $\chi = 0$ ,  $|\widehat{U}(\xi, t)|$  doesn't converge to zero when time  $t$  goes to infinity.*

(ii) *There exist  $c, \widetilde{c} > 0$  such that (4.1) holds true with the following  $f$ :*

**Case**  $(\tau_1, \tau_2, \tau_3) = (1, 0, 0)$  **and**  $\chi \neq 0$ :

$$f(\xi) = \frac{\xi^{4+2\epsilon_0}}{\widetilde{f}(\xi)} \quad \text{and} \quad \widetilde{f}(\xi) = \begin{cases} 1 + \xi^{10} & \text{for (5.1),} \\ 1 + \xi^8 & \text{for (5.2).} \end{cases} \quad (5.5)$$

**Case**  $(\tau_1, \tau_2, \tau_3) \in \{(0, 1, 0), (0, 0, 1)\}$ :

$$f(\xi) = \frac{\xi^{2+2\epsilon_0}}{\widetilde{f}(\xi)} \quad \text{and} \quad \widetilde{f}(\xi) = \begin{cases} 1 + \xi^6 & \text{for (5.1) under (1.12),} \\ 1 + \xi^4 & \text{for (5.2) under (1.12),} \\ 1 + \xi^{10} & \text{for (5.1) without (1.12),} \\ 1 + \xi^8 & \text{for (5.2) without (1.12).} \end{cases} \quad (5.6)$$

*Proof.* The proof is very similar to the one given in Sections 3 and 4 with some small modifications related to the coupling terms (1.14). We give here a brief idea of the proof.

We see that, for (5.4), the expressions (3.1)-(3.5) and (3.7)-(3.10) are satisfied with  $\tau_j \gamma \widehat{\eta}$  instead of  $i\tau_j \gamma \xi \widehat{\eta}$ , and (3.6) holds true if we replace  $i\gamma \xi(\tau_1 \widehat{u} + \tau_2 \widehat{y} + \tau_3 \widehat{\theta})$  by  $-\gamma(\tau_1 \widehat{u} + \tau_2 \widehat{y} + \tau_3 \widehat{\theta})$ .

Now, we distinguish the cases  $(\tau_1, \tau_2, \tau_3) = (1, 0, 0)$ ,  $(\tau_1, \tau_2, \tau_3) = (0, 1, 0)$  and  $(\tau_1, \tau_2, \tau_3) = (0, 0, 1)$ .

**Case 1.1:**  $(\tau_1, \tau_2, \tau_3) = (1, 0, 0)$  and  $\chi \neq 0$ . We start by modifying the expressions (4.3)-(4.8) (according to (5.4)). Multiplying (5.4)<sub>2</sub> and (5.4)<sub>8</sub> by  $-\frac{|\gamma|}{\gamma}\xi^2\bar{\eta}$  and  $-\frac{|\gamma|}{\gamma}\xi^2\bar{u}$ , respectively, then adding the resulting equations, taking the real part and using (2.11), we obtain

$$\begin{aligned} \frac{d}{dt} \operatorname{Re}\left(-\frac{|\gamma|}{\gamma}\xi^2\widehat{u\eta}\right) &= |\gamma|\xi^2(|\widehat{\eta}|^2 - |\widehat{u}|^2) - \frac{|\gamma|}{\gamma}k_4\xi^2 \operatorname{Re}(i\xi\widehat{\sigma u}) \\ &\quad - \frac{|\gamma|}{\gamma}k_1\xi^2 \operatorname{Re}(i\xi\widehat{v\eta}) + \frac{|\gamma|}{\gamma}k_5\xi^{2\epsilon_0+2} \operatorname{Re}(\widehat{\eta u}). \end{aligned} \tag{5.7}$$

Multiplying (5.4)<sub>6</sub> and (5.4)<sub>8</sub> by  $-i\xi\bar{\eta}$  and  $i\xi\bar{\theta}$ , respectively, adding the resulting equations, taking the real part and using (2.11), we find

$$\begin{aligned} \frac{d}{dt} \operatorname{Re}(i\xi\widehat{\eta\theta}) &= \gamma \operatorname{Re}(i\xi\widehat{u\theta}) - k_4\xi^2 \operatorname{Re}(\widehat{\sigma\theta}) - k_5\xi^{2\epsilon_0} \operatorname{Re}(i\xi\widehat{\eta\theta}) + k_3\xi^2 \operatorname{Re}(\widehat{\phi\eta}) + k_1 \operatorname{Re}(i\xi\widehat{v\eta}). \end{aligned} \tag{5.8}$$

Also, multiplying (5.4)<sub>4</sub> and (5.4)<sub>8</sub> by  $-i\xi\bar{\eta}$  and  $i\xi\bar{y}$ , respectively, adding the resulting equations, taking the real part and using (2.11), we obtain

$$\begin{aligned} \frac{d}{dt} \operatorname{Re}(i\xi\widehat{\eta y}) &= \gamma \operatorname{Re}(i\xi\widehat{u y}) - k_4\xi^2 \operatorname{Re}(\widehat{\sigma y}) - k_5\xi^{2\epsilon_0} \operatorname{Re}(i\xi\widehat{\eta y}) \\ &\quad + k_2\xi^2 \operatorname{Re}(\widehat{z\eta}) + k_1 \operatorname{Re}(i\xi\widehat{v\eta}). \end{aligned} \tag{5.9}$$

Multiplying (5.4)<sub>1</sub> and (5.4)<sub>7</sub> by  $\bar{\sigma}$  and  $\bar{v}$ , respectively, adding the resulting equations, taking the real part and using (2.11), we infer that

$$\frac{d}{dt} \operatorname{Re}(\widehat{v\sigma}) = -\operatorname{Re}(i\xi\widehat{\sigma u}) + \operatorname{Re}(i\xi\widehat{v\eta}) + \operatorname{Re}(\widehat{y\sigma}) + \operatorname{Re}(\widehat{\theta\sigma}). \tag{5.10}$$

Similarly, multiplying (5.4)<sub>3</sub> and (5.4)<sub>7</sub> by  $i\xi\bar{\sigma}$  and  $-i\xi\bar{z}$ , respectively, then adding the resulting equations, taking the real part and using (2.11), we arrive at

$$\frac{d}{dt} \operatorname{Re}(i\xi\widehat{z\sigma}) = -\xi^2 \operatorname{Re}(\widehat{\sigma y}) + \xi^2 \operatorname{Re}(\widehat{z\eta}). \tag{5.11}$$

Multiplying (5.4)<sub>5</sub> and (5.4)<sub>7</sub> by  $i\xi\bar{\sigma}$  and  $-i\xi\bar{\phi}$ , respectively, adding the resulting equations, taking the real part and using (2.11), we entail

$$\frac{d}{dt} \operatorname{Re}(i\xi\widehat{\phi\sigma}) = -\xi^2 \operatorname{Re}(\widehat{\sigma\theta}) + \xi^2 \operatorname{Re}(\widehat{\phi\eta}). \tag{5.12}$$

We put  $\tilde{F}_0(\xi, t) = \xi^2 F_0(\xi, t)$ , where  $F_0$  is defined in (4.9). Multiplying (3.1)-(3.10) (with the modifications cited above) by  $\lambda_1, \dots, \lambda_5, 1, \lambda_6, \dots, \lambda_9$ , respectively, and adding the obtained expressions, we find (instead of (4.10))

$$\begin{aligned} \frac{d}{dt} \tilde{F}_0(\xi, t) &= -\xi^4(k_3\lambda_3|\widehat{\phi}|^2 + (\lambda_5 - \lambda_1)|\widehat{y}|^2 + (\lambda_4 - \lambda_3)|\widehat{\theta}|^2 \\ &\quad + (k_1\lambda_2 - k_1\lambda_4 - k_1\lambda_5)|\widehat{v}|^2) - \xi^4(k_2\lambda_1|\widehat{z}|^2 + k_4|\widehat{\sigma}|^2) \\ &\quad + I_1\xi^2 \operatorname{Re}(i\xi\widehat{\theta u}) + I_2\xi^2 \operatorname{Re}(i\xi\widehat{y u}) - \gamma\xi^2 \operatorname{Re}(i\xi\widehat{\sigma u}) + \xi^4(\lambda_2|\widehat{u}|^2 \\ &\quad + |\widehat{\eta}|^2) + \xi^2 \operatorname{Re}\left(\gamma\lambda_8\widehat{\eta z} + \gamma\lambda_9\widehat{\eta\phi} - ik_5\xi^{2\epsilon_0+1}\widehat{\eta\sigma} - i\gamma\lambda_2\xi\widehat{\eta v}\right), \end{aligned} \tag{5.13}$$

where  $I_1$  and  $I_2$  are defined in (4.11). We put

$$I_3 = -\frac{k_4}{\gamma}(I_1 + |\gamma|\lambda_0)\xi^2 - \gamma, \quad I_4 = -\frac{k_4}{\gamma}(I_2 + |\gamma|\lambda_0)\xi^2 - \gamma, \quad I_5 = -\gamma - \frac{|\gamma|}{\gamma}k_4\lambda_0\xi^2,$$

and introduce the functional

$$F_1(\xi, t) = \tilde{F}_0(\xi, t) - \frac{|\gamma|}{\gamma}\lambda_0\xi^4 \operatorname{Re}(\widehat{u}\widehat{\eta}) + \frac{1}{\gamma}\xi^2 \operatorname{Re}(iI_1\xi\widehat{\eta}\widehat{\theta} + iI_2\xi\widehat{\eta}\widehat{y}) + I_5\xi^2 \operatorname{Re}(\widehat{v}\widehat{\sigma}) + I_4 \operatorname{Re}(i\xi\widehat{z}\widehat{\sigma}) + I_3 \operatorname{Re}(i\xi\widehat{\phi}\widehat{\sigma}). \tag{5.14}$$

Multiplying (5.7)-(5.12) by  $\lambda_0\xi^2$ ,  $\frac{1}{\gamma}I_1\xi^2$ ,  $\frac{1}{\gamma}I_2\xi^2$ ,  $I_5\xi^2$ ,  $I_4$  and  $I_3$ , respectively, and then adding the obtained equations and (5.13), we arrive at

$$\begin{aligned} \frac{d}{dt}F_1(\xi, t) &= -\xi^4 \left( k_2\lambda_1|\widehat{z}|^2 + k_3\lambda_3|\widehat{\phi}|^2 + (\lambda_5 - \lambda_1)|\widehat{y}|^2 + (\lambda_4 - \lambda_3)|\widehat{\theta}|^2 \right. \\ &\quad \left. + (k_1\lambda_2 - k_1\lambda_4 - k_1\lambda_5)|\widehat{v}|^2 \right) - \xi^4 \left( (|\gamma|\lambda_0 - \lambda_2)|\widehat{u}|^2 + k_4|\widehat{\sigma}|^2 \right) \\ &\quad + (|\gamma|\lambda_0 + 1)\xi^4|\widehat{\eta}|^2 + \xi^2 \operatorname{Re} \left( \frac{|\gamma|}{\gamma}k_5\lambda_0\xi^{2\epsilon_0+2}\widehat{\eta}\widehat{u} \right. \\ &\quad \left. - ik_5\xi^{2\epsilon_0+1}\widehat{\eta}\widehat{\sigma} \right) + \xi^2 \operatorname{Re} \left[ i\left(\frac{k_1}{\gamma}(I_1 + I_2) + \gamma\lambda_2 \right. \right. \\ &\quad \left. \left. + I_5 - \frac{|\gamma|}{\gamma}k_1\lambda_0\xi^2\right)\xi\widehat{v}\widehat{\eta} + i\frac{k_5}{\gamma}I_1\xi^{2\epsilon_0+1}\widehat{\theta}\widehat{\eta} + i\frac{k_5}{\gamma}I_2\xi^{2\epsilon_0+1}\widehat{y}\widehat{\eta} \right] \\ &\quad + \xi^2 \operatorname{Re} \left[ \left(\gamma\lambda_8 + I_4 + \frac{k_2}{\gamma}I_2\xi^2\right)\widehat{\eta}\widehat{z} + (\gamma\lambda_9 + I_3 + \frac{k_3}{\gamma}I_1\xi^2)\widehat{\eta}\widehat{\phi} \right]. \end{aligned} \tag{5.15}$$

Now, we consider  $\tilde{f}$  and  $f$  defined in (5.5), and introduce the functionals

$$F(\xi, t) = \xi^{2\epsilon_0}F_1(\xi, t) \quad \text{and} \quad L(\xi, t) = \lambda\widehat{E}(\xi, t) + \frac{1}{\tilde{f}(\xi)}F(\xi, t). \tag{5.16}$$

Applying Young's inequality, (5.15) implies (4.16). So, the proof can be completed as for Lemma 4.1.

**Case 1.2:**  $(\tau_1, \tau_2, \tau_3) = (1, 0, 0)$  and  $\chi = 0$ . To prove that  $|\widehat{U}(\xi, t)|$  does not converge to zero when time  $t$  goes to infinity, it is enough to prove (4.30), where (according to (5.4))

$$\lambda I - A = \begin{pmatrix} \lambda & -i\xi & 0 & -1 & 0 & -1 & 0 & 0 \\ -ik_1\xi & \lambda & 0 & 0 & 0 & 0 & 0 & \gamma \\ 0 & 0 & \lambda & -i\xi & 0 & 0 & 0 & 0 \\ k_1 & 0 & -ik_2\xi & \lambda & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \lambda & -i\xi & 0 & 0 \\ k_1 & 0 & 0 & 0 & -ik_2\xi & \lambda & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \lambda & -i\xi \\ 0 & -\gamma & 0 & 0 & 0 & 0 & -ik_4\xi & k_5\xi^{2\epsilon_0} + \lambda \end{pmatrix}.$$

A direct computation shows that

$$\begin{aligned} \det(\lambda I - A) &= 2k_1\lambda^2(\lambda^2 + k_2\xi^2)[\lambda(\lambda + k_5\xi^{2\epsilon_0}) + k_4\xi^2 + \gamma^2] + k_4\xi^2(\lambda^2 + k_1\xi^2)(\lambda^2 + k_2\xi^2)^2 \\ &\quad + \lambda(\lambda^2 + k_2\xi^2)^2[\lambda^2(\lambda + k_5\xi^{2\epsilon_0}) + \gamma^2\lambda + k_1\xi^2(\lambda + k_5\xi^{2\epsilon_0})]. \end{aligned}$$

Then, the conclusions indicated in the proof of Theorem 4.4 are valid for (5.4).

**Case 2:**  $(\tau_1, \tau_2, \tau_3) = (0, 1, 0)$ . First, we modify the expressions (4.32)-(4.37) according to (5.4). Multiplying (5.4)<sub>4</sub> and (5.4)<sub>8</sub> by  $-\frac{|\gamma|}{\gamma}\xi^2\bar{\eta}$  and  $-\frac{|\gamma|}{\gamma}\xi^2\bar{y}$ , respectively, then adding the resulting equations, taking the real part and using (2.11), we obtain

$$\begin{aligned} & \frac{d}{dt} \operatorname{Re} \left( -\frac{|\gamma|}{\gamma} \xi^2 \bar{y} \bar{\eta} \right) \\ &= |\gamma| \xi^2 (|\hat{\eta}|^2 - |\hat{y}|^2) - \frac{|\gamma|}{\gamma} k_4 \xi^2 \operatorname{Re}(i \xi \bar{\sigma} \bar{y}) + \frac{|\gamma|}{\gamma} k_1 \xi^2 \operatorname{Re}(\hat{\eta} \bar{v}) \\ & \quad - \frac{|\gamma|}{\gamma} k_2 \xi^2 \operatorname{Re}(i \xi \bar{z} \bar{\eta}) + \frac{|\gamma|}{\gamma} k_5 \xi^{2\epsilon_0+2} \operatorname{Re}(\hat{\eta} \bar{y}). \end{aligned} \tag{5.17}$$

Multiplying (5.4)<sub>1</sub> and (5.4)<sub>7</sub> by  $i\xi\bar{\sigma}$  and  $-i\xi\bar{v}$ , respectively, adding the resulting equations, taking the real part and using (2.11), we find

$$\frac{d}{dt} \operatorname{Re}(i\xi\bar{v}\bar{\sigma}) = -\operatorname{Re}(i\xi\bar{\sigma}\bar{y}) - \operatorname{Re}(i\xi\bar{\sigma}\bar{\theta}) - \xi^2 \operatorname{Re}(\hat{u}\bar{\sigma}) + \xi^2 \operatorname{Re}(\hat{\eta}\bar{v}). \tag{5.18}$$

Also, multiplying (5.4)<sub>3</sub> and (5.4)<sub>7</sub> by  $\bar{\sigma}$  and  $\bar{z}$ , respectively, adding the resulting equations, taking the real part and using (2.11), we obtain

$$\frac{d}{dt} \operatorname{Re}(\hat{z}\bar{\sigma}) = -\operatorname{Re}(i\xi\bar{\sigma}\bar{y}) + \operatorname{Re}(i\xi\hat{\eta}\bar{z}). \tag{5.19}$$

Multiplying (5.4)<sub>2</sub> and (5.4)<sub>8</sub> by  $-i\xi\bar{\eta}$  and  $i\xi\bar{u}$ , respectively, adding the resulting equations, taking the real part and using (2.11), we infer that

$$\frac{d}{dt} \operatorname{Re}(-i\xi\bar{u}\bar{\eta}) = \gamma \operatorname{Re}(i\xi\bar{y}\bar{u}) + k_1 \xi^2 \operatorname{Re}(\hat{v}\bar{\eta}) - k_4 \xi^2 \operatorname{Re}(\bar{\sigma}\bar{u}) - k_5 \xi^{2\epsilon_0} \operatorname{Re}(i\xi\hat{\eta}\bar{u}). \tag{5.20}$$

Multiplying (5.4)<sub>5</sub> and (5.4)<sub>7</sub> by  $\bar{\sigma}$  and  $\bar{\phi}$ , respectively, adding the resulting equations, taking the real part and using (2.11), we see that

$$\frac{d}{dt} \operatorname{Re}(\hat{\phi}\bar{\sigma}) = -\operatorname{Re}(i\xi\bar{\sigma}\bar{\theta}) + \operatorname{Re}(i\xi\hat{\eta}\bar{\phi}). \tag{5.21}$$

Finally, multiplying (5.4)<sub>6</sub> and (5.4)<sub>8</sub> by  $\bar{\eta}$  and  $\bar{\theta}$ , respectively, adding the resulting equations, taking the real part and using (2.11), it follows that

$$\begin{aligned} & \frac{d}{dt} \operatorname{Re}(\hat{\eta}\bar{\theta}) \\ &= \gamma \operatorname{Re}(\hat{y}\bar{\theta}) + k_4 \operatorname{Re}(i\xi\bar{\sigma}\bar{\theta}) - k_3 \operatorname{Re}(i\xi\hat{\eta}\bar{\phi}) - k_5 \xi^{2\epsilon_0} \operatorname{Re}(\hat{\eta}\bar{\theta}) - k_1 \operatorname{Re}(\hat{v}\bar{\eta}). \end{aligned} \tag{5.22}$$

We define the functional  $F_0$  by (4.38), and we obtain (instead of (4.39))

$$\begin{aligned} & \frac{d}{dt} F_0(\xi, t) \\ &= -\xi^2 (k_3 \lambda_3 |\hat{\phi}|^2 + \lambda_2 |\hat{u}|^2 + (\lambda_4 - \lambda_3) |\hat{\theta}|^2 + (k_1 \lambda_5 - k_1 \lambda_4 - k_1 \lambda_2) |\hat{v}|^2) \\ & \quad - \xi^2 (k_2 \lambda_1 |\hat{z}|^2 + k_4 |\hat{\sigma}|^2) + I_1 \operatorname{Re}(i\xi\bar{y}\bar{u}) + I_2 \xi^2 \operatorname{Re}(\hat{y}\bar{\theta}) - \gamma \operatorname{Re}(i\xi\bar{\sigma}\bar{y}) \\ & \quad + \xi^2 ((\lambda_1 + \lambda_5) |\hat{y}|^2 + |\hat{\eta}|^2) + \operatorname{Re}(-i\gamma \lambda_1 \xi \hat{\eta} \bar{z} + i\gamma \lambda_7 \xi \hat{\eta} \bar{\phi}) \\ & \quad - ik_5 \xi^{2\epsilon_0+1} \hat{\eta} \bar{\sigma} - \gamma \lambda_5 \xi^2 \hat{\eta} \bar{v}. \end{aligned} \tag{5.23}$$

We put

$$I_3 = -\frac{|\gamma|}{\gamma} k_4 \lambda_0 \xi^2 - \frac{k_4}{\gamma} I_1 - \gamma \quad \text{and} \quad I_4 = -\frac{k_4}{\gamma} (I_2 \xi^2 + I_1),$$

and introduce the functional

$$\begin{aligned}
 F_1(\xi, t) &= F_0(\xi, t) - \frac{|\gamma|}{\gamma} \lambda_0 \xi^2 \operatorname{Re}(\widehat{y\eta}) + \frac{k_4}{\gamma} I_1 \operatorname{Re}(i\xi \widehat{v\sigma}) + I_3 \operatorname{Re}(\widehat{z\sigma}) \\
 &\quad + \frac{1}{\gamma} I_1 \operatorname{Re}(i\xi \widehat{u\eta}) + I_4 \operatorname{Re}(\widehat{\phi\sigma}) - \frac{1}{\gamma} I_2 \xi^2 \operatorname{Re}(\widehat{\eta\theta}).
 \end{aligned}
 \tag{5.24}$$

Multiplying (5.17)-(5.22) and (5.23) by  $\lambda_0, \frac{k_4}{\gamma} I_1, I_3, -\frac{1}{\gamma} I_1, I_4, -\frac{1}{\gamma} I_2 \xi^2$  and 1, respectively, and adding the obtained expressions, we arrive at

$$\begin{aligned}
 &\frac{d}{dt} F_1(\xi, t) \\
 &= -\xi^2 (k_2 \lambda_1 |\widehat{z}|^2 + k_3 \lambda_3 |\widehat{\phi}|^2 + \lambda_2 |\widehat{u}|^2 + (\lambda_4 - \lambda_3) |\widehat{\theta}|^2 + (k_1 \lambda_5 - k_1 \lambda_4 \\
 &\quad - k_1 \lambda_2) |\widehat{v}|^2) - \xi^2 ((|\gamma| \lambda_0 - \lambda_1 - \lambda_5) |\widehat{y}|^2 + k_4 |\widehat{\sigma}|^2) + (|\gamma| \lambda_0 + 1) \xi^2 |\widehat{\eta}|^2 \\
 &\quad + \xi \operatorname{Re} \left[ (I_5 \widehat{v} + i I_6 \widehat{z} + i I_7 \widehat{\phi} - i k_5 \xi^{2\epsilon_0} \widehat{\sigma} + \frac{|\gamma|}{\gamma} k_5 \lambda_0 \xi^{2\epsilon_0+1} \widehat{y} \right. \\
 &\quad \left. + \frac{k_5}{\gamma} \xi^{2\epsilon_0+1} I_2 \widehat{\theta} + i \frac{k_5}{\gamma} \xi^{2\epsilon_0} I_1 \widehat{u} \widehat{\eta} \right],
 \end{aligned}
 \tag{5.25}$$

where

$$\begin{aligned}
 I_5 &= \left( \frac{|\gamma|}{\gamma} k_1 \lambda_0 - \gamma \lambda_5 + \frac{k_1}{\gamma} (I_2 - I_1) + \frac{k_4}{\gamma} I_1 \right) \xi, \\
 I_6 &= \frac{|\gamma|}{\gamma} k_2 \lambda_0 \xi^2 - \gamma \lambda_1 + I_3, \quad I_7 = \frac{k_3}{\gamma} I_2 \xi^2 + I_4 + \gamma \lambda_7.
 \end{aligned}$$

Because (4.42) is still satisfied, we infer that, for  $\tilde{f}$  defined in (5.6),

$$\begin{aligned}
 &\frac{d}{dt} F_1(\xi, t) \\
 &\leq -\xi^2 ((k_2 \lambda_1 - \epsilon) |\widehat{z}|^2 + (k_3 \lambda_3 - \epsilon) |\widehat{\phi}|^2 + (\lambda_2 - \epsilon) |\widehat{u}|^2 + (\lambda_4 - \lambda_3 - \epsilon) |\widehat{\theta}|^2) \\
 &\quad - \xi^2 ((k_1 \lambda_5 - k_1 \lambda_4 - k_1 \lambda_2 - \epsilon) |\widehat{v}|^2 + (|\gamma| \lambda_0 - \lambda_1 - \lambda_5 - \epsilon) |\widehat{y}|^2 \\
 &\quad + (k_4 - \epsilon) |\widehat{\sigma}|^2) + C_{\epsilon, \lambda_0, \dots, \lambda_9} \tilde{f}(\xi) |\widehat{\eta}|^2.
 \end{aligned}
 \tag{5.26}$$

Therefore, we introduce the functionals  $F$  and  $L$  defined in (4.46) and consider the same choices of  $\lambda_0, \dots, \lambda_5$  and  $\epsilon$ , we arrive at

$$\frac{d}{dt} F(\xi, t) \leq -c_1 \xi^{2+2\epsilon_0} \widehat{E}(\xi, t) + C \tilde{f}(\xi) \xi^{2\epsilon_0} |\widehat{\eta}|^2.
 \tag{5.27}$$

Hence, the proof can be completed as for Lemma 4.1.

**Case 3:**  $(\tau_1, \tau_2, \tau_3) = (0, 0, 1)$ . This case can be treated using very similar modifications to the ones considered for the case  $(\tau_1, \tau_2, \tau_3) = (0, 1, 0)$ ; we omit the details here.  $\square$

**Theorem 5.2.** *Let  $N, \ell \in \mathbb{N}$  such that  $\ell \leq N, U_0 \in H^N(\mathbb{R}) \cap L^1(\mathbb{R})$  and  $U$  be the solution of (2.3). Then for any  $j \in \{0, \dots, N - \ell\}$ , there exist  $c_0, \tilde{c}_0 > 0$  such that, for any  $t \in \mathbb{R}_+$ ,*

(i) **Case**  $(\tau_1, \tau_2, \tau_3) = (1, 0, 0)$  **and**  $\chi \neq 0$ :

$$\|\partial_x^j U\|_{L^2(\mathbb{R})} \leq c_0 (1+t)^{-1/12-j/6} \|U_0\|_{L^1(\mathbb{R})} + c_0 (1+t)^{-\ell/4} \|\partial_x^{j+\ell} U_0\|_{L^2(\mathbb{R})}$$

for (5.1), and

$$\|\partial_x^j U\|_{L^2(\mathbb{R})} \leq c_0(1+t)^{-1/8-j/4} \|U_0\|_{L^1(\mathbb{R})} + c_0(1+t)^{-\ell/4} \|\partial_x^{j+\ell} U_0\|_{L^2(\mathbb{R})}$$

for (5.2).

(ii) **Case**  $(\tau_1, \tau_2, \tau_3) \in \{(0, 1, 0), (0, 0, 1)\}$  and  $k_1 = k_2 = k_3$ :

$$\|\partial_x^j U\|_{L^2(\mathbb{R})} \leq c_0(1+t)^{-1/8-j/4} \|U_0\|_{L^1(\mathbb{R})} + c_0(1+t)^{-\ell/2} \|\partial_x^{j+\ell} U_0\|_{L^2(\mathbb{R})}$$

for (5.1), and

$$\|\partial_x^j U\|_{L^2(\mathbb{R})} \leq c_0(1+t)^{-1/4-j/2} \|U_0\|_{L^1(\mathbb{R})} + c_0(1+t)^{-\ell/2} \|\partial_x^{j+\ell} U_0\|_{L^2(\mathbb{R})}$$

for (5.2).

(iii) **Case**  $(\tau_1, \tau_2, \tau_3) \in \{(0, 1, 0), (0, 0, 1)\}$  and  $k_1 = k_2 = k_3$ :

$$\|\partial_x^j U\|_{L^2(\mathbb{R})} \leq c_0(1+t)^{-1/8-j/4} \|U_0\|_{L^1(\mathbb{R})} + c_0(1+t)^{-\ell/6} \|\partial_x^{j+\ell} U_0\|_{L^2(\mathbb{R})}$$

for (5.1), and

$$\|\partial_x^j U\|_{L^2(\mathbb{R})} \leq c_0(1+t)^{-1/4-j/2} \|U_0\|_{L^1(\mathbb{R})} + c_0(1+t)^{-\ell/6} \|\partial_x^{j+\ell} U_0\|_{L^2(\mathbb{R})}$$

for (5.2).

The proof of the above theorem is identical to the one of Theorem 4.6; therefore we omit it.

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