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DUALITY ARGUMENTS FOR WELL-POSEDNESS OF HISTORY-DEPENDENT VARIATIONAL INEQUALITIES

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Abstract. In this article we introduce a concept of dual problems in metric spaces. Then we state and prove an equivalence result concerning their well-posedness with respect to appropriate Tykhonov triples. We exemplify this result in the study of a history-dependent variational inequality with time-dependent constraints, for which the dual problem is in a form of a history-dependent inclusion. This allows us to deduce a convergence result which provides the continuous dependence of the solution with respect to the data. We end this paper with an example which represents an evidence of our abstract results.

1. Introduction

The concept of well-posedness with respect to a Tykhonov triple was introduced in [14]. It extends the concept of well-posedness for a minimization problem, introduced in the pioneering work [12] as well as the concepts of well-posedness used in [1, 8, 15] for various optimization problems and [2, 3, 4, 5, 6, 7, 13] for various classes of inequalities. This abstract concept was applied in [10] in the study of variational inequalities governed by a history-dependent operator, the so-called history-dependent variational inequalities. There, the well-posedness with respect to several Tykhonov triples was studied and a strategy which allows to deduce convergence results was discussed. An existence and uniqueness result for a class of history-dependent inclusions was obtained in the recent paper [9].

This article represents a continuation of [9, 10, 14]. Here, we complete the theoretical study initiated in [14] by studying the well-posedness of a couple of problems in duality. The idea is to deduce the well-posedness of a problem $\mathcal{P}$ by using the well-posedness of a different problem $\mathcal{Q}$, called the dual of $\mathcal{P}$. The interest in this method arises in the fact that in several cases we have a number of results in the study of Problem $\mathcal{Q}$ which can be useful in the analysis of Problem $\mathcal{P}$. This represents the first trait of novelty of the current paper. The second novelty is that we use these arguments in the study of a history-dependent variational inequality which is more general than the inequality in [10]. Indeed, in contrast with [10], the inequality we consider in this paper involves a time-dependent set of constraints. As a consequence, the dual problem of this inequality is given by a history-dependent inclusion, already studied in [9].

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The rest of this article is structured as follows. In Section 2 we introduce the concept of dual problems in metric spaces. Then we state and prove an equivalence result, Theorem 2.2. In Section 3 we introduce a history-dependent variational inequality $\mathcal{P}$, then we use Theorem 2.2 to prove its well-posedness. We complete our study in Section 4 where we prove a convergence result which states the continuous dependence of the solution of $\mathcal{P}$ with respect to the data.

We end this section by recalling some basic definitions which will be crucial in the rest of the paper. Consider an abstract mathematical object $\mathcal{M}$, called generic "problem", defined in a metric space (Z, d). Problem $\mathcal{M}$ could be an equation, a minimization problem, a fixed point problem, an inclusion or an inequality problem, for instance. We associate to Problem $\mathcal{M}$ the concept of "solution" which follows from the context. The concept of well-posedness for Problem $\mathcal{M}$ is provided by the following definition.

Definition 1.1.

(a) A Tykhonov triple is a mathematical object of the form $\mathcal{T} = (I, \Omega, \mathcal{C})$ where I is a given nonempty set, $\Omega : I \to 2^Z$ is a nonempty set-valued operator and $\mathcal{C}$ is a nonempty subset of sequences with elements in I.

(b) Given a Tykhonov triple $\mathcal{T} = (I, \Omega, \mathcal{C})$, a sequence $\{z_n\} \subset Z$ is called a $\mathcal{T}$-approximating sequence if there exists a sequence $\{\theta_n\} \in \mathcal{C}$ such that $z_n \in \Omega(\theta_n)$ for each $n \in \mathbb{N}$.

(c) Given a Tykhonov triple $\mathcal{T} = (I, \Omega, \mathcal{C})$, Problem $\mathcal{M}$ is said to be $\mathcal{T}$-well-posed if it has a unique solution and every $\mathcal{T}$-approximating sequence converges in Z to its solution.

For a Tykhonov triple $\mathcal{T} = (I, \Omega, \mathcal{C})$ we refer to I as the set of parameters; the family of sets $\{\Omega(\theta)\}_{\theta \in I}$ represents the family of approximating sets; besides, we say that $\mathcal{C}$ defines the criterion of convergence. We remark that approximating sequences always exist since, by assumption, $\mathcal{C} \neq \emptyset$ and, moreover, for any sequence $\{\theta_n\} \in \mathcal{C}$ and any $n \in \mathbb{N}$, the set $\Omega(\theta_n)$ is not empty. In addition, we recall that the concept of approximating sequence depends on the Tykhonov triple $\mathcal{T}$ and, for this reason, we use the terminology "$\mathcal{T}$-approximating sequence". As a consequence, the concept of well-posedness depends on $\mathcal{T}$ and, therefore, we refer to it as "well-posedness with respect to $\mathcal{T}$" or "$\mathcal{T}$-well-posedness", for short.

2. An abstract equivalence result

Consider two abstract problems $\mathcal{P}$ and $\mathcal{Q}$ formulated in the metric spaces (U, d_U) and (Σ, d_Σ), respectively. We start with the following definition.

Definition 2.1. Problems $\mathcal{P}$ and $\mathcal{Q}$ are said to be dual of each other if there exists a mapping $D : U \to \Sigma$ such that:

(a) D is bijective;

(b) both $D : U \to \Sigma$ and its inverse $D^{-1} : \Sigma \to U$ are continuous;

(c) $u \in U$ is solution of Problem $\mathcal{P}$ if and only if $\sigma = Du$ is solution of Problem $\mathcal{Q}$.

If (a)–(c) hold we say that Problem $\mathcal{Q}$ is a dual problem of Problem $\mathcal{P}$ (with D) and, conversely, Problem $\mathcal{P}$ is a dual problem of Problem $\mathcal{Q}$ (with D^{-1}). In this framework we consider two Tykhonov triples $\mathcal{T}_P = (I, \Omega_P, \mathcal{C})$ and $\mathcal{T}_Q = (I, \Omega_Q, \mathcal{C})$

in which the approximating sets $\Omega_P : I \to 2^U$ and $\Omega_Q : I \to 2^\Sigma$ are such that

$$\Omega_P(\theta) = \{u \in U : Du \in \Omega_Q(\theta)\}, \quad \Omega_Q(\theta) = \{\sigma \in \Sigma : D^{-1}\sigma \in \Omega_P(\theta)\}, \tag{2.1}$$

for all $\theta \in I$. The interest in considering these Tykhonov triples follows from the following equivalence result.

Theorem 2.2. *Let $\mathcal{P}$ and $\mathcal{Q}$ be dual problems (with D and D^{-1}, respectively). Then Problem $\mathcal{P}$ is $\mathcal{T}_P$-well-posed if and only if Problem $\mathcal{Q}$ is $\mathcal{T}_Q$-well-posed.*

Proof. Assume that Problem $\mathcal{P}$ is $\mathcal{T}_P$-well-posed. This implies that Problem $\mathcal{P}$ has a unique solution $u \in U$. Moreover, it follows from properties (a) and (c) in Definition 2.1 that $\sigma = Du$ is the unique solution of Problem $\mathcal{Q}$.

Let $\{\sigma_n\} \subset \Sigma$ be a $\mathcal{T}_Q$-approximating sequence for Problem $\mathcal{Q}$. Then there exists a sequence $\{\theta_n\} \in \mathcal{C}$ such that, for each $n \in \mathbb{N}$, $\sigma_n \in \Omega_Q(\theta_n)$, i.e., $D^{-1}\sigma_n \in \Omega_P(\theta_n)$, which means that $\{D^{-1}\sigma_n\} \subset U$ is a $\mathcal{T}_P$-approximating sequence for Problem $\mathcal{P}$. Therefore, using the $\mathcal{T}_P$-well-posedness of Problem $\mathcal{P}$ and the continuity of operator D to deduce that the sequence $\{\sigma_n\}$ converges to the unique solution $\sigma \in \Sigma$ of Problem Q. We conclude from above that Problem $\mathcal{Q}$ is $\mathcal{T}_Q$-well-posed.

Similar arguments show that if Problem $\mathcal{Q}$ is $\mathcal{T}_Q$-well-posed then Problem $\mathcal{P}$ is $\mathcal{T}_P$-well-posed, which completes the proof. □

3. Problem statement and its well-posedness

Everywhere in the rest of this article V will be a real Hilbert space. We use $(\cdot,\cdot)_V$ and $\|\cdot\|_V$ for the inner product and the associated norm of space V. For any nonempty closed convex set $K \subset X$ we denote by $P_K : V \to K$ and $N_K : V \to 2^V$ the projection operator on K and the outward normal cone of K, respectively. Moreover, we use notation $X = C([0,T];V)$ for the space of continuous functions on $[0,T]$ with values in V, equipped with the norm of the uniform convergence. Finally, we mention that, unless stated otherwise, all the limits below are considered as $n \to \infty$, even if we do not mention it explicitly.

Let $K : [0,T] \to 2^V$, $A \colon V \to V$, $\mathcal{S} : C([0,T];V) \to C([0,T];V)$ and $f : [0,T] \to V$. With these data we consider the following time-dependent inequality.

Problem $\mathcal{P}$. *Find a function $u \in C([0,T];V)$ such that the following inequality holds: $u(t) \in K(t)$ and*

$$(Au(t), v - u(t))_V + (\mathcal{S}u(t), v - u(t))_V \geq (f(t), v - u(t))_V \tag{3.1}$$

for all $v \in K(t)$ and $t \in [0,T]$.

Note that here and below when no confusion arises, we use the shorthand notation $\mathcal{S}u(t)$ to represent the value of the function $\mathcal{S}u$ at the point t, i.e., $\mathcal{S}u(t) = (\mathcal{S}u)(t)$. Moreover, for an element $v \in V$ we shall still write v for the constant function $t \mapsto v$ for all $t \in [0,T]$ and, therefore, notation $\mathcal{S}v$ used below in this section defines an element of X.

Our aim in what follows is to associate Problem $\mathcal{P}$ with a dual problem $\mathcal{Q}$ and to deduce its well-posedness with respect to a specific Tykhonov triple. To this end, we consider the following hypotheses.

(H1) (a) $K : [0,T] \to 2^V$ has nonempty closed and convex values.
(b) For each $u \in V$, $t \in [0,T]$ and each sequence $\{t_n\} \subset [0,T]$, $t_n \to t$ implies $P_{K(t_n)}u \to P_{K(t)}u$ in V.

(H2) $A\colon V\to V$ is a linear continuous and coercive operator, i.e. there exist L_A, $m_A>0$ such that
$$\|Au\|_V\le L_A\|v\|_V,\ (Au,u)_V\ge m_A\|u\|_V^2\quad\forall\,u\in V.$$
(H3) $\mathcal{S}\colon C([0,T];V)\to C([0,T];V)$ is a history-dependent operator, i.e., there exists $L_S>0$ such that
$$\|\mathcal{S}u(t)-\mathcal{S}v(t)\|_V\le L_S\int_0^t\|u(s)-v(s)\|_V\,ds\quad\forall\,u,v\in C([0,T];V),\ t\in[0,T].$$
(H4) $f\in C([0,T];V)$.

Note that assumption (H2) implies that the operator A is invertible and its inverse $A^{-1}:V\to V$ satisfies the inequalities
$$\|A^{-1}u\|_V\le\frac{1}{m_A}\,\|u\|_V,\quad (A^{-1}u,u)_V\ge\frac{m_A}{L_A^2}\,\|u\|_V^2\quad\forall\,u\in V. \tag{3.2}$$
Next, we consider the Tykhonov triple $\mathcal{T}_P=(I,\Omega_P,\mathcal{C})$ defined as follows:
$$I=\mathbb{R}_+,\quad\mathcal{C}=\{\{\theta_n\}_n:\theta_n\in I\ \forall\,n\in\mathbb{N},\quad\theta_n\to0\text{ as }n\to\infty\} \tag{3.3}$$
and, for each $\theta\ge0$, the set $\Omega_P(\theta)$ is defined as follows:
$$\begin{aligned}\Omega_P(\theta)=\big\{&u\in C([0,T];V):u(t)\in K(t),\\ &(Au(t),v-u(t))_V+(\mathcal{S}u(t),v-u(t))_V+\theta\ge(f(t),v-u(t))_V\\ &\forall\,v\in K(t),\ t\in[0,T]\big\}.\end{aligned} \tag{3.4}$$
Note that $\Omega_P(\theta)\ne\emptyset$ for each $\theta\in\mathbb{R}_+$, as it will result from the proof of Theorem 3.2 below.

To proceed, we need the following preliminary result.

Proposition 3.1. *Under assumptions* (H2)–(H4)*, the operator $D:C([0,T];V)\to C([0,T];V)$ defined by*
$$Du(t)=Au(t)+\mathcal{S}u(t)-f(t)\quad\forall\,u\in C([0,T];V),\ t\in[0,T] \tag{3.5}$$
is bijective and has inverse of the form $A^{-1}+\mathcal{R}$, where $\mathcal{R}\colon C([0,T];V)\to C([0,T];V)$ is a history-dependent operator with constant $L_R>0$.

A proof of Proposition 3.1 can be found in [11, p.55]. Based on Proposition 3.1 we consider the following problem.

Problem $\mathcal{Q}$. *Find a function $\sigma\in C([0,T];V)$ such that the following inclusion holds:*
$$-\sigma(t)\in N_{K(t)}(A^{-1}\sigma(t)+\mathcal{R}\sigma(t))\quad\forall\,t\in[0,T]. \tag{3.6}$$
Our main result in this section is the following.

Theorem 3.2. *Assume* (H1)–(H4) *hold. Then Problem $\mathcal{P}$ is $\mathcal{T}_P$-well-posed.*

Proof. The proof will be carried out in several steps, as follows.

Step 1. We prove that Problems $\mathcal{P}$ and $\mathcal{Q}$ are dual problems. Indeed, since
$$\|v\|_X=\max_{t\in[0,T]}\|v(t)\|_V\quad\forall\,v\in C([0,T];V),$$
it is easy to see that any history-dependent operator $\mathcal{H}:C([0,T];V)\to C([0,T];V)$ is continuous. Therefore, Proposition 3.1 implies that the operator D defined by (3.5) satisfies conditions (a) and (b) in Definition 2.1 with $U=\Sigma=C([0,T];V)$.

Assume now that $u \in C([0,T];V)$ is a solution of Problem $\mathcal{P}$, i.e., $u(t) \in K(t)$ and

$$(f(t) - Au(t) - \mathcal{S}u(t), v - u(t))_V \le 0 \quad \forall\, v \in K(t),\ t \in [0,T]. \tag{3.7}$$

This inequality is equivalent to the inclusion

$$f(t) - Au(t) - \mathcal{S}u(t) \in N_{K(t)}(u(t)) \quad \forall\, t \in [0,T]. \tag{3.8}$$

We now take $\sigma(t) = Du(t)$ for all $t \in [0,T]$, then we use definition (3.5) and Proposition 3.1 to see that

$$\sigma(t) = Au(t) + \mathcal{S}u(t) - f(t), \quad u(t) = A^{-1}\sigma(t) + \mathcal{R}\sigma(t)\ \forall\, t \in [0,T]. \tag{3.9}$$

Therefore, from (3.8) and (3.9) we deduce that

$$-\sigma(t) \in N_{K(t)}(A^{-1}\sigma(t) + \mathcal{R}\sigma(t)) \quad \forall\, t \in [0,T], \tag{3.10}$$

which shows that σ is a solution of Problem Q.

Conversely, if $\sigma \in C([0,T];V)$ is a solution of (3.10) then it is easy to see that the function $u(t) = D^{-1}\sigma(t)$ for all $t \in [0,T]$ is a solution of inequality (3.7).

We deduce from here that condition (c) in Definition 2.1 is also satisfied. Therefore, $\mathcal{P}$ and $\mathcal{Q}$ are dual problems, which concludes step 1.

Step 2. We now construct a Tykhonov triple $\mathcal{T}_Q$ and prove the $\mathcal{T}_Q$-well-posedness of Problem $\mathcal{Q}$. First, we recall that the existence of a unique solution to Problem $\mathcal{Q}$ follows from a recent result proved in [9]. Next, we use notation (3.3) and define the Tykhonov triple $\mathcal{T}_Q = (I, \Omega_Q, \mathcal{C})$ as follows:

$$\begin{aligned}\Omega_Q(\theta) = \big\{&\sigma \in C([0,T];V) : A^{-1}\sigma(t) + \mathcal{R}\sigma(t) \in K(t),\\ &(A^{-1}\sigma(t) + \mathcal{R}\sigma(t) - v, \sigma(t))_V \le \theta \quad \forall\, v \in K(t),\ t \in [0,T]\big\}\end{aligned} \tag{3.11}$$

for each $\theta \ge 0$. Let $\sigma \in C([0,T];V)$ be the solution of Problem $\mathcal{Q}$. Then

$$\begin{gathered}A^{-1}\sigma(t) + \mathcal{R}\sigma(t) \in K(t),\\ (A^{-1}\sigma(t) + \mathcal{R}\sigma(t) - v, \sigma(t))_V \le 0 \quad \forall\, v \in K(t),\ t \in [0,T].\end{gathered} \tag{3.12}$$

This implies that $\sigma \in \Omega_Q(\theta)$ for each $\theta \in \mathbb{R}_+$ and, therefore, $\mathcal{T}_Q$ is a Tykhonov triple.

Let $t \in [0,T]$ and let $\{\sigma_n\} \subset C([0,T];V)$ be a $\mathcal{T}_Q$-approximating sequence for Problem $\mathcal{Q}$. Then there exists a sequence $\{\theta_n\} \in \mathcal{C}$ such that, for each $n \in \mathbb{N}$, $\sigma_n \in \Omega_Q(\theta_n)$. We now use (3.11) and (3.12) to deduce that

$$(A^{-1}\sigma(t) - A^{-1}\sigma_n(t), \sigma(t) - \sigma_n(t))_V \le \theta_n + (\mathcal{R}\sigma(t) - \mathcal{R}\sigma_n(t), \sigma_n(t) - \sigma(t))_V.$$

Using (3.2) and the history-dependence of the operator $\mathcal{R}$ we find that

$$m_{A^{-1}}\|\sigma(t) - \sigma_n(t)\|_V^2 \le \theta_n + L_R\Big(\int_0^t \|\sigma(s) - \sigma_n(s)\|_V\, ds\Big)\|\sigma(t) - \sigma_n(t)\|_V$$

where, here and below, $m_{A^{-1}} = \frac{m_A}{L_A^2}$. Therefore, the elementary inequality

$$x^2 \le ax + b \implies x \le a + \sqrt{b} \quad \forall\, x, a, b > 0$$

implies that

$$\|\sigma(t) - \sigma_n(t)\|_V \le \Big(\frac{\theta_n}{m_{A^{-1}}}\Big)^{1/2} + \frac{L_R}{m_{A^{-1}}}\int_0^t \|\sigma(s) - \sigma_n(s)\|_V\, ds$$

and, using Gronwall's argument, we find that

$$\|\sigma(t)-\sigma_n(t)\|_V \le \Big(\frac{\theta_n}{m_{A^{-1}}}\Big)^{1/2} e^{\frac{L_R}{m_{A^{-1}}}t}.$$

Next, since $\theta_n \to 0$ we deduce that $\sigma_n \to \sigma$ in $C([0,T];V)$. We conclude from here that Problem $\mathcal{Q}$ is $\mathcal{T}_Q$-well-posed.

Step 3. Completion of the proof. Let $\theta \in \mathbb{R}_+$, $u \in C([0,T],V)$ and let $\sigma = Du$. Then, using arguments similar to those used to prove that σ satisfies inequality (3.10) if and only if u satisfies (3.7), it is easy to see that

$$u \in \Omega_P(\theta) \iff \sigma \in \Omega_Q(\theta).$$

We conclude from here that $\Omega_P(\theta) \neq \emptyset$ and, moreover, (2.1) holds. It follows now from Steps 1 and 2 that we are in a position to use Theorem 2.2 to conclude the proof of Theorem 3.2. □

4. A convergence result

In this section we use the well-posedness of Problem $\mathcal{P}$ with respect to the Tykhonov triple $\mathcal{T}_P$ to deduce a continuous dependence result of the solution with respect to the data. To this end we assume that (H1) holds and we consider three sequences $\{A_n\}$, $\{\mathcal{S}_n\}$ and $\{f_n\}$ such that, for each $n \in \mathbb{N}$, the following conditions hold.

(H5) $A_n : V \to V$ satisfies condition (H2) with $m_n > 0$.

(H6) $\mathcal{S}_n : C([0,T];V) \to C([0,T];V)$ is a history-dependent operator, i.e., there exists $L_n > 0$ such that

$$\|\mathcal{S}_n u(t) - \mathcal{S}_n v(t)\|_V \le L_n \int_0^t \|u(s)-v(s)\|_V\,ds$$

for all $u, v \in C([0,T];V)$ and $t \in [0,T]$.

(H7) $f_n \in C([0;T];V)$.

Then we consider the following variational problem.

Problem $\mathcal{P}_n$. *Find a function $u_n \in C([0,T];V)$ such that the following inequality holds: $u_n(t) \in K(t)$, and*

$$(A_n u_n(t), v-u_n(t))_V + (\mathcal{S}_n u_n(t), v-u_n(t))_V \ge (f_n(t), v-u_n(t))_V \tag{4.1}$$

for all $v \in K(t)$ and $t \in [0,T]$.

Then the arguments in Section 2 imply that Problem $\mathcal{P}_n$ has a unique solution, for each $n \in \mathbb{N}$. Assume now that

(H8) There exists $u_0 \in V$ such that $u_0 \in K(t)$ for all $t \in [0,T]$.

(H9) (a) For each $n \in \mathbb{N}$ there exists $\alpha_n > 0$ such that

$$\|A_n u - Au\|_V \le \alpha_n(\|u\|_V + 1) \quad \forall\, u \in V.$$

(b) $\alpha_n \to 0$ as $n \to \infty$.

(H10) (a) For each $n \in \mathbb{N}$ there exists $\beta_n > 0$ such that

$$\|\mathcal{S}_n u(t) - \mathcal{S}u(t)\|_V \le \beta_n\Big(\int_0^t \|u(s)\|_V\,ds + 1\Big)$$

for all $u \in C([0,T];V)$ and $t \in [0,T]$.

(b) $\beta_n \to 0$ as $n \to \infty$.

(H11) $f_n \to f$ in $C([0,T];V)$.

The main result of this section is the following.

Theorem 4.1. *Assume* (H1)–(H11) *hold. Then, the solution u_n of Problem $\mathcal{P}_n$ converges to the solution u of Problem $\mathcal{P}$, i.e.,*

$$u_n \to u \quad \text{in } C([0,T];V). \tag{4.2}$$

Proof. The proof is carried out in three steps, as follows.

Step 1. We prove that there exists $M>0$ such that

$$\|u_n(t)\|_V \le M \quad \forall\, n\in\mathbb{N},\ t\in[0,T]. \tag{4.3}$$

Let $n\in\mathbb{N}$ and $t\in[0,T]$. Then, using (4.1) with $v=u_0\in K(t)$ we find that

$$(A_nu_n(t),u_n(t)-u_0)_V \le (\mathcal{S}_nu_n(t),u_0-u_n(t))_V + (f_n(t),u_n(t)-u_0)_V. \tag{4.4}$$

We write

$$\begin{aligned}
&(A_nu_n(t),u_n(t)-u_0)_V\\
&= (A_nu_n(t)-Au_n(t),u_n(t)-u_0)_V + (Au_n(t)-Au_0,u_n(t)-u_0)_V\\
&\quad + (Au_0,u_n(t)-u_0)_V,
\end{aligned}$$

then we use Cauchy-Schwarz inequality and assumptions (H2), (H9)(a) to see that

$$\begin{aligned}
(A_nu_n(t),u_n(t)-u_0)_V \ge{}& -(\alpha_n(\|u_n(t)\|_V+1))\|u_n(t)-u_0\|_V\\
&+ m_A\|u_n(t)-u_0\|_V^2 - \|Au_0\|_V\|u_n(t)-u_0\|_V.
\end{aligned} \tag{4.5}$$

Note that

$$\begin{aligned}
&(\mathcal{S}_nu_n(t),u_0-u_n(t))_V\\
&= (\mathcal{S}_nu_n(t)-\mathcal{S}u_n(t),u_0-u_n(t))_V\\
&\quad + (\mathcal{S}u_n(t)-\mathcal{S}u_0(t),u_0-u_n(t))_V + (\mathcal{S}u_0(t),u_0-u_n(t))_V.
\end{aligned}$$

Then, it follows from assumptions (H3) and (H10)(a) that

$$\begin{aligned}
&(\mathcal{S}_nu_n(t),u_0-u_n(t))_V\\
&\le \beta_n\Big(\int_0^t \|u_n(s)\|_V\,ds+1\Big)\|u_n(t)-u_0\|_V\\
&\quad + L_S\Big(\int_0^t \|u_n(s)-u_0\|_V\,ds\Big)\|u_n(t)-u_0\|_V + \|\mathcal{S}u_0(t)\|_V\|u_n(t)-u_0\|_V.
\end{aligned} \tag{4.6}$$

Moreover, we have

$$\begin{aligned}
&(f_n(t),u_n(t)-u_0)_V\\
&= (f_n(t)-f(t),u_n(t)-u_0)_V + (f(t),u_n(t)-u_0)_V\\
&\le \|f_n(t)-f(t)\|_V\|u_n(t)-u_0\|_V + \|f(t)\|_V\|u_n(t)-u_0\|_V.
\end{aligned} \tag{4.7}$$

We now combine inequalities (4.4)–(4.7) to see that

$$\begin{aligned}
&m_A\|u_n(t)-u_0\|_V\\
&\le \beta_n\Big(\int_0^t \|u_n(s)\|_V\,ds+1\Big) + L_S\int_0^t \|u_n(s)-u_0\|_V\,ds\\
&\quad + \|\mathcal{S}u_0(t)\|_V + \|f_n(t)-f(t)\|_V + \|f(t)\|_V + \alpha_n(\|u_n(t)\|_V+1) + \|Au_0\|_V,
\end{aligned}$$

which implies that

$$(m_A-\alpha_n)\|u_n(t)\|_V$$

$$\begin{aligned}&\le \beta_n\Big(\int_0^t \|u_n(s)\|_V\,ds+1\Big)+L_S\int_0^t \|u_n(s)\|_V\,ds+L_ST\|u_0\|_V\\&\quad+\|\mathcal{S}u_0(t)\|_V+\|f_n(t)-f(t)\|_V+\|f(t)\|_V+\alpha_n+\|Au_0\|_V+m_A\|u_0\|_V.\end{aligned}$$

Using assumptions (H9)(b), (H10)(b) and (H11), there exist $N_0\in\mathbb{N}$ and two positive constants C_0 and C_1 such that

$$\|u_n(t)\|_V\le C_0+C_1\int_0^t \|u_n(s)\|_V\,ds$$

for all $n\ge N_0$. It follows from Gronwall's inequality that

$$\|u_n(t)\|_V\le C_0e^{C_1t}.$$

This inequality completes the proof of (4.3).

Step 2. We prove that the sequence $\{u_n\}\subset C([0,T];V)$ is a $\mathcal{T}_P$-approximating sequence for Problem $\mathcal{P}$. Let $n\in\mathbb{N}$, $t\in[0,T]$ and $v\in K(t)$. We write

$$\begin{aligned}&(Au_n(t),v-u_n(t))_V+(\mathcal{S}u_n(t),v-u_n(t))_V-(f(t),v-u_n(t))_V\\&=(A_nu_n(t),v-u_n(t))_V+(\mathcal{S}_nu_n(t),v-u_n(t))_V-(f_n(t),v-u_n(t))_V\\&\quad+(Au_n(t)-A_nu_n(t),v-u_n(t))_V+(\mathcal{S}u_n(t)-\mathcal{S}_nu_n(t),v-u_n(t))_V\\&\quad+(f_n(t)-f(t),v-u_n(t))_V,\end{aligned}$$

then we use inequality (4.1) to find that

$$\begin{aligned}&(Au_n(t),v-u_n(t))_V+(\mathcal{S}u_n(t),v-u_n(t))_V\\&\ge(f(t),v-u_n(t))_V-\|Au_n(t)-A_nu_n(t)\|_V\|v-u_n(t)\|_V\\&\quad-\|\mathcal{S}u_n(t)-\mathcal{S}_nu_n(t)\|_V\|v-u_n(t)\|_V-\|f_n(t)-f(t)\|_V\|v-u_n(t)\|_V.\end{aligned}$$

Therefore, (H9)(a), (H10)(a) and (4.3) imply that there exists a constant $C>0$ such that

$$\begin{aligned}&(Au_n(t),v-u_n(t))_V+(\mathcal{S}u_n(t),v-u_n(t))_V\\&+C(\alpha_n(M+1)+\beta_n(MT+1)+\|f_n(t)-f(t)\|_V)\\&\ge(f(t),v-u_n(t))_V.\end{aligned}\tag{4.8}$$

We define

$$\theta_n=C(\alpha_n(M+1)+\beta_n(MT+1)+\|f_n(t)-f(t)\|_V),\tag{4.9}$$

and combine (4.8), (4.9), (3.4) to see that $u_n\in\Omega_P(\theta_n)$. Moreover, (H9)(b), (H10)(b) and (H11) imply that $\theta_n\to0$ as $n\to\infty$. It follows from here that $\{u_n\}$ is a $\mathcal{T}_P$-approximating sequence for Problem $\mathcal{P}$.

Step 3. Completion of the proof. We use Theorem 3.2 and Definition 1.1(c) to deduce the convergence (4.2), which concludes the proof. □

We end this section with the following example in which $V=\mathbb{R}$.

Example 4.2. Consider Problem $\mathcal{P}$ in the particular case when

$$K(t)=[0,2-e^{-t}]\quad\forall\,t\in[0,T],\quad Au=u\quad\forall\,u\in V,$$

$$\mathcal{S}u(t)=\int_0^t u(s)\,ds\ \forall\,t\in[0,T],\ u\in C([0,T];V),\quad f(t)=t\quad\forall\,t\in[0,T].$$

Note that in this case inequality (3.1) becomes: $u(t) \in K(t)$ and

$$\Big(u(t) + \int_0^t u(s)\,ds - t\Big)\big(v - u(t)\big) \geq 0 \quad \forall\, v \in K(t),\ t \in [0,T]. \tag{4.10}$$

Assume that

$$A_n u = \frac{n+1}{n}\,u \quad \forall\, u \in V,$$
$$\mathcal{S}_n u(t) = \frac{n+1}{n}\int_0^t u(s)\,ds \quad \forall\, t \in [0,T],\ u \in C([0,T];V),$$
$$f_n(t) = t + \frac{1}{n} \quad \forall\, t \in [0,T],$$

for each $n \in \mathbb{N}$. Then, inequality (4.1) becomes: $u_n(t) \in K(t)$ and

$$\Big(\frac{n+1}{n}u_n(t) + \frac{n+1}{n}\int_0^t u_n(s)\,ds - t - \frac{1}{n}\Big)\big(v - u_n(t)\big) \geq 0 \tag{4.11}$$

for all $v \in K(t)$ and $t \in [0,T]$. It is easy to see that in this particular case assumptions (H1)–(H11) are satisfied. Therefore, Theorems 3.2 and 4.1 guarantee the unique solvability of inequalities (4.10) and (4.11) as well as the convergence (4.2).

This convergence can be proved directly. Indeed, consider the integral equation

$$u(t) + \int_0^t u(s)\,ds = t \quad \forall\, t \in [0,T].$$

The solution of this equation is

$$u(t) = 1 - e^{-t} \quad \forall\, t \in [0,T]. \tag{4.12}$$

and, since $0 \leq 1 - e^{-t} \leq 2 - e^{-t}$ for all $t \in [0,T]$, we deduce that the function (4.12) is also the solution of the history-dependent inequality (4.10). Next, using a similar argument based on the solvability of the integral equation

$$\frac{n+1}{n}u_n(t) + \frac{n+1}{n}\int_0^t u_n(s)\,ds = t + \frac{1}{n} \quad \forall\, t \in [0,T],$$

we find that the solution of the history-dependent variational inequality (4.11) is

$$u_n(t) = \frac{n}{n+1} - \frac{n-1}{n+1}\,e^{-t} \quad \forall\, t \in [0,T]. \tag{4.13}$$

Then, a simple calculation shows that

$$|u_n(t) - u(t)| \leq \frac{3}{n+1} \quad \forall\, t \in [0,T],$$

which implies the convergence (4.2).

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