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PULLBACK ATTRACTORS FOR NON-AUTONOMOUS BRESSE SYSTEMS

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ABSTRACT. This article concerns the asymptotic behavior of solutions of non-autonomous Bresse systems. We establish the existence of pullback attractor and upper semicontinuity of attractors as a non-autonomous perturbations tend to zero. In addition we study the continuity of attractors with respect to a parameter in a residual dense set.

1. Introduction

An important problem in dynamical systems is the study of the asymptotic behavior of evolution processes associated with mathematical models that appear in many applications to natural sciences. Our attention will be in evolution processes associated with non-autonomous problems, where the pullback attractor theory has been applied quite successfully. This theory is an extension of the autonomous concept of global attractor. A good presentation about this theory can be found in the book by Carvalho, Langa and Robinson [6].

Let $L > 0$ be given. This article concerns the long-time dynamics of the non-autonomous Bresse system for vibrations of curved beams,

$$\rho_1\varphi_{tt} - k(\varphi_x + \psi + lw)_x - k_0 l(w_x - l\varphi) + g_1(\varphi_t) + f_1(\varphi,\psi,w) = h_1(x,t), \quad (1.1)$$

$$\rho_2\psi_{tt} - b\psi_{xx} + k(\varphi_x + \psi + lw) + g_2(\psi_t) + f_2(\varphi,\psi,w) = h_2(x,t), \quad (1.2)$$

$$\rho_1 w_{tt} - k_0(w_x - l\varphi)_x + kl(\varphi_x + \psi + lw) + g_3(w_t) + f_3(\varphi,\psi,w) = h_3(x,t), \quad (1.3)$$

defined in $(0,L)\times\mathbb{R}$, where φ, ψ, w represent, vertical displacement, shear angle, and longitudinal displacement, respectively. The coefficients ρ_1, ρ_2, b, k, k_0, l are positive constants, $g_1(\varphi_t)$, $g_2(\psi_t)$, $g_3(w_t)$ are nonlinear damping terms, $f_i(\varphi,\psi,w)$, $i = 1,2,3$, are nonlinear forces and $h_i(x,t)$, $i = 1,2,3$, are time-dependent perturbations. To this system we add the Dirichlet boundary condition

$$\varphi(0,t) = \varphi(L,t) = \psi(0,t) = \psi(L,t) = w(0,t) = w(L,t) = 0, \quad t \geqslant \tau, \quad (1.4)$$

and the initial condition (for $t = \tau$),

$$\begin{gathered}\varphi(\cdot,\tau) = \varphi_0^\tau, \quad \varphi_t(\cdot,\tau) = \varphi_1^\tau, \quad \psi(\cdot,\tau) = \psi_0^\tau, \\ \psi_t(\cdot,\tau) = \varphi_1^\tau, \quad w(\cdot,\tau) = w_0^\tau, \quad w_t(\cdot,\tau) = w_1^\tau.\end{gathered} \quad (1.5)$$

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In recent years, the Bresse system has been studied by many authors. There are several works concerned with decay rates of solutions. For the linear case, see for instance [1, 2, 8, 9], and for the nonlinar case see [7, 12]. On the other hand, the long-time dynamics in autonomous case ($h_i(x,t) = 0$) was discussed in [13], where the authors prove the existence of a global attractor and compare the Bresse system with the Timoshenko system, in the sense of upper-semicontinuity of their attractors as $l \to 0$. The non-autonomous case was not discussed there. Motivated by this, we study the long-time dynamics of the non-autonomous Bresse system (1.1)-(1.3) characterized by pullback attractors. To the best of our knowledge the first work concerned with Bresse systems with non-autonomous forces was [3], where the authors establish uniform decay rates of the energy.

Since our problem has damping terms in all of the equations we shall not assume any equal wave speeds assumption which is a remarkable stability criteria to the Bresse systems. For more details see [3, 13].

The content of the paper is as follows:

- (i) Under appropriate assumptions on the forcing terms and on h_i, $i = 1, 2, 3$, we establish the existence of a pullback attractor. Our main result is presented in Theorem 3.1.
- (ii) We show the upper-semicontinuity of a family of pullback attractors as the non-autonomous perturbations tend to zero. To this end, we replace h_i by ϵh_i and let $\epsilon \to 0$. See Theorem 4.1.
- (iii) We apply a recent result by Hoang, Olson and Robinson [11] to study the continuity of a family of pullback attractors $\mathcal{A}_\epsilon$ when $\epsilon \in J$, a residual dense set of $[0, 1]$. See Theorem 5.1.

2. Preliminaries

Let the L^p norm be denoted by $\|u\|_p$ when $p \neq 2$, and by $\|u\|$ when $p = 2$. In the Sobolev space $H^1(0, L)$ we have

$$\|u\| \leqslant \frac{L}{\pi}\|u_x\| \tag{2.1}$$

and $\|u\|_{H_0^1} = \|u_x\|$. We consider weak solutions in the phase space $H = H_0^1(0, L)^3 \times L^2(0, L)^3$, equipped with the norm

$$\|y\|_H^2 = \|\varphi_x\|^2 + \|\psi_x\|^2 + \|w_x\|^2 + \|\tilde{\varphi}\| + \|\tilde{\psi}\|^2 + \|\tilde{w}\|^2,$$

where $y = (\varphi, \psi, w, \tilde{\varphi}, \tilde{\psi}, \tilde{w})$. For convenience, we use the equivalent norm

$$\|y\|_{\mathcal{H}}^2 = \rho_1\|\tilde{\varphi}\|^2 + \rho_2\|\tilde{\psi}\|^2 + \rho_1\|\tilde{w}\|^2 + b\|\varphi_x\| + k\|\varphi_x + \psi + lw\|^2 + k_0\|w_x - l\varphi\|^2.$$

In fact, as proved in [13], there exists constants $\gamma_1 > 0$, $\gamma_2 > 0$ and $\gamma_3 > 0$ such that

$$\|y\|_{\mathcal{H}}^2 \leqslant \gamma_1\|y\|_H^2, \tag{2.2}$$

$$\|y\|_H^2 \leqslant \gamma_2\|y\|_{\mathcal{H}}^2, \tag{2.3}$$

$$\|\varphi_x\|^2 + \|\psi_x\|^2 + \|w_x\|^2 \leqslant \gamma_3\big(b\|\psi_x\|^2 + k\|\varphi_x + \psi + lw\|^2 + k_0\|w_x - l\varphi\|^2\big). \tag{2.4}$$

Next we present some definitions and results about pullback attractors, taken from [6] and [15]. Let (X, d) a metric space and $\{U(t, \tau) : t \geqslant \tau \in \mathbb{R}\}$ a evolution process in X.

Definition 2.1. Let A and B be non-empty subsets of X. We define the Hausdorff semidistance

$$\operatorname{dist}_X(A,B)=\sup_{a\in A}\inf_{b\in B} d(a,b).$$

The Hausdorff metric in X is

$$d_X(A,B)=\max\{\operatorname{dist}_X(A,B),\operatorname{dist}_X(B,A)\}.$$

Definition 2.2. A process $\{U(t,\tau): t\geqslant\tau\in\mathbb{R}\}$ in X is a two-parameter family of maps $U(t,\tau):X\to X$ such that

(i) $U(t,t)=I$, where I is the identity operator in X, for all $t\in\mathbb{R}$,
(ii) $U(t,s)\circ U(s,\tau)=U(t,\tau)$, for all $t\geqslant s\geqslant\tau$ in $\mathbb{R}$,
(iii) $(t,\tau,x)\mapsto U(t,\tau)x$ is continuous for all $t\geqslant\tau$ and $x\in X$.

Definition 2.3. A family of compact sets $\mathcal{A}=\{A(t):t\in\mathbb{R}\}$ in X is the pullback attractor for the process $U(\cdot,\cdot)$ if

(i) $\mathcal{A}$ is invariant: $U(t,\tau)A(\tau)=A(t)$, for all $t\geqslant\tau$,
(ii) $\mathcal{A}$ is pullback atracting: for any bounded set $D\subset X$ and $t\in\mathbb{R}$,
$$\operatorname{dist}_X(U(t,\tau)D,A(t))\to 0\quad\text{as } \tau\to-\infty,$$
(iii) $\mathcal{A}$ is minimal, that is, if $\{C(t)\}$ is any other family of compact sets that satisfies (i) and (ii), then $A(t)\subset C(t)$, for all $t\in\mathbb{R}$.

The following definitions are useful to guarantee the existence of pullback attractor for a process.

Definition 2.4. A family $\{B(t)\}$ of non-empty subset of X is called pullback absorbing for the process $U(\cdot,\cdot)$ if given $t\in\mathbb{R}$, $\epsilon>0$ and bounded subset D of X, there exists $\tau_\epsilon\leqslant t$ such that

$$U(t,\tau)D\subset B(t),$$

for all $\tau\leqslant\tau_\epsilon$.

Definition 2.5. An evolution process $U(\cdot,\cdot)$ is called pullback asymptotically compact if, for each $t\in\mathbb{R}$, each sequence $\{\tau_k\}\subset(-\infty,t]$, and each bounded sequence $\{x_k\}\subset X$ such that

(i) $\tau_k\to-\infty$ as $k\to\infty$, and
(ii) $\{U(t,\tau_k)x_k\}$ is bounded,

it follows that the sequence $\{U(t,\tau_k)x_k\}$ has a convergent subsequence.

The following result gives conditions for obtaining the existence of pullback attractors for an evolution processes.

Theorem 2.6. *Let $\{U(t,\tau);t\geqslant\tau\in\mathbb{R}\}$ be an evolution process in a metric space X. Assume that the process $\{U(t,\tau)\}_{t\geqslant\tau}$ is pullback asymptotically compact and possesses a pullback absorbing family $\hat{B}=\{B(t)\}_{t\in\mathbb{R}}$. Then the family $\hat{\mathcal{A}}=\{A(t)\}_{t\in\mathbb{R}}$ given by*

$$A(t)=\overline{\cup_{\substack{D\subset X,\\ \text{bounded}}}\omega(D,t)}$$

is a minimal pullback attractor for the process $U(\cdot,\cdot)$, where

$$\omega(D,t)=\cap_{s\leqslant t}\overline{\cup_{\tau\leqslant s}U(t,\tau)D(\tau)}.$$

Next, we present a sufficient condition for pullback asymptotic compactness of an evolution process.

Definition 2.7. Let X be a Banach space. Then, one says that the function $\Psi : X\times X \to \mathbb{R}$ is contractive on a bounded subset B of X if given a sequence $\{x_n\} \subset B$ there exists a subsequence $\{x_{n_k}\}$ such that $\lim_{k\to\infty}\lim_{l\to\infty}\Psi(x_{n_k},x_{n_l}) = 0$.

Theorem 2.8. *Let X be a Banach space and $U(\cdot,\cdot)$ be a process which admits a pullback absorbing family of bounded subsets of X, $\hat{B} = \{B(t)\}_{t\in\mathbb{R}}$. Suppose that for all $t \in \mathbb{R}$ and $\epsilon > 0$ there exist $\tau_\epsilon \leqslant t$ and a contractive function Ψ_ϵ on $B(\tau_\epsilon)$, such that*

$$\|U(t,\tau_\epsilon)x - U(t,\tau_\epsilon)y\|_X \leqslant \epsilon + \Psi_\epsilon(x,y), \quad \textit{for all } , x,y \in B(\tau_\epsilon).$$

Then $\{U(t,\tau) : t \geqslant \tau \in \mathbb{R}\}$ is pullback asymptotically compact.

Let $\mathcal{A}_\epsilon$ a family of pullback attractors for a process $U_\epsilon(\cdot,\cdot)$, with ϵ in a metric space Λ. We establish conditions ensuring the continuity of pullback attractor with respect to a parameter.

Definition 2.9. We say that a family of pullback attractors $\{\mathcal{A}_\epsilon\}_{\epsilon\in\Lambda}$ is upper semicontinuous at $\epsilon_0 \in \Lambda$ if

$$\lim_{\epsilon\to\epsilon_0} \operatorname{dist}_X(\mathcal{A}_\epsilon(t),\mathcal{A}(t)) = 0,$$

for all $t \in \mathbb{R}$.

Analogously, if we interchange A_ϵ and A_{ϵ_0} in the above limit, then we say that A_ϵ is lower semicontinuous at $\epsilon_0 \in \Lambda$.

Then, A_ϵ is continous at $\epsilon \in \Lambda$ if

$$\lim_{\epsilon\to\epsilon_0} d_X(\mathcal{A}_\epsilon,\mathcal{A}_{\epsilon_0}) = 0.$$

Proposition 2.10. *Let $U_\epsilon : X \to X$ be a family of parametrized processes with $\epsilon \in [0,1)$. Suppose that*

(i) *U_ϵ has a pullback attractor $\mathcal{A}_\epsilon$ for all $\epsilon \in [0,1)$,*
(ii) *For every $t \in \mathbb{R}$, $T \geqslant 0$ and bounded subsets $D \subset X$,*
$$\sup_{\tau\in[0,T],u_0\in D} d(U_\epsilon(t+\tau,t)u_0, U_0(t+\tau,t)u_0) \to 0, \quad \textit{as } \epsilon\to 0,$$
(iii) *There exists δ and $t_0 \in \mathbb{R}$ such that $\cup_{\epsilon\in(0,\delta)}\cup_{s\leqslant t_0}\mathcal{A}_\epsilon(s)$ is bounded.*

Then the family of processes $U_\epsilon(t,\tau) : X \to X$ is upper semicontinuous at $\epsilon = 0$.

We shall use a recent result in [11] for the continuity of pullback attractors with respect to a parameter.

Theorem 2.11. *Let $U_\epsilon(\cdot,\cdot)$ a family of processes on X with ϵ in a metric space Λ. Assume that $U_\epsilon(\cdot,\cdot)$ has a pullback attractor for every ϵ and*

(i) *There exists a bounded set $B \subset X$ such that $A_\epsilon(t) \subset D$, for every $\epsilon \in \Lambda$ and $t \in \mathbb{R}$,*
(ii) *For every $\tau \in \mathbb{R}$ and $t \geqslant \tau$, $U_\epsilon(t,\tau)x$ is continuous in ϵ, uniformly for x in bounded subsets of X,*
(iii) *For every $\epsilon_0 \in \Lambda$ and $t \in \mathbb{R}$, there exists $\delta > 0$ such that*
$$\overline{\cup_{B_\Lambda(\epsilon_0,\delta)} A_\epsilon(t)} \tag{2.5}$$
is compact.

Then there exists a residual set $J \subset \Lambda$ such that for every $t \in \mathbb{R}$ the function $\epsilon \mapsto A_\epsilon(t)$ is continuous at each $\epsilon \in J$.

2.1. **Assumptions.** Our hypothesis are similar to those in [13]. We consider f_1, f_2, f_3 are locally Lipschitz and gradient type. Let us assume there exists a C^2 function $F:\mathbb{R}^3\to\mathbb{R}$ such that

$$\nabla F=(f_1,f_2,f_3), \tag{2.6}$$

and satisfies the following conditions: There exist β, $m_F\geqslant 0$ such that

$$F(u,v,w)\geqslant -\beta(|u|^2+|v|^2+|w|^2)-m_F,\quad \forall u,v,w\in\mathbb{R}, \tag{2.7}$$

where

$$0\leqslant\beta\leqslant\frac{\pi^2}{2\gamma_3L^2}, \tag{2.8}$$

and there exist $p\geqslant 1$ and $C_f>0$ such that, for $i=1,2,3$,

$$|\nabla f_i(u,v,w)|\leqslant C_f(1+|u|^{p-1}+|v|^{p-1}+|w|^{p-1}),\quad \forall u,v,w\in\mathbb{R}. \tag{2.9}$$

In particular, there exists $C_F>0$ such that

$$F(u,v,w)\leqslant C_F(1+|u|^{p+1}+|v|^{p+1}+|w|^{p+1}),\quad \forall u,v,w\in\mathbb{R}.$$

Furthermore, we assume that for all $u,v,w\in\mathbb{R}$,

$$\nabla F(u,v,w)\cdot(u,v,w)-F(u,v,w)\geqslant -\beta(|u|^2+|v|^2+|w|^2)-m_F. \tag{2.10}$$

With respect to the damping functions $g_i\in C^1(\mathbb{R})$, $i=1,2,3$, we assume that g_i is increasing, $g_i(0)=0$ and there exist constants m_i, $M_i>0$ such that

$$m_i\leqslant g_i'(s)\leqslant M_i,\forall s\in\mathbb{R}. \tag{2.11}$$

Furthermore, we assume h_1, h_2, $h_3\in L^2_{\rm loc}(\mathbb{R};L^2(0,L))$ and satisfy, for some constant $C_h>0$,

$$\int_{-\infty}^t e^{-\sigma(t-s)}\big(\|h_1(s)\|_2^2+\|h_2(s)\|_2^2+\|h_3(s)\|_2^2\big)ds<C_h,\quad \forall t\in\mathbb{R}, \tag{2.12}$$

with $\sigma\in[0,\sigma_0]$, where $\sigma_0>0$ is a constant dependent only on the parameters ρ_1, ρ_2, b, k, l, β, and will be defined later.

2.2. **Energy of the system.** The energy of the system along a solution $y(t)=(\varphi(t),\psi(t),w(t),\varphi_t(t),\psi_t(t),w_t(t))$, $t\geqslant\tau$, is

$$E(t)=\frac{1}{2}\|(\varphi(t),\psi(t),w(t),\varphi_t(t),\psi_t(t),w_t(t))\|_{\mathcal{H}}^2 \tag{2.13}$$

and

$$\mathcal{E}(t)=E(t)+\int_0^L F(\varphi,\psi,w)dx. \tag{2.14}$$

Then, multiplying (1.1)-(1.3) by φ_t, ψ_t, w_t, respectively, we obtain by integration over $[0,L]$,

$$\begin{aligned}\frac{d}{dt}\mathcal{E}(t)=&-\int_0^L\big(g_1(\varphi_t)\varphi_t+g_2(\psi_t)\psi_t+g_3(w_t)w_t\big)dx\\&+\int_0^L\big(h_1(t)\varphi_t(t)+h_2(t)\psi_t(t)+h_3(t)w_t(t)\big)dx,\quad\forall t\geqslant\tau.\end{aligned} \tag{2.15}$$

The following inequalities will be useful in next sections.

Lemma 2.12. *There exists $\beta_0 > 0$, $k_F > 0$ such that*

$$\mathcal{E}(t) \geqslant \beta_0 \|U(t,\tau)z\|_H^2 - Lm_F, \quad \forall t \geqslant \tau, z; \tag{2.16}$$

$$\mathcal{E}(t) \leqslant k_F(1 + \|U(t,\tau)z\|_H^{p+1}), \quad \forall t \geqslant \tau \tag{2.17}$$

Proof. From (2.7) and (2.14),

$$\begin{aligned} \mathcal{E}(t) &\geqslant E(t) - \beta(\|\varphi\|^2 + \|\psi\|^2 + \|w\|^2) - Lm_F \\ &\geqslant \Big(1 - \frac{2\beta\gamma_3 L^2}{\pi^2}\Big) E_l(t) - Lm_F. \end{aligned}$$

Then, we take $\beta_0 = 1 - \frac{2\beta\gamma_3 L^2}{\pi^2}$. Now, using (2.9) and the embedding of $H^1(0,L)$ in $L^p(0,L)$ we obtain (2.17). □

2.3. **Well-possedness.** We shall write the system (1.1)-(1.5) as an abstract Cauchy problem,

$$\frac{d}{dt} y(t) - (B_1 + B_2) y(t) = \mathcal{F}(y(t), t), \quad y(\tau) = y_\tau, \tag{2.18}$$

where

$$y(t) = (\varphi(t), \psi(t), w(t), \varphi'(t), \psi'(t), w'(t)) \in H, \quad \varphi' = \varphi_t, \quad \psi' = \psi_t, \quad w' = w_t,$$

and

$$y_\tau = (\varphi_\tau, \psi_\tau, w_\tau, \varphi'_\tau, \psi'_\tau, w'_\tau) \in H$$

is the initial condition. We let $B_1 : D(B_1) \subset H \to H$ be defined by

$$B_1 y = \begin{bmatrix} \varphi' \\ \psi' \\ w' \\ \frac{k}{\rho_1}(\varphi_x + \psi + lw)_x + \frac{k_0 l}{\rho_1}(w_x - l\varphi) \\ \frac{b}{\rho_2}\psi_{xx} - \frac{k}{\rho_2}(\varphi_x + \psi + lw) \\ \frac{k_0}{\rho_1}(w_x - l\varphi)_x - \frac{kl}{\rho_1}(\varphi_x + \psi + lw) \end{bmatrix},$$

with domain $D(B_1) = (H^2(0,L) \cap H_0^1(0,L))^3 \times H_0^1(0,L)^3$, and $B_2 : H \to H$, is given by

$$B_2 y = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -\frac{g_1(\varphi')}{\rho_1} \\ -\frac{g_2(\varphi')}{\rho_2} \\ -\frac{g_3(w')}{\rho_1} \end{bmatrix},$$

with domain $D(B_2) = H$. Also, we have the nonlinear function $\mathcal{F} : \mathcal{H} \to \mathcal{H}$ defined by

$$\mathcal{F}(y,t) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -\frac{f_1(\varphi,\psi,w)}{\rho_1} + \frac{h_1}{\rho_1} \\ -\frac{f_2(\varphi,\psi,w)}{\rho_2} + \frac{h_2}{\rho_2} \\ -\frac{f_3(\varphi,\psi,w)}{\rho_1} + \frac{h_3}{\rho_1} \end{bmatrix}.$$

We establish the following result on the existence and uniqueness of a solution.

Theorem 2.13. *Assume that* (2.6)-(2.12) *hold and* h_1, h_2 $h_3 \in L^2_{\rm loc}(\mathbb{R}; L^2(0,L))$. *Then for each initial time* $\tau \in \mathbb{R}$ *and data* $y_\tau \in H$, *problem* (1.1)-(1.5) *has a unique weak solution* $y = (\varphi, \psi, w, \varphi_t, \psi_t, w_t)$ *satisfying*

$$y \in C([\tau, \infty); H), \quad y(\tau) = y_\tau.$$

In addition, if $y_\tau \in D(B_1)$, *then the solution is strong. Also the weak solutions depend continuously on the initial data.*

Proof. It follows from [7, Theorem 2.2] that $B_1 + B_2$ is maximal monotone in H. Then the Cauchy problem

$$\frac{d}{dt}y(t) - (B_1 + B_2)y(t) = 0, \quad y(\tau) = y_\tau \tag{2.19}$$

has a unique solution. We will show that (2.18) is a locally Lipschitz perturbation of (2.19). Then from classical results in [5], we obtain a local solution defined in an interval $[\tau, t_{\max})$, and if $t_{\max} < \infty$, then

$$\lim_{t \to t_{\max}^-} \|y(t)\|_H = +\infty. \tag{2.20}$$

To show that $F(y,t)$ is locally Lipschitz in y, for each t, let y^1, $y^2 \in H$,

$$y^1 = (\varphi^1, \psi^1, w^1, \varphi^1_t, \psi^1_t, w^1_t), \quad y^2 = (\varphi^2, \psi^2, w^2, \varphi^2_t, \psi^2_t, w^2_t).$$

From (2.10), there exists a constant $C_r > 0$ depending of $r = \max\{\|y^1\|_H, \|y^2\|_H\}$, such that

$$\int_0^L |f_j(y^1) - f_j(y^2)|^2 dx \leqslant C_r \|y^1 - y^2\|^2_H.$$

Then

$$\|\mathcal{F}(y^1) - \mathcal{F}(y^2)\|^2 \leqslant 3C_r \|y^1 - y^2\|^2_H,$$

which shows that $\mathcal{F}$ is locally Lipschitz on H.

To verify that the solution is global, that is, $t_{\max} = \infty$, we consider the energy of system defined by (2.14). By density argument, we can assume that y is a strong solution. Then, of (2.15), we obtain

$$\begin{aligned} \frac{d}{dt}\mathcal{E}(t) &\leqslant -m_1\|\varphi_t(t)\|^2 - m_2\|\psi_t(t)\|^2 - m_3\|w_t(t)\|^2 + \frac{\|h_1(t)\|^2}{m_1} + \frac{\|h_2(t)\|^2}{m_2} \\ &\quad + \frac{\|h_3(t)\|^2}{m_3} + \frac{m_1}{2}\|\varphi_t(t)\|^2 + \frac{m_2}{2}\|\psi_t(t)\|^2 + \frac{m_3}{2}\|w_t(t)\|^2 \\ &\leqslant \frac{\|h_1(t)\|^2}{m_1} + \frac{\|h_2(t)\|^2}{m_2} + \frac{\|h_3(t)\|^2}{m_3}. \end{aligned} \tag{2.21}$$

It follows that

$$\mathcal{E}(t) \leqslant \mathcal{E}(\tau) + \frac{1}{\min\{m_1, m_2, m_3\}} \int_\tau^t \big(\|h_1(s)\|^2 + \|h_2(s)\|^2 + \|h_3(s)\|^2\big) ds, \quad \forall t \geqslant \tau.$$

By Lemma 2.12 we have that there exists a constant $C > 0$, independent of t, such that

$$\|y(t)\|^2_H \leqslant C\mathcal{E}(t) + C, \quad \forall t \geqslant \tau.$$

Consequently, $\|y(t)\|_H < \infty$, $t \geqslant \tau$, which shows that $t_{\max} = \infty$.

Finally, using (2.9) and (2.11) we can check that for any solution of (1.1)-(1.5),

$$y^1 = (\varphi^1, \psi^1, w^1, \varphi^1_t, \psi^1_t, w^1_t), \quad y^2 = (\varphi^2, \psi^2, w^2, \varphi^2_t, \psi^2_t, w^2_t),$$

with corresponding initial data $y^1_\tau,\ y^2_\tau \in H$, respectively, there exists constant $C>0$, such that

$$\|y^1(t)-y^2(t)\|_H^2 \leqslant C\|y^1_\tau - y^2_\tau\|_H^2, \quad \forall t\in[\tau,T],$$

for all $T\geqslant\tau$. This shows the continuous dependence of solutions on the initial data. □

Theorem 2.13 shows that the solution operator $U(t,\tau): H\to H$, given by

$$U(t,\tau)y = (\varphi(t),\psi(t),w(t),\varphi_t(t),\psi_t(t),w_t(t)), \quad t\geqslant\tau$$

defines a continuous evolution process.

3. Pullback attractor

Our main result reads as follows.

Theorem 3.1. *If* (2.6)-(2.12) *hold, then the evolution process generated by the problem* (1.1)-(1.5) *possesses a pullback attractor* $\mathcal{A}=\{A(t)\}_{t\in\mathbb{R}}$ *in the phase space* $\mathcal{H}$.

The proof of the above theorem will be completed after the following two lemmas.

Lemma 3.2. *The process generated by problem* (1.1)-(1.5) *possesses a pullback absorbing family.*

Proof. From (2.15), we have

$$\begin{aligned}\frac{d}{dt}\mathcal{E}(t) = &-\int_0^L \big(g_1(\varphi_t)\varphi_t + g_2(\psi_t)\psi_t + g_3(w_t)w_t\big)dx \\ &+\int_0^L \big(h_1(t)\varphi_t(t) + h_2(t)\psi_t(t) + h_3(t)w_t(t)\big)dx.\end{aligned}\tag{3.1}$$

We define the perturbed energy

$$\mathcal{E}_\alpha(t) = \mathcal{E}(t) + \alpha\Phi(t) \quad (\alpha>0),\tag{3.2}$$

where

$$\Phi(t) = \rho_1\int_0^L \varphi(t)\varphi_t(t)dx + \rho_2\int_0^L \psi(t)\psi_t(t)dx + \rho_1\int_0^L w(t)w_t(t)dx.\tag{3.3}$$

Using Holder's and Young's inequalities, (2.1) and (2.3), we can estimate (3.3):

$$|\Phi(t)| \leqslant \frac{1}{2}\max\Big\{\rho_1,\rho_2,\rho_1\frac{L}{\pi},\rho_2\frac{L}{\pi}\Big\}\gamma_2\|y\|^2_{\mathcal{H}}.\tag{3.4}$$

Using Lemma 2.12 and choosing $\alpha_0 = (\max\{\rho_1,\rho_2,\rho_1\frac{L}{\pi},\rho_2\frac{L}{\pi}\})^{-1}\frac{\beta_0}{2}\gamma_2^{-1}$, we obtain

$$\alpha|\Phi(t)| \leqslant \alpha_0|\Phi(t)| \leqslant \frac{1}{2}(\mathcal{E}(t)+LM_F),$$

since $\alpha\leqslant\alpha_0$. So that

$$\mathcal{E}_\alpha(t) = \mathcal{E} + \alpha\Phi(t) \leqslant \frac{3}{2}\mathcal{E}(t) + \frac{L}{2}M_F,\tag{3.5}$$

$$\mathcal{E}_\alpha(t) = \mathcal{E} + \alpha\Phi(t) \geqslant \frac{1}{2}\mathcal{E}(t) - \frac{L}{2}M_F,\tag{3.6}$$

for each $\alpha\in[0,\alpha_0]$.

Now we estimating $\Phi'(t)$,

$$
\begin{aligned}
\Phi'(t) &= \rho_1\|\varphi_t\|^2 + \rho_2\|\psi_t\|^2 + \rho_1\|w_t\|^2 \\
&\quad + \int_0^L \varphi\big[h_1 + k(\varphi_x+\psi+lw)_x + k_0 l(w_x - l\varphi) - g_1(\varphi_t) - f_1\big]dx \\
&\quad + \int_0^L \psi\big[h_2 + b\psi_{xx} - k(\varphi_x+\psi+lw) - g_2(\psi_t) - f_2\big]dx \\
&\quad + \int_0^L w\big[h_3 + k_0(w_x - l\varphi)_x - kl(\varphi_x+\psi+lw) - g_3(w_t) - f_3\big]dx.
\end{aligned} \tag{3.7}
$$

From Green's Formula and using (2.6) in (3.7) we obtain

$$
\begin{aligned}
\Phi'(t) &= \rho_1\|\varphi_t\|^2 + \rho_2\|\psi_t\|^2 + \rho_1\|w_t\|^2 - k\|\varphi_x+\psi+w\|^2 - k_0\|w_x - l\varphi\|^2 \\
&\quad - b\|\psi_x\|^2 - \int_0^L \varphi g_1(\varphi_t)dx - \int_0^L \psi g_2(\psi_t)dx - \int_0^L w g_3(w_t)dx \\
&\quad - \int_0^L \nabla F(\varphi,\psi,w)\cdot(\varphi,\psi,w)dx + \int_0^L \varphi h_1 dx \\
&\quad + \int_0^L \psi h_2 dx + \int_0^L w h_3 dx
\end{aligned} \tag{3.8}
$$

Using $\mathcal{E}(t)$ in (3.8), we have

$$
\begin{aligned}
&\Phi'(t) \\
&= -\mathcal{E}(t) - \frac{1}{2}\big(k\|\varphi_x+\psi+w\|^2 + k_0\|w_x - l\varphi\|^2 + b\|\psi_x\|^2\big) + \frac{3}{2}\rho_1\|\varphi_t\|^2 \\
&\quad + \frac{3}{2}\rho_2\|\psi_t\|^2 + \frac{3}{2}\rho_1\|w_t\|^2 + \int_0^L \big[F(\varphi,\psi,w) - \nabla F(\varphi,\psi,w)\cdot(\varphi,\psi,w)\big]dx \\
&- \int_0^L \varphi g_1(\varphi_t)dx - \int_0^L \psi g_2(\psi_t)dx - \int_0^L w g_3(w_t)dx + \int_0^L \varphi h_1 dx \\
&\quad + \int_0^L \psi h_2 dx + \int_0^L w h_3 dx
\end{aligned} \tag{3.9}
$$

From (2.1), (2.4), (2.10), (2.11) and Holder's and Young's inequalities, we can write

$$
\begin{aligned}
&\int_0^L \big[F(\varphi,\psi,w) - \nabla F(\varphi,\psi,w)\cdot(\varphi,\psi,w)\big]dx \\
&\leqslant \frac{\beta L^2\gamma_3}{\pi^2}\big(b\|\psi_x\|^2 + k\|\varphi_x+\psi+lw\|^2 + k_0\|w_x - l\varphi\|^2\big) + M_F,
\end{aligned} \tag{3.10}
$$

$$
\begin{aligned}
&\int_0^L |\varphi g_1(\varphi_t) + \psi g_2(\psi_t) + w g_3(w_t)|dx \\
&\leqslant \frac{L^2\gamma_3}{\beta_0\pi^2}\big(\max\{M_1,M_2,M_3\}\big)^2\big(\|\varphi_t\|^2 + \|\psi_t\|^2 + \|w_t\|^2\big) \\
&\quad + \frac{\beta_0}{4}\big(b\|\psi_x\|^2 + k\|\varphi_x+\psi+lw\|^2 + k_0\|w_x - l\varphi\|^2\big),
\end{aligned} \tag{3.11}
$$

$$
\begin{aligned}
&\int_0^L |\varphi h_1 + \psi h_2 + w h_3|dx \leqslant \frac{L^2\gamma_3}{\beta_0\pi^2}\big(\|h_1\|^2 + \|h_2\|^2 + \|h_3\|^3\big) \\
&+ \frac{\beta_0}{4}\big(b\|\psi_x\|^2 + k\|\varphi_x+\psi+lw\|^2 + k_0\|w_x - l\varphi\|^2\big).
\end{aligned} \tag{3.12}
$$

Using (3.10)-(3.12) in (3.9), we find that

$$
\begin{aligned}
&\Phi'(t)\\
&\leqslant -\mathcal{E}(t) + \Big[\frac{3\rho_1}{2} + \frac{L^2\gamma_3}{\beta_0\pi^2}(\max\{M_1, M_2, M_3\})^2\Big]\|\varphi_t\|^2\\
&\quad + \Big[\frac{3\rho_2}{2} + \frac{L^2\gamma_3}{\beta_0\pi^2}(\max\{M_1, M_2, M_3\})^2\Big]\|\psi_t\|^2\\
&\quad + \Big[\frac{3\rho_1}{2} + \frac{L^2\gamma_3}{\beta_0\pi^2}(\max\{M_1, M_2, M_3\})^2\Big]\|w_t\|^2\\
&\quad + \frac{L^2\gamma_3}{\beta_0\pi^2}\big(\|h_1\|^2 + \|h_2\|^2 + \|h_3\|^2\big) + M_F\\
&\quad + \Big[\frac{1}{2} + \frac{\beta L^2\gamma_3}{\pi^2} + \frac{\beta_0}{2}\Big]\big(b\|\psi_x\|^2 + k\|\varphi_x + \psi + lw\|^2 + k_0\|w_x - l\varphi\|^2\big).
\end{aligned}
\tag{3.13}
$$

As $[\frac{1}{2} + \frac{\beta L^2\gamma_3}{\pi^2} + \frac{\beta_0}{2}] = 0$, by (3.13) we obtain

$$
\begin{aligned}
\Phi'(t) &\leqslant -\mathcal{E}(t) + c_1\|\varphi_t\|^2 + c_2\|\psi_t\|^2 + c_1\|w_t\|^2\\
&\quad + c_3(\|h_1\|^2 + \|h_2\|^2 + \|h_3\|^2) + M_F,
\end{aligned}
\tag{3.14}
$$

where

$$
\begin{gathered}
c_1 = \Big[\frac{3\rho_1}{2} + \frac{L^2\gamma_3}{\beta_0\pi^2}\big(\max\{M_1, M_2, M_3\}\big)^2\Big]\\
c_2 = \Big[\frac{3\rho_2}{2} + \frac{L^2\gamma_3}{\beta_0\pi^2}\big(\max\{M_1, M_2, M_3\}\big)^2\Big]\\
c_3 = \frac{L^2\gamma_3}{\beta_0\pi^2}.
\end{gathered}
$$

From (2.15), (3.2), and (3.14), taking $c_4 = \max\{c_1, c_2\}$, we obtain

$$
\begin{aligned}
\frac{d}{dt}\mathcal{E}_\alpha(t) &\leqslant -\frac{m_1}{2}\|\varphi_t\|^2 - \frac{m_2}{2}\|\psi_t\|^2 - \frac{m_3}{2}\|w_t\|^2\\
&\quad + \Big(\frac{2}{\min\{m_1, m_2, m_3\}} + \alpha c_3\Big)\big(\|h_1\|^2 + \|h_2\|^2 + \|h_3\|^2\big)\\
&\quad + \alpha c_4(\|\varphi_t\|^2 + \|\psi_t\|^2 + \|w_t\|^2) + \alpha M_F - \alpha\mathcal{E}(t).
\end{aligned}
\tag{3.15}
$$

Setting $\alpha = \min\{\alpha_0, 1, \frac{1}{c_4\min\{m_1, m_2, m_3\}}\}$ and using (3.5), we obtain

$$
\frac{d}{dt}\mathcal{E}_\alpha(t) \leqslant -\frac{2}{3}\alpha\mathcal{E}_\alpha(t) + c_5\big(\|h_1\|^2 + \|h_2\|^2 + \|h_3\|^2\big) + \alpha\Big(\frac{L+3}{3}\Big)M_F, \tag{3.16}
$$

where $c_5 = \frac{2}{\min\{m_1, m_2, m_3\}} + \alpha c_3$.

Using Gronwall's inequality and (3.5), (3.6) in (3.16),

$$
\begin{aligned}
\mathcal{E}(t) &\leqslant 3\mathcal{E}(\tau)e^{-\sigma_0(t-\tau)} + 2c_5\int_\tau^t e^{-\sigma_0(t-\tau)}\big(\|h_1\|^2 + \|h_2\|^2 + \|h_3\|^2\big)ds\\
&\quad + (3L+3)M_F,
\end{aligned}
\tag{3.17}
$$

with

$$
\sigma_0 = \frac{2\alpha}{3}. \tag{3.18}
$$

From (2.12) we can rewrite (3.17) as

$$
\mathcal{E}(t) \leqslant 3\mathcal{E}(\tau)e^{-\sigma_0(t-\tau)} + 2c_5c_h + (3L+3)M_F. \tag{3.19}
$$

Using (2.17) and (3.19) we can write, for $y \in H$, that

$$\|U(t,\tau)y\|_{\mathcal{H}}^2 \leqslant \frac{3}{\beta_0}\big[k_F\big(1+\|y\|_{\mathcal{H}}^{p+1}\big)\big]e^{-\sigma_0(t-\tau)} + \frac{c}{\beta_0} + \frac{LM_F}{\beta_0}, \tag{3.20}$$

where $c = 2c_5c_h + (3L+3)M_F$.

So, there exists a uniformly bounded pullback absorbing family $B(t) = \overline{B_{\mathcal{H}}(0,R)}$, with $R^2 > \frac{c}{\beta_0} + \frac{LM_F}{\beta_0}$, for all $t \in \mathbb{R}$. □

Lemma 3.3. *The process generated by system* (1.1)-(1.5) *is pullback asymptotically compact.*

Proof. Let $y = y_1 - y_2 = (\varphi,\psi,w,\varphi_t,\psi_t,w_t)$ be a solution of the problem

$$\begin{gathered}
\rho_1\varphi_{tt} - k(\varphi_x+\psi+lw)_x - k_0l(w_x - l\varphi) \\
= -\big[g_1(\varphi_t^1) - g_1(\varphi_t^2)\big] + \big[f_1(\varphi^2,\psi^2,w^2) - f_1(\varphi^1,\psi^1,w^1)\big], \\
\rho_2\psi_{tt} - b\psi_{xx} + k(\varphi_x+\psi+lw) \\
= -\big[g_2(\psi_t^1) - g_2(\psi_t^2)\big] + \big[f_2(\varphi^2,\psi^2,w^2) - f_2(\varphi^1,\psi^1,w^1)\big], \\
\rho_1 w_{tt} - k_0(w_x - l\varphi)_x + kl(\varphi_x+\psi+lw) \\
= -\big[g_3(\varphi_t^1) - g_3(\varphi_t^2)\big] + \big[f_3(\varphi^2,\psi^2,w^2) - f_3(\varphi^1,\psi^1,w^1)\big],
\end{gathered}$$

with initial condition

$$(\varphi(\tau),\psi(\tau),w(\tau),\varphi_t(\tau),\psi_t(\tau),w_t(\tau)) = y_\tau^1 - y_\tau^2\big(y_\tau^1, \quad y_\tau^2 \in B(t)\big).$$

Multiplying the above equations by φ_t, ψ_t and w_t, respectively, and integrating in $[0,L]$, we obtain

$$\begin{aligned}
&\frac{d}{dt}G(t) \\
&= -\int_0^L \big[g_1(\varphi_t^1) - g_1(\varphi_t^2)\big]\varphi_t - \int_0^L \big[g_2(\psi_t^1) - g_2(\psi_t^2)\big]\psi_t \\
&\quad - \int_0^L \big[g_3(w_t^1) - g_3(w_t^2)\big]w_t + \int_0^L \big[f_1(\varphi^2,\psi^2,w^2) - f_1(\varphi^1,\psi^1,w^1)\big]\varphi_t \\
&\quad + \int_0^L \big[f_2(\varphi^2,\psi^2,w^2) - f_2(\varphi^1,\psi^1,w^1)\big]\psi_t \\
&\quad + \int_0^L \big[f_3(\varphi^2,\psi^2,w^2) - f_3(\varphi^1,\psi^1,w^1)\big]w_t,
\end{aligned} \tag{3.21}$$

where

$$\begin{aligned}
G(t) &= \frac{\rho_1}{2}\|\varphi_t\|^2 + \frac{\rho_2}{2}\|\psi_t\|^2 + \frac{\rho_1}{2}\|w_t\|^2 + \frac{b}{2}\|\psi_x\|^2 + \frac{k}{2}\|\varphi_x+\psi+lw\|^2 \\
&\quad + \frac{k_0}{2}\|w_x - l\varphi\|^2.
\end{aligned}$$

We define

$$G_\eta(t) = G(t) + \eta\Psi(t), \quad \eta \geqslant 0, \tag{3.22}$$

with

$$\Psi(t) = \rho_1\int_0^L \varphi\varphi_t dx + \rho_2\int_0^L \psi\psi_t dx + \rho_1\int_0^L ww_t dx.$$

We observe that

$$|\Psi(t)| \leqslant \gamma_2 \max\big\{\rho_1,\rho_2,\rho_1\frac{L}{\pi},\rho_2\frac{L}{\pi}\big\}G(t).$$

Choosing $\eta_0 = \frac{1}{2}\gamma_2^{-1}\big(\max\{\rho_1, \rho_2, \rho_1\frac{L}{\pi}, \rho_2\frac{L}{\pi}\}\big)^{-1}$, since $\eta \in [0, \eta_0]$, we have

$$\frac{1}{2}G(t) \leqslant G_\eta(t) \leqslant \frac{3}{2}G(t) \quad \forall t \geqslant \tau. \tag{3.23}$$

Now we estimate $\Psi'(t)$,

$$\begin{aligned}
&\Psi'(t)\\
&= \rho_1\|\varphi_t\|^2 + \rho_2\|\psi_t\|^2 + \rho_1\|w_t\|^2 - b\|\psi_x\|^2 - k\|\varphi_x + \psi + lw\|^2\\
&\quad - k_0\|w_x - l\varphi\|^2 - \int_0^L \big[g_1(\varphi_t^1) - g_1(\varphi_t^2)\big]\varphi - \int_0^L \big[g_2(\psi_t^1) - g_2(\psi_t^2)\big]\psi\\
&\quad - \int_0^L \big[g_3(w_t^1) - g_3(w_t^2)\big]w + \int_0^L \big[f_1(\varphi^2, \psi^2, w^2) - f_1(\varphi^1, \psi^1, w^1)\big]\varphi\\
&\quad + \int_0^L \big[f_2(\varphi^2, \psi^2, w^2) - f_2(\varphi^1, \psi^1, w^1)\big]\psi\\
&\quad + \int_0^L \big[f_3(\varphi^2, \psi^2, w^2) - f_3(\varphi^1, \psi^1, w^1)\big]w\\
&= -G(t) - \frac{1}{2}\big(k\|\varphi_x + \psi + lw\|^2 + k_0\|w_x - l\varphi\|^2 + b\|\psi_x\|^2\big) + \frac{3\rho_1}{2}\|\varphi_t\|^2 \qquad (3.24)\\
&\quad + \frac{3\rho_2}{2}\|\psi_t\|^2 + \frac{3\rho_1}{2}\|w_t\|^2 + \int_0^L \big[g_1(\varphi_t^1) - g_1(\varphi_t^2)\big]\varphi\\
&\quad - \int_0^L \big[g_2(\psi_t^1) - g_2(\psi_t^2)\big]\psi - \int_0^L \big[g_3(w_t^1) - g_3(w_t^2)\big]w\\
&\quad + \int_0^L \big[f_1(\varphi^2, \psi^2, w^2) - f_1(\varphi^1, \psi^1, w^1)\big]\varphi\\
&\quad + \int_0^L \big[f_2(\varphi^2, \psi^2, w^2) - f_2(\varphi^1, \psi^1, w^1)\big]\psi\\
&\quad + \int_0^L \big[f_3(\varphi^2, \psi^2, w^2) - f_3(\varphi^1, \psi^1, w^1)\big]w
\end{aligned}$$

We observe that

$$-\int_0^L \big[g_1(\varphi_t^1) - g_1(\varphi_t^2)\big]\varphi_t \leqslant -m_1\|\varphi_t\|^2, \tag{3.25}$$

$$-\int_0^L \big[g_2(\psi_t^1) - g_2(\psi_t^2)\big]\psi_t \leqslant -m_2\|\psi_t\|^2, \tag{3.26}$$

$$-\int_0^L \big[g_3(\varphi_t^1) - g_3(w_t^2)\big]w_t \leqslant -m_3\|w_t\|^2, \tag{3.27}$$

$$-\int_0^L \big[g_1(\varphi_t^1) - g_1(\varphi_t^2)\big]\varphi \leqslant \frac{L\gamma_3 M_1^2}{2\pi}\|\varphi_t\|^2 + \frac{\pi}{2L\gamma_3}\|\varphi\|^2, \tag{3.28}$$

$$-\int_0^L \big[g_2(\psi_t^1) - g_1(\psi_t^2)\big]\psi \leqslant \frac{L\gamma_3 M_2^2}{2\pi}\|\psi_t\|^2 + \frac{\pi}{2L\gamma_3}\|\psi\|^2, \tag{3.29}$$

$$-\int_0^L \big[g_1(w_t^1) - g_1(w_t^2)\big]w \leqslant \frac{L\gamma_3 M_3^2}{2\pi}\|w_t\|^2 + \frac{\pi}{2L\gamma_3}\|w\|^2. \tag{3.30}$$

Replacing (3.24)-(3.30) in (3.22), we see that

$$
\begin{aligned}
\frac{d}{dt}G_\eta(t) \leqslant -\eta G(t) &+ \Big(- m_1 + \frac{3\rho_1}{2}\eta + \frac{L\gamma_3 M_1^2}{2\pi}\eta\Big)\|\varphi_t\|^2 \\
&+ \Big(- m_2 + \frac{3\rho_2}{2}\eta + \frac{L\gamma_3 M_2^2}{2\pi}\eta\Big)\|\psi_t\|^2 \\
&+ \Big(- m_3 + \frac{3\rho_1}{2}\eta + \frac{L\gamma_3 M_3^2}{2\pi}\eta\Big)\|w_t\|^2 \\
&+ \int_0^L \big[f_1(\varphi^2,\psi^2,w^2) - f_1(\varphi^1,\psi^1,w^1)\big]\varphi_t \\
&+ \int_0^L \big[f_2(\varphi^2,\psi^2,w^2) - f_2(\varphi^1,\psi^1,w^1)\big]\psi_t \\
&+ \int_0^L \big[f_3(\varphi^2,\psi^2,w^2) - f_3(\varphi^1,\psi^1,w^1)\big]w_t \\
&+ \int_0^L \big[f_1(\varphi^2,\psi^2,w^2) - f_1(\varphi^1,\psi^1,w^1)\big]\varphi \\
&+ \int_0^L \big[f_2(\varphi^2,\psi^2,w^2) - f_2(\varphi^1,\psi^1,w^1)\big]\psi \\
&+ \int_0^L \big[f_3(\varphi^2,\psi^2,w^2) - f_3(\varphi^1,\psi^1,w^1)\big]w
\end{aligned}
\tag{3.31}
$$

Taking

$$
\eta = \min\Big\{\eta_0, 1, \frac{2\pi m_1}{3\rho_1\pi + L\gamma_3 M_1^2}, \frac{2\pi m_2}{3\rho_2\pi + L\gamma_3 M_2^2}, \frac{2\pi m_3}{3\rho_1\pi + L\gamma_3 M_3^2}\Big\},
$$

from (3.31) we have

$$
\begin{aligned}
\frac{d}{dt}G_\eta(t) \leqslant -\eta G(t) &+ \int_0^L \big[f_1(\varphi^2,\psi^2,w^2) - f_1(\varphi^1,\psi^1,w^1)\big]\varphi_t \\
&+ \int_0^L \big[f_2(\varphi^2,\psi^2,w^2) - f_2(\varphi^1,\psi^1,w^1)\big]\psi_t \\
&+ \int_0^L \big[f_3(\varphi^2,\psi^2,w^2) - f_3(\varphi^1,\psi^1,w^1)\big]w_t \\
&+ \int_0^L \big[f_1(\varphi^2,\psi^2,w^2) - f_1(\varphi^1,\psi^1,w^1)\big]\varphi \\
&+ \int_0^L \big[f_2(\varphi^2,\psi^2,w^2) - f_2(\varphi^1,\psi^1,w^1)\big]\psi \\
&+ \int_0^L \big[f_3(\varphi^2,\psi^2,w^2) - f_3(\varphi^1,\psi^1,w^1)\big]w
\end{aligned}
\tag{3.32}
$$

Using (2.9), and Holder's and Young's inequalities, we can estimate the terms on the right-hand side of (3.32),

$$\begin{aligned}
&\int_0^L \big[f_1(\varphi^2,\psi^2,w^2)-f_1(\varphi^1,\psi^1,w^1)\big]\varphi\\
&+\int_0^L \big[f_2(\varphi^2,\psi^2,w^2)-f_2(\varphi^1,\psi^1,w^1)\big]\psi\\
&+\int_0^L \big[f_3(\varphi^2,\psi^2,w^2)-f_3(\varphi^1,\psi^1,w^1)\big]w\\
&\leqslant C_1\big(1+\|y_1\|_{\mathcal{H}}^{p-1}+\|y_2\|_{\mathcal{H}}^{p-1}\big)\big(\|\varphi\|_{2p}^2+\|\psi\|_{2p}^2+\|w\|_{2p}^2\big)\\
&=: k_1(\tau,t)\big(\|\varphi\|_{2p}^2+\|\psi\|_{2p}^2+\|w\|_{2p}^2\big),
\end{aligned}\tag{3.33}$$

for some constant $C_1>0$; and

$$\begin{aligned}
&\int_0^L \big[f_1(\varphi^2,\psi^2,w^2)-f_1(\varphi^1,\psi^1,w^1)\big]\varphi_t\\
&+\int_0^L \big[f_2(\varphi^2,\psi^2,w^2)-f_2(\varphi^1,\psi^1,w^1)\big]\psi_t\\
&+\int_0^L \big[f_3(\varphi^2,\psi^2,w^2)-f_3(\varphi^1,\psi^1,w^1)\big]w_t\\
&\leqslant \frac{C_2}{\epsilon}\big(1+\|y_1\|_{\mathcal{H}}^{2(p-1)}+\|y_2\|_{\mathcal{H}}^{2(p-1)}\big)\big(\|\varphi\|_{2p}^2+\|\psi\|_{2p}^2+\|w\|_{2p}^2\big)\\
&\quad +\epsilon\Big(\frac{\rho_1}{2}\|\varphi_t\|^2+\frac{\rho_2}{2}\|\psi_t\|^2+\frac{\rho_1}{2}\|w_t\|^2\Big)\\
&=: k_2(\tau,t)\big(\|\varphi\|_{2p}^2+\|\psi\|_{2p}^2+\|w\|_{2p}^2\big)\\
&\quad +\epsilon\Big(\frac{\rho_1}{2}\|\varphi_t\|^2+\frac{\rho_2}{2}\|\psi_t\|^2+\frac{\rho_1}{2}\|w_t\|^2\Big),
\end{aligned}\tag{3.34}$$

for some constants $C_2>0$ and $\epsilon>0$.

Replacing (3.23), (3.33) and (3.34) in (3.32), we obtain

$$\frac{d}{dt}G_\eta(t)\leqslant\Big(-\frac{2}{3}\eta+2\epsilon\Big)G_\eta(t)+k(\tau,t)\big(\|\varphi\|_{2p}^2+\|\psi\|_{2p}^2+\|w\|_{2p}^2\big),\tag{3.35}$$

with $k(\tau,t)=\max\{k_1(\tau,t),k_2(\tau,t)\}$.

Now, we take $\epsilon>0$ sufficiently small such that $\sigma_1=-\frac{2}{3}\eta+2\epsilon<0$ and $\sigma_0\leqslant\sigma_1$, where σ_0 was given in (3.18). Using Gronwall's inequality in $(3.35) and from (3.23)$, we infer that

$$G(t)\leqslant 3G(\tau)e^{-\sigma_1(t-\tau)}+2\sup_{s\in[\tau,t]}k(\tau,s)\int_\tau^t e^{-\sigma_1(t-s)}\big(\|\varphi\|_{2p}+\|\psi\|_{2p}+\|w\|_{2p}\big)ds$$

As $G(t)=\frac{1}{2}\|U(t,\tau)y_\tau^1-U(t,\tau)y_\tau^2\|_{\mathcal{H}}^2$ and $e^{-\sigma(t-s)}<1$, for each $s\in[\tau,t]$, since y_τ^1, $y_\tau^2\in B=B(t)\subset\overline{B_{\mathcal{H}}(0,R)}$ (defined in the Lemma 3.2), we have

$$\begin{aligned}
&\|U(t,\tau)y_\tau^1-U(t,\tau)y_\tau^2\|_{\mathcal{H}}^2\\
&\leqslant 3\|y_\tau^1-y_\tau^2\|_{\mathcal{H}}^2e^{-\sigma_1(t-\tau)}+4\sup_{s\in[\tau,t]}k(\tau,s)\int_\tau^t\big(\|\varphi\|_{2p}+\|\psi\|_{2p}+\|w\|_{2p}\big)ds\\
&\leqslant 3R^2e^{-\sigma_1(t-\tau)}+4\sup_{s\in[\tau,t]}k(\tau,s)\int_\tau^t\big(\|\varphi\|_{2p}+\|\psi\|_{2p}+\|w\|_{2p}\big)ds.
\end{aligned}\tag{3.36}$$

Let $t \in \mathbb{R}$ and $\epsilon > 0$. Then there exists $\tau_\epsilon = \tau(t, B, \epsilon)$ such that

$$3R^2 e^{-\sigma_1(t-\tau_\epsilon)} < \epsilon^2.$$

We defining Ψ_ϵ on $B(\tau_\epsilon)$ by

$$\Psi_\epsilon(z_1, z_2) = 2\Big(C_{t,\tau_\epsilon} \int_{\tau_\epsilon}^{t} \big(\|\varphi\|_{2p}^2 + \|\psi\|_{2p}^2 + \|w\|_{2p}^2\big)\Big)^{1/2}.$$

So, we can rewrite (3.36) as

$$\|U(t, \tau_\epsilon)y - U(t, \tau_\epsilon)z\|_{\mathcal{H}}^2 \leqslant \epsilon + \Psi_\epsilon(y, z)$$

for any $y,\, z \in B(\tau_\epsilon)$.

Let us prove that Ψ_ϵ is a contractive function. In fact, let $\{z_n\}$ be a sequence in $B(\tau_\epsilon)$. We have

$$\|U(s, \tau_\epsilon)z_n\|_{\mathcal{H}} \leqslant R < \infty$$

for any $s \in [\tau_\epsilon, t]$. So, since $U(s, \tau_\epsilon)z_n = (\varphi^n, \psi^n, w^n, \varphi_t^n, \psi_t^n, w_t^n)$, we obtain

$$\begin{gathered}(\varphi^n, \psi^n, w^n) \text{ is bounded in } L^2(\tau_\epsilon, t; H_0^1(\Omega)^3),\\ (\varphi_t^n, \psi_t^n, w_t^n) \text{ is bounded in } L^2(\tau_\epsilon, t; L^2(\Omega)^3)\end{gathered}$$

From the embedding $H_0^1(0, L) \xhookrightarrow{c} L^{2p}(0, L) \hookrightarrow L^2(0, L)$ there exist a function z and a subsequence $\{z_{n_k}\}$ such that

$$z_{n_k} \to z \quad \text{strongly in } L^2(\tau_\epsilon, t; L^{2p}(\Omega)^3).$$

Then, Ψ_ϵ is contractive on $B(\tau_\epsilon)$. Consequently, the process generated by (1.1)-(1.5) is pullback asymptotically compact. $\square$

4. Upper semicontinuity

We consider (1.1)-(1.5), with $h_i(x, t)$ replaced by $\epsilon h_i(x, t)$, with $i = 1, 2, 3$; that is,

$$\begin{gathered}\rho_1\varphi_{tt} - k(\varphi_x + \psi + lw)_x - k_0 l(w_x - l\varphi) + g_1(\varphi_t) + f_1(\varphi, \psi, w) = \epsilon h_1(x, t),\\ \rho_2\psi_{tt} - b\psi_{xx} + k(\varphi_x + \psi + lw) + g_2(\psi_t) + f_2(\varphi, \psi, w) = \epsilon h_2(x, t),\\ \rho_1 w_{tt} - k_0(w_x - l\varphi)_x + kl(\varphi_x + \psi + lw) + g_3(w_t) + f_3(\varphi, \psi, w) = \epsilon h_3(x, t),\end{gathered}$$

As we will take $\epsilon \to 0$, we can assume without loss of generality, that $\epsilon \in [0, 1]$. The same procedure used before yields an evolution process $U_\epsilon(t, \tau) : H \to H$, for each $\epsilon \in [0, 1]$.

Noting that for $\epsilon = 0$, the above problem becomes autonomous and has a C^0 semigroup associated $S(t)$ on H. As proved in [13], $S(t)$ admits a global attractor A_0. Furthermore, we can see that $S(t)$ is a evolution process defined by $U_0(t, \tau) = S(t - \tau)$. Then the constant family $\mathcal{A}_0 = \{A_0\}$, for any $t \in \mathbb{R}$, is the pullback attractor by the process $U_0(t, \tau)$.

As the R from the absorbing ball, given in the Lemma 3.2, is independent of ϵ and t, we have $\{B_\epsilon(t)\}$ with $B_\epsilon(t) = \overline{B(0, R)}$ is a absorbing family by the process $U_\epsilon(t, \tau)$ that absorbs all bounded sets in H.

Theorem 4.1. *The pullback attractors family $\mathcal{A}_\epsilon$ is upper-semicontinuous as $\epsilon \to 0$, that is,*

$$\lim_{\epsilon \to 0} \operatorname{dist}(A_\epsilon(t), A_0) = 0, \quad \forall t \in \mathbb{R}.$$

Proof. From Lemma 3.2 there exists a uniformly bounded family $B_0(t) = \overline{B(0,R)}$. From the invariance of the attractor, we have that $A_\epsilon(t) \subset \overline{B(0,R)}$, for all $t \in \mathbb{R}$ and $\epsilon \in [0,1]$. And this prove (ii) from the Preposition 2.10.

Given $z \in D$ ($D \subset H$ bounded) and $\tau \leqslant t$, let

$$
\begin{aligned}
U_\epsilon(t,\tau)z = (\varphi^1, \psi^1, w^1, \varphi_t^1, \psi_t^1, w_t^1) = u^1,\\
U_0(t,\tau)z = (\varphi^2, \psi^2, w^2, \varphi_t^2, \psi_t^2, w_t^2) = u^2.
\end{aligned}
$$

Then, we can see that $u = u_1 - u_2$ solves the problem

$$
\begin{aligned}
&\rho_1\varphi_{tt} - k(\varphi_x + \psi + lw)_x - k_0 l(w_x - l\varphi)\\
&= -\big[g_1(\varphi_t^1) - g_1(\varphi_t^2)\big] + \big[f_1(\varphi^2,\psi^2,w^2) - f_1(\varphi^1,\psi^2,w^1)\big] + \epsilon h_1,\\
&\rho_2\psi_{tt} - b\psi_{xx} + k(\varphi_x + \psi + lw)\\
&= -\big[g_2(\psi_t^1) - g_2(\psi_t^2)\big] + \big[f_2(\varphi^2,\psi^2,w^2) - f_2(\varphi^1,\psi^2,w^1)\big] + \epsilon h_2,\\
&\rho_1 w_{tt} - k_0(w_x - l\varphi)_x + kl(\varphi_x + \psi + lw)\\
&= -\big[g_3(w_t^1) - g_3(w_t^2)\big] + \big[f_1(\varphi^2,\psi^2,w^2) - f_1(\varphi^1,\psi^2,w^1)\big] + \epsilon h_3,
\end{aligned}
$$

with null initial conditions. Multiplying the first equation by φ_t, the second equation by ψ_t, and the third equation by w_t, we obtain

$$
\begin{aligned}
&\frac{d}{dt}\big[\rho_1\|\varphi_t\|^2 + \rho_2\|\psi\|^2 + \rho_1\|w_t\|^2 + b\|\psi_x\|^2 + k\|\varphi_x + \psi + lw\|^2 + k_0\|w_x - l\varphi\|^2\big]\\
&= -2\int_0^L \big[g_1(\varphi_t^1) - g_1(\varphi_t^2)\big]\varphi_t dx - 2\int_0^L \big[g_2(\psi_t^1) - g_1(\psi_t^2)\big]\psi_t dx\\
&\quad - 2\int_0^L \big[g_3(w_t^1) - g_3(w_t^2)\big]w_t dx + 2\int_0^L \big[f_1(\varphi^2,\psi^2,w^2) - f_1(\varphi^1,\psi^1,w^1)\big]\varphi_t\\
&\quad + 2\int_0^L \big[f_2(\varphi^2,\psi^2,w^2) - f_2(\varphi^1,\psi^1,w^1)\big]\psi_t\\
&\quad + 2\int_0^L \big[f_3(\varphi^2,\psi^2,w^2) - f_3(\varphi^1,\psi^1,w^1)\big]w_t + 2\epsilon\int_0^L \big[h_1\varphi_t + h_2\psi_t + h_2 w_t\big]dx.
\end{aligned}
\tag{4.1}
$$

Estimating the terms on the right-hand side of (4.1), we obtain

$$
\begin{aligned}
-2\int_0^L \big[g_1(\varphi_t^1) - g_1(\varphi_t^2)\big]\varphi_t dx \leqslant -2m_1\|\varphi_t\|^2 \leqslant 0,\\
-2\int_0^L \big[g_2(\psi_t^1) - g_1(\psi_t^2)\big]\psi_t dx \leqslant -2m_2\|\psi_t\|^2 \leqslant 0,\\
-2\int_0^L \big[g_3(w_t^1) - g_3(w_t^2)\big]w_t dx \leqslant -2m_3\|w_t\|^2 \leqslant 0,
\end{aligned}
$$

$$
\begin{aligned}
&2\int_0^L \big[f_1(\varphi^2,\psi^2,w^2)-f_1(\varphi^1,\psi^1,w^1)\big]\varphi_t \\
&+2\int_0^L \big[f_2(\varphi^2,\psi^2,w^2)-f_2(\varphi^1,\psi^1,w^1)\big]\psi_t \\
&+2\int_0^L \big[f_3(\varphi^2,\psi^2,w^2)-f_3(\varphi^1,\psi^1,w^1)\big]w_t \\
&\leqslant C\big(1+\|u_1\|_{\mathcal{H}}^{2(p-1)}+\|u_2\|_{\mathcal{H}}^{2(p-1)}\big)\big(\|\varphi_x\|^2+\|\psi_x\|^2+\|w_x\|^2\big) \\
&\quad +\frac{1}{2}(\|\varphi_t\|^2+\|\psi_t\|^2+\|w_t\|^2)
\end{aligned}
$$

for some constant $C>0$. Also

$$
\begin{aligned}
&2\epsilon\int_0^L \big[h_1\varphi_t+h_2\psi_t+h_2w_t\big]dx \\
&\leqslant 2\epsilon^2\big(\|h_1\|^2+\|h_2\|^2+\|h_3\|^2\big)+\frac{1}{2}\big(\|\varphi_t\|^2+\|\psi_t\|^2+\|w_t\|^2\big).
\end{aligned}
$$

Replacing this estimates in (4.1), we can write

$$
\begin{aligned}
&\frac{d}{dt}\big[\rho_1\|\varphi_t\|^2+\rho_2\|\psi\|^2+\rho_1\|w_t\|^2+b\|\psi_x\|^2+k\|\varphi_x+\psi+lw\|^2 \\
&+k_0\|w_x-l\varphi\|^2\big] \\
&\leqslant C(t,T,D)\big[\rho_1\|\varphi_t\|^2+\rho_2\|\psi\|^2+\rho_1\|w_t\|^2+b\|\psi_x\|^2+k\|\varphi_x+\psi+lw\|^2 \\
&\quad +k_0\|w_x-l\varphi\|^2\big]+2\epsilon^2\big[\|h_1\|^2+\|h_2\|^2+\|h_3\|^2\big],
\end{aligned}
\tag{4.2}
$$

where $C(t,T,D)$ depends on t, T and D.

Applying Gronwall's inequality in (4.2), we obtain

$$
\|u_1-u_2\|_{\mathcal{H}}^2\leqslant 2\epsilon^2\int_\tau^t e^{C(t,T,D)(t-s)}\big[\|h_1(s)\|^2+\|h_2(s)\|^2+\|h_3(s)\|^2\big]ds
$$

and, thus

$$
\begin{aligned}
&\|U_\epsilon(t,\tau)z-U_0(t,\tau)z\|_{\mathcal{H}}^2 \\
&\leqslant 2\epsilon^2\int_{t-T}^t e^{C(t,T,D)(t-s)}\big[\|h_1(s)\|^2+\|h_2(s)\|^2+\|h_3(s)\|^2\big]ds,
\end{aligned}
$$

for all $\tau\in[t-T,t]$ and $z\in D$. As h_1, h_2, and h_3 are locally integrable, we verify (iii) of Preposition 2.10. Consequently, we prove the upper semicontinuity of A_ϵ when $\epsilon\to 0$. □

5. Continuity of attractors

Our last result establishes the continuity of pullback attractors with respect to some parameter.

Theorem 5.1. *Let $t\in\mathbb{R}$. In the context of Theorem 3.1 there exists a set J dense in $[0,1]$ such that $\mathcal{A}_\epsilon$ is continuous with respect to any parameter $\epsilon_0\in J$, that is,*

$$
\lim_{\epsilon\to\epsilon_0} d_H(\mathcal{A}_\epsilon,\mathcal{A}_{\epsilon_0})=0\quad \forall \epsilon_0\in J.
$$

Proof. We can apply Theorem 2.11. Assumptions (i) and (ii) of Theorem 2.11 hold because of Theorem 3.1 and Lemma 3.2.

Let $D \subset H$ bounded. Given ϵ_1, $\epsilon_2 \in [0,1]$, $z \in D$ and $\tau \leqslant t$, let us denote

$$
\begin{aligned}
U_{\epsilon_1}(t,\tau)z = (\varphi^1, \psi^1, w^1, \varphi_t^1, \psi_t^1, w_t^1) = u_1,\\
U_{\epsilon_2}(t,\tau)z = (\varphi^2, \psi^2, w^2, \varphi_t^2, \psi_t^2, w_t^2) = u_2.
\end{aligned}
$$

Then, $u = u_1 - u_2$ is solution of

$$
\begin{aligned}
&\rho_1\varphi_{tt} - k(\varphi_x + \psi + lw)_x - k_0 l(w_x - l\varphi)\\
&= -\big[g_1(\varphi_t^1) - g_1(\varphi_t^2)\big] + \big[f_1(\varphi^2,\psi^2,w^2) - f_1(\varphi^1,\psi^2,w^1)\big] + (\epsilon_1 - \epsilon_2)h_1\\
&\rho_2\psi_{tt} - b\psi_{xx} + k(\varphi_x + \psi + lw)\\
&= -\big[g_2(\psi_t^1) - g_2(\psi_t^2)\big] + \big[f_2(\varphi^2,\psi^2,w^2) - f_2(\varphi^1,\psi^2,w^1)\big] + (\epsilon_1 - \epsilon_2)h_2\\
&\rho_1 w_{tt} - k_0(w_x - l\varphi)_x + kl(\varphi_x + \psi + lw)\\
&= -\big[g_3(w_t^1) - g_3(w_t^2)\big] + \big[f_1(\varphi^2,\psi^2,w^2) - f_1(\varphi^1,\psi^2,w^1)\big] + (\epsilon_1 - \epsilon_2)h_3
\end{aligned}
$$

Multiplying the first equation by φ_t, the second equation by ψ_t, and the third equation by w_t, we can write

$$
\begin{aligned}
&\frac{d}{dt}\big[\rho_1\|\varphi_t\|^2 + \rho_2\|\psi\|^2 + \rho_1\|w_t\|^2 + b\|\psi_x\|^2 + k\|\varphi_x + \psi + lw\|^2 + k_0\|w_x - l\varphi\|^2\big]\\
&= -2\int_0^L \big[g_1(\varphi_t^1) - g_1(\varphi_t^2)\big]\varphi_t dx - 2\int_0^L \big[g_2(\psi_t^1) - g_1(\psi_t^2)\big]\psi_t dx\\
&\quad - 2\int_0^L \big[g_3(w_t^1) - g_3(w_t^2)\big]w_t dx + 2\int_0^L \big[f_1(\varphi^2,\psi^2,w^2) - f_1(\varphi^1,\psi^1,w^1)\big]\varphi_t\\
&\quad + 2\int_0^L \big[f_2(\varphi^2,\psi^2,w^2) - f_2(\varphi^1,\psi^1,w^1)\big]\psi_t\\
&\quad + 2\int_0^L \big[f_3(\varphi^2,\psi^2,w^2) - f_3(\varphi^1,\psi^1,w^1)\big]w_t\\
&\quad + 2(\epsilon_1 - 2\epsilon_2)\int_0^L \big[h_1\varphi_t + h_2\psi_t + h_2 w_t\big]dx.
\end{aligned}
\tag{5.1}
$$

Estimating the terms on the right-hand side of (5.1) we have

$$
\begin{aligned}
-2\int_0^L \big[g_1(\varphi_t^1) - g_1(\varphi_t^2)\big]\varphi_t dx \leqslant -2m_1\|\varphi_t\|^2 \leqslant 0,\\
-2\int_0^L \big[g_2(\psi_t^1) - g_1(\psi_t^2)\big]\psi_t dx \leqslant -2m_2\|\psi_t\|^2 \leqslant 0,\\
-2\int_0^L \big[g_3(w_t^1) - g_3(w_t^2)\big]w_t dx \leqslant -2m_3\|w_t\|^2 \leqslant 0,
\end{aligned}
$$

$$
\begin{aligned}
&2\int_0^L \big[f_1(\varphi^2,\psi^2,w^2)-f_1(\varphi^1,\psi^1,w^1)\big]\varphi_t \\
&\quad + 2\int_0^L \big[f_2(\varphi^2,\psi^2,w^2)-f_2(\varphi^1,\psi^1,w^1)\big]\psi_t \\
&\quad + 2\int_0^L \big[f_3(\varphi^2,\psi^2,w^2)-f_3(\varphi^1,\psi^1,w^1)\big]w_t \\
&\leqslant C\big(1+\|u_1\|_{\mathcal{H}}^{2(p-1)}+\|u_2\|_{\mathcal{H}}^{2(p-1)}\big)\big(\|\varphi_x\|^2+\|\psi_x\|^2+\|w_x\|^2\big) \\
&\quad + \frac{1}{2}\big(\|\varphi_t\|^2+\|\psi_t\|^2+\|w_t\|^2\big)
\end{aligned}
$$

for some constant $C>0$. Also

$$
\begin{aligned}
&2(\epsilon_1-\epsilon_2)\int_0^L \big[h_1\varphi_t+h_2\psi_t+h_2w_t\big]dx \\
&\leqslant 2|\epsilon_1-\epsilon_2|^2\big(\|h_1\|^2+\|h_2\|^2+\|h_3\|^2\big)+\frac{1}{2}\big(\|\varphi_t\|^2+\|\psi_t\|^2+\|w_t\|^2\big).
\end{aligned}
$$

Replacing these estimates in (5.1), we obtain

$$
\begin{aligned}
&\frac{d}{dt}\big[\rho_1\|\varphi_t\|^2+\rho_2\|\psi\|^2+\rho_1\|w_t\|^2+b\|\psi_x\|^2+k\|\varphi_x+\psi+lw\|^2 \\
&\quad + k_0\|w_x-l\varphi\|^2\big] \\
&\leqslant C(t,T,D)\big[\rho_1\|\varphi_t\|^2+\rho_2\|\psi\|^2+\rho_1\|w_t\|^2+b\|\psi_x\|^2+k\|\varphi_x+\psi+lw\|^2 \\
&\quad + k_0\|w_x-l\varphi\|^2\big]+2|\epsilon_1-\epsilon_2|^2\big[\|h_1\|^2+\|h_2\|^2+\|h_3\|^2\big],
\end{aligned}
\tag{5.2}
$$

where $C(t,T,D)$ depends on t, T and D.

Applying Gronwall's inequality in (5.2), we obtain

$$
\|u_1-u_2\|_{\mathcal{H}}^2 \leqslant 2|\epsilon_1-\epsilon_2|^2\int_\tau^t e^{C(t,T,D)(t-s)}\big[\|h_1(s)\|^2+\|h_2(s)\|^2+\|h_3(s)\|^2\big]ds
$$

and, therefore

$$
\begin{aligned}
&\|U_\epsilon(t,\tau)z-U_0(t,\tau)z\|_{\mathcal{H}}^2 \\
&\leqslant 2|\epsilon_1-\epsilon_2|^2\int_{t-T}^t e^{C(t,T,D)(t-s)}\big[\|h_1(s)\|^2+\|h_2(s)\|^2+\|h_3(s)\|^2\big]ds,
\end{aligned}
$$

for all $\tau\in[t-T,t]$ and $z\in D$. As h_1, h_2, and h_3 are locally integrable, from (ii) of Theorem 2.11. This completes the proof. □

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