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DYNAMICS OF A NON-AUTONOMOUS STOCHASTIC WEAKLY DAMPED PLATE MODEL WITH CRITICAL EXPONENT

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Abstract. In this article, we study the long-time behavior of the non-autonomous stochastic weakly damped plate model with critical exponent. By decomposing the solutions of the system and estimating the bounds of solutions in a more regular space, we obtain random attractors, when the external term is time-dependent and the nonlinearity has a critical growth.

1. Introduction

In this article, we consider the non-autonomous stochastic weakly damped plate equation with critical nonlinearity and additive white noise,

$$
\begin{gathered}
u_{tt}+\alpha u_t+\Delta^2 u+\lambda u+f(u,x)=g(x,t)+h(x)\dot{W}(t),\quad x\in U, t>\tau,\tau\in\mathbb{R},\\
u(x,t)|_{\partial U}=\frac{\partial}{\partial\nu}u|_{\partial U}=0,\quad t\geq\tau,\tau\in\mathbb{R},\\
u(x,\tau)=u_\tau(x),\quad u_t(x,\tau)=u_{1\tau}(x),\quad x\in U,\ \tau\in\mathbb{R},
\end{gathered}
\tag{1.1}
$$

where U is an open bounded set of $\mathbb{R}^n$ with a smooth boundary ∂U, $u(t)=u(x,t)$ is a real-valued function on $U\times[\tau,\infty)$, $\tau\in\mathbb{R}$, $\lambda>0$, $\alpha>0$, the external term $g(\cdot,t)\in C_b(\mathbb{R},H_0^2(U))$, $C_b(\mathbb{R},H_0^2(U))$ denotes the set of continuous bounded functions from $\mathbb{R}$ into $H_0^2(U)$, $h\in H_0^2(U)$. W is a two-sided real-valued Wiener process on a probability space $(\Omega,\mathcal{F},\mathbb{P})$, where $\Omega=\{\omega\in C(\mathbb{R},\mathbb{R}):\omega(0)=0\}$, the Borel σ-algebra $\mathcal{F}$ on Ω is generated by the compact open topology, $\mathbb{P}$ is the corresponding Wiener measure on $\mathcal{F}$. For any $t\in\mathbb{R}$, we can define a mapping θ_t on Ω by $\theta_t(\cdot)=\omega(t+\cdot)-\omega(\cdot)$ for $\omega\in\Omega$, then $(\Omega,\mathcal{F},\mathbb{P},(\theta_t)_{t\in\mathbb{R}})$ is an ergodic metric dynamical system.

The nonlinear term f satisfies the following assumptions:

(A1) $f(u,x)=f_1(u,x)+f_2(u,x)$ and there exist positive constants c_0, c_1, c_2, c_3, c_4, c_5, and functions $\beta_i\in L^1(U)$, $i=1,2$, such that for $x\in U$, $u\in\mathbb{R}$,

$$
\begin{gathered}
0\leq G_1(u,x)\leq c_0uf_1(u,x)\leq c_1G_1(u,x),\\
G_1(u,x)=\int_0^u f_1(r,x)dr,\\
f_1\in C^2(\mathbb{R},\mathbb{R}),\quad f_{1,u}'(0,x)=0,\quad |f_{1,u}''(u,x)|\leq c_2|u|^{q-2},
\end{gathered}
\tag{1.2}
$$

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where $2 < q < \infty$ if $n \le 4$, and $2 < q \le \frac{n}{n-4}$ if $n \ge 5$;

$$f_2(\cdot, x) \in C^1(\mathbb{R}, \mathbb{R}), \quad |f'_{2,u}(u,x)| \le c_3\left(1 + |u|^p\right), \quad 0 \le p < q-1, \tag{1.3}$$
$$c_4|u|^{q+1} - \beta_1(x) \le G(u,x) \le c_5 u f(u,x) + \delta_0 u^2 + \beta_2(x),$$
$$G(u,x) = \int_0^u f(r,x)dr,$$

for some $0 \le \delta_0 \le \frac{c_5}{8\lambda_1^2}$, where λ_1 is the first eigenvalue of operator $A = -\triangle$.

The plate equation arises in the nonlinear theory of oscillations, and our problem (1.1) has strong background in mathematical physics, the main motivation of the study comes not only from applications, but also from the mathematics, here the time-dependent external forcing and critical nonlinearity require more complicated techniques to some extent. In recent years the asymptotic behavior of the plate equation has been considered extensively in many papers(see, e.g., [1, 4, 11, 12, 14, 25]). For the deterministic plate equation without noise (i.e., $h \equiv 0$), [11, 12] established the existence of global attractors for localized damping, in [1, 25], the authors dealt with the plate equation with nonlinear damping; Von Karman equation is also one of the most important plate models (see [5, 6, 13] for details).

As for the autonomous stochastic system (where the external term g is independent of t), for the wave equation, if the nonlinearity f has a subcritical exponent, the existence of random attractors have been investigated in [10, 28]. When f has a critical exponent, the existence of random attractors have been considered in [16, 26]. For the autonomous stochastic plate system, the authors in [17] proved the existence of random attractors for plate model with strongly damping, [18] showed the random attractors for plate equation with linear memory.

When the forcing term g is time-dependent, for the non-autonomous stochastic wave equation, if f has a subcritical growth, the existence of random attractors were studied in [15, 20, 22], and [29, 30] showed the upper bound of fractal dimension of random attractors. When $n = 3$ and f has a critical exponent, the existence and boundedness of fractal dimension of random attractors have been successfully obtained for both additive noise and multiplicative noise, see [23, 24]. However, to the best of our knowledge, the non-autonomous stochastic weakly damped plate equation is less discussed, especially for the non-autonomous external term.

In this article, inspired by the ideas in [23, 24], we analyze the dynamical behavior of the non-autonomous stochastic weakly damped plate equation. Under the assumptions that the external term g is time-dependent, and the nonlinear team f has a critical growth, by decomposing the solutions of system through two different modes, we estimate the bounds of solutions in a higher regular space, and then establish the existence of random attractors. For the existence of random attractors, some kind of compactness of the process is a key ingredient, when verifying the pullback compactness, it is important to deal with the critical nonlinearity.

This article is organized as follows. In section 2, we give some preparations for our consideration. In section 3, we estimate the bounds of solutions. In section 4, we decompose the solutions of the equation into two parts: one part decays exponentially, another part is bounded in a higher regular space. In section 5, we obtain the existence of random attractors of the system (1.1).

2. Preliminaries

In this section, we recall some basic concepts of pullback random attractors [2, 7, 8].

We know that self-adjoint positive linear sectorial operator $A=-\triangle$ has eigenvalues $\{\lambda_i\}_{i\in N}$ such that $0<\lambda_1\le\lambda_2\le\dots\le\lambda_m\le\dots,\lambda_m\to+\infty$ as $m\to+\infty$. For $r\in\mathbb{R}$, the r-th powers A^r of A can be defined. Denote $V_{2r}=D(A^r)$, and it is a Hilbert space with inner product $(u,v)_{2r}=(A^ru,A^rv)$. The injection $V_{r_1}\hookrightarrow V_{r_2}$ is compact for any $r_1>r_2$ and $V_0=L^2(U),V_2=H_0^2(U)$. Write $E^r=D(A^{r+1})\times D(A^r)$ for $r\in\mathbb{R}$, and let $E=H_0^2(U)\times L^2(U)$. Denote the inner products and norms of $L^2(U)$, $H_0^2(U)$ and E by

$$
\begin{gathered}
(u,v)=\int_U uv\,dx,\quad \|u\|^2=(u,u),\quad \forall u,v\in L^2(U),\\
(u,v)_2=\int_U \Delta u\Delta v\,dx,\quad \|u\|_2^2=(u,u)_2,\quad \forall u,v\in H_0^2(U),\\
(y_1,y_2)_E=(u_1,u_2)_2+(v_1,v_2),\quad \forall y_i=(u_i,v_i)^{\mathrm{T}}\in E,\ i=1,2,\\
\|y\|_E^2=\|u\|_2^2+\|v\|^2,\quad \forall y=(u,v)^{\mathrm{T}}\in E.
\end{gathered}
$$

First of all, we transfer the stochastic differential equation (1.1) into a random system without noise term. Write $z(\theta_t\omega):=-\alpha\int_{-\infty}^0 e^{\alpha s}(\theta_t\omega)(s)ds$ $(t\in\mathbb{R})$ as an Ornstein-Uhlenbeck stationary process which can solve the equation $dz+\alpha zdt=dW(t)$. From [2, 3, 9, 30], we know that $t\mapsto z(\theta_t\omega)$ is continuous in t for almost every $\omega\in\Omega$ and

$$
\lim_{t\to+\infty}e^{-\gamma t}|z(\theta_{-t}\omega)|=0,\quad \forall\gamma>0;\ \mathrm{E}[|z(\theta_t\omega)|^r]=\frac{\Gamma(\frac{1+r}{2})}{\sqrt{\pi\alpha^r}},\tag{2.1}
$$

for all $r>0$ and $t\in\mathbb{R}$, where Γ is Gamma function. Actually, as $z(\theta_t\omega)$ is Gauss stationary process with expectation 0 and square variance $\frac{1}{2\alpha}$ [2, 9, 20]. Then for any $r>0$ and $t\in\mathbb{R}$, by [23],

$$
\mathrm{E}[|z(\theta_t\omega)|^r]=\frac{1}{\sqrt{\pi\alpha^r}}\int_0^{+\infty}\xi^{\frac{1+r}{2}-1}e^{-\xi}d\xi=\frac{\Gamma(\frac{1+r}{2})}{\sqrt{\pi\alpha^r}}.
$$

For simplicity, in this paper, we write a.e. $\omega\in\Omega$ as $\omega\in\Omega$; all the numbers c_i $(i\in\mathbb{N})$ below are independent of (ω,τ,t). Let

$$
v=u_t+\varepsilon u-h(x)z(\theta_t\omega),\quad \varepsilon=\frac{\lambda_1^2\alpha}{\alpha^2+3\lambda_1^2},\quad t\ge\tau,\ \tau\in\mathbb{R}.
$$

Then (1.1) can be changed into the stochastic system in the Hilbert space E,

$$
\begin{gathered}
\dot\varphi+\Lambda\varphi=F(\varphi,\theta_t\omega,t),\quad t\ge\tau,\\
\varphi(\tau,\omega)=\varphi_\tau(\omega)=(u_\tau,u_{1,\tau}+\varepsilon u_\tau-h(x)z(\theta_\tau\omega))^{\mathrm{T}},
\end{gathered}\tag{2.2}
$$

$\tau\in R$, where

$$
\begin{gathered}
\varphi=\begin{pmatrix}u\\ v\end{pmatrix},\quad \Lambda=\begin{pmatrix}\varepsilon I & -I\\ A^2-\varepsilon(\alpha-\varepsilon)I & (\alpha-\varepsilon)I\end{pmatrix}\\
F(\varphi,\theta_t\omega,t)=\begin{pmatrix}h(x)z(\theta_t\omega)\\ -f(u,x)-\lambda u+g(x,t)+\varepsilon h(x)z(\theta_t\omega)\end{pmatrix}.
\end{gathered}
$$

Following the arguments in [16, 17, 18, 20, 26], it can be proved that for any $\varphi_\tau=\varphi(\tau,\omega)\in E$, problem (2.2) is well-posed in E; that is, the (weak) solution

$\varphi(\cdot,\tau,\omega,\varphi_\tau)$ of (2.2) exists uniquely and globally for $t\in[\tau,\infty)$, and $\varphi(\cdot,\tau,\omega,\varphi_\tau)\in C([\tau,\infty);E)$ can define a continuous cocycle on E, $\Phi:\mathbb{R}^+\times\mathbb{R}\times\Omega\times E\to E$, $(t,\tau,\omega,\varphi_\tau)\mapsto\Phi(t,\tau,\omega)\varphi_\tau(\omega)$ by

$$\begin{aligned}\Phi(t,\tau,\omega)\varphi_\tau(\omega)&=\varphi(t+\tau,\tau,\theta_{-\tau}\omega,\varphi_\tau(\theta_{-\tau}\omega))\\&=\begin{pmatrix}u(t+\tau,\tau,\theta_{-\tau}\omega,\varphi_\tau(\theta_{-\tau}\omega))\\u_t(t+\tau,\tau,\theta_{-\tau}\omega,\varphi_\tau)+\varepsilon u(t+\tau,\tau,\theta_{-\tau}\omega,\varphi_\tau)-h(x)z(\theta_t\omega)\end{pmatrix}\end{aligned}$$

over $\mathbb{R}$ and $(\Omega,\mathcal{F},\mathbb{P},(\theta_t)_{t\in\mathbb{R}})$, where $\Phi(0,\tau,\omega)\varphi_\tau(\omega)=\varphi_\tau(\theta_{-\tau}\omega)$ and

$$\Phi(t,\tau-t,\theta_{-t}\omega)\varphi_{\tau-t}(\theta_{-t}\omega)=\varphi(\tau,\tau-t,\theta_{-\tau}\omega,\varphi_{\tau-t}(\theta_{-\tau}\omega)).$$

We notice that the initial data $u_\tau(x),u_{1\tau}(x)$ of (1.1) is independent of ω, conversely, $u(t,\tau,\omega,x)$ and $u_t(t,\tau,\omega,x)$ depend on ω for $t>\tau$.

Next, we recall the definition of random attractor and the existence criterion of random attractor for cocycle Φ.

Definition 2.1. A family $K=\{K(\tau,\omega):\tau\in\mathbb{R},\omega\in\Omega\}\in\mathcal{D}(E)$ of nonempty subsets of E is called a measurable $\mathcal{D}(E)$-pullback attracting set for Φ if

(i) K is a measurable with respect to $\mathcal{F}$ in Ω;

(ii) for all $\tau\in\mathbb{R},\omega\in\Omega$, and for every $B\in\mathcal{D}(E)$,
$$\lim_{t\to+\infty}d_H\left(\Phi(t,\tau-t,\theta_{-t}\omega,B(\tau-t,\theta_{-t}\omega)),K(\tau,\omega)\right)=0,$$
where $d_H(\cdot,\cdot)$ denotes the Hausdorff semi-distance between two subsets of E.

Definition 2.2. A family $A=\{A(\tau,\omega):\tau\in\mathbb{R},\omega\in\Omega\}\in\mathcal{D}(E)$ of nonempty subsets of E is called a measurable $\mathcal{D}(E)$-pullback random attractor for Φ if

(i) A is a measurable in ω and compact in E for any $\tau\in\mathbb{R}$, $\omega\in\Omega$;

(ii) A is invariant, i.e., for any $\tau\in\mathbb{R}$, $\omega\in\Omega$, $t\geq0$, $\Phi(t,\tau,\omega,A(\tau,\omega))=A(\tau+t,\theta_t\omega)$;

(iii) A is an attracting set in $\mathcal{D}(E)$.

Lemma 2.3. [2, 3] *If Φ has a compact measurable (with respect to $\mathcal{F}$) $\mathcal{D}(E)$-pullback attracting set K in $\mathcal{D}(E)$, then Φ has a unique $\mathcal{D}(E)$-pullback attractor $\mathcal{A}$ in $\mathcal{D}(E)$ given by: for each $\tau\in\mathbb{R}$ and $\omega\in\Omega$,*
$$\mathcal{A}(\tau,\omega)=\cap_{r\geq0}\overline{\cup_{t\geq r}\Phi(t,\tau-t,\theta_{-t}\omega,K(\tau-t,\theta_{-t}\omega))}.$$

3. Boundedness of solutions

In this section, we obtain bounds for solutions. Define $\mathcal{D}(E)$ as the collection of all tempered families of nonempty subsets of E with respect to $(\theta_t)_{t\in\mathbb{R}}$ [21], which means, for every $B=\{B(\tau,\omega)\subset E:\tau\in\mathbb{R},\omega\in\Omega\}\in\mathcal{D}(E)$, it holds that for any $\gamma>0$, $\omega\in\Omega$, $\lim_{t\to\infty}e^{-\gamma|t|}\|B(\tau+t,\theta_t\omega)\|_E=0$, where $\|B(\tau,\omega)\|_E=\sup_{x\in B(\tau,\omega)}\|x\|$.

Lemma 3.1. *For any $\tau\in\mathbb{R}$, $\omega\in\Omega$, there exists a tempered variable $M_0(\omega)$(independent of τ) such that for any set $B\in\mathcal{D}(E)$, there exists $T(\tau,\omega,B)\geq0$ such that the solution $\varphi(\tau,\tau-t,\theta_{-\tau}\omega,\varphi_{\tau-t}(\theta_{-\tau}\omega))$ of (2.2) with $\varphi_{\tau-t}(\theta_{-\tau}\omega)\in B(\tau-t,\theta_{-t}\omega)$ satisfies*
$$\|\varphi(\tau,\tau-t,\theta_{-\tau}\omega,\varphi_{\tau-t}(\theta_{-\tau}\omega))\|_E\leq M_0(\omega),\quad\forall t\geq T(\tau,\omega,B),$$

that is, the closed tempered measurable ball $B_0(\omega)=\{\varphi\in E:\|\varphi\|_E\le M_0(\omega)\}$ *of* E *satisfies*

$$\Phi(t,\tau-t,\theta_{-t}\omega,B(\tau-t,\theta_{-t}\omega))\subseteq B_0(\omega),\quad \forall t\ge T(\tau,\omega,B). \tag{3.1}$$

Proof. For any $\tau\in\mathbb{R},\omega\in\Omega,t\ge 0$, let $\varphi(r)=\varphi(r,\tau-t,\theta_{-\tau}\omega,\varphi_{\tau-t}(\theta_{-\tau}\omega))=(u(r),v(r))^{\mathrm{T}}\in E$ $(r\ge\tau-t)$ is a solution of (2.2) with $\varphi(\tau-t)=\varphi_{\tau-t}(\theta_{-\tau}\omega)=(u_{\tau-t},u_{1,\tau-t}+\varepsilon u_{\tau-t}-h(x)z(\theta_{-t}\omega))^{\mathrm{T}}\in E$. From (A1) we obtain

$$\begin{gathered}(f(u,x),u)\ge\frac{1}{c_5}\bar G(r)-\frac18\|u\|_2^2-\frac{\bar\beta_2}{c_5},\\ \int_U|u|^{q+1}dx\le\frac{1}{c_4}[\bar G(r)+\bar\beta_1],\quad |f(u,x)|\le c_6(1+|u|)^q,\end{gathered} \tag{3.2}$$

here $\bar G(r)=\int_U G(u(r,x),x)dx$, $\bar\beta_i(x)=\int_U\beta_i(x)dx$, $i=1,2$. Taking the inner product of (2.2) in E with $\varphi(r)$, for $r\ge\tau-t$, we have

$$\begin{aligned}&\frac12\frac{d}{dt}\left[\|\varphi(r)\|_E^2+2\bar G(r)+2\bar\beta_1+\lambda\|u\|^2\right]+(\Lambda\varphi,\varphi)_E+\varepsilon(f(u,x)+\lambda u,u)\\&=(f(u,x)+\lambda u,h(x)z(\theta_{r-\tau}\omega))+(h(x)z(\theta_{r-\tau}\omega),u)_2\\&\quad+(g(x,r)+\varepsilon h(x)z(\theta_{r-\tau}\omega),v)\,.\end{aligned} \tag{3.3}$$

From [9, 30] we obtain

$$(\Lambda\varphi,\varphi)_E\ge\frac{\varepsilon}{2}(\|u\|_2^2+\|v\|^2)+\frac{\alpha}{2}\|v\|^2. \tag{3.4}$$

So we can deduce that

$$\begin{aligned}&(f(u,x)+\lambda u,h(x)z(\theta_{r-\tau}\omega))\\&\le c_7|z(\theta_{r-\tau}\omega)|\Big(\|h\|+(\int_U|u|^{q+1}dx)^{\frac{q}{q+1}}\|h\|_{L^{q+1}}\\&\quad+\lambda(\int_U|u|^{q+1}dx)^{\frac{1}{q+1}}\|h\|_{L^{\frac{q+1}{q}}}\Big)\\&\le c_7|z(\theta_{r-\tau}\omega)|\Big(\|h\|+\Big(\frac{1}{c_4}[\bar G(r)+\bar\beta_1]\Big)^{\frac{q}{q+1}}\|h\|_{L^{q+1}}\\&\quad+\lambda\Big(\frac{1}{c_4}[\bar G(r)+\bar\beta_1]\Big)^{\frac{1}{q+1}}\|h\|_{L^{\frac{q+1}{q}}}\Big)\\&\le c_7\|h\||z(\theta_{r-\tau}\omega)|+c_8\Big(\frac{1}{c_4}[\bar G(r)+\bar\beta_1]\Big)^{\frac{q}{q+1}}\|h\|_{L^{q+1}}|z(\theta_{r-\tau}\omega)|\\&\quad+c_8\lambda\Big(\frac{1}{c_4}[\bar G(r)+\bar\beta_1]\Big)^{\frac{1}{q+1}}\|h\|_{L^{\frac{q+1}{q}}}|z(\theta_{r-\tau}\omega)|\\&\le c_7\|h\||z(\theta_{r-\tau}\omega)|+\frac{\varepsilon}{2c_5}\bar G(r)+c_9\bar\beta_1+c_{10}\|h\|_2^{q+1}|z(\theta_{r-\tau}\omega)|^{q+1}\\&\quad+c_{11}\|h\|_2^{\frac{q+1}{q}}|z(\theta_{r-\tau}\omega)|^{\frac{q+1}{q}},\end{aligned} \tag{3.5}$$

where c_{11} depends on λ,

$$(h(x)z(\theta_{r-\tau}\omega),u)_2\le\frac{2}{\varepsilon}z^2(\theta_{r-\tau}\omega)\|h\|_2^2+\frac{\varepsilon}{8}\|u\|_2^2, \tag{3.6}$$

and

$$(g(x,r)+\varepsilon h(x)z(\theta_{r-\tau}\omega),v)\le\frac{1}{\alpha}\left[\|g\|^2+\varepsilon^2z^2(\theta_{r-\tau}\omega)\|h\|^2\right]+\frac{\alpha}{2}\|v\|^2, \tag{3.7}$$

where $\|g\|^2 = \sup_{r\in\mathbb{R}} \|g(\cdot,r)\|^2 < \infty$. From (3.3)-(3.7), we obtain that

$$\frac{d}{dt}y(r) + \rho y(r) \le q(\theta_{r-\tau}\omega), \quad \forall r \ge \tau - t, \tag{3.8}$$

where $\rho = \min\{\frac{\varepsilon}{2}, \frac{\varepsilon}{2c_5}\}$ and

$$\begin{aligned} y(r) &= \|\varphi(r)\|_E^2 + 2\bar{G}(u) + 2\bar{\beta}_1 + \lambda\|u\|^2 \ge \|\varphi(r)\|_E^2, \\ q(\theta_{r-\tau}\omega) &= \frac{4}{\varepsilon}z^2(\theta_{r-\tau}\omega)\|h\|_2^2 + \frac{2}{\alpha}\left[\|g\|^2 + \varepsilon^2 z^2(\theta_{r-\tau}\omega)\|h\|^2\right] \\ &\quad + c_7\|h\||z(\theta_{r-\tau}\omega)| + 2c_9\bar{\beta}_1 + c_{10}\|h\|_2^{q+1}|z(\theta_{r-\tau}\omega)|^{q+1} \\ &\quad + c_{11}\|h\|_2^{\frac{q+1}{q}}|z(\theta_{r-\tau}\omega)|^{\frac{q+1}{q}} + \frac{4\varepsilon}{c_5}\bar{\beta}_2 + 2\rho\bar{\beta}_1 \\ &\le c_{12} + c_{13}|z(\theta_{r-\tau}\omega)|^{q+1}, \end{aligned}$$

where c_{13} depends on λ. By (3.2), we have

$$-\bar{\beta}_1 \le \bar{G}(r) + \frac{\lambda}{2}\|u\|^2 \le c_{14}\Big(1 + \int_U |u|^{q+1}dx\Big) + \lambda\|u\|^2 \le c_{15}\big(1 + \|u\|_2^{q+1}\big),$$

where c_{15} depends on λ. Using the Gronwall's inequality to (3.8) on $[\tau - t, r]$ ($r \ge \tau - t$), we can deduce that for $r \ge \tau - t$,

$$\begin{aligned} &y\left(r, \tau - t, \theta_{-\tau}\omega, \varphi_{\tau-t}(\theta_{-\tau}\omega)\right) \\ &\le y\left(\tau - t, \tau - t, \theta_{-\tau}\omega, \varphi_{\tau-t}(\theta_{-\tau}\omega)\right) e^{-\rho(r+t-\tau)} \\ &\quad + \int_{\tau-t}^{r} q(\theta_{s-\tau}\omega)e^{-\rho(r-s)}ds, \end{aligned} \tag{3.9}$$

where

$$y(\tau - t, \tau - t, \theta_{-\tau}\omega, \varphi_{\tau-t}(\theta_{-\tau}\omega)) \le \|\varphi_{\tau-t}(\theta_{-\tau}\omega)\|_E^2 + 2c_{15}(1 + \|u_{\tau-t}\|_2^{q+1}) + 2\bar{\beta}_1,$$

and

$$\int_{\tau-t}^{r} q(\theta_{s-\tau}\omega)e^{-\rho(r-s)}ds \le \frac{c_{12}}{\rho} + c_{13}\int_{\tau-t}^{r} |z(\theta_{s-\tau}\omega)|^{q+1}e^{-\rho(r-s)}ds.$$

From (3.9) we obtain

$$\begin{aligned} &\|\varphi(r, \tau - t, \theta_{-\tau}\omega, \varphi_{\tau-t}(\theta_{-\tau}\omega))\|_E^2 \\ &\le \Big(\|\varphi_{\tau-t}(\theta_{-\tau}\omega)\|_E^2 + 2c_{15}(1 + \|u_{\tau-t}\|_2^{q+1})\Big)e^{-\rho(r+t-\tau)} \\ &\quad + c_{16} + c_{13}\int_{\tau-t}^{r} |z(\theta_{s-\tau}\omega)|^{q+1}e^{-\rho(r-s)}ds, \quad \forall r \ge \tau - t. \end{aligned}$$

So we have

$$\begin{aligned} &\|\varphi(\tau, \tau - t, \theta_{-\tau}\omega, \varphi_{\tau-t}(\theta_{-\tau}\omega))\|_E^2 \\ &\le \Big(\|\varphi_{\tau-t}(\theta_{-\tau}\omega)\|_E^2 + 2c_{15}(1 + \|u_{\tau-t}\|_2^{q+1})\Big)e^{-\rho t} \\ &\quad + c_{16} + c_{13}\int_{-\infty}^{0} |z(\theta_s\omega)|^{q+1}e^{\rho s}ds. \end{aligned} \tag{3.10}$$

For any set $B(\tau,\omega) \in B \in \mathcal{D}(E)$,

$$\varphi_{\tau-t}(\theta_{-\tau}\omega) = (u_{\tau-t}, u_{1,\tau-t} + \varepsilon u_{\tau-t} - h(x)z(\theta_{-t}\omega))^{\mathrm{T}} \in B(\tau - t, \theta_{-t}\omega) \in \mathcal{D}(E),$$

we obtain

$$\lim\sup_{t\to+\infty}\Big(\|\varphi_{\tau-t}(\theta_{-\tau}\omega)\|_E^2+2c_{15}(1+\|u_{\tau-t}\|_2^{q+1})\Big)e^{-\rho t}=0. \tag{3.11}$$

Writing

$$M_0^2(\omega)=2\Big(c_{16}+c_{13}\int_{-\infty}^0|z(\theta_s\omega)|^{q+1}e^{\rho s}ds\Big)<\infty, \tag{3.12}$$

this is a tempered random variable, then from (3.10) and (3.11), there exists $T(\tau,\omega,B)\ge 0$ such that $\varphi(\tau,\tau-t,\theta_{-\tau}\omega,\varphi_{\tau-t}(\theta_{-\tau}\omega))\in B_0(\omega)$ for all $t\ge T(\tau,\omega,B)$, i.e., (3.1) holds. □

By (3.1), then there exists $T(\tau,\omega,B_0)\ge 0$ such that

$$\varphi(r,\tau-t,\theta_{-\tau}\omega,B_0(\theta_{-t}\omega))\in B_0(\theta_{r-\tau}\omega),\quad \forall t\ge T(\tau,\omega,B_0),\ \tau-t\le r\le\tau. \tag{3.13}$$

From (2.1) and (3.12), for any $\tau\in\mathbb{R}$, we have $\mathrm{E}\left(M_0^2(\theta_\tau\omega)\right)=2\big(c_{16}+c_{13}\frac{1}{\rho}\frac{\Gamma(\frac{q+2}{2})}{\sqrt{\pi\alpha^{q+1}}}\big)$ and for $k>1$,

$$\begin{aligned}
&\mathrm{E}(M_0^{2k}(\theta_\tau\omega))\\
&\le 2^{2k}\Big[c_{16}^k+c_{13}^k\Big(\int_{-\infty}^0 e^{\frac{k}{2(k-1)}\rho s}ds\Big)^{k-1}\mathrm{E}\Big(\int_{-\infty}^0 e^{\frac{k}{2}\rho s}|z(\theta_{s+\tau}\omega)|^{(q+1)k}ds\Big)\Big]\\
&=2^{2k}\Big[c_{16}^k+c_{13}^k\Big(\frac{2(k-1)}{k\rho}\Big)^{k-1}\frac{2}{k\rho}\frac{\Gamma(\frac{1+(q+1)k}{2})}{\sqrt{\pi}\alpha^{\frac{(q+1)k}{2}}}\Big]<\infty.
\end{aligned} \tag{3.14}$$

4. Decomposition of solutions

In this section, we decompose the solution of (2.2) into two parts, one of them decays exponentially and another one is ultimately pullback bounded in a more regular space. For this goal, we make two methods of decomposing the solutions of (2.2) with different initial data.

For any $\tau\in\mathbb{R}$ and $\omega\in\Omega$, let

$$B_1(\tau,\omega)=\cup_{t\ge T(\tau,\omega,B_0)}\varphi(\tau,\tau-t,\theta_{-\tau}\omega,B_0(\theta_{-t}\omega))\subseteq B_0(\omega).$$

Let $\varphi(r)=\varphi(r,\tau-t,\theta_{-\tau}\omega,\varphi_{\tau-t}(\theta_{-\tau}\omega))(r\ge\tau-t,t\ge 0)$ be a solution of (2.2) with $\varphi_{\tau-t}(\theta_{-\tau}\omega)\in B_1(\tau-t,\theta_{-t}\omega)\subseteq B_0(\theta_{-t}\omega)$, then by (3.13) we know that $\varphi(r)\in B_0(\theta_{r-\tau}\omega)$ for all $r\ge\tau-t$,

$$\|\varphi(r,\tau-t,\theta_{-\tau}\omega,\varphi_{\tau-t}(\theta_{-\tau}\omega))\|_E\le M_0(\theta_{r-\tau}\omega). \tag{4.1}$$

4.1. Decomposition of solution I.

First, we decompose $\varphi(r)=\varphi_1(r)+\varphi_2(r)$. Here $\varphi_1(r)=(u_1,v_1)^{\mathrm{T}}$ and $\varphi_2(r)=(u_2,v_2)^{\mathrm{T}}$ satisfy

$$\begin{gathered}
\dot\varphi_1+\Lambda\varphi_1+F_1(\varphi_1,x)=0,\quad r>\tau-t,\\
\varphi_1(\tau-t,\tau-t,\theta_{-\tau}\omega,\varphi_{\tau-t}(\theta_{-\tau}\omega))=\varphi_{\tau-t}(\theta_{-\tau}\omega),
\end{gathered} \tag{4.2}$$

and

$$\begin{gathered}
\dot\varphi_2+\Lambda\varphi_2+F_2(\varphi,\varphi_1,x)=F_3(\theta_{r-\tau}\omega,r),\quad r>\tau-t,\\
\varphi_2(\tau-t,\tau-t,\theta_{-\tau}\omega,\varphi_{\tau-t}(\theta_{-\tau}\omega))=(0,0)^{\mathrm{T}},
\end{gathered} \tag{4.3}$$

where

$$v_1=u_{1,t}+\varepsilon u_1,\quad v_2=u_{2,t}+\varepsilon u_2-h(x)z(\omega),$$

$$F_1(\varphi_1,x)=\begin{pmatrix}0\\ f_1(u_1,x)+\lambda u_1\end{pmatrix},\quad F_2(\varphi,\varphi_1,x)=\begin{pmatrix}0\\ f(u,x)-f_1(u_1,x)+\lambda u_2\end{pmatrix},$$

$$F_3(\omega, r) = \begin{pmatrix} h(x)z(\omega) \\ g(x,r) + \varepsilon h(x)z(\omega) \end{pmatrix}.$$

Next we appraise the part φ_1.

Lemma 4.1. *For any $\tau \in \mathbb{R}$, $\omega \in \Omega$, $t \geq 0$, there exist a constant $\sigma_1 > 0$ and a tempered random variable $M_1(\omega) > 0$ (independent of t and τ) such that the solution $\varphi_1(r) = \varphi_1(r, \tau - t, \theta_{-\tau}\omega, \varphi_{\tau-t}(\theta_{-\tau}\omega))$ of (4.2) satisfies*

$$\|\varphi_1(r, \tau - t, \theta_{-\tau}\omega, \varphi_{\tau-t}(\theta_{-\tau}\omega))\|_E \leq M_1(\theta_{-t}\omega)e^{-\sigma_1(t+r-\tau)}, \quad \forall\, r \geq \tau - t. \tag{4.4}$$

Proof. As for (3.3), we take the inner product $(\cdot,\cdot)_E$ of (4.2) with $\varphi_1 = (u_1, v_1)^{\mathrm{T}}$. Then from (1.2) and (3.4), the following inequality holds for $r \geq \tau - t$,

$$\frac{d}{dt}\left[\|\varphi_1(r)\|_E^2 + 2\bar{G}_1(r) + \lambda\|u_1\|^2\right] + \varepsilon(\|u_1\|_2^2 + \|v_1\|^2) + \frac{2\varepsilon}{c_0}(\bar{G}_1(r) + \lambda\|u_1\|^2) \leq 0,$$

here $\bar{G}_1(r) = \int_U G_1(u_1(r,x),x)dx \geq 0$. Therefore,

$$\frac{d}{dt}y_1(r) + 2\sigma_1 y_1(r) \leq 0, \quad \forall r \geq \tau - t, \tag{4.5}$$

where

$$y_1(r) = \|\varphi_1(r)\|_E^2 + 2\bar{G}_1(u_1) + \lambda\|u_1\|^2 \geq \|\varphi_1(r)\|_E^2, \quad \sigma_1 = \min\{\frac{\varepsilon}{2}, \frac{\varepsilon}{c_0}\}.$$

Using the Gronwall's inequality in (4.5), we obtain that

$$\begin{aligned} \|\varphi_1(r)\|_E^2 &\leq y_1(r) \leq y_1(\tau - t)e^{-2\sigma_1(t+r-\tau)} \\ &\leq \Big(c_{17}\|\varphi_{\tau-t}(\theta_{-\tau}\omega)\|_E^2 + 2c_{15}(1 + \|u_{\tau-t}\|_2^{q+1})\Big)e^{-2\sigma_1(t+r-\tau)} \\ &\leq c_{18}\Big(1 + M_0^{q+1}(\theta_{-t}\omega)\Big)e^{-2\sigma_1(t+r-\tau)} \\ &= M_1^2(\theta_{-t}\omega)e^{-2\sigma_1(t+r-\tau)}, \quad \forall\, r \geq \tau - t. \end{aligned} \tag{4.6}$$

□

For the part φ_2, we have the following estimate.

Lemma 4.2. *For any $\tau \in \mathbb{R}$, $\omega \in \Omega$, $t \geq 0$, there exists a positive-value random variable $M_2(t,\omega) > 0$ such that for $r \geq \tau - t$, the solution $\varphi_2(r) = (u_2, v_2)^{\mathrm{T}}$ of* (4.3) *satisfies*

$$\|A^{\nu+1}u_2(\tau, \tau - t, \theta_{-\tau}\omega, \varphi_{\tau-t})\|^2 + \|A^{\nu}v_2(\tau, \tau - t, \theta_{-\tau}\omega, \varphi_{\tau-t})\|^2 \leq M_2(t,\omega) \tag{4.7}$$

with a positive constant

$$\nu = \min\left\{\frac{1}{2}, \frac{n-4}{4}, \frac{4-(n-4)p}{4}\right\} > 0. \tag{4.8}$$

Proof. Taking the inner product $(\cdot,\cdot)_E$ with $A^{2\nu}\varphi_2=(A^{2\nu}u_2,A^{2\nu}v_2)^{\rm T}$ of (4.3), we obtain

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\Big(\|A^{\nu+1}u_2\|^2+\|A^{\nu}v_2\|^2+2\int_U[f(u,x)-f_1(u_1,x)+\lambda u_2]A^{2\nu}u_2dx\Big)\\
&+(\Lambda\varphi_2,A^{2\nu}\varphi_2)+\varepsilon\int_U[f(u,x)-f_1(u_1,x)+\lambda u_2]\,A^{2\nu}u_2dx\\
&-\int_U\big[\big(f'_{1,u}(u,x)-f'_{1,u}(u_1,x)\big)u_{1,t}+f'_{2,u}(u,x)u_{1,t}\\
&+f'_u(u,x)u_{2,t}+\lambda u_{2,t}]A^{2\nu}u_2dx\\
&=\big(f(u,x)-f_1(u_1,x)+\lambda u_2,A^{2\nu}h(x)z(\theta_{r-\tau}\omega)\big)+\big(h(x)z(\theta_{r-\tau}\omega),A^{2\nu}u_2\big)_2\\
&\quad+\big(g(x,r)+\varepsilon h(x)z(\theta_{r-\tau}\omega),A^{2\nu}v_2\big),\quad\forall r\ge\tau-t.
\end{aligned}\tag{4.9}
$$

Similar to (3.4)-(3.7), we obtain that for $r\ge\tau-t$,

$$
\begin{gathered}
(\Lambda\varphi_2,A^{2\nu}\varphi_2)\ge\frac{\varepsilon}{2}\|A^{\nu+1}u_2\|^2+\frac{\varepsilon}{2}\|A^{\nu}v_2\|^2+\frac{\alpha}{2}\|A^{\nu}v_2\|^2,\\
\big(f(u,x)-f_1(u_1,x)+\lambda u_2,A^{2\nu}h(x)z(\theta_{r-\tau}\omega)\big)\\
\le c_{19}\Big(1+M_0^{2q}(\theta_{r-\tau}\omega)+M_0^2(\theta_{r-\tau}\omega)+z^2(\theta_{r-\tau}\omega)\Big),\\
\big(h(x)z(\theta_{r-\tau}\omega),A^{2\nu}u_2\big)_2\le\frac{4}{\varepsilon}z^2(\theta_{r-\tau}\omega)\|h\|_2^2+\frac{\varepsilon}{16}\|A^{\nu+1}u_2\|^2,\\
\big(g(x,r)+\varepsilon h(x)z(\theta_{r-\tau}\omega),A^{2\nu}v_2\big)\le\frac{2}{\alpha}[\|g\|_2^2+\varepsilon^2z^2(\theta_{r-\tau}\omega)\|h\|_2^2]+\frac{\alpha}{4}\|A^{\nu}v_2\|^2,
\end{gathered}
$$

where c_{19} depends on λ and $\|g\|_2^2=\sup_{r\in\mathbb{R}}\|g(\cdot,r)\|_2^2<\infty$. Using Hölder's inequality, (1.2), (1.3), (4.1), and (4.4), we have the following estimates for $r\ge\tau-t$,

$$
\begin{aligned}
&\int_U f'_u(u,x)u_{2,t}\cdot A^{2\nu}u_2\,dx\\
&\le c_{20}\int_U\big(1+|u|^{q-1}\big)|u_{2,t}||A^{2\nu}u_2|dx\\
&\le c_{21}\Big(\int_U(1+|u|^{q-1})^{\frac{n}{2}}dx\Big)^{2/n}\Big(\int_U|A^{2\nu}u_2|^{\frac{2n}{n-4+4\nu}}dx\Big)^{\frac{n-4+4\nu}{2n}}\Big(\int_U|u_{2,t}|^{\frac{2n}{n-4\nu}}dx\Big)^{\frac{n-4\nu}{2n}}\\
&\le c_{22}\Big(1+\|u\|_2^{q-1}\Big)\|A^{\nu+1}u_2\|\,\big[\|A^{\nu}v_2\|+\|\varepsilon A^{\nu}u_2+z(\theta_{r-\tau}\omega)A^{\nu}h\|\big]\\
&\le\frac{1}{2}c_{22}\Big(1+M_0^{q-1}(\theta_{r-\tau}\omega)\Big)(\|A^{\nu+1}u_2\|^2+\|A^{\nu}v_2\|^2)\\
&\quad+c_{23}\Big(1+M_0^{4q-4}(\theta_{r-\tau}\omega)+z^4(\theta_{r-\tau}\omega)\Big)+\frac{\varepsilon}{16}\|A^{\nu+1}u_2\|^2,
\end{aligned}
$$

$$
\begin{aligned}
&\int_U f'_{2,u}(u,x)u_{1,t}\cdot A^{2\nu}u_2dx\\
&\le c_3\int_U|u_{1,t}|(1+|u|^p)|A^{2\nu}u_2|dx\\
&\le c_3\Big(\int_U|u_{1,t}|^2dx\Big)^{1/2}\Big(\int_U(1+|u|^p)^{\frac{2n}{4-4\nu}}dx\Big)^{\frac{4-4\nu}{2n}}\Big(\int_U|A^{2\nu}u_2|^{\frac{2n}{n+4\nu-4}}dx\Big)^{\frac{n+4\nu-4}{2n}}\\
&\le c_{24}\left(\|v_1\|+\varepsilon\|u_1\|\right)\left(1+\|u\|_2^p\right)\|A^{\nu+1}u_2\|\\
&\le c_{25}\big[1+M_0^{4p}(\theta_{r-\tau}\omega)+M_1^4(\theta_{-t}\omega)e^{-4\sigma_1(t+r-\tau)}\big]+\frac{\varepsilon}{16}\|A^{\nu+1}u_2\|^2,
\end{aligned}
$$

and

$$\begin{aligned}
&\int_U [f'_{1,u}(u,x) - f'_{1,u}(u_1,x)]u_{1,t}A^{2\nu}u_2dx\\
&\le c_{26}\int_U |u_{1,t}|\left(|u_1|^{q-2}+|u|^{q-2}\right)|u_2||A^{2\nu}u_2|dx\\
&\le c_{26}\Big(\int_U |u_{1,t}|^{\frac{2n}{4q-nq+2n}}dx\Big)^{\frac{4q-nq+2n}{2n}}\Big(\int_U (|u_1|^{q-2}+|u|^{q-2})^{\frac{2n}{(n-4)(q-2)}}dx\Big)^{\frac{(n-4)(q-2)}{2n}}\\
&\times\Big(\int_U |u_2|^{\frac{2n}{n-4-4\nu}}dx\Big)^{\frac{n-4-4\nu}{2n}}\Big(\int_U |A^{2\nu}u_2|^{\frac{2n}{n-4+4\nu}}dx\Big)^{\frac{n-4+4\nu}{2n}}\\
&\le c_{27}(1+\frac{\varepsilon}{\sqrt{\lambda_1}})M_1(\theta_{-t}\omega)e^{-\sigma_1(t+r-\tau)}\\
&\times\Big(M_1^{q-2}(\theta_{-t}\omega)e^{-(q-2)\sigma_1(t+r-\tau)}+M_0^{q-2}(\theta_{r-\tau}\omega)\Big)\|A^{\nu+1}u_2\|^2\\
&\le c_{28}\big[M_0^{q-1}(\theta_{r-\tau}\omega)+M_1^{q-1}(\theta_{-t}\omega)e^{-(q-1)\sigma_1(t+r-\tau)}\big]\|A^{\nu+1}u_2\|^2.
\end{aligned}$$

Also we have

$$\int_U \lambda u_{2t}A^{2\nu}u_2dx \le \lambda c_{29}\left(\|A^{\nu+1}u_2\|^2+\|A^\nu v_2\|^2+z^2(\theta_{r-\tau}\omega)\right).$$

By the above inequalities and (4.9), we obtain

$$\frac{d}{dt}y_2(r)\le m_1(\theta_{r-\tau}\omega)y_2(r)+q_1(\theta_{r-\tau}\omega),\quad \forall r\ge\tau-t, \tag{4.10}$$

where

$$y_2=\|A^{\nu+1}u_2\|^2+\|A^\nu v_2\|^2+2\int_U[f(u,x)-f_1(u_1,x)+\lambda u_2]A^{2\nu}u_2dx, \tag{4.11}$$

$$m_1(\theta_{r-\tau}\omega)=c_{30}[1+M_0^{q-1}(\theta_{r-\tau}\omega)+M_1^{q-1}(\theta_{-t}\omega)e^{-\frac{4}{n-4}\sigma_1(t+r-\tau)}]-\frac{\varepsilon}{4}, \tag{4.12}$$

$$q_1(\theta_{r-\tau}\omega)=c_{31}[1+M_0^{4q-4}(\theta_{r-\tau}\omega)+z^4(\theta_{r-\tau}\omega)+M_1^{2q}(\theta_{-t}\omega)e^{-2q\sigma_1(t+r-\tau)}],$$

$$y_2\left(\tau-t,\tau-t,\theta_{-\tau}\omega,\varphi_{\tau-t}(\theta_{-\tau}\omega)\right)=(0,0)^{\rm T},$$

where c_{30}, c_{31} depend on λ. By using the Gronwall's inequality to (4.10) on $[\tau-t,r]$ $(r\ge\tau-t)$, we have

$$y_2(r,\tau-t,\theta_{-\tau}\omega,\varphi_{\tau-t}(\theta_{-\tau}\omega))\le\int_{\tau-t}^r q_1(\theta_{\xi-\tau}\omega)e^{\int_\xi^r m_1(r,\omega)dr}d\xi, \tag{4.13}$$

for all $r\ge\tau-t$.

Note that

$$\begin{aligned}
\Big|\int_U f_2(u,x)\cdot A^{2\nu}u_2dx\Big| &\le c_{32}\int_U\Big(1+|u|^{p+1}\Big)|A^{2\nu}u_2|dx\\
&\le c_{32}\Big(\int_U\left(1+|u|^{p+1}\right)^2dx\Big)^{1/2}\Big(\int_U|A^{2\nu}u_2|^2dx\Big)^{1/2}\\
&\le c_{32}\big[1+M_0^{2p+2}(\theta_{r-\tau}\omega)+M_1^2(\theta_{-t}\omega)e^{-2\sigma_1(t+r-\tau)}\big],
\end{aligned}$$

and

$$\begin{aligned}
&\int_U\big|[f_1(u,x)-f_1(u_1,x)+\lambda u_2]A^{2\nu}u_2\big|dx\\
&\le c_{33}\int_U\left(1+|u_2|^q+|u_1|^q+\lambda|u_2|\right)|A^{2\nu}u_2|dx
\end{aligned}$$

$$\begin{aligned}&\le c_{33}\Big(\int_U\big(1+|u_2|^q+|u_1|^q+\lambda|u_2|\big)^2dx\Big)^{1/2}\Big(\int_U|A^{2\nu}u_2|^2dx\Big)^{1/2}\\&\le c_{34}\big[1+M_0^{2q}(\theta_{r-\tau}\omega)+M_1^{2q}(\theta_{-t}\omega)e^{-2q\sigma_1(t+r-\tau)}\big].\end{aligned}$$

Therefore,

$$\begin{aligned}&\int_U\left[f(u,x)-f_1(u_1,x)+\lambda u_2\right]A^{2\nu}u_2dx\\&=\int_U[f_2(u,x)+f_1(u,x)-f_1(u_1,x)+\lambda u_2]A^{2\nu}u_2dx\\&\le c_{35}[1+M_0^{2q}(\theta_{r-\tau}\omega)+M_1^{2q}(\theta_{-t}\omega)e^{-2q\sigma_1(t+r-\tau)}],\quad\forall r\ge\tau-t,\end{aligned}\tag{4.14}$$

where c_{34}, c_{35} depend on λ. Thus, according to (4.11), (4.13) and (4.14), we obtain that for $r\ge\tau-t$,

$$\begin{aligned}&\|A^{\nu+1}u_2(r,\tau-t,\theta_{-\tau}\omega,\varphi_{\tau-t}(\theta_{-\tau}\omega))\|^2+\|A^{\nu}v_2(r,\tau-t,\theta_{-\tau}\omega,\varphi_{\tau-t}(\theta_{-\tau}\omega))\|^2\\&\le 2y_2\left(r,\tau-t,\theta_{-\tau}\omega,\varphi_{\tau-t}(\theta_{-\tau}\omega)\right)\\&\quad+2c_{35}[1+M_0^{2q}(\theta_{r-\tau}\omega)+M_1^{2q}(\theta_{-t}\omega)e^{-2q\sigma_1(t+r-\tau)}]\\&\le 2c_{35}[1+M_0^{2q}(\theta_{r-\tau}\omega)+M_1^{2q}(\theta_{-t}\omega)e^{-2q\sigma_1(t+r-\tau)}]\\&\quad+2c_{31}\int_{\tau-t}^{r}[1+M_0^{4q-4}(\theta_{\xi-\tau}\omega)+z^4(\theta_{\xi-\tau}\omega)\\&\quad+M_1^{2q}(\theta_{-t}\omega)e^{-2q\sigma_1(t+\xi-\tau)}]e^{\int_\xi^r m_1(\theta_{s-\tau}\omega)dr}d\xi.\end{aligned}\tag{4.15}$$

Denoting

$$\begin{aligned}M_2(t,\omega)&=2c_{35}[1+M_0^{2q}(\omega)+M_1^{2q}(\theta_{-t}\omega)e^{-2q\sigma_1t}]\\&\quad+2c_{31}\int_{-t}^{0}[1+M_0^{4q-4}(\theta_\xi\omega)+z^4(\theta_\xi\omega)\\&\quad+M_1^{2q}(\theta_{-t}\omega)e^{-2q\sigma_1(t+\xi)}]e^{\int_\xi^0 m_1(\theta_r\omega)dr}d\xi,\end{aligned}\tag{4.16}$$

then we can obtain (4.7) from (4.15) and (4.16). □

Motivated by [27, Proposition 1.4], Lemmas 4.1 and 4.2, we obtain the following decomposition of solutions for (2.2).

Lemma 4.3. *Let $\tau\in\mathbb{R}$, $\omega\in\Omega$ and $t\ge0$. Then for any $T>0$, the following statements are valid.*

(i) There exist two positive constants $\bar K$, K_1 and the solution $\varphi(r)$ of (2.2) *has a decomposition: $\varphi(r)=\phi_1(r)+\phi_2(r)$, where $\phi_1(r)$, $\phi_2(r)$ satisfy*

$$\begin{aligned}&\int_r^\tau\|\phi_1\left(\xi,\tau-t,\theta_{-\tau}\omega,\varphi_{\tau-t}(\theta_{-\tau}\omega)\right)\|_E^{q-1}d\xi\le\frac{K_0}{\sigma_1T}(\tau-r)+\bar K,\\&\|\phi_2(r,\tau-t,\theta_{-\tau}\omega,\varphi_{\tau-t}(\theta_{-\tau}\omega))\|_E^{q-1}\le K_1M_2(T,\omega),\quad\forall r\ge\tau-t.\end{aligned}\tag{4.17}$$

(ii) The first component $u(r)$ of solution $\varphi(r)$ of (2.2) *has a decomposition that $u(r)=w_1(r)+w_2(r)$, where $w_1(r)$, $w_2(r)$ satisfy*

$$\begin{aligned}&\int_r^\tau\|w_1(\xi,\tau-t,\theta_{-\tau}\omega,\varphi_{\tau-t}(\theta_{-\tau}\omega))\|_2^{q-1}d\xi\le\frac{K_0}{\sigma_1T}(\tau-r)+\bar K,\\&\|A^{\nu+1}w_2(r,\tau-t,\theta_{-\tau}\omega,\varphi_{\tau-t}(\theta_{-\tau}\omega))\|^{q-1}\le K_1M_2(T,\omega),\quad\forall r\ge\tau-t,\end{aligned}\tag{4.18}$$

where ν is as in (4.8)*, $K_0=\mathrm{E}[M_1^{q-1}(\omega)]$.*

Proof. From (4.6), we know $M_1^2(\omega) = c_{18}\Big(1 + M_0^{q+1}(\omega)\Big)$, then we can find a positive constant c_{36} that

$$M_1^{q-1}(\omega) = c_{36}\Big(1 + M_0^{\frac{q^2-1}{2}}(\omega)\Big).$$

As $(\theta_t)_{t\in\mathbb{R}}$ is measure-preserving and ergodic on $(\Omega, \mathcal{F}, \mathbb{P})$, from the Birkhoff ergodic Theorem [19], we obtain

$$\begin{aligned}
&\mathrm{E}[M_1^{q-1}(\theta_s\omega)] = \mathrm{E}[M_1^{q-1}(\omega)]\\
&= c_{36}\Big(1 + \mathrm{E}\Big[M_0^{\frac{q^2-1}{2}}(\omega)\Big]\Big)\\
&= c_{36}\Big(1 + 2^{\frac{q^2-1}{2}}\Big[c_{16}^{\frac{q^2-1}{4}} + c_{13}^{\frac{q^2-1}{4}}\big(\frac{2q^2-10}{(q^2-1)\rho}\big)^{\frac{q^2-5}{4}}\frac{8}{(q^2-1)\rho}\frac{\Gamma(\frac{4+(q+1)(q^2-1)}{8})}{\sqrt{\pi}\alpha^{\frac{(q+1)(q^2-1)}{8}}}\Big]\Big)\\
&= K_0 < \infty, \quad \forall s \in \mathbb{R},
\end{aligned}$$

and for any fixed $T > 0$, $s \in \mathbb{R}$ and $\omega \in \Omega$ (in fact for $a.e. \omega \in \Omega$),

$$\frac{1}{k}\sum_{l=1}^{k} M_1^{q-1}(\theta_{s+lT}\omega) \to \mathrm{E}[M_1^{q-1}(\theta_s\omega)] = K_0, \quad k \to \infty.$$

Therefore, for $\omega \in \Omega$, there exists a large integer $k_0(\omega) < \infty$ satisfying

$$\frac{K_0}{2} \le \frac{1}{k}\sum_{l=1}^{k} M_1^{q-1}(\theta_{s+lT}\omega) \le \frac{3K_0}{2}, \quad \forall k \ge k_0(\omega), \quad \forall s \in \mathbb{R},$$

and

$$\frac{K_0}{2}k_0(\omega) \le \sum_{l=1}^{k_0(\omega)} M_1^{q-1}(\theta_{s+lT}\omega) \le \frac{3K_0}{2}k_0(\omega), \quad \forall s \in \mathbb{R}. \tag{4.19}$$

Taking the expectation on (4.19), we obtain

$$\frac{1}{2}\,\mathrm{E}[k_0] \le k_0(\omega) \le \frac{3}{2}\,\mathrm{E}[k_0], \quad k_0(\omega) < \infty,$$

so, we have $\mathrm{E}[k_0] < \infty$.

(i) We construct functions $\phi_1(r)$ and $\phi_2(r)$. When $T > 0$ and $k \in \mathbb{N}$, consider equations (4.2) and (4.3) at the interval $[\tau - t + (k-1)T, \tau - t + kT]$. Note

$$\begin{gathered}
\phi_1 = (w_1, \tilde{w}_1)^{\mathrm{T}} = \varphi_1, \quad \phi_1(\tau - t + (k-1)T) = \varphi(\tau - t + (k-1)T),\\
\phi_2 = (w_2, \tilde{w}_2)^{\mathrm{T}} = \varphi_2, \quad \phi_2(\tau - t + (k-1)T) = (0,0)^{\mathrm{T}}.
\end{gathered}$$

By (4.4), as $s \ge 0$, $r \in [\tau - t + s + (k-1)T, \tau - t + s + kT]$, we obtain

$$\begin{aligned}
\|\phi_1(r)\|_E^{q-1} &= \|w_1\|_2^{q-1} + \|\tilde{w_1}\|^{q-1}\\
&\le M_1^{q-1}(\theta_{-t+s+(k-1)T}\omega)e^{-(q-1)\sigma_1(t-s-(k-1)T+r-\tau)},
\end{aligned} \tag{4.20}$$

and

$$\begin{aligned}
&\int_{\tau-t+s+(k-1)T}^{\tau-t+s+kT} \|\phi_1(\xi)\|_E^{q-1}d\xi\\
&\le \int_{\tau-t+s+(k-1)T}^{\tau-t+s+kT} M_1^{q-1}(\theta_{-t+s+(k-1)T}\omega)e^{-(q-1)\sigma_1(t-s-(k-1)T+\xi-\tau)}d\xi
\end{aligned}$$

$$\le \frac{M_1^{q-1}(\theta_{-t+s+(k-1)T}\omega)}{(q-1)\sigma_1}.$$

By (4.20), as $k \ge k_0(\omega)$ and $s \ge 0$,

$$\begin{aligned}
&\int_{\tau-t+s}^{\tau-t+s+kT} \|\phi_1(\xi)\|_E^{q-1} d\xi \\
&\le \Big(\int_{\tau-t+s}^{\tau-t+s+T} + \int_{\tau-t+s+T}^{\tau-t+s+2T} + \dots + \int_{\tau-t+s+(k-1)T}^{\tau-t+s+kT}\Big) \|\phi_1(\xi)\|_E^{q-1} d\xi \\
&\le \frac{1}{(q-1)\sigma_1}\Big[M_1^{q-1}(\theta_{-t+s}\omega) + M_1^{q-1}(\theta_{-t+s+T}\omega) + \dots + M_1^{q-1}(\theta_{-t+s+(k-1)T}\omega)\Big] \\
&\le \frac{K_0}{\sigma_1} k.
\end{aligned}$$

Thus, when $\tau - t \le r \le \tau$ and $\tau - r = mT + \bar r$, $m \in \mathbb{Z}_+$, $\bar r \in [0, T)$, we obtain:

(a) If $m \ge k_0(\omega)$, then

$$\begin{aligned}
\int_r^\tau \|\phi_1(\xi)\|_E^{q-1} d\xi &\le (\int_r^{r+T} + \int_{r+T}^{r+2T} + \dots + \int_{r+mT}^{r+(m+1)T}) \|\phi_1(\xi)\|_E^{q-1} d\xi \\
&\le \frac{K_0}{\sigma_1 T}(\tau - r) + \frac{K_0}{\sigma_1}.
\end{aligned}$$

(b) If $0 < m < k_0(\omega)$, then

$$\begin{aligned}
\int_r^\tau \|\phi_1(\xi)\|_E^{q-1} d\xi &\le \Big(\int_{\tau-k_0T}^{\tau-(k_0-1)T} + \int_{\tau-(k_0-1)T}^{\tau-(k_0-2)T} + \dots + \int_{\tau-t}^{\tau}\Big) \|\phi_1(\xi)\|_E^{q-1} d\xi \\
&\le \frac{K_0}{\sigma_1} k_0(\omega).
\end{aligned}$$

Thus

$$\int_r^\tau \|\phi_1(\xi)\|_E^{q-1} d\xi \le \frac{K_0}{\sigma_1 T}(\tau - r) + \bar K,$$

where $\bar K = \frac{3K_0}{2\sigma_1}\,\mathrm{E}[k_0] + \frac{K_0}{\sigma_1}$.

For $t \ge T$, $\varphi_2(r)$ is the solution of (4.3) on the interval $[r-T, r]$ with $\varphi_2(r-T) = (0,0)^{\mathrm{T}}$, by (4.7), we obtain

$$\begin{aligned}
\|\varphi_2(r, r-T, \theta_{-\tau}\omega, \varphi_{r-T})\|_{E^\nu}^{q-1} &\le c_{37}\|\varphi_2(r, r-T, \theta_{-\tau}\omega, \varphi_{r-T})\|_{E^\nu}^2 \\
&\le K_1 M_2(T, \omega), \quad \forall r \ge \tau - t.
\end{aligned} \tag{4.21}$$

Thus, combining (4.7) and (4.21), for any $t \ge 0$ and $r \ge \tau - t$, we can choose ϕ_2 like this, then (4.17) holds.

(ii) can be obtained directly from (i). □

4.2. **Decomposition of solutions II.** Let $\varphi(r)$ be the solution of (2.2), and write $\varphi(r) = \varphi_L(r) + \varphi_N(r)$. Here $\varphi_L(r) = (u_L, v_L)^{\mathrm{T}}$ and $\varphi_N(r) = (u_N, v_N)^{\mathrm{T}}$ satisfy

$$\begin{gathered}
\dot\varphi_L + \Lambda\varphi_L + F_1(\varphi_L, x) = 0, \quad r > \tau - t, \\
\varphi_L(\tau - t, \tau - t, \theta_{-\tau}\omega, \varphi_{\tau-t}(\theta_{-\tau}\omega)) = \varphi_{L,\tau-t} = (u_{\tau-t}, u_{1,\tau-t} + \varepsilon u_{\tau-t})^{\mathrm{T}},
\end{gathered} \tag{4.22}$$

and

$$\begin{gathered}
\dot\varphi_N + \Lambda\varphi_N + F_2(\varphi, \varphi_L, x) = \tilde F_2(\theta_{r-\tau}\omega, r), \quad r > \tau - t, \\
\varphi_N(\tau - t, \tau - t, \theta_{-\tau}\omega, \varphi_{\tau-t}(\theta_{-\tau}\omega)) = (0, -h(x)z(\theta_{-t}\omega))^{\mathrm{T}}, \quad t \ge 0,
\end{gathered} \tag{4.23}$$

where $u_{\tau-t}$ and $u_{1,\tau-t}$ are independent of ω, and $v_L = u_{L,t} + \varepsilon u_L$, $v_N = u_{N,t} + \varepsilon u_N - h(x)z(\omega)$. Next, we estimate the component φ_L.

Lemma 4.4. *For any $\tau \in \mathbb{R}$ and $\omega \in \Omega$, $t \geq 0$, there exists a constant $M_L > 0$ (independent of ω, t and τ) such that the solution $\varphi_L(r) = \varphi_L(r, \tau - t, \varphi_{L,\tau-t})$ of* (4.22) *satisfies*

$$\begin{aligned} \|\varphi_L(r, \tau - t, \varphi_{L,\tau-t})\|_E &= \big(\|u_L(r)\|_2^2 + \|v_L(r)\|^2\big)^{1/2} \\ &\leq M_L e^{-\sigma_1(t+r-\tau)}, \quad \forall r \geq \tau - t. \end{aligned} \tag{4.24}$$

Proof. Note that $\varphi_{L,\tau-t} = \varphi_{\tau-t}(\theta_{-\tau}\omega) + (0, h(x)z(\theta_{-t}\omega))^{\mathrm{T}} \in B_0(\theta_{-t}\omega)$ and $\varphi_{L,\tau-t}$ is independent of ω, we replace $\theta_t\omega$ by ω, so we have

$$\|\varphi_{L,\tau-t}\|_E^2 \leq 2M_0^2(\omega) + 2\|h(x)\|^2 |z(\omega)|^2, \quad \forall \tau \in \mathbb{R},\ \omega \in \Omega,\ t \geq 0.$$

Taking the expectation with respect to $\omega \in \Omega$, we obtain that

$$\begin{aligned} \|\varphi_{L,\tau-t}\|_E^2 &= \|u_{\tau-t}\|_2^2 + \|u_{1,\tau-t} + \varepsilon u_{\tau-t}\|^2 \\ &\leq 2\,\mathrm{E}[M_0^2(\omega)] + 2\|h(x)\|^2 \cdot \mathrm{E}[|z(\omega)|^2] \\ &\leq 4\Big(c_{16} + c_{13}\frac{1}{\rho}\frac{\Gamma(\frac{q+2}{2})}{\sqrt{\pi\alpha^{q+1}}}\Big) + 2\|h(x)\|^2 \frac{\Gamma(\frac{1+2}{2})}{\sqrt{\pi}\alpha} \\ &= 4\Big(c_{16} + c_{13}\frac{1}{\rho}\frac{\Gamma(\frac{q+2}{2})}{\sqrt{\pi\alpha^{q+1}}}\Big) + \frac{1}{\alpha}\|h(x)\|^2. \end{aligned}$$

As in (4.6) for $r \geq \tau - t$, we obtain

$$\|\varphi_L(r)\|_E^2 \leq [\|\varphi_{L,\tau-t}\|_E^2 + 2c_{15}(1 + \|u_{\tau-t}\|_2^{q+1})] e^{-2\sigma_1(t+r-\tau)} = M_L^2 e^{-2\sigma_1(t+r-\tau)},$$

where

$$\begin{aligned} M_L^2 &= 4c_{16} + 4c_{13}\frac{1}{\rho}\frac{\Gamma(\frac{q+2}{2})}{\sqrt{\pi\alpha^{q+1}}} + \frac{1}{\alpha}\|h(x)\|^2 + c_{15} \\ &\quad + c_{15}\Big(4c_{16} + 4c_{13}\frac{1}{\rho}\frac{\Gamma(\frac{q+2}{2})}{\sqrt{\pi\alpha^{q+1}}} + \frac{1}{\alpha}\|h(x)\|^2\Big)^{\frac{q+1}{2}}. \end{aligned}$$

□

The following estimate is same as in Lemma 4.2.

Lemma 4.5. *For any $\tau \in \mathbb{R}$, $\omega \in \Omega$, there exist a random variable $t_\nu(\omega) \geq 0$ and a tempered random variable $M_\nu(\omega) > 0$ (independent of t and τ) such that the solution $\varphi_N(\tau, \tau - t, \theta_{-\tau}\omega, \varphi_{\tau-t}(\theta_{-\tau}\omega))$ of* (4.23) *satisfies: for $t \geq t_\nu(\omega)$,*

$$\|A^{\nu+1} u_N(\tau, \tau - t, \theta_{-\tau}\omega, \varphi_{\tau-t})\|^2 + \|A^\nu v_N(\tau, \tau - t, \theta_{-\tau}\omega, \varphi_{\tau-t})\|^2 \leq M_\nu^2(\omega), \tag{4.25}$$

where ν is as in (4.8).

Proof. Similar to (4.9), we take the inner product $((4.23), A^{2\nu}\varphi_N)_E$, then

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\Big(\|A^{\nu+1}u_N\|^2 + \|A^\nu v_N\|^2 + 2\int_U [f(u,x) - f_1(u_L,x) + \lambda u_N]A^{2\nu}u_N dx\Big)\\
&+ (\Lambda\varphi_N, A^{2\nu}\varphi_N) + \varepsilon\int_U [f(u,x) - f_1(u_L,x) + \lambda u_N]A^{2\nu}u_N dx\\
&- \int_U [(f_1'(u,x) - f_1'(u_L,x))u_{L,t} + f_2'(u,x)u_{L,t} + f'(u,x)u_{N,t} + \lambda u_{N,t}]A^{2\nu}u_N dx\\
&= \big(f(u,x) - f_1(u_L,x) + \lambda u_N, A^{2\nu}h(x)z(\theta_{r-\tau}\omega)\big) + \big(h(x)z(\theta_{r-\tau}\omega), A^{2\nu}u_N\big)_2\\
&\quad + (g(x,r) + \varepsilon h(x)z(\theta_{r-\tau}\omega), A^{2\nu}v_N), \quad \forall r \ge \tau - t.
\end{aligned} \tag{4.26}$$

From $\varphi_N(r) = \varphi(r) - \varphi_L(r)$ and (4.1), (4.24),

$$\|\varphi_N(r)\|_E \le M_L + M_0(\theta_{r-\tau}\omega), \quad \forall r \ge \tau - t.$$

Using Hölder's inequality, (A1) and Lemma 4.3, we obtain that: for $r \ge \tau - t$,

$$\begin{aligned}
&\int_U [f_1'(u,x) - f_1'(u_L,x)]\, u_{L,t}A^{2\nu}u_N dx\\
&\le c_2\int_U |u_{L,t}|\left(|u_L|^{q-2} + |u|^{q-2}\right)|u_N||A^{2\nu}u_N|dx\\
&\le c_2\Big(\int_U |u_{L,t}|^{\frac{2n}{4q-nq+2n}}dx\Big)^{\frac{4q-nq+2n}{2n}}\Big(\int_U \left(|u_L|^{q-2}\right)^{\frac{2n}{(n-4)(p-2)}}dx\Big)^{\frac{(n-4)(p-2)}{2n}}\\
&\quad\times\Big(\int_U |u_N|^{\frac{2n}{n-4-4\nu}}dx\Big)^{\frac{n-4-4\nu}{2n}}\Big(\int_U |A^{2\nu}u_N|^{\frac{2n}{n-4+4\nu}}dx\Big)^{\frac{n-4+4\nu}{2n}}\\
&\quad + c_2\Big(\int_U |u_{L,t}|^{\frac{2n}{4q-nq+2n}}dx\Big)^{\frac{4q-nq+2n}{2n}}\Big(\int_U \left(|w_1|^{q-2}\right)^{\frac{2n}{(n-4)(p-2)}}dx\Big)^{\frac{(n-4)(p-2)}{2n}}\\
&\quad\times\Big(\int_U |u_N|^{\frac{2n}{n-4-4\nu}}dx\Big)^{\frac{n-4-4\nu}{2n}}\Big(\int_U |A^{2\nu}u_N|^{\frac{2n}{n-4+4\nu}}dx\Big)^{\frac{n-4+4\nu}{2n}}\\
&\quad + c_2\Big(\int_U |u_{L,t}|^{\frac{2n}{4q-nq+2n}}dx\Big)^{\frac{4q-nq+2n}{2n}}\Big(\int_U |u_N|^{\frac{2n}{(n-4)(p-2)}}dx\Big)^{\frac{(n-4)(p-2)}{2n}}\\
&\quad\times\Big(\int_U \left(|w_2|^{q-2}\right)^{\frac{2n}{n-4-4\nu}}dx\Big)^{\frac{n-4-4\nu}{2n}}\Big(\int_U |A^{2\nu}u_N|^{\frac{2n}{n-4+4\nu}}dx\Big)^{\frac{n-4+4\nu}{2n}}\\
&\le c_{38}\Big(\|u_L\|_2^{q-2} + \|w_1\|_2^{q-2}\Big)\|A^{\nu+1}u_N\|^2 + c_{39}\big[1 + M_0^4(\theta_{r-\tau}\omega)\\
&\quad + \|A^{\nu+1}w_2\|^{4q-8}\big] + \frac{\varepsilon}{16}\|A^{\nu+1}u_N\|^2,
\end{aligned}$$

and

$$\begin{aligned}
&\int_U [f'(u,x)u_{N,t} + \lambda u_{N,t}]\, A^{2\nu}u_N dx\\
&\le c_{20}\int_U \big[\left(1 + |u|^{q-1}\right)|u_{N,t}| + \lambda u_{N,t}\big]\,|A^{2\nu}u_N|dx\\
&\le c_{20}\Big(\int_U |u_{N,t}|^2dx\Big)^{1/2}\Big(\int_U |A^{2\nu}u_N|^2dx\Big)^{1/2}\\
&\quad + c_{20}\Big(\int_U |w_1|^{\frac{n(q-1)}{2}}dx\Big)^{2/n}\Big(\int_U |A^{2\nu}u_N|^{\frac{2n}{n-4+4\nu}}dx\Big)^{\frac{n-4+4\nu}{2n}}\Big(\int_U |u_{N,t}|^{\frac{2n}{n-4\nu}}dx\Big)^{\frac{n-4\nu}{2n}}
\end{aligned}$$

$$
\begin{aligned}
&+ c_{20}\Big(\int_U |u_{N,t}|^2 dx\Big)^{1/2}\Big(\int_U \big(|w_2|^{q-1}\big)^{\frac{2n}{4-4\nu}}\,dx\Big)^{\frac{4-4\nu}{2n}}\\
&\times\Big(\int_U |A^{2\nu}u_N|^{\frac{2n}{n+4\nu-4}}dx\Big)^{\frac{n+4\nu-4}{2n}}\\
&+ c_{20}\Big(\int_U |\lambda|^{\frac{n}{2}}\Big)^{2/n}\Big(\int_U |u_{N,t}|^{\frac{2n}{n-4\nu}}dx\Big)^{\frac{n-4\nu}{2n}}\Big(\int_U |A^{2\nu}u_N|^{\frac{2n}{n+4\nu-4}}dx\Big)^{\frac{n+4\nu-4}{2n}}\\
&\le c_{40}\|w_1\|_2^{q-1}\left(\|A^{\nu+1}u_N\|^2+\|A^\nu v_N\|^2\right)+\frac{\varepsilon}{16}\|A^{\nu+1}u_N\|^2\\
&\quad + c_{41}\big[M_0^4(\theta_{r-\tau}\omega)+z^4(\theta_{r-\tau}\omega)+\|w_1\|_2^{4q-4}+\|A^{\nu+1}w_2\|^{4q-4}\big],
\end{aligned}
$$

where c_{40} is dependent of λ;

$$
\begin{aligned}
&\int_U f_2'(u,x)u_{L,t}\cdot A^{2\nu}u_N dx\\
&\le c_{42}\int_U |u_{L,t}|\left(1+|u|^p\right)|A^{2\nu}u_N|dx\\
&\le c_{43}\Big(\int_U |u_{L,t}|^2dx\Big)^{1/2}\Big(\int_U (1+|u|^p dx)^{\frac{2n}{4-4\nu}}\,dx\Big)^{\frac{4-4\nu}{2n}}\Big(\int_U |A^{2\nu}u_N|^{\frac{2n}{n+4\nu-4}}dx\Big)^{\frac{n+4\nu-4}{2n}}\\
&\le c_{44}\big[1+M_0^{2p}(\theta_{r-\tau}\omega)\big]+\frac{\varepsilon}{16}\|A^{\nu+1}u_N\|^2.
\end{aligned}
$$

From (4.26) and similar to (4.10), we have

$$
\frac{d}{dt}y_N(r)+m_2(r)y_N(r)\le q_2(\theta_{r-\tau}\omega),\quad \forall r\ge\tau-t, \tag{4.27}
$$

where

$$
y_N=\|A^{\nu+1}u_N\|^2+\|A^\nu v_N\|^2+2\int_U\left[f(u,x)-f_1(u_L,x)+\lambda u_2\right]A^{2\nu}u_N dx, \tag{4.28}
$$

$$
\begin{gathered}
m_2(r)=\frac{\varepsilon}{2}-c_{45}\|w_1(r)\|_2^{q-1}-c_{45}\|u_L(r)\|_2^{q-2},\\
q_2(\theta_{r-\tau}\omega)=c_{46}[1+M_0^{2q}(\theta_{r-\tau}\omega)+z^4(\theta_{r-\tau}\omega)+\|w_1\|_2^{4q-4}+\|A^{\nu+1}w_2\|^{4q-4}],\\
y_N\left(\tau-t,\tau-t,\theta_{-\tau}\omega,\varphi_{\tau-t}(\theta_{-\tau}\omega)\right)\le\|h\|_2^2\cdot z^2(\theta_{-t}\omega),\\
2\int_U\left[f(u,x)-f_1(u_L,x)+\lambda u_N\right]A^{2\nu}u_N dx\le c_{47}[1+M_0^{q+1}(\theta_{r-\tau}\omega)],
\end{gathered}
$$

where c_{45}, c_{46}, and c_{47} are dependent of λ. Using the Gronwall's inequality in (4.27), on $[\tau-t,r]$ $(r\ge\tau-t)$, we have

$$
y_N(r)\le y_N(\tau-t)e^{-\int_{\tau-t}^r m_2(s)ds}+\int_{\tau-t}^r q_2(\theta_{\xi-\tau}\omega)e^{-\int_\xi^r m_2(s)ds}d\xi. \tag{4.29}
$$

Let

$$
T=T_1=\frac{4c_{45}K_0}{\sigma_1\varepsilon}
$$

in (4.18), for $\tau-t\le r\le\tau$, we have

$$
\begin{gathered}
\int_r^\tau\|w_1(s)\|_2^{q-1}ds\le\frac{\varepsilon}{4c_{45}}(\tau-r)+\bar K,\\
\|A^{\nu+1}w_2(r)\|^{q-1}\le K_1M_2(T_1,\omega).
\end{gathered} \tag{4.30}
$$

For $r \geq \xi \geq \tau - t$, from (4.30) and (4.32), we obtain that

$$\begin{aligned} \int_{\xi}^{\tau} m_2(s)ds &= \int_{\xi}^{\tau} \frac{\varepsilon}{2} - c_{45}\|w_1(s)\|_2^{q-1} - c_{45}\|u_L(s)\|_2^{q-2} ds, \\ &\geq \frac{\varepsilon}{4}(\tau - \xi) - c_{48}, \end{aligned} \tag{4.31}$$

where $c_{48} = c_{45}\bar{K} + \frac{c_{45}}{\sigma_1(q-2)} M_L^{q-2}$. So we have

$$y_N(\tau - t)e^{-\int_{\tau-t}^{r} m_2(s)ds} \leq \|h\|_2^2 z^2(\theta_{-t}\omega)e^{-\frac{\varepsilon}{4}t + c_{48}} \overset{t\to+\infty}{\longrightarrow} 0. \tag{4.32}$$

From (4.20), we obtain

$$\begin{aligned} &\int_{-mT_1}^{-(m-1)T_1} \|w_1(r)\|_2^{4q-4} e^{\frac{\varepsilon}{4}r} dr \\ &\leq \int_{-mT_1}^{-(m-1)T_1} M_1^{4q-4}(\theta_{-mT_1}\omega)e^{-(4q-4)\sigma_1(r+mT_1)} e^{\frac{\varepsilon}{4}r} dr \\ &\leq \frac{4}{\varepsilon} e^{\frac{\varepsilon}{4}T_1} M_1^{4q-4}(\theta_{-mT_1}\omega)e^{-\frac{\varepsilon}{4}mT_1}, \quad \forall m \geq 1, \end{aligned}$$

and

$$\begin{aligned} &\int_{-\infty}^{0} \|w_1(r)\|_2^{4q-4} e^{\frac{\varepsilon}{4}r} dr \\ &= \Big(\cdots + \int_{-mT_1}^{-(m-1)T_1} + \cdots + \int_{-2T_1}^{-T_1} + \int_{-T_1}^{0} \Big) \|w_1\|_2^{4q-4} e^{\frac{\varepsilon}{4}r} dr \\ &\leq \frac{4}{\varepsilon} e^{\frac{\varepsilon}{4}T_1} \sum_{m=1}^{+\infty} M_1^{4q-4}(\theta_{-mT_1}\omega)e^{-\varepsilon mT_1/4}. \end{aligned}$$

From [2], we know that $M_1^{4q-4}(\omega)$ is tempered and $M_1^{4q-4}(\theta_t\omega)$ is continuous in t, so we can find a tempered random variable $\varsigma(\omega)$ (> 0) such that

$$M_1^{4q-4}(\theta_{-mT_1}\omega) \leq \varsigma(\omega)e^{\frac{\varepsilon}{4q-4}mT_1}, \quad \forall m \geq 1,$$

then we obtain

$$\int_{-\infty}^{0} \|w_1(r)\|_2^{4q-4} e^{\frac{\varepsilon}{4}r} dr \leq \varsigma(\omega)\frac{4}{\varepsilon} e^{\frac{\varepsilon}{4}T_1} \sum_{m=1}^{+\infty} e^{\frac{\varepsilon(2-q)}{4q-4}mT_1} = \frac{4}{\varepsilon} \frac{e^{\frac{\varepsilon}{4q-4}T_1}}{1 - e^{\frac{\varepsilon(2-q)}{4q-4}T_1}} \varsigma(\omega) < \infty.$$

Hence

$$\begin{aligned} &\int_{\tau-t}^{\tau} q_2(\theta_{\xi-\tau}\omega)e^{-\int_{\xi}^{\tau} m_2(s)ds} d\xi \\ &\leq c_{49} \int_{\tau-t}^{\tau} \big[1 + M_0^{2q}(\theta_{r-\tau}\omega) + z^4(\theta_{r-\tau}\omega) + \|w_1\|_2^{4q-4} + M_2^4(T_1,\omega)\big] e^{-\frac{\varepsilon}{4}(\tau-\xi)} d\xi \\ &\leq c_{50}\Big(1 + M_2^4(T_1,\omega) + \int_{-\infty}^{0} \big(M_0^{2q}(\theta_r\omega) + z^4(\theta_r\omega) + \|w_1(r)\|_2^{4q-4}\big) e^{\frac{\varepsilon}{4}r} dr\Big) < \infty, \end{aligned}$$

where c_{49} and c_{50} depend on λ.

From (4.28), (4.29) and (4.32), for $t \geq 0$, we can deduce that

$$\begin{aligned} &\|A^{\nu+1}u_N(\tau,\tau-t,\theta_{-\tau}\omega,\varphi_{\tau-t})\|^2 + \|A^{\nu}v_N(\tau,\tau-t,\theta_{-\tau}\omega,\varphi_{\tau-t})\|^2 \\ &\leq 2y_N(\tau,\tau-t,\theta_{-\tau}\omega,\varphi_{\tau-t}(\theta_{-\tau}\omega)) + 2c_{47}[1+M_0^{q+1}(\omega)] \\ &\leq 2\|h\|_2^2 \cdot z^2(\theta_{-t}\omega)e^{-\frac{\varepsilon}{4}t+c_{48}} + 2c_{51}\Big[1+M_0^{q+1}(\omega)\Big] + c_{52}(1+M_2^4(T_1,\omega). \\ &\quad + \int_{-\infty}^0 \Big(M_0^{2q}(\theta_r\omega) + z^4(\theta_r\omega) + \|w_1(r)\|_2^{4q-4}\Big) e^{\frac{\varepsilon}{4}r}dr). \end{aligned} \tag{4.33}$$

By (4.32), there exists a random variable $t_\nu(\omega) \geq 0$ such that

$$0 \leq 2\|h\|_2^2 \cdot z^2(\theta_{-t}\omega)e^{-\frac{\varepsilon}{4}t+c_{48}} \leq 1, \quad \forall t \geq t_\nu(\omega). \tag{4.34}$$

We denote

$$\begin{aligned} &M_\nu^2(\omega) \\ &= c_{52}\Big(1+M_2^4(T_1,\omega) + \int_{-\infty}^0 \Big(M_0^{2q}(\theta_r\omega) + z^4(\theta_r\omega) + \|w_1(r)\|_2^{4q-4}\Big) e^{\frac{\varepsilon}{4}r}dr\Big) \\ &\quad + 1 + 2c_{51}\Big[1+M_0^{q+1}(\omega)\Big], \end{aligned} \tag{4.35}$$

which is tempered, then we have (4.25) from (4.33) and (4.34). □

Lemma 4.6. *For any $\tau \in \mathbb{R}$, $\omega \in \Omega$, $t \geq 0$, assume that $B_\nu(\tau,\omega) \subseteq B_1(\tau,\omega) \subseteq B_0(\omega)$ and $B_\nu(\tau,\omega) \in \mathcal{D}(E^\nu)$ where ν is same as in (4.8). Then there exist a random variable $\tilde{t}_\nu(\omega) > 0$ and a tempered random variable $\tilde{M}_\nu(\omega) > 0$ (independent of t and τ) such that the solution $\varphi(\tau,\tau-t,\theta_{-\tau}\omega,\varphi_{\tau-t}(\theta_{-\tau}\omega))$ of (2.2) with $\varphi_{\tau-t}(\theta_{-\tau}\omega) \in B_\nu(\tau-t,\theta_{-t}\omega) \subseteq B_0(\theta_{-t}\omega) \cap \mathcal{D}(E^\nu)$ satisfies*

$$\begin{aligned} &\|\varphi(\tau,\tau-t,\theta_{-\tau}\omega,\varphi_{\tau-t}(\theta_{-\tau}\omega))\|_{E^\nu}^2 \\ &= \|A^{\nu+1}u(\tau,\tau-t,\theta_{-\tau}\omega,\varphi_{\tau-t})\|^2 + \|A^\nu v(\tau,\tau-t,\theta_{-\tau}\omega,\varphi_{\tau-t})\|^2 \\ &\leq \tilde{M}_\nu^2(\omega), \quad \forall t \geq \tilde{t}_\nu(\omega). \end{aligned} \tag{4.36}$$

Proof. We take the inner product $((2.2), A^{2\nu}\varphi)_E$ with $A^{2\nu}\varphi = (A^{2\nu}u, A^{2\nu}v)^{\mathrm{T}}$, we obtain

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\Big(\|A^{\nu+1}u\|^2 + \|A^\nu v\|^2 + 2\int_U [f(u,x)+\lambda u]A^{2\nu}udx\Big) \\ &+ (\Lambda\varphi, A^{2\nu}\varphi) + \varepsilon\int_U [f(u,x)+\lambda u]\,A^{2\nu}udx - \int_U [f_u'(u,x)u_t + \lambda u_t]\,A^{2\nu}udx \\ &= \big(f(u,x)+\lambda u, A^{2\nu}h(x)z(\theta_{r-\tau}\omega)\big) + \big(h(x)z(\theta_{r-\tau}\omega), A^{2\nu}u\big)_2 \\ &\quad + \big(g(x,r)+\varepsilon h(x)z(\theta_{r-\tau}\omega), A^{2\nu}v\big), \quad \forall r \geq \tau-t. \end{aligned} \tag{4.37}$$

Then, for $r \geq \tau - t$,

$$\begin{aligned} &\int_U [f'(u,x)u_t + \lambda u_t]\,A^{2\nu}udx \\ &\leq c_{43}\|w_1\|_2^{q-1}\big(\|A^{\nu+1}u\|^2 + \|A^\nu v\|^2\big) + \frac{\varepsilon}{16}\|A^{\nu+1}u\|^2 \\ &\quad + c_{45}\big[M_0^4(\theta_{r-\tau}\omega) + z^4(\theta_{r-\tau}\omega) + \|w_1\|_2^{4q-4} + \|A^{\nu+1}w_2\|^{4q-4}\big], \end{aligned}$$

$$\int_U [f(u,x)+\lambda u]\,A^{2\nu}udx \leq c_{53}\big[1+M_0^{q+1}(\theta_{r-\tau}\omega)\big],$$

where c_{53} depends on λ. Similar to (4.27) by using (4.37), we obtain

$$\frac{d}{dt}\tilde{y}_1(r)+\tilde{m}_1(r)\tilde{y}_1(r)\le \tilde{q}_1(\theta_{r-\tau}\omega),\quad \forall r\ge \tau-t, \tag{4.38}$$

where

$$\begin{gathered}\tilde{y}_1(r)=\|A^{\nu+1}u\|^2+\|A^\nu v\|^2+2\int_U [f(u,x)+\lambda u]\,A^{2\nu}u\,dx,\\ \tilde{q}_1(\theta_{r-\tau}\omega)=c_{54}\big[1+M_0^4(\theta_{r-\tau}\omega)+z^4(\theta_{r-\tau}\omega)+\|w_1\|_2^{4q-4}+M_2^4(T_1,\omega)\big],\\ \tilde{m}_1(r)=\frac{\varepsilon}{2}-c_{47}\|w_1(r)\|_2^{q-1}.\end{gathered}$$

Using the Gronwall's inequality in (4.38) on $[\tau-t,r]$ ($r\ge\tau-t$), we have

$$\tilde{y}_1(r)\le \tilde{y}_1(\tau-t)e^{-\int_{\tau-t}^r\tilde{m}_1(s)ds}+\int_{\tau-t}^r\tilde{q}_1(\theta_{\xi-\tau}\omega)e^{-\int_\xi^r\tilde{m}_1(s)ds}d\xi. \tag{4.39}$$

Same as (4.31), for $\tau\ge\xi\ge\tau-t$, we obtain that

$$\int_\xi^\tau \tilde{m}_1(s)ds\ge\frac{\varepsilon}{4}(\tau-\xi)-c_{50},$$

so

$$\begin{aligned}&\int_{\tau-t}^\tau \tilde{q}_1(\theta_{\xi-\tau}\omega)e^{-\int_\xi^\tau\tilde{m}_1(s)ds}d\xi\\ &\le c_{55}\Big(1+M_2^4(T_1,\omega)+\int_{-\infty}^0\Big(M_0^4(\theta_r\omega)+z^4(\theta_r\omega)+\|w_1(r)\|_2^{4q-4}\Big)\,e^{\frac{\varepsilon}{4}r}dr\Big).\end{aligned}$$

From $\varphi_{\tau-t}(\theta_{-\tau}\omega)\in B_0(\theta_{-t}\omega)\cap\mathcal{D}(E^\nu)$, we obtain

$$\begin{aligned}&\tilde{y}_1(\tau-t)e^{-\int_{\tau-t}^r\tilde{m}_1(s)ds}\\ &\le\Big(\|A^{\nu+1}u_{\tau-t}\|^2+\|A^\nu v_{\tau-t}\|^2+2c_{53}\Big[1+M_0^{q+1}(\theta_{r-\tau}\omega)\Big]\Big)\,e^{-\frac{\varepsilon}{4}t+c_{50}}\overset{t\to+\infty}{\longrightarrow}0.\end{aligned}$$

Hence, there exists a random variable $\tilde{t}_\nu(\omega)\ge0$ satisfying

$$0\le\tilde{y}_1(\tau-t)e^{-\int_{\tau-t}^\tau\tilde{m}_1(s)ds}\le1,\quad\forall t\ge\tilde{t}_\nu(\omega).$$

Based on (4.39), we obtain

$$\begin{aligned}&\|A^{\nu+1}u(\tau,\tau-t,\theta_{-\tau}\omega,\varphi_{\tau-t})\|^2+\|A^\nu v(\tau,\tau-t,\theta_{-\tau}\omega,\varphi_{\tau-t})\|^2\\ &\le2\tilde{y}_1(\tau,\tau-t,\theta_{-\tau}\omega,\varphi_{\tau-t}(\theta_{-\tau}\omega))+2c_{53}[1+M_0^{q+1}(\omega)]\\ &\le c_{55}\Big(1+M_2^4(T_1,\omega)+\int_{-\infty}^0\big(M_0^4(\theta_r\omega)+z^4(\theta_r\omega)+\|w_1(r)\|_2^{4q-4}\big)e^{\frac{\varepsilon}{4}r}dr\Big)\\ &\quad+1+2c_{53}\Big[1+M_0^{q+1}(\omega)\Big]\\ &=\tilde{M}_\nu^2(\omega),\quad\forall t\ge\tilde{t}_\nu(\omega).\end{aligned}\tag{4.40}$$

□

Using Lemmas 4.1-4.6, by recursion we obtain the following result.

Lemma 4.7. *For any $\tau\in\mathbb{R}$, $\omega\in\Omega$, $t\ge0$, let $B_\kappa(\tau,\omega)\subseteq B_1(\tau,\omega)$, $B_\kappa(\tau,\omega)\in\mathcal{D}(E^\kappa)$ and $\varphi_{\tau-t}(\theta_{-\tau}\omega)\in B_\kappa(\tau-t,\theta_{-t}\omega)$. Then there exist $T_\kappa(\omega)\ge0$ and tempered random variables $\tilde{M}_\kappa(\omega)>0$, $M_\kappa(\omega)>0$ (independent of t and τ) such that for $t\ge T_\kappa(\omega)$, the solution $\varphi(\tau,\tau-t,\theta_{-\tau}\omega,\varphi_{\tau-t}(\theta_{-\tau}\omega))$ of (2.2) satisfies* *(i) for* $\nu \le \kappa \le 1$,

$$\begin{aligned}&\|\varphi(\tau,\tau-t,\theta_{-\tau}\omega,\varphi_{\tau-t}(\theta_{-\tau}\omega))\|_{E^\kappa}^2\\&=\|A^{\kappa+1}u(\tau,\tau-t,\theta_{-\tau}\omega,\varphi_{\tau-t}(\theta_{-\tau}\omega))\|^2+\|A^\kappa v(\tau,\tau-t,\theta_{-\tau}\omega,\varphi_{\tau-t}(\theta_{-\tau}\omega))\|^2\\&\le \tilde{M}_\kappa^2(\omega);\end{aligned}$$

(ii) for $\nu \le \kappa \le 1-\nu$,

$$\begin{aligned}&\|\varphi_N(\tau,\tau-t,\theta_{-\tau}\omega,\varphi_{\tau-t}(\theta_{-\tau}\omega))\|_{E^{\kappa+\nu}}^2\\&=\|A^{\kappa+\nu+1}u_N(\tau,\tau-t,\theta_{-\tau}\omega,\varphi_{\tau-t})\|^2+\|A^{\kappa+\nu}v_N(\tau,\tau-t,\theta_{-\tau}\omega,\varphi_{\tau-t})\|^2\\&\le M_\kappa^2(\omega),\end{aligned}\tag{4.41}$$

where ν *is same as in* (4.8).

5. Existence of a random attractor

In this section, by applying [29, Lemma 3.7] and Lemma 4.7, show the existence of a random attractor for (1.1). Firstly, we consider the Lipschitz property of cocycle Φ on $B_1(\tau,\omega)$. For any $\tau\in\mathbb{R}$, $\omega\in\Omega$ and any $\varphi_{j\tau}(\omega)=(u_{j\tau}(\omega),v_{j\tau}(\omega))\in B_1(\tau,\omega)$, $j=1,2$, let

$$\varphi_j(r)=\varphi_j(r,\tau,\omega,\varphi_{j\tau}(\omega))=(u_j(r),v_j(r)),\quad r\ge\tau,$$

be solutions of (2.2) with initial data $\varphi_{j\tau}(\omega)$, $j=1,2$, respectively, and let

$$\psi(r)=\varphi_1(r)-\varphi_2(r)=(u_1(r)-u_2(r),v_1(r)-v_2(r))=(\xi(r),\eta(r)).$$

So that

$$\begin{gathered}\dot{\psi}+\Lambda\psi=F(\varphi_1,\theta_r\omega,r)-F(\varphi_2,\theta_r\omega,r),\quad r\ge\tau,\\ \psi_\tau(\omega)=(\xi_\tau,\eta_\tau)=(\xi_\tau,u_{1,1,\tau}-u_{2,1,\tau}+\varepsilon\xi_\tau)=(u_{1\tau}-u_{2\tau},v_{1\tau}-v_{2\tau}).\end{gathered}\tag{5.1}$$

Lemma 5.1. *There exists a tempered random variable* $C_1(\omega)>0$ *such that for any* $\tau\in\mathbb{R}$, $t\ge0$, *and* $\omega\in\Omega$, *it holds*

$$\begin{aligned}&\|\varphi(t+\tau,\tau,\theta_{-\tau}\omega,\varphi_{1\tau}(\theta_{-\tau}\omega))-\varphi(t+\tau,\tau,\theta_{-\tau}\omega,\varphi_{2\tau}(\theta_{-\tau}\omega))\|_E\\&\le e^{\int_0^t C_1(\theta_s\omega)ds}\|\varphi_{1\tau}-\varphi_{2\tau}\|_E.\end{aligned}\tag{5.2}$$

Proof. From (4.1), for $r\ge\tau$, we obtain $\|\varphi_1(r)\|_E\le M_0(\theta_r\omega)$, $\|\varphi_2(r)\|_E\le M_0(\theta_r\omega)$. Based on (5.1), we have the following inequality by taking the inner product $(\cdot,\cdot)_E$ with $\psi(r)$,

$$\frac{d}{dt}\|\psi(r)\|_E^2\le\Big(\frac{c_{56}}{2\alpha}\Big(1+M_0^{2q-2}(\theta_{r-\tau}\omega)\Big)-\varepsilon+\lambda\Big)\|\psi(r)\|_E^2,\quad\forall r\ge\tau.\tag{5.3}$$

Using Gronwall's inequality on (5.3), we obtain

$$\|\varphi_1(r)-\varphi_2(r)\|_E^2\le\|\varphi_{1\tau}-\varphi_{2\tau}\|_E^2e^{\int_0^{r-\tau}\big(\frac{c_{56}}{2\alpha}\big(1+M_0^{2q-2}(\theta_{\tau+s}\omega)\big)-\varepsilon+\lambda\big)ds},\quad\forall r\ge\tau.$$

As $\omega\to\theta_{-\tau}\omega$ and $r\to t+\tau$, (5.2) holds with

$$\begin{gathered}C_1(\omega)=\frac{c_{56}}{2\alpha}\Big(1+M_0^{2q-2}(\omega)\Big)+\lambda,\\ 0<\mathrm{E}(C_1(\omega))=\frac{c_{56}}{2\alpha}\Big(1+\mathrm{E}[M_0^{2q-2}(\omega)]\Big)+\lambda\le\frac{c_{56}}{2\alpha c_{18}}K_0+\lambda<\infty.\end{gathered}$$

□

Lemma 5.2. *For any $\tau \in \mathbb{R}$, $\omega \in \Omega$, there exist a $T_\nu(\omega) \geq 0$, a random bounded ball $\tilde{B}_1(\omega)$ of E^1 with radius $b_1(\omega)$, a positive number $\tilde{\sigma}$ and a tempered random variable $\tilde{Q}(\omega) > 0$ such that the solution $\varphi(\tau, \tau - t, \theta_{-\tau}\omega, \varphi_{\tau-t}(\theta_{-\tau}\omega))$ of* (2.2) *with initial data $\varphi_{\tau-t}(\theta_{-\tau}\omega) \in B_1(\tau - t, \theta_{-t}\omega)$, it holds that*

$$d_E\left(\varphi(\tau, \tau - t, \theta_{-\tau}\omega, B_1(\tau - t, \theta_{-t}\omega)), \tilde{B}_1(\omega)\right) \leq \tilde{Q}(\theta_{-t}\omega)e^{-\tilde{\sigma}t}, \quad t \geq T_\nu(\omega).$$

Proof. Now we assume $\varphi_{\tau-t}(\theta_{-\tau}\omega) \in B_1(\tau - t, \theta_{-t}\omega)$, and $\tilde{K}_\nu(\omega) \subset E^\nu \subset E$ is the random ball of E^ν with radius $M_\nu(\omega)$ defined in (4.35), here ν is same as in (4.8). By Lemma 4.4, we can deduce that for any $t \geq 0$,

$$d_E\left(\varphi(\tau, \tau - t, \theta_{-\tau}\omega, B_1(\tau - t, \theta_{-t}\omega)), \tilde{K}_\nu(\omega)\right) \leq M_L e^{-\sigma_1 t}. \tag{5.4}$$

For $\varphi_{\tau-t}(\theta_{-\tau}\omega) \in \tilde{K}_\nu(\theta_{-t}\omega)$, according to Lemmas 4.4–4.7, there exist $t_{1\nu}(\omega) \geq 0$ and a random ball $\tilde{K}_{2\nu}(\omega)$ of $E^{2\nu}$ with the radius $M_{2\nu}(\omega)$ and satisfies

$$d_E\left(\varphi(\tau, \tau - t, \theta_{-\tau}\omega, \tilde{K}_\nu(\theta_{-t}\omega)), \tilde{K}_{2\nu}(\omega)\right) \leq P_{1\nu}(\theta_{-t}\omega)e^{-\sigma_1 t}, \quad t \geq t_{1\nu}(\omega), \tag{5.5}$$

where $P_{1\nu}^2(\theta_{-t}\omega) = \tilde{c_1}\left[1 + M_\nu^4(\theta_{-t}\omega)\right] + 2\|h\|^2 \cdot |z(\theta_{-t}\omega)|^2$. From [29, Lemma 3.7], Lemma 5.1 and (5.4) and (5.5), there exists $T_{1\nu}(\omega) \geq t_{1\nu}(\omega) \geq 0$ (independent of τ), such that for $t \geq T_{1\nu}(\omega)$,

$$d_E\left(\varphi(\tau, \tau - t, \theta_{-\tau}\omega, B_1(\tau - t, \theta_{-t}\omega)), \tilde{K}_{2\nu}(\omega)\right) \leq P_{2\nu}(\theta_{-t}\omega)e^{-\frac{a_1}{2}\sigma_1 t},$$

where $P_{2\nu}(\theta_{-t}\omega) = M_L + P_{1\nu}(\theta_{-(1-a_1)t}\omega)$ is tempered and

$$0 < a_1 \leq \frac{\sigma_1}{3\,\mathrm{E}[C_1(\omega)] + 4\sigma_1}.$$

There exists an integer $\tilde{k} \geq 1$ such that $1 - \nu \leq (\tilde{k} - 1)\nu < 1$ when $\nu > 0$ is a fixed positive constant. Repeating $\tilde{k}$ ($\leq [\frac{1}{\nu}] + 2$) steps as above recursion, we can obtain that there exist $T_{\tilde{k}\nu}(\omega) > 0$ (independent of τ) and a random ball $\tilde{B}_1(\omega)$ of E^1 with radius $b_1(\omega)$ (defined in (4.41)) such that for $t \geq T_{\tilde{k}\nu}(\omega)$,

$$d_E\left(\varphi(\tau, \tau - t, \theta_{-\tau}\omega, B_1(\tau - t, \theta_{-t}\omega)), \tilde{B}_1(\omega)\right) \leq P_{\tilde{k}\nu}(\theta_{-t}\omega)e^{-\frac{a_{\tilde{k}-1}}{2}\sigma_1 t}, \tag{5.6}$$

where

$$0 < a_j \leq \frac{\frac{a_{j-1}}{2}\sigma_1}{3\,\mathrm{E}[C_1(\omega)] + \frac{a_{j-1}}{2}\sigma_1 + 3\sigma_1} < \infty, \quad j = 2, \cdots, \tilde{k} - 1,$$
$$P_{\tilde{k}\nu}(\theta_{-t}\omega) = M_L + P_{(\tilde{k}-1)\nu}(\theta_{-(1-a_{\tilde{k}-1})t}\omega),$$

is tempered. □

Combining Lemmas 2.3 and 5.2, the existence of a random attractor for the RDS Φ can be proved.

Theorem 5.3. *The cocycle Φ associated with* (2.2) *possesses a $\mathcal{D}(E)$-pullback random attractor $\mathcal{A} \in \mathcal{D}(E)$ such that for any $\tau \in \mathbb{R}$, $\omega \in \Omega$, $\mathcal{A}(\tau, \omega) \subseteq \tilde{B}_1(\omega) \cap B_0(\omega)$ and*

$$\|\mathcal{A}(\tau, \omega)\|_{E^1} = \sup_{\varphi \in \mathcal{A}(\tau, \omega)} \|\varphi\|_{E^1} \leq b_1(\omega),$$

where $b_1(\omega)$ is the radius of the bounded ball $\tilde{B}_1(\omega) \subset E^1$.

Proof. Since $E^1 \hookrightarrow E$ is a compact embedding, from (5.6) we know that $\tilde{B}_1(\omega)$ in Lemma 5.2 is a compact measurable $\mathcal{D}(E)$-pullback attracting ball in E for any $\tau \in \mathbb{R}$ and $\omega \in \Omega$. According to Lemma 2.3, the RDS Φ possesses a $\mathcal{D}(E)$-pullback random attractor $\mathcal{A} \in \mathcal{D}(E)$, that is for any $\tau \in \mathbb{R}$, $\omega \in \Omega$, and $\mathcal{A}(\tau, \omega) \subseteq \tilde{B}_1(\omega) \cap B_0(\omega)$. Based on (4.35), (4.40) and (4.41), the radius $b_1(\omega)$ of $\tilde{B}_1(\omega) \subset E^1$ is given by

$$\begin{aligned} b_1^2(\omega) &= c_{57}\Big(1 + M_2^{4\tilde{k}}(T_1, \omega) + M_0^{(q+1)\tilde{k}}(\omega)\Big) \\ &\quad + c_{58}\int_{-\infty}^0 \Big(M_0^{4\tilde{k}}(\theta_r\omega) + |z(\theta_r\omega)|^{4\tilde{k}} + \|w_1(r)\|_2^{(4q-4)\tilde{k}}\Big) e^{\varepsilon r/4} dr. \end{aligned} \tag{5.7}$$

□

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