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REMARKS ON THE SECOND NEUMANN EIGENVALUE

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Abstract. This work reviews some basic features on the second (first non-trivial) eigenvalue λ_2 to the Neumann problem

$$-\Delta_p u = \lambda |u|^{p-2}u \quad x \in \Omega$$
$$|\nabla u|^{p-2}\frac{\partial u}{\partial \nu} = 0 \quad x \in \partial\Omega,$$

where Ω is a bounded Lipschitz domain of $\mathbb{R}^N$, ν is the outer unit normal, and $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ is the p-Laplacian operator. We are mainly concerned with the variational characterization of λ_2 and place emphasis on the range $1 < p < 2$, where the nonlinearity $|u|^{p-2}u$ becomes non smooth. We also address the corresponding result for the p-Laplacian in graphs.

1. Introduction

The analysis of the eigenvalues of the p-Laplacian operator under different types of boundary conditions is one of the most interesting issues in nonlinear analysis [13, 14, 17]. Here, we focuss on the Neumann eigenvalue problem

$$\begin{aligned} -\Delta_p u &= \lambda |u|^{p-2}u \quad x \in \Omega \\ |\nabla u|^{p-2}\frac{\partial u}{\partial \nu} &= 0 \quad x \in \partial\Omega, \end{aligned} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$ is a $C^{0,1}$ bounded domain, ν stands for its outer unit normal on $\partial\Omega$ and $p > 1$.

We recall that $u \in W^{1,p}(\Omega) \setminus \{0\}$ is said to be a weak eigenfunction to (1.1) associated with the eigenvalue $\lambda \in \mathbb{R}$ if the equality

$$\int_\Omega |\nabla u|^{p-2}\nabla u \nabla v \, dx = \lambda \int_\Omega |u|^{p-2}uv \, dx, \tag{1.2}$$

holds for arbitrary test functions v in $W^{1,p}(\Omega)$.

From the definition of eigenvalue it follows by choosing $v = u$ in (1.2) that eigenvalues λ must be nonnegative. Thus $\lambda_1 = 0$ becomes the "first" (lowest) eigenvalue whose eigenfunctions are constant functions. As all those eigenfunctions are a multiple of $u = 1$, this amounts to say that λ_1 is simple (in a proper sense). Moreover, $\lambda_1 = 0$ is an isolated eigenvalue as it is going to be checked below (Theorem 1.1). It should be stressed that for the Dirichlet boundary condition $u = 0$ on $\partial\Omega$, proving the simplicity of the first (lowest) eigenvalue $\lambda_1^{\mathcal{D}}$ turned out to

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be a quite hard question. A successful answer was given in [3], where the isolation of $\lambda_1^{\mathcal{D}}$ was furthermore shown (a sharpened result was later proved in [15, 16]). Moreover, the existence of a second Dirichlet eigenvalue $\lambda_2^{\mathcal{D}}$ is a consequence of the latter assertion. In addition, it is worth remarking that obtaining a variational characterization of $\lambda_2^{\mathcal{D}}$ is by no means an easy task. Reader is referred to [4, 11, 12] for different variational expressions of $\lambda_2^{\mathcal{D}}$. Just to grasp an insight, the corresponding one in [11] is

$$\lambda_2^{\mathcal{D}} = \inf_{\gamma} \max_{u\in\gamma(I)} \frac{\int_\Omega |\nabla u|^p \, dx}{\int_\Omega |u|^p \, dx}, \tag{1.3}$$

where γ varies in the set of all continuous curves $\gamma : I = [0,1] \to W_0^{1,p}(\Omega)\setminus\{0\}$ such that $\gamma(0) = \phi_1$, $\gamma(1) = -\phi_1$, ϕ_1 being a fixed normalized eigenfunction associated with $\lambda_1^{\mathcal{D}}$. See also [5, 6] for related results in a "nonsymmetric" version of the Dirichlet and Neumann eigenvalue problems.

As mentioned above, $\lambda_1 = 0$ is isolated and so the Neumann problem (1.1) admits a second eigenvalue λ_2. More interestingly, I came across reference [7] when searching for sharp lower estimates for λ_2 in a general domain Ω. I was amazed by the existence of the elegant characterization of this eigenvalue, freely used there. The expression is considerably much simpler than its Dirichlet counterpart (1.3), and seems indeed closer to the familiar 'Rayleigh quotient' for $-\Delta$. Namely,

$$\lambda_2 = \inf \frac{\int_\Omega |\nabla u|^p \, dx}{\int_\Omega |u|^p \, dx}, \tag{1.4}$$

the infimum being extended over all those nonvanishing functions $u \in W^{1,p}(\Omega)$ satisfying the "null average" type condition (see Section 2),

$$\int_\Omega |u|^{p-2} u \, dx = 0. \tag{1.5}$$

In fact, by setting $v = 1$ as a test function in (1.2) it follows that the eigenfunctions $u \in W^{1,p}(\Omega)$ associated with all possible eigenvalues $\lambda \neq 0$ must satisfy (1.5).

On the other hand, it is more or less straightforward for the specialist to check (1.4) in the case $p \geq 2$. On the contrary, the proof for the complementary range $1 < p < 2$ is far from obvious and this brief note is just devoted to this goal. Since I have not been able to find a proof, I decided to publish one of my own. It should be mentioned that the omission of the case $1 < p < 2$ is striking in some references (see [14, Chapter IV, §2]).

It should be also remarked that $C^* = \lambda_2^{-1/p}$ can be regarded as the optimum constant C in Poincaré's inequality

$$\inf_{t\in\mathbb{R}} \|u - t\|_{L^p(\Omega)} \leq C \| |\nabla u| \|_{L^p(\Omega)}, \tag{1.6}$$

where $u \in W^{1,p}(\Omega)$. In spite of reference [18] containing general versions of this inequality, a connection between (1.6) and λ_2 is not reported there.

Our main result reads as follows.

Theorem 1.1. *Let $\Omega \subset \mathbb{R}^N$ be a bounded $C^{0,1}$ domain and set*

$$\hat{\lambda} = \inf_{u\in\mathcal{M}_0\setminus\{0\}} \frac{\int_\Omega |\nabla u|^p \, dx}{\int_\Omega |u|^p \, dx}, \tag{1.7}$$

where

$$\mathcal{M}_0 = \{u \in W^{1,p}(\Omega) : \int_\Omega |u|^{p-2}u\,dx = 0\}.$$

Then the following properties hold

(i) *The infimum in* (1.7) *is achieved at some* $u \in \mathcal{M}_0$ *and thus* $\hat{\lambda} > 0$.
(ii) *Every eigenvalue* $\lambda \neq 0$ *to the Neumann problem* (1.1) *satisfies* $\lambda \geq \hat{\lambda}$. *In particular,* $\lambda = 0$ *is an isolated eigenvalue.*
(iii) $\hat{\lambda}$ *is an* eigenvalue *and therefore*

$$\lambda_2 = \hat{\lambda}. \tag{1.8}$$

Note that (i) and (ii) are well-known and that (iii) can be proved in a standard way when $p \geq 2$ (see Remark 3.2). Therefore we focuss on showing (1.8) when p falls in the 'singular' range $1 < p < 2$.

Moving now to a further scenario, the version of the p-Laplacian operator $-\Delta_p$ for graphs $\mathcal{G}$ has been recently studied in [2], where its second eigenvalue λ_2 has been introduced and analyzed in detail (see also [8, 9, 10]. The latter specifically concerns a variety of features on the spectrum of $-\Delta$). Such eigenvalue plays an important rôle in the so-called 'clustering problem' for undirected graphs [8]. Our next result extends [2, Theorem 1] to the range $p > 1$. In fact, the proof contained there is only valid in the case $p \geq 2$, and the same remark applies to [8, Theorem 3.2] (see also [9]). The reader is referred to Section 4 for the necessary background material concerning graph theory.

Theorem 1.2. *Let* $\mathcal{G} = (\mathcal{V}, E)$, $\mathcal{V} = \{v_1, \dots, v_n\}$, *be a connected graph with weights matrix* $A = (\omega_{ij})_{1\leq i\leq n}$, $\omega_{ij} = \omega_{ji}$, $\omega_{ij} \geq 0$, $\omega_{ii} = 0$, $A \neq 0$. *Consider the eigenvalue problem*

$$-\Delta_p(f) = \lambda\nu\phi_p(f), \tag{1.9}$$

where $f = (f_i) \in \mathbb{R}^n$, $\nu = (\nu_i) \in \mathbb{R}^n_+$, $\phi_p(f) = (|p_i|^{p-2}p_i)$ *and* $-\Delta_p$ *is the* p-*Laplacian in* $\mathcal{G}$. *Then the following properties hold*

(i) $\lambda_1 = 0$ *is the first eigenvalue which is isolated, simple and has* $\operatorname{span}\{1\}$ *as the set of associated eigenfunctions.*
(ii) *The second eigenvalue* λ_2 *is expressed as*

$$\lambda_2 = \inf_{f\in\mathcal{M}_0\setminus\{0\}} \frac{1}{2}\frac{\sum_{i,j}\omega_{ij}|f_i - f_j|^p}{\sum_i \nu_i|f_i|^p}, \tag{1.10}$$

where $\mathcal{M}_0 = \{f : \sum_i \nu_i|f_i|^{p-2}f_i = 0\}$.
(iii) *The maximum eigenvalue* λ^* *is provided by the expression*

$$\lambda^* = \sup_{f\in\mathcal{M}_0\setminus\{0\}} \frac{1}{2}\frac{\sum_{i,j}\omega_{ij}|f_i - f_j|^p}{\sum_i \nu_i|f_i|^p}. \tag{1.11}$$

Remark 1.3. In this work we denote by $-\Delta_p$ what is commonly defined as the p-Laplacian operator in graphs, usually designated as Δ_p. We have proceeded in this way to preserve the analogies with partial differential equations (see Section 4).

This work is organized as follows. Section 2 analyzes the differentiability properties of the variance functional in $L^p(X, \mu)$, where (X, μ) is a measurable space.

The results attained seem to be new and are instrumental in the subsequent sections. Section 3 is devoted to proving Theorem 1.1. There, we are dealing with the slightly more general version of (1.1),

$$\begin{gathered} -\Delta_p u = \lambda m(x)|u|^{p-2}u \quad x \in \Omega \\ |\nabla u|^{p-2}\frac{\partial u}{\partial \nu} = 0 \quad x \in \partial\Omega, \end{gathered} \tag{1.12}$$

where $m \in L^r(\Omega)$ is a weight function such that $m(x) > 0$ a.e. in Ω, while exponent $r \geq 1$ is suitably chosen.

The tools developed in Section 2 turn also out to be useful for studying the nonlinear diffusion problem

$$\begin{gathered} -\Delta_p u = \lambda m(x)|u|^{q-2}u \quad x \in \Omega \\ |\nabla u|^{p-2}\frac{\partial u}{\partial \nu} = 0 \quad x \in \partial\Omega, \end{gathered} \tag{1.13}$$

where $1 \leq q < p^*$ with $p^* = \frac{pN}{N-p}$ if $p < N$, $p^* = \infty$ otherwise, $m \in L^r(\Omega)^+$ and r varies in a convenient range. It should be pointed out that the Dirichlet counterpart of (1.13) was discussed in full detail in [13] where $m \in L^\infty(\Omega)^+$. In Section 4 we analyze the main existence issues concerning (1.13). Finally Section 5 contains the proof of Theorem 1.2.

2. Variance functional in $L^p(X, \mu)$

Our forthcoming results will be obtained in the general framework of a measurable space $(X, \mathcal{A}, \mu)$ where μ is a finite measure defined in some fixed σ-algebra $\mathcal{A}$ in X. As usual, $L^p(X, \mu)$ stands for the space of measurable functions f such that $\int_X |f|^p \, d\mu < \infty$. We are proceeding in this way for the sake of completeness. We begin by reviewing some elementary features which permit us refreshing the notion of variance.

Lemma 2.1. *Let $(X, \mathcal{A})$ be a measurable space endowed with a finite measure μ. Then, for every $u \in L^p(X, \mu)$ there exists a unique $\tilde{u} \in \mathbb{R}$ such that*

$$\int_X |u - \tilde{u}|^{p-2}(u - \tilde{u}) \, d\mu = 0.$$

Moreover, $\tilde{u}$ is variationally characterized by the expression

$$\int_X |u - \tilde{u}|^p \, d\mu = \inf_{t \in \mathbb{R}} \int_X |u - t|^p \, d\mu. \tag{2.1}$$

Furthermore, $\tilde{u}$ defines a continuous functional of $u \in L^p(X, \mu)$.

Definition 2.2. For a function $u \in L^p(X, \mu)$ its variance is defined as

$$V_p(u) = \int_X |u - \tilde{u}|^p \, d\mu,$$

while $\tilde{u}$ is said to be the average of u relative to $L^p(X, \mu)$.

Remark 2.3. This is of course the standard definition when $p = 2$. In fact,

$$\tilde{u} = \frac{1}{|X|} \int_X u \, d\mu, \quad |X| := \int_X d\mu = \mu(X),$$

if $p = 2$ and $\tilde{u}$ coincides with the standard average of u in X.

Proof of Lemma 2.1. The function $g(t) = \int_X |u-t|^{p-2}(u-t)\,d\mu$ is continuous and decreasing while $\lim_{t\to\pm\infty} g(t) = \mp\infty$. Hence, the existence and uniqueness assertions follow.

On the other hand, $f(t) = \int_X |u-t|^p\,d\mu$ is a convex coercive function such that $f' = -pg$. This implies (2.1).

As for the continuity assertion assume that $u_n \to u$ in $L^p(X,\mu)$. The inequalities

$$|X||\tilde{u}|^p \le 2^{p-1}\int_X \{|u-\tilde{u}|^p + |u|^p\}\,d\mu \le 2^p \int_X |u|^p\,d\mu$$

imply

$$|\tilde{u}| \le \frac{2}{|X|^{1/p}}\|u\|_p, \tag{2.2}$$

with $\|u\|_p = \|u\|_{L^p(X,\mu)}$. Thus $\tilde{u}_n$ is bounded and a convergent subsequence $\tilde{u}_{n'} \to \tilde{u}'$ can be extracted from $\tilde{u}_n$. Since $|u_{n'} - \tilde{u}_{n'}|^{p-2}(u_{n'} - \tilde{u}_{n'}) \to |u-\tilde{u}'|^{p-2}(u-\tilde{u}')$ in $L^{p'}(X,\mu)$, it follows that

$$\int_X |u-\tilde{u}'|^{p-2}(u-\tilde{u}')\,d\mu = 0.$$

This means that $\tilde{u}' = \tilde{u}$ and the continuity is shown. □

The main result of this section is the next one. It states the differentiability of the variance functional V_p and seems new at the best of our knowledge. In this regard, it should be remarked that V_p is defined in a variational way and so this is not a straightforward issue.

Theorem 2.4. *Under the assumptions of Lemma 2.1, the variance functional V_p is Fréchet differentiable in $L^p(X,\mu)$ for all $p>1$. Moreover, its differential $DV_p(u)$ at u is represented as*

$$\langle DV_p(u), v\rangle = p\int_X |u-\tilde{u}|^{p-2}(u-\tilde{u})v\,d\mu, \quad v \in L^p(X,\mu).$$

The proof of Theorem 2.4 relies on the next crucial lemma.

Lemma 2.5. *Let $u \in L^p(X,\mu)$ be fixed. Then the Gâteaux derivative $dV_p(u,v)$ of the Variance V_p at u in the direction $v \in L^p(X,\mu)$ exists and is given by*

$$dV_p(u,v) = p\int_X |u-\tilde{u}|^{p-2}(u-\tilde{u})v\,d\mu. \tag{2.3}$$

Proof. Fix $u, v \in L^p(X,\mu)$. For $t \in \mathbb{R}$ define

$$u_t = u + tv, \quad \tilde{u}_t = \widetilde{u_t} = \widetilde{u+tv},$$

together with

$$V(t) = V_p(u_t) = V_p(u+tv).$$

By setting

$$f_t(\sigma) = \int_X |u+tv-\sigma|^p\,d\mu = \int_X |u_t-\sigma|^p\,d\mu,$$

it holds that

$$V(t) = f_t(\tilde{u}_t) = \inf_{\sigma\in\mathbb{R}} f_t(\sigma), \quad V(0) = f_0(\tilde{u}) = \inf_{\sigma\in\mathbb{R}} f_0(\sigma).$$

On the other hand,

$$f_t(\tilde{u}_t) - f_0(\tilde{u}_t) \le V(t) - V(0) \le f_t(\tilde{u}) - f_0(\tilde{u}). \tag{2.4}$$

The first term in inequality (2.4) can be written as

$$f_t(\tilde{u}_t) - f_0(\tilde{u}_t) = pt \int_X \Big\{ \int_0^1 |u + tsv - \tilde{u}_t|^{P-2}(u + tsv - \tilde{u}_t)\, ds \Big\} v\, d\mu.$$

By using (2.2) and assuming that $|t| \le 1$, the integrand in the last term can be estimated as

$$\begin{aligned} &\Big| \int_0^1 |u + tsv - \tilde{u}_t|^{p-2}(u + tsv - \tilde{u}_t)\, ds \Big| \\ &\le \int_0^1 |u + tsv - \tilde{u}_t|^{p-1}\, ds \\ &\le C\{|u|^{p-1} + |v|^{p-1} + \|u\|_p^{p-1} + \|v\|_p^{p-1}\}, \end{aligned}$$

where $C > 0$ is a constant only depending on p and $|X|$. Since

$$\int_0^1 |u + tsv - \tilde{u}_t|^{p-2}(u + tsv - \tilde{u}_t)\, ds \to |u - \tilde{u}|^{p-2}(u - \tilde{u}),$$

a.e. in X as $t \to 0$, the Lebesgue dominated convergence theorem implies that

$$\lim_{t\to 0} \frac{1}{t}\big(f_t(\tilde{u}_t) - f_0(\tilde{u}_t)\big) = p \int_X |u - \tilde{u}|^{p-2}(u - \tilde{u}) v\, d\mu.$$

An identical argument shows that

$$\lim_{t\to 0} \frac{1}{t}\big(f_t(\tilde{u}) - f_0(\tilde{u})\big) = p \int_X |u - \tilde{u}|^{p-2}(u - \tilde{u}) v\, d\mu,$$

and the desired result follows from dividing the three terms in (2.4) by $t \neq 0$ and passing to the limit as $t \to 0$. □

Proof of Theorem 2.4. The Gâteaux derivative $dV_p(u, v)$ is linear continuous in $v \in L^p(X, \mu)$ for u fixed. On the other hand, mapping $u \to dV_p(u, \cdot)$ regarded as taking values in $(L^p(X, \mu))^* = L^{p'}(X, \mu)$ is continuous. This entails that V_p is Fréchet differentiable at u (see for instance [1, Chapter 1]). □

We next single out two special cases where Theorem 2.4 is applied. In the first one, $X = \Omega$ is a bounded set of $\mathbb{R}^n$, endowed with the measure $d\mu = m(x)dx$ where $m \in L^1(\Omega)$, $m(x) > 0$ a.e. in Ω.

Corollary 2.6. *Let V_p be the variance functional defined in $L^p(\Omega, mdx)$. Then, at any $u \in L^p(\Omega, mdx)$ we have*

$$\langle DV_p(u), v\rangle = p \int_\Omega |u - \tilde{u}|^{p-2}(u - \tilde{u}) v\, mdx, \quad v \in L^p(\Omega, mdx).$$

In the second example X is a finite set $\mathcal{V} = \{v_1, \dots, v_n\}$ where the measure is

$$\mu = \sum_{i=1}^n \nu_i \delta(x - v_i),$$

while δ is the Dirac's delta and $\nu_i > 0$, $1 \le i \le n$. Functions $f : \mathcal{V} \to \mathbb{R}$ are identified to vectors in $\mathbb{R}^n$ by means of the expression $f = (f_i)$ with $f_i = f(i)$. Then

$$V_p(f) = \sum_i |f_i - \tilde{f}|^p \nu_i = \inf_{t\in\mathbb{R}} \sum_i |f_i - t|^p \nu_i,$$

where $t = \tilde{f}$ is the unique number so that $\sum_i |f_i - t|^{p-2}(f_i - t)\nu_i = 0$.

Corollary 2.7. *Function V_p is differentiable at any $f \in \mathbb{R}^n$ and*

$$\langle DV_p(f), g\rangle = p\sum_i |f_i - \tilde{f}|^{p-2}(f_i - \tilde{f})g_i\nu_i, \quad g \in \mathbb{R}^n. \tag{2.5}$$

3. Proof of Theorem 1.1

In this section we prove a slightly more general version of Theorem 1.1 that can be stated as follows.

Theorem 3.1. *Assume that $\Omega \subset \mathbb{R}^n$ is class $C^{0,1}$ bounded domain, $m \in L^r(\Omega)$, $m(x) > 0$ a.e. in Ω, where*

$$r \begin{cases} > \left(\frac{p^*}{p}\right)' = Np & \text{if } 1 < p < N, \\ > 1 & \text{if } p = N, \\ = 1 & \text{if } p > N. \end{cases}$$

We define

$$\hat{\lambda}(m) = \inf_{u \in \mathcal{M}_0 \setminus \{0\}} \frac{\int_\Omega |\nabla u|^p \, dx}{\int_\Omega |u|^p \, m dx}, \tag{3.1}$$

with $\mathcal{M}_0 = \{u \in W^{1,p}(\Omega) : \int_\Omega |u|^{p-2} u \, m dx = 0\}$. Then eigenvalue problem (1.12) *satisfies the assertions (i)–(iii) in Theorem 1.1 with λ_2 replaced by $\lambda_2(m)$, the second eigenvalue to* (1.12).

Proof. To show Theorem 3.1 for problem (1.12) we first notice that

$$\hat{\lambda}(m) = \inf_{u \in \mathcal{M}_0 \setminus \{0\}} \frac{\int_\Omega |\nabla u|^p \, dx}{\int_\Omega |u|^p \, m dx} = \inf_{u \in \mathcal{M}_1} \mathcal{J}(u),$$

where $\mathcal{J}(u) = \int_\Omega |\nabla u|^p \, dx$, and

$$\mathcal{M}_0 = \big\{u \in W^{1,p}(\Omega) : \int_\Omega |u|^{p-2} u \; m dx = 0\big\}, \quad \mathcal{M}_1 = \mathcal{M}_0 \cap \big\{ \int_\Omega |u|^p \, m dx = 1\big\}.$$

Functional $\mathcal{J}$ is coercive, i.e. $\mathcal{J}(u) \to \infty$ as $\|u\|_{W^{1,p}(\Omega)} \to \infty$, and weakly lower semicontinuous, while the election of r entails that $\mathcal{M}_1$ is weakly closed in $W^{1,p}(\Omega)$. Thus, a well-known result in Calculus of Variations ([19, Chapter I]) ensures us the existence of a global minimizer $u_1 \in \mathcal{M}_1$. Hence

$$0 < \hat{\lambda}(m) = \mathcal{J}(u_1) = \inf_{u \in \mathcal{M}_0 \setminus \{0\}} \frac{\int_\Omega |\nabla u|^p \, dx}{\int_\Omega |u|^p \; m dx}.$$

Since every possible eigenfunction u associated with a nonzero eigenvalue λ to (1.12) lies in $\mathcal{M}_0$ then $\lambda \geq \hat{\lambda}(m)$.

Thus the main feature remains to be proved. Namely that $\hat{\lambda}(m)$ is actually an eigenvalue. The next proof, relying on the properties of the variance functional V_p presented in Section 2, can be applied to cover both cases $p \geq 2$ and $1 < p < 2$. First observe that

$$\mathcal{M}_0 = \{u - \tilde{u} : u \in W^{1,p}(\Omega)\},$$

where the notation of Section 2 has been used. Hence,

$$\hat{\lambda}(m) = \inf_{v \in \mathcal{M}_0 \setminus \{0\}} \frac{\int_\Omega |\nabla v|^p \, dx}{\int_\Omega |v|^p \; m dx} = \inf_{u \notin \operatorname{span}\{1\}} \frac{\int_\Omega |\nabla u|^p \, dx}{\int_\Omega |u - \tilde{u}|^p \; m dx},$$

since $\mathcal{M}_0 \setminus \{0\} = \{u - \tilde{u} : u \in W^{1,p}(\Omega),\, u \notin \operatorname{span}\{1\}\}$. Thus, an alternative expression for $\hat{\lambda}(m)$ reads as follows

$$\hat{\lambda}(m) = \inf_{u \notin \operatorname{span}\{1\}} \frac{\int_\Omega |\nabla u|^p \, dx}{V_p(u)}.$$

Next we observe that the quotient can be differentiated regardless the value of $p > 1$. Accordingly, by using the expression for the Gâteaux derivative of V_p stated in Lemma 2.5 (Corollary 2.6), evaluated at a minimizer u_1 and in the direction $v \in W^{1,p}(\Omega)$ we easily arrive to

$$p \int_\Omega |\nabla u_1|^{p-2} \nabla u_1 \nabla v \, dx = p\hat{\lambda}(m) \langle DV_p(u), v \rangle,$$

equivalently,

$$\int_\Omega |\nabla u_1|^{p-2} \nabla u_1 \nabla v \, dx = \hat{\lambda}(m) \int_\Omega |u_1 - \tilde{u}_1|^{p-2} (u_1 - \tilde{u}_1) v \, m dx.$$

Since $v \in W^{1,p}(\Omega)$ is arbitrary, this means that $\hat{\lambda}(m)$ is an eigenvalue with associated eigenfunction $u_1 - \tilde{u}_1$. $\square$

Remark 3.2. An alternative proof of the fact that $\hat{\lambda}(m)$ is an eigenvalue can be given in the case $p \geq 2$. In this case, Lagrange's multiplier rule can be employed in a standard way. Indeed, the constraint defining $\mathcal{M}_0$ involves the functional $\mathcal{I}(v) = \int_\Omega |v|^{p-2} v$ which is C^1 provided $p \geq 2$.

4. A further subcritical problem

The results of Section 2 can be still employed to study the following nonlinear problems

$$\begin{gathered} -\Delta_p u = \lambda m(x) |u|^{q-2} u \quad x \in \Omega \\ |\nabla u|^{p-2} \frac{\partial u}{\partial \nu} = 0 \quad x \in \partial\Omega, \end{gathered} \tag{4.1}$$

where

$$1 \leq q < p^*, \tag{4.2}$$

with $p^* = \frac{pN}{N-p}$ if $p < N$, $p^* = \infty$ otherwise. In addition, $m \in L^r(\Omega)$ is positive a.e. in Ω and

$$r \begin{cases} > \left(\frac{p^*}{q}\right)' = \frac{Np}{N(p-q)+pq} & \text{if } 1 < p < N, \\ > 1 & \text{if } p = N, \\ = 1 & \text{if } p > N. \end{cases} \tag{4.3}$$

Theorem 4.1. *Assume that the exponents q and r satisfy* (4.2) *and* (4.3)*, respectively. Then problem* (4.1) *admits a nontrivial (nonconstant) solution $u \in W^{1,p}(\Omega)$ if and only if $\lambda > 0$. These nontrivial solutions satisfy the average condition*

$$\int_\Omega |u|^{q-2} u \, m dx = 0. \tag{4.4}$$

Proof. Solutions $u \in W^{1,p}(\Omega)$ are understood in weak sense, that is, equality

$$\int_\Omega |\nabla u|^{p-2} \nabla u \nabla v \, dx = \lambda \int_\Omega |u|^{q-2} u v \, m dx, \tag{4.5}$$

holds for all $v \in W^{1,p}(\Omega)$. Observe that the latter integrand lays in $L^1(\Omega)$ because of conditions (4.3) satisfied by m. Thus, it is readily deduced that nonconstant

nontrivial solutions are only possible when $\lambda > 0$ while in addition, every solution must satisfy (4.4) (use $v = 1$ as a test function).

On the other hand, solving (4.1) for a specific $\lambda = \lambda_0 > 0$ amounts to solving it for every $\lambda > 0$. In fact, if u_0 is a non trivial solution for $\lambda = \lambda_0$ then

$$u_\lambda = \Big(\frac{\lambda}{\lambda_0}\Big)^{\frac{1}{p-q}} u_0,$$

defines a solution to (4.1) corresponding to any $\lambda > 0$. Thus it is enough with finding out a solution at some fixed value of $\lambda_0 > 0$.

We now mimic the proof of Theorem 1.1 and minimize $\mathcal{J}(u) = \int_\Omega |\nabla u|^p$ on $\mathcal{M}_{1,q} = \mathcal{M}_{0,q} \cap \big\{u \in W^{1,p}(\Omega) : \int_\Omega |u|^q \ m dx = 1\big\}$, $\mathcal{M}_{0,q}$ being the functions of $W^{1,p}(\Omega)$ satisfying

$$\int_\Omega |u|^{q-2} u \, m dx = 0.$$

By similar reasons as in Section 3, $\mathcal{J}$ achieves a minimum at some $u_1 \in \mathcal{M}_{1,q}$ and so

$$\begin{aligned} 0 < \mu_1 := \mathcal{J}(u_1) &= \inf_{\mathcal{M}_{1,q}} \mathcal{J} \\ &= \inf_{u \in \mathcal{M}_{0,q} \setminus \{0\}} \frac{\int_\Omega |\nabla u|^p \, dx}{\big(\int_\Omega |u|^q \ m dx\big)^{p/q}} = \inf_{u \notin \operatorname{span}\{1\}} \frac{\int_\Omega |\nabla u|^p \, dx}{V_q(u)^{p/q}}. \end{aligned} \tag{4.6}$$

In the above expression we used that $\mathcal{M}_{0,q} \setminus \{0\} = \{u - \tilde{u}_q : u \in W^{1,p}(\Omega),\ u \notin \operatorname{span}\{1\}\}$ where $\tilde{u}_q$ means the average of u relative to $L^q(\Omega, m dx)$ (Section 2). By taking the directional derivative of the latter quotient at $u = u_1$ in the direction $v \in W^{1,p}(\Omega)$, and then equaling to zero we obtain

$$\langle -\Delta_p u_1, v\rangle = V_q(u_1)^{\frac{p}{q}-1} \mu_1 \big\langle \frac{1}{q} DV_q(u_1), v\big\rangle,$$

for all $v \in W^{1,p}(\Omega)$. By using Lemma 2.5 this implies that $u_0 = u_1 - \widetilde{u_1}$ solves (4.1) for the special value:

$$\lambda_0 = V_q(u_1)^{\frac{p}{q}-1} \mu_1.$$

As already pointed out this permits us obtaining a solution for any positive value $\lambda > 0$. □

Remark 4.2. As in the case of Poincaré's inequality (1.6), $\mu_1^{-\frac{1}{q}}$ defines the optimum constant C in the inequality:

$$\inf_{t \in \mathbb{R}} \|u - t\|_{L^q(\Omega, m dx)} \le C \| |\nabla u| \|_{L^p(\Omega)}, \tag{4.7}$$

associated with the embedding $W^{1,p}(\Omega) \subset L^q(\Omega, m dx)$. See [18, Lemma V.2.3–2] for the case $m = 1$.

5. p-Laplacian on graphs

An order n graph $\mathcal{G}$ is defined through a couple $(\mathcal{V}, E)$ where $\mathcal{V} = \{v_1, \dots, v_n\}$ is a set with $n \ge 2$ elements, the *vertices* of the graph, together with a family E of two–elements subsets $e = \{u, v\}$ of $\mathcal{V}$. Members of E define the *edges* of $\mathcal{G}$ and it is said that u is adjacent (or 'connected') to v when $\{u, v\} \in E$. It is often convenient to associate a weight $\omega > 0$ to every edge $e = \{u, v\} \in E$, which could be understood as the 'connection intensity' between the vertices u and v. Observe that no order is prescribed in the edges. A possible way of simultaneously defining both

the edges and their weights consists in introducing the weights matrix $A = (\omega_{ij})$ of $\mathcal{G}$. Such a matrix A is always chosen nonnegative and symmetric, $\omega_{ij} = \omega_{ji}$, $\omega_{ij} \geq 0$, while $\omega_{ij} > 0$ both means that $\{v_i, v_j\} \in E$ and that has weight ω_{ij} as an edge of E. Observe that, from the definition of E, $\omega_{ii} = 0$ for every $1 \leq i \leq n$. Matrix A is termed as the 'adjacency' matrix when the weights $\omega_{ij} \in \{0, 1\}$.

A subset $\{v_{i_0}, v_{i_1}, \dots, v_{i_m}\}$ of $m+1$ vertices so that $\{v_{i_{k-1}}, v_{i_k}\} \in E$ for $k = 1, \dots, m$ is defined to be a *path* of length m connecting v_{i_0} to v_{i_m}. Accordingly, a graph $\mathcal{G}$ is said to be *connected* if every couple of distinct vertices $x, y \in \mathcal{V}$ can be joined through a path.

Let us next define the p-Laplacian operator $-\Delta_p$ on $\mathcal{G}$. As pointed out at the end of Section 2, the set $\mathcal{H}$ of real functions $f : \mathcal{V} \to \mathbb{R}$ can be identified with $\mathbb{R}^n$ ($f = (f_i)$, $f_i = f(i)$). The p-Laplacian in $\mathcal{G}$ is defined as a mapping from $\mathcal{H}$ into itself according the next definition. The eigenvalue problem for $-\Delta_p$ is also introduced there. Warning: to keep the similarities with the partial differential equations setting, a minus sign preceding the operator is employed.

Definition 5.1. Let $\mathcal{G} = (\mathcal{V}, E)$ be a graph with weights matrix $A = (\omega_{ij})$. The p-Laplacian operator $-\Delta_p : \mathcal{H} \to \mathcal{H}$ is defined as

$$-\Delta_p(f)(i) = \sum_j \omega_{ij} |f_i - f_j|^{p-2}(f_i - f_j), \quad 1 \leq i \leq n. \tag{5.1}$$

It is said that $\lambda \in \mathbb{R}$ is an eigenvalue to $-\Delta_p$ with associated eigenvector $f \in \mathcal{H} \backslash \{0\}$ provided that

$$-\Delta_p(f) = \lambda \nu \phi_p(f), \tag{5.2}$$

where $\nu = (\nu_i) \in \mathbb{R}^n_+$ is a given weight function, $\phi_p(f)(i) = |f_i|^{p-2} f_i$, $1 \leq i \leq n$.

Remark 5.2. When $p = 2$, $-\Delta_p$ becomes the linear operator (the Laplacian in $\mathcal{G}$):

$$-\Delta_2 f = (D - A) f,$$

A being the weights matrix of $\mathcal{G}$ and $D = \operatorname{diag}(d_1, \dots, d_n)$ where $d_i = \sum_j \omega_{ij}$. In this case the spectrum of $-\Delta_2$ corresponding to the weight $\nu = 1$ consists of the eigenvalues of $D - A$. In some cases the interest is focussed on the *normalized* eigenvalues of $-\Delta_2$ (respectively, $-\Delta_p$). These are the eigenvalues corresponding to the choice $\nu = (d_i)$ as a weight function in (5.2) (see [10, 8]).

Proof of Theorem 1.2. We first review some preliminary well known features [2, 8]. By using the Euclidean scalar product $\langle \cdot, \cdot \rangle_2$ of $\mathbb{R}^n$ it is found that

$$\begin{aligned} \langle -\Delta_p(f), f \rangle_2 &= \sum_i \sum_j \omega_{ij} |f_i - f_j|^{p-2}(f_i - f_j) f_i \\ &= -\sum_i \sum_j \omega_{ij} |f_i - f_j|^{p-2}(f_i - f_j) f_j, \end{aligned}$$

which implies

$$\langle -\Delta_p(f), f \rangle_2 = \frac{1}{2} \sum_{i,j} \omega_{ij} |f_i - f_j|^p =: \mathcal{D}(f). \tag{5.3}$$

As a consequence, all possible eigenvalues λ of $-\Delta_p$ must be nonnegative since the weight function ν is positive. Another implication of (5.3) is the fact that $\lambda = 0$ is a simple eigenvalue since its set of associated eigenfunctions is $\operatorname{span}\{1\}$, 1 standing for the function $f = 1$. In this regard, the connectedness of $\mathcal{G}$ is employed to show that for every associated eigenfunction f it holds $f_i = f_j$ for every couple of indices

$1 \le i, j \le n$. In fact, the constant value $f = f_i$ is 'propagated' through the path connecting v_i to v_j.

Next observe that

$$\langle -\Delta_p(f), 1\rangle_2 = \sum_i (-\Delta_p(f))_i = \sum_i \sum_j \omega_{ij} |f_i - f_j|^{p-2}(f_i - f_j) = 0.$$

That is why any eigenfunction f associated with an eigenvalue $\lambda \neq 0$ must satisfy the condition equivalent to (1.5), i. e.,

$$\sum_i |f_i|^{p-2} f_i \nu_i = 0. \tag{5.4}$$

Set now $\mathcal{M}_1 = \big\{ f \in \mathbb{R}^n : f \text{ satisfies (5.4) and } \sum_i |f_i|^p \nu_i = 1 \big\}$. By compactness, function $\mathcal{D}$ achieves its minimum at some $f_1 \in \mathcal{M}_1$. In addition,

$$0 < \hat{\lambda} := \mathcal{D}(f_1) = \min_{f \notin \operatorname{span}\{1\}} \frac{\mathcal{D}(f)}{V_p(f)},$$

since

$$\big\{ f \in \mathbb{R}^n : \sum |f_i|^{p-2} f_i \nu_i = 0 \big\} \setminus \{0\} = \big\{ f - \tilde{f} : f \in \mathbb{R}^n \setminus \operatorname{span}\{1\} \big\}.$$

Next observe that

$$\nabla \mathcal{D}(f) = p(-\Delta_p(f)).$$

Hence, it readily follows by differentiating $\frac{\mathcal{D}(u)}{V_p(u)}$ and employing (2.5) that

$$-\Delta_p(f_1 - \tilde{f}_1) = \hat{\lambda} \nu \phi_p(f_1 - \tilde{f}_1).$$

This means that $\hat{\lambda}$ defines the second eigenvalue λ_2. Observe that Corollary 2.7 has permitted us handling both the case $p \ge 2$ and the singular one $1 < p < 2$.

By arguing in the same way one obtains that

$$\lambda^* := \max_{f \notin \operatorname{span}\{1\}} \frac{\mathcal{D}(f)}{V_p(f)},$$

constitutes the maximum eigenvalue of $-\Delta_p$ in $\mathcal{G}$. □

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