Electronic Journal of Differential Equations, Vol. 2022 (2022), No. 14, pp. 1–10.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu

COMPACTNESS OF THE SET OF SOLUTIONS TO ELLIPTIC EQUATIONS IN 2 DIMENSIONS

SAMY SKANDER BAHOURA

Abstract. We study the behavior of solutions to elliptic equations in 2 dimensions. In particular, we show that the set of solutions is compact under a Lipschitz condition.

1. Introduction

Let us define the operator

$$e^{L}_{\epsilon} := \Delta + \epsilon(x_1\partial_1 + x_2\partial_2) = \frac{\operatorname{div}[a_\epsilon(x)\nabla]}{a_\epsilon(x)}, \quad \text{with} \quad a_\epsilon(x) = e^{\epsilon|x|^2/2}.$$

We consider the equation

$$\begin{gathered} -\Delta u - \epsilon(x_1\partial_1 u + x_2\partial_2 u) = -L_\epsilon u = Ve^u \quad \text{in } \Omega \subset \mathbb{R}^2, \\ u = 0 \quad \text{in } \partial\Omega\,, \end{gathered} \tag{1.1}$$

where Ω is a starshaped set, $u \in W_0^{1,1}(\Omega)$, $e^u \in L^1(\Omega)$, $0 \le V \le b$, $1 \ge \epsilon \ge 0$.

For $\epsilon = 0$ equation (1.1) has been studied by many authors with and without the boundary condition. This equation also has been studied in Riemann surfaces; see [1]–[20], where one can find some existence and compactness results. Also we have a nice formulation in the sense of the distributions of this problem in [7]. Among the known results we find the following Theorem.

Theorem 1.1 (Brezis-Merle [6])**.** *If (u_i) and (V_i) are two sequences of functions in problem (1.1) with $\epsilon = 0$, and*

$$0 < a \le V_i \le b < +\infty,$$

then for all compact subset K of Ω it holds

$$\sup_K u_i \le c,$$

with c depending on a, b, K and Ω.

We can find an interior estimate if we assume $a = 0$, but we need an assumption on the integral of e^{u_i}.

2020 *Mathematics Subject Classification.* 35J60, 35B44, 35B45.
Key words and phrases. Blow-up; compactness; boundary; elliptic equation; Lipschitz condition; starshaped domain.

Submitted September 4, 2021. Published February 21, 2022.

Theorem 1.2 (Brezis-Merle [6]). *Let (u_i) and (V_i) two sequences of functions in problem* (1.1) *with*

$$0 \le V_i \le b < +\infty \quad and \quad \int_\Omega e^{u_i} dy \le C.$$

Then, for all compact subset K of Ω it holds

$$\sup_K u_i \le c,$$

with c depending on b, C, K and Ω.

The condition $\int_\Omega e^{u_i} dy \le C$ is a necessary in Problem (1.1) as showed by the following statement for $\epsilon = 0$.

Theorem 1.3 (Brezis-Merle [6]). *There are sequences (u_i) and (V_i) in problem* (1.1) *with*

$$0 \le V_i \le b < +\infty, \quad \int_\Omega e^{u_i} dy \le C,$$

such that $\sup_\Omega u_i \to +\infty$.

To obtain Theorems 1.1 and 1.2 Brezis and Merle used an inequality [6, Theorem 1] obtained by an approximation argument, Fatou's lemma, and the maximum principle in $W_0^{1,1}(\Omega)$, which arises from Kato's inequality. Also this weak form of the maximum principle is used to prove the local uniform boundedness result by comparing a certain function and the Newtonian potential. We refer the reader to [5] for information about the weak form of the maximum principle.

Note that for problem (1.1), by using the Pohozaev identity, we can prove that $\int_\Omega e^{u_i}$ is uniformly bounded when $0 < a \le V_i \le b < +\infty$, $\|\nabla V_i\|_{L^\infty} \le A$, and Ω starshaped. When $a = 0$ and $\nabla \log V_i$ is uniformly bounded, we can find a uniform bound for $\int_\Omega V_i e^{u_i}$.

Ma-Wei [17] proved that those results remain valid for all open sets not necessarily starshaped when $a > 0$. Chen-Li [9] proved that if $a = 0$, $\int_\Omega e^{u_i}$ is uniformly bounded, and $\nabla \log V_i$ is uniformly bounded, then (u_i) is bounded near the boundary and we have directly the compactness result for the problem (1.1). Ma-Wei [17] extend this result in the case where $a > 0$.

When $\epsilon = 0$ and if we assume V more regular we can have another type of estimates called $\sup + \inf$ type inequalities. It was proved by Shafrir [19] that, if $(u_i), (V_i)$ are two sequences of solutions to Problem (1.1), without assumption on the boundary and $0 < a \le V_i \le b < +\infty$, then it holds

$$C\Big(\frac{a}{b}\Big) \sup_K u_i + \inf_\Omega u_i \le c = c(a, b, K, \Omega).$$

We find in [10] the explicit value $C(a/b) = \sqrt{a/b}$. In his proof, Shafrir [19] used the blow-up function, the Stokes formula and an isoperimetric inequality. Chen-Lin [10] used the blow-up analysis combined with some geometric type inequality for obtaining the integral curvature.

Now, if we assume (V_i) is uniformly Lipschitzian with constant A, then $C(a/b) = 1$ and $c = c(a, b, A, K, \Omega)$ see Brezis-Li-Shafrir [4]. This result was extended for Hölderian sequences (V_i) by Chen-Lin [10]. Also we have in [15], an extension of the Brezis-Li-Shafrir result to compact Riemannian surfaces without boundary. One can see in [17] an explicit form, ($8\pi m, m \in \mathbb{N}^*$ exactly), for the numbers in front of the Dirac masses when the solutions blow-up. Here the notion of isolated blow-up point is used. Also one can find in [11] refined estimates near the isolated blow-up points and the bubbling behavior of the blow-up sequences.

Here we study the behavior of the blow-up points on the boundary, and give a compactness result with Lipschitz condition. Note that our problem is an extension of the Brezis-Merle Problem.

Brezis-Merle Problem [6]**.** Suppose that $V_i \to V$ in $C^0(\bar{\Omega})$ with $0 \leq V_i$, and consider a sequence of solutions (u_i) of (1.1) relative to (V_i) such that

$$\int_\Omega e^{u_i}\,dx \leq C.$$

Is it possible to have

$$\|u_i\|_{L^\infty} \leq C = C(b, C, V, \Omega)\,?$$

Here we give a blow-up analysis on the boundary when V_i are nonnegative and bounded (similar to the prescribed curvature when $\epsilon = 0$). On the other hand, if we add the assumption that these functions (similar to the prescribed curvature) are uniformly Lipschitzian, we have a compactness of the solutions of problem (1.1) for ϵ small enough. (In particular we can take a sequence of ϵ_i tending to 0).

For the behavior of the blow-up points on the boundary, the following condition is sufficient,

$$0 \leq V_i \leq b,$$

The condition $V_i \to V$ in $C^0(\bar{\Omega})$ is not necessary. But for the compactness of the solutions we add the condition

$$\|\nabla V_i\|_{L^\infty} \leq A.$$

Our main results read as follows.

Theorem 1.4. *Assume that* $\max_\Omega u_i \to +\infty$*, where* (u_i) *are solutions of* (1.1) *with* $\epsilon = \epsilon_i$ *and*

$$0 \leq V_i \leq b, \quad \int_\Omega e^{u_i}\,dx \leq C, \quad \epsilon_i \to 0\,.$$

Then, after passing to a subsequence, there are a function u*, a number* $N \in \mathbb{N}$*, and* N *points* $x_1, \dots, x_N \in \partial\Omega$*, such that*

$$\partial_\nu u_i \to \partial_\nu u + \sum_{j=1}^N \alpha_j \delta_{x_j}, \quad \alpha_j \geq 4\pi,$$

in the sense of measures on $\partial\Omega$*, and*

$$u_i \to u \quad \text{in } C^1_{\rm loc}(\bar{\Omega} - \{x_1, \dots, x_N\}).$$

Theorem 1.5. *Assume that* (u_i) *are solutions of* (1.1) *with* $\epsilon = \epsilon_i$*, and*

$$0 \leq V_i \leq b, \quad \|\nabla V_i\|_{L^\infty} \leq A, \quad \int_\Omega e^{u_i} \leq C, \quad \epsilon_i \to 0.$$

Then

$$\|u_i\|_{L^\infty} \leq c(b, A, C, \Omega)\,.$$

2. Proofs of main results

Proof of Theorem 1.4. First we remark that

$$\begin{gathered} -\Delta u_i = \epsilon_i(x_1\partial_1 u_i + x_2\partial_2 u_i) + V_i e^{u_i} \in L^1(\Omega) \quad \text{in } \Omega \subset \mathbb{R}^2, \\ u_i = 0 \quad \text{in } \partial\Omega. \end{gathered} \tag{2.1}$$

and $u_i \in W_0^{1,1}(\Omega)$.

By [6, Corollary 1] we have $e^{u_i} \in L^k(\Omega)$ for all $k > 2$ and the elliptic estimates of Agmon and the Sobolev embedding see [1] imply that

$$u_i \in W^{2,k}(\Omega) \cap C^{1,\epsilon}(\bar{\Omega}).$$

Also we remark that for two positive constants $C_q = C(q, \Omega)$ and $C_1 = C_1(\Omega)$, we have

$$\|\nabla u_i\|_{L^q} \le C_q \|\Delta u_i\|_{L^1} \le (C'_q + \epsilon C_1 \|\nabla u_i\|_{L^1}), \quad \forall i \text{ and } 1 < q < 2.$$

(see [7]). Thus, if $\epsilon > 0$ is small enough and by Holder's inequality,

$$\|\nabla u_i\|_{L^q} \le C''_q, \quad \forall i \text{ and } 1 < q < 2.$$

Step 1: Interior estimate. First we consider the equation

$$\begin{gathered} -\Delta w_i = \epsilon_i(x_1\partial_1 u_i + x_2\partial_2 u_i) \in L^q, \quad 1 < q < 2 \quad \text{in } \Omega \subset \mathbb{R}^2, \\ w_i = 0 \quad \text{in } \partial\Omega. \end{gathered} \tag{2.2}$$

If we consider v_i as the Newtonnian potential of $\epsilon_i(x_1\partial_1 u_i + x_2\partial_2 u_i)$, we have

$$v_i \in C^0(\bar{\Omega}), \quad \Delta(w_i - v_i) = 0.$$

By the maximum principle $w_i - v_i \in C^0(\bar{\Omega})$ and thus $w_i \in C^0(\bar{\Omega})$.

Also we have by elliptic estimates that $w_i \in W^{2,1+\epsilon} \subset L^\infty$, and we can write the equation of the Problem as

$$\begin{gathered} -\Delta(u_i - w_i) = \tilde{V}_i e^{u_i - w_i} \quad \text{in } \Omega \subset \mathbb{R}^2, \\ u_i - w_i = 0 \quad \text{in } \partial\Omega, \end{gathered} \tag{2.3}$$

with

$$0 \le \tilde{V}_i = V_i e^{w_i} \le \tilde{b}, \quad \int_\Omega e^{u_i - w_i} \le \tilde{C}.$$

We apply the Brezis-Merle theorem to $u_i - w_i$ to have $u_i - w_i \in L^\infty_{\rm loc}(\Omega)$, and, thus $u_i \in L^\infty_{\rm loc}(\Omega)$.

Step2: Boundary estimate. Let $\partial_\nu u_i$ be the inner derivative of u_i. By the maximum principle $\partial_\nu u_i \ge 0$. Then we have

$$\int_{\partial\Omega} \partial_\nu u_i d\sigma \le C.$$

We have the existence of a nonnegative Radon measure μ such that

$$\int_{\partial\Omega} \partial_\nu u_i \phi d\sigma \to \mu(\phi), \quad \forall \phi \in C^0(\partial\Omega).$$

We take an $x_0 \in \partial\Omega$ such that $\mu(x_0) < 4\pi$. Set $B(x_0, \epsilon) \cap \partial\Omega := I_\epsilon$. We choose a function η_ϵ such that

$$\begin{gathered} \eta_\epsilon \equiv 1, \quad \text{on } I_\epsilon,\ 0 < \epsilon < \delta/2, \\ \eta_\epsilon \equiv 0, \quad \text{outside } I_{2\epsilon}, \end{gathered}$$

$$0 \le \eta_\epsilon \le 1,$$
$$\|\nabla \eta_\epsilon\|_{L^\infty(I_{2\epsilon})} \le \frac{C_0(\Omega, x_0)}{\epsilon}.$$

We take a $\tilde{\eta}_\epsilon$ such that

$$-\Delta \tilde{\eta}_\epsilon = 0 \quad \text{in } \Omega \subset \mathbb{R}^2,$$
$$\tilde{\eta}_\epsilon = \eta_\epsilon \quad \text{in } \partial\Omega.$$

Remark 2.1. We use the following steps in the construction of $\tilde{\eta}_\epsilon$, taking a cutoff function η_0 in $B(0,2)$ or in $B(x_0,2)$:
(1) We set $\eta_\epsilon(x) = \eta_0(|x-x_0|/\epsilon)$ in the case of the unit disk it is sufficient.
(2) Or, in the general case: we use a chart $(f, \tilde{\Omega})$ with $f(0) = x_0$ and we take $\mu_\epsilon(x) = \eta_0(f(|x|/\epsilon))$ to have connected sets I_ϵ and we take $\eta_\epsilon(y) = \mu_\epsilon(f^{-1}(y))$. Because f, f^{-1} are Lipschitz, $|f(x)-x_0| \le k_2|x| \le 1$ for $|x| \le 1/k_2$ and $|f(x)-x_0| \ge k_1|x| \ge 2$ for $|x| \ge 2/k_1 > 1/k_2$, the support of η is in $I_{(2/k_1)\epsilon}$.

$$\eta_\epsilon \equiv 1, \quad \text{on } f(I_{(1/k_2)\epsilon}),\ 0 < \epsilon < \delta/2,$$
$$\eta_\epsilon \equiv 0, \quad \text{outside } f(I_{(2/k_1)\epsilon}),$$
$$0 \le \eta_\epsilon \le 1,$$
$$\|\nabla \eta_\epsilon\|_{L^\infty(I_{(2/k_1)\epsilon})} \le \frac{C_0(\Omega, x_0)}{\epsilon}.$$

(3) Also, we can take: $\mu_\epsilon(x) = \eta_0(|x|/\epsilon)$ and $\eta_\epsilon(y) = \mu_\epsilon(f^{-1}(y))$, we extend it by 0 outside $f(B_1(0))$. We have $f(B_1(0)) = D_1(x_0)$, $f(B_\epsilon(0)) = D_\epsilon(x_0)$ and $f(B_\epsilon^+) = D_\epsilon^+(x_0)$ with f and f^{-1} smooth diffeomorphism.

$$\eta_\epsilon \equiv 1, \quad \text{on the connected set } J_\epsilon = f(I_\epsilon),\ 0 < \epsilon < \delta/2,$$
$$\eta_\epsilon \equiv 0, \quad \text{outside } J'_\epsilon = f(I_{2\epsilon}),$$
$$0 \le \eta_\epsilon \le 1,$$
$$\|\nabla \eta_\epsilon\|_{L^\infty(J'_\epsilon)} \le \frac{C_0(\Omega, x_0)}{\epsilon}.$$

And $H_1(J'_\epsilon) \le C_1 H_1(I_{2\epsilon}) = C_1 4\epsilon$, because f is Lipschitz. Here H_1 is the Hausdorff measure. We solve the Dirichlet Problem

$$\Delta \bar{\eta}_\epsilon = \Delta \eta_\epsilon \quad \text{in } \Omega \subset \mathbb{R}^2,$$
$$\bar{\eta}_\epsilon = 0 \quad \text{in } \partial\Omega.$$

and finally we set $\tilde{\eta}_\epsilon = -\bar{\eta}_\epsilon + \eta_\epsilon$. Also, by the maximum principle and the elliptic estimates we have

$$\|\nabla \tilde{\eta}_\epsilon\|_{L^\infty} \le C(\|\eta_\epsilon\|_{L^\infty} + \|\nabla \eta_\epsilon\|_{L^\infty} + \|\Delta \eta_\epsilon\|_{L^\infty}) \le \frac{C_1}{\epsilon^2},$$

with C_1 depending on Ω.

As we said in the beginning, see also [3, 7, 13, 20], we have

$$\|\nabla u_i\|_{L^q} \le C_q, \quad \forall i,\ 1 < q < 2.$$

We deduce from the above estimate that, (u_i) converge weakly in $W_0^{1,q}(\Omega)$, almost everywhere to a function $u \ge 0$ and $\int_\Omega e^u < +\infty$ (by Fatou lemma). Also, V_i

weakly converge to a nonnegative function V in L^∞. The function u is in $W_0^{1,q}(\Omega)$ solution of

$$-\Delta u = Ve^u \in L^1(\Omega) \quad \text{in } \Omega \subset \mathbb{R}^2,$$
$$u = 0 \quad \text{in } \partial\Omega.$$

According to [6, Ccorollary 1], we have $e^{ku} \in L^1(\Omega), k > 1$. By the elliptic estimates, we have $u \in W^{2,k}(\Omega) \cap C^{1,\epsilon}(\bar{\Omega})$.

We denote by $f \cdot g$ the inner product of any two vectors f and g of $\mathbb{R}^2$. Then we can write

$$-\Delta((u_i - u)\tilde{\eta}_\epsilon) = (V_i e^{u_i} - Ve^u)\tilde{\eta}_\epsilon - 2\nabla(u_i - u) \cdot \nabla\tilde{\eta}_\epsilon + \epsilon_i(\nabla u_i \cdot x)\tilde{\eta}_\epsilon. \tag{2.4}$$

We use the interior estimate in Brezis-Merle [6].

Step 1: Estimate of the integral of the first term of the right-hand side of (2.4). We use Green's formula between $\tilde{\eta}_\epsilon$ and u, to obtain

$$\int_\Omega Ve^u \tilde{\eta}_\epsilon \, dx = \int_{\partial\Omega} \partial_\nu u \eta_\epsilon \le C\epsilon = O(\epsilon) \tag{2.5}$$

then we have

$$-\Delta u_i - \epsilon_i \nabla u_i \cdot x = V_i e^{u_i} \quad \text{in } \Omega \subset \mathbb{R}^2,$$
$$u = 0 \quad \text{in } \partial\Omega.$$

We use Green's formula between u_i and $\tilde{\eta}_\epsilon$ to have

$$\begin{aligned}\int_\Omega V_i e^{u_i} \tilde{\eta}_\epsilon \, dx &= \int_{\partial\Omega} \partial_\nu u_i \eta_\epsilon d\sigma - \epsilon_i \int_\Omega (\nabla u_i \cdot x)\tilde{\eta}_\epsilon \\ &= \int_{\partial\Omega} \partial_\nu u_i \eta_\epsilon d\sigma + o(1) \\ &\to \mu(\eta_\epsilon) \le \mu(J'_\epsilon) \le 4\pi - \epsilon_0, \quad \epsilon_0 > 0\end{aligned} \tag{2.6}$$

From (2.5) and (2.6) we have that for all $\epsilon > 0$ there is i_0 such that, for $i \ge i_0$,

$$\int_\Omega |(V_i e^{u_i} - Ve^u)\tilde{\eta}_\epsilon| \, dx \le 4\pi - \epsilon_0 + C\epsilon \tag{2.7}$$

Step 2.1: Estimate of integral of the second term of the right hand side of (2.4). Let $\Sigma_\epsilon = \{x \in \Omega, d(x, \partial\Omega) = \epsilon^3\}$ and $\Omega_{\epsilon^3} = \{x \in \Omega, d(x, \partial\Omega) \ge \epsilon^3\}$, $\epsilon > 0$. Then, for ϵ small enough, Σ_ϵ is an hypersurface.

The measure of $\Omega - \Omega_{\epsilon^3}$ is $k_2\epsilon^3 \le meas(\Omega - \Omega_{\epsilon^3}) = \mu_L(\Omega - \Omega_{\epsilon^3}) \le k_1\epsilon^3$.

Remark 2.2. For the unit ball $\bar{B}(0,1)$, our new manifold is $\bar{B}(0, 1 - \epsilon^3)$. To prove this fact, we consider consider $d(x, \partial\Omega) = d(x, z_0), z_0 \in \partial\Omega$, which implies that $(d(x, z_0))^2 \le (d(x, z))^2$ for all $z \in \partial\Omega$. This is equivalent to $(z - z_0) \cdot (2x - z - z_0) \le 0$ for all $z \in \partial\Omega$. Let us consider a chart around z_0 and $\gamma(t)$ a curve in $\partial\Omega$, we have $(\gamma(t) - \gamma(t_0) \cdot (2x - \gamma(t) - \gamma(t_0)) \le 0$ if we divide by $(t - t_0)$ (with the sign and tend t to t_0), we have $\gamma'(t_0) \cdot (x - \gamma(t_0)) = 0$. This implies that $x = z_0 - s\nu_0$ where ν_0 is the outward normal of $\partial\Omega$ at z_0)

From the above remark, we can say that

$$S = \{x, d(x, \partial\Omega) \le \epsilon\} = \{x = z_0 - s\nu_{z_0}, z_0 \in \partial\Omega, \ -\epsilon \le s \le \epsilon\}.$$

It is sufficient to work on $\partial\Omega$. Let us consider charts $(z, D = B(z, 4\epsilon_z), \gamma_z)$ with $z \in \partial\Omega$ such that $\cup_z B(z, \epsilon_z)$ is cover of $\partial\Omega$. One can extract a finite cover

$(B(z_k,\epsilon_k)), k=1,\dots,m$, by the area formula the measure of $S\cap B(z_k,\epsilon_k)$ is less than a $k\epsilon$ (a ϵ-rectangle). For the reverse inequality, it is sufficient to consider one chart around one point of the boundary). We write

$$\int_\Omega |\nabla(u_i-u)\cdot\nabla\tilde\eta_\epsilon|\,dx = \int_{\Omega_{\epsilon^3}} |\nabla(u_i-u)\cdot\nabla\tilde\eta_\epsilon|\,dx + \int_{\Omega-\Omega_{\epsilon^3}} |\nabla(u_i-u)\cdot\nabla\tilde\eta_\epsilon|\,dx. \tag{2.8}$$

Step 2.1.1: Estimate of $\int_{\Omega-\Omega_{\epsilon^3}} |\nabla(u_i-u)\cdot\nabla\tilde\eta_\epsilon|\,dx$. First, we know from elliptic estimates that $\|\nabla\tilde\eta_\epsilon\|_{L^\infty}\le C_1/\epsilon^2$, C_1 depends on Ω.

We know that $(|\nabla u_i|)_i$ is bounded in L^q, $1<q<2$, we can extract from this sequence a subsequence which converge weakly to $h\in L^q$. But, we know that we have locally the uniform convergence to $|\nabla u|$ (by Brezis-Merle's theorem), then, $h=|\nabla u|$ a.e. Let q' be the conjugate of q.

We have that for all $f\in L^{q'}(\Omega)$,

$$\int_\Omega |\nabla u_i| f\,dx \to \int_\Omega |\nabla u| f\,dx$$

If we take $f=1_{\Omega-\Omega_{\epsilon^3}}$, for each $\epsilon>0$ there exists $i_1=i_1(\epsilon)\in\mathbb{N}$, such that $i\ge i_1$ implies

$$\int_{\Omega-\Omega_{\epsilon^3}} |\nabla u_i| \le \int_{\Omega-\Omega_{\epsilon^3}} |\nabla u| + \epsilon^3.$$

Then, for $i\ge i_1(\epsilon)$,

$$\int_{\Omega-\Omega_{\epsilon^3}} |\nabla u_i| \le \mathrm{meas}(\Omega-\Omega_{\epsilon^3})\|\nabla u\|_{L^\infty}+\epsilon^3 = \epsilon^3(k_1\|\nabla u\|_{L^\infty}+1)=O(\epsilon^3).$$

Thus, we obtain

$$\int_{\Omega-\Omega_{\epsilon^3}} |\nabla(u_i-u)\cdot\nabla\tilde\eta_\epsilon|\,dx \le \epsilon C_1(2k_1\|\nabla u\|_{L^\infty}+1)=O(\epsilon) \tag{2.9}$$

The constant C_1 does not depend on ϵ but on Ω.

Step 2.1.2: Estimate of $\int_{\Omega_{\epsilon^3}} |\nabla(u_i-u)\cdot\nabla\tilde\eta_\epsilon|\,dx$. We know that, $\Omega_\epsilon\subset\subset\Omega$, and (because of Brezis-Merle's interior estimates) $u_i\to u$ in $C^1(\Omega_{\epsilon^3})$. We have

$$\|\nabla(u_i-u)\|_{L^\infty(\Omega_{\epsilon^3})}\le\epsilon^3, \text{ for } i\ge i_3.$$

We write

$$\int_{\Omega_{\epsilon^3}} |\nabla(u_i-u)\cdot\nabla\tilde\eta_\epsilon|\,dx \le \|\nabla(u_i-u)\|_{L^\infty(\Omega_{\epsilon^3})}\|\nabla\tilde\eta_\epsilon\|_{L^\infty}=C_1\epsilon=O(\epsilon)$$

for $i\ge i_3$. For $\epsilon>0$, and $i\in\mathbb{N}$, with $i\ge i'$, we have

$$\int_\Omega |\nabla(u_i-u)\cdot\nabla\tilde\eta_\epsilon|\,dx \le \epsilon C_1(2k_1\|\nabla u\|_{L^\infty}+2)=O(\epsilon) \tag{2.10}$$

From (2.7) and (2.10), for $\epsilon>0$, there is i'' such that $i\ge i''$, we have

$$\int_\Omega |\Delta[(u_i-u)\tilde\eta_\epsilon]|dx \le 4\pi-\epsilon_0+\epsilon 2C_1(2k_1\|\nabla u\|_{L^\infty}+2+C)=4\pi-\epsilon_0+O(\epsilon) \tag{2.11}$$

Now we choose $\epsilon>0$ small enough to have a good estimate of (2.4). Indeed, we have

$$-\Delta[(u_i-u)\tilde\eta_\epsilon]=g_{i,\epsilon} \quad textin\Omega\subset\mathbb{R}^2,$$
$$(u_i-u)\tilde\eta_\epsilon=0 \quad \text{in } \partial\Omega.$$

with $\|g_{i,\epsilon}\|_{L^1(\Omega)} \leq 4\pi - \epsilon_0/2$.

We can use [6, Theorem 1] to conclude that there are $q \geq \tilde{q} > 1$ such that

$$\int_{V_\epsilon(x_0)} e^{\tilde{q}|u_i-u|}\,dx \leq \int_\Omega e^{q|u_i-u|\tilde{\eta}_\epsilon}\,dx \leq C(\epsilon,\Omega),$$

where, $V_\epsilon(x_0)$ is a neighborhood of x_0 in $\bar{\Omega}$. Here we have used that in a neighborhood of x_0 by the elliptic estimates, $1 - C\epsilon \leq \tilde{\eta}_\epsilon \leq 1$.

Thus, for each $x_0 \in \partial\Omega - \{\bar{x}_1,\dots,\bar{x}_m\}$ there is $\epsilon_0 > 0, q_0 > 1$ such that

$$\int_{B(x_0,\epsilon_0)} e^{q_0 u_i}\,dx \leq C, \quad \forall i.$$

By elliptic estimates see [14], we have

$$\|u_i\|_{C^{1,\theta}[B(x_0,\epsilon)]} \leq c_3 \quad \forall i.$$

We have proved that there is a finite number of points $\bar{x}_1,\dots,\bar{x}_m$ such that the sequence (u_i) is locally uniformly bounded in $C^{1,\theta}, (\theta > 0)$ on $\bar{\Omega} - \{\bar{x}_1,\dots,\bar{x}_m\}$.

Proof of theorem 1.5. The Pohozaev identity gives

$$\int_{\partial\Omega} \frac{1}{2}(x\cdot\nu)(\partial_\nu u_i)^2 d\sigma + \epsilon \int_\Omega (x\cdot\nabla u_i)^2\,dx + \int_{\partial\Omega}(x\cdot\nu)V_i e^{u_i} d\sigma = \int_\Omega (x\cdot\nabla V_i + 2V_i)e^{u_i}\,dx\,.$$

We use the boundary condition, that Ω is starshaped, and that $\epsilon > 0$ to have

$$\int_{\partial\Omega} (\partial_\nu u_i)^2\,dx \leq c_0(b,A,C,\Omega). \tag{2.12}$$

Thus we can use the weak convergence in $L^2(\partial\Omega)$ to have a subsequence $\partial_\nu u_i$, such that

$$\int_{\partial\Omega} \partial_\nu u_i \phi\,dx \to \int_{\partial\Omega} \partial_\nu u \phi\,dx, \quad \forall \phi \in L^2(\partial\Omega),$$

Thus, $\alpha_j = 0$, $j = 1,\dots,N$ and (u_i) is uniformly bounded.

Remark 2.3. If we assume the open set bounded starshaped and V_i uniformly Lipschitzian and between two positive constants we can bound, by using the inner normal derivative $\int_\Omega e^{u_i}$.

If we assume the open set bounded starshaped and $\nabla \log V_i$ uniformly bounded, by the previous Pohozaev identity (we consider the inner normal derivative) one can bound $\int_\Omega V_i e^{u_i}$ uniformly.

One can consider the problem on the unit ball and an ellipse. These two problems are different, because:

(1) if we use a linear transformation, $(y_1, y_2) = (x_1/a, x_2/b)$, the Laplcian is not invariant under this map.
(2) If we use a conformal transformation, by a Riemann theorem, the quantity $x \cdot \nabla u$ is not invariant under this map.

We can not use, after using those transformations, the Pohozaev identity.

3. A counterexample

We start with the notation of the counterexample of Brezis and Merle. The domain Ω is the unit ball centered in $x_0 = (1, 0)$. Consider z_i (obtained by the variational method), such that

$$-\Delta z_i - \epsilon_i (x - x_0) \cdot \nabla z_i = -\tilde{L}_{\epsilon_i}(z_i) = f_{\epsilon_i},$$

with Dirichlet condition. By the regularity theorem, $z_i \in C^1(\bar{\Omega})$. Then we have

$$\|f_{\epsilon_i}\|_1 = 4\pi A.$$

Thus by the duality theorem of Stampacchia or Brezis-Strauss, we have

$$\|\nabla z_i\|_q \leq C_q,\ 1 \leq q < 2.$$

We solve

$$-\Delta w_i = \epsilon_i (x - x_0) \cdot \nabla z_i,$$

with Dirichlet boundary condition.

By elliptic estimates, $w_i \in C^1(\bar{\Omega})$ and $w_i \in C^0(\bar{\Omega})$ uniformly. By the maximum principle we have

$$z_i - w_i \equiv u_i.$$

Where u_i is the function of the counterexemple of Brezis Merle. Then we write

$$-\Delta z_i - \epsilon_i (x - x_0) \cdot \nabla z_i = f_{\epsilon_i} = V_i e^{z_i}.$$

Thus, we have

$$\int_\Omega e^{z_i} \leq C_1, \quad 0 \leq V_i \leq C_2,$$

$$z_i(a_i) \geq u_i(a_i) - C_3 \to +\infty, \quad a_i \to O\,.$$

To have a counterexample on the unit disk, we do a translation $x \to x - x_0$ in the previous counterexample.

Acknowledgments. The author would like to tank the anonymous reviewers.

References

[1] T. Aubin; *Some Nonlinear Problems in Riemannian Geometry.* Springer-Verlag 1998.

[2] C. Bandle; *Isoperimetric Inequalities and Applications.* Pitman, 1980.

[3] L. Boccardo, T. Gallouet; *Nonlinear elliptic and parabolic equations involving measure data.* J. Funct. Anal. 87 no. 1, (1989), 149–169.

[4] H. Brezis, Y.Y. Li, I. Shafrir; *A sup+inf inequality for some nonlinear elliptic equations involving exponential nonlinearities.* J. Funct. Anal. 115 (1993), 344–358.

[5] H. Brezis, M. Marcus, A. C. Ponce; *Nonlinear elliptic equations with measures revisited. Mathematical aspects of nonlinear dispersive equations*, 55–109, Ann. of Math. Stud., 163, Princeton Univ. Press, Princeton, NJ, 2007.

[6] H. Brezis, F. Merle; *Uniform estimates and Blow-up behavior for solutions of $-\Delta u = V(x)e^u$ in two dimension.* Comm. Part. Diff. Equations, 16 (8 and 9) (1991), 1223–1253.

[7] H. Brezis, W. A. Strauss; *Semi-linear second-order elliptic equations in L1.* J. Math. Soc. Japan 25 (1973), 565–590.

[8] A. Sun-Yung Chang, Matthew J. Gursky, Paul C. Yang; *Scalar curvature equation on* 2- *and* 3-*spheres.* Calc. Var. Partial Differential Equations 1 (1993), no. 2, 205–229.

[9] W. Chen, C. Li; *A priori estimates for solutions to nonlinear elliptic equations.* Arch. Rational. Mech. Anal., 122 (1993) 145–157.

[10] C.-C. Chen, C-S. Lin; *A sharp sup+inf inequality for a nonlinear elliptic equation in* $\mathbb{R}^2$. Comm. Anal. Geom. 6, no.1 (1998), 1–19.

[11] C.-C.Chen, C-S. Lin; *Sharp estimates for solutions of multi-bubbles in compact Riemann surfaces.* Comm. Pure Appl. Math. 55 (2002), no. 6, 728–771.

[12] D. G. De Figueiredo, P. L. Lions, R. D. Nussbaum; *A priori Estimates and Existence of Positive Solutions of Semilinear Elliptic Equations*, J. Math. Pures et Appl., vol 61, 1982, pp.41–63.

[13] W. Ding, J. Jost, J. Li, G. Wang; *The differential equation* $\Delta u = 8\pi - 8\pi h e^u$ *on a compact Riemann surface.* Asian J. Math. 1 (1997), no. 2, 230–248.

[14] D. Gilbarg, N. S, Trudinger; *Elliptic Partial Differential Equations of Second order*, Berlin Springer-Verlag.

[15] Y. Y. Li; *Harnack Type Inequality: the method of moving planes.* Commun. Math. Phys. 200 (1999), 421–444.

[16] Y. Y. Li, I. Shafrir; *Blow-up analysis for solutions of* $-\Delta u = V e^u$ *in dimension two.* Indiana. Math. J. Vol 3, no. 4 (1994). 1255–1270.

[17] L. Ma, J.-C. Wei; *Convergence for a Liouville equation.* Comment. Math. Helv. 76 (2001), 506–514.

[18] K. Nagasaki, T. Suzuki; *Asymptotic analysis for two-dimensional elliptic eigenvalue problems with exponentially dominated nonlinearities.* Asymptotic Anal. 3 (1990), no. 2, 173–188.

[19] I. Shafrir; *A sup+inf inequality for the equation* $-\Delta u = V e^u$. C. R. Acad.Sci. Paris Sér. I Math. 315 (1992), no. 2, 159–164.

[20] G. Tarantello; *Multiple condensate solutions for the Chern-Simons-Higgs theory.* J. Math. Phys. 37 (1996), no. 8, 3769–3796.

Samy Skander Bahoura

Department of Mathematics, Pierre et Marie Curie University, 4 Place Jussieu, 75005, Paris, France

Email address: samybahoura@gmail.com