Electronic Journal of Differential Equations, Vol. 2022 (2022), No. 17, pp. 1–26.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu

(ω, c)-PERIODIC SOLUTIONS FOR NON-INSTANTANEOUS IMPULSIVE SYSTEMS WITH UNBOUNDED TIME-VARYING COEFFICIENTS

KUI LIU, MICHAL FEČKAN, DONAL O'REGAN, JINRONG WANG

Abstract. In this article, we study (ω, c)-periodic solutions for non-instantaneous impulsive systems and the time-varying coefficient $A(t)$ is a family of unbounded linear operators. We show the existence and uniqueness of (ω, c)-periodic solutions using a fixed point theorem. An example is given to illustrate our results.

1. Introduction

Non-instantaneous impulsive can characterize drug absorption, diffusion and metabolism of drugs in the body and was considered in 2013 by Hernández and O'Regan [10]. Existence and uniqueness of non-instantaneous impulsive solutions in various situations can be found in [1, 3, 9, 14, 20, 21, 24, 25, 26, 32, 33].

The so-called time-varying system refers to the system whose characteristics change with time, it is also called the variable coefficient system. The characteristic of a time-varying system is that its output waveform is not only related to the input waveform, but also to the time when the input signal is added and some results were obtained at pulse time-varying systems. In [28, 29, 30] the authors constructed corresponding impulse evolution operators, considered the appropriate Poincáre operator method, introduced an appropriate Gronwall inequality and studied a class of impulsive periodic systems with time-varying generation operators, linear, nonlinear and integro-differential, and also the authors studied periodic PC-mild solutions. The authors in [18] studied a class of hybrid nonlinear impulse integral differential equations with time-varying generation operators and existence of a PC-mild solution is established, and existence of an optimal pair of a class of mixed pulse integral differential equations was discussed, and in [16, 17, 19] the authors constructed the corresponding evolution system generated by an operator matrix, by introducing an appropriate solution of the second-order nonlinear pulse, and discussed the existence of optimal control for the second-order nonlinear pulse evolution differential equation system with unbounded operator perturbation, the second-order nonlinear pulse evolution differential equation system, and the Lagrange problem with second-order nonlinear mixed pulse integral differential equations.

2020 *Mathematics Subject Classification.* 34A37.
Key words and phrases. Non-instantaneous impulsive systems; (ω, c)-periodic solutions; existence; uniqueness.

Submitted December 1, 2021. Published March 4, 2022.

Because of the universality of periodic phenomena the study of periodicity was considered; see [6, 8, 12, 23, 31, 34] and the references therein. The authors in [5] studied a class of (ω, c)-periodic functions containing periodic, anti-periodic, Bloch and unbounded functions and gave several properties on this class of functions, in [2] the authors studied uniqueness and existence of (ω, c)-periodic solutions for semilinear evolution equation $x' = Ax + f(t, x)$ in complex Banach spaces, where A is a bounded or unbounded linear operator, in [11] the authors studied existence and uniqueness of (ω, c)-periodic solutions for a class of impulsive differential systems by constructing Green functions and adjoint systems, and in [27] the authors presented existence and uniqueness results for a class of (ω, c)-periodic time varying impulsive differential equations.

Recently, Wang et al. [25] introduced new linear non-instantaneous impulsive differential equations and derived the representation of the solution and the asymptotic stability of linear and nonlinear problems. Motivated by [2, 5, 11, 25, 27], we study (ω, c)-periodic solutions for the following homogeneous non-instantaneous impulsive system with time-varying generating operators:

$$\begin{gathered} y'(t) = A(t)y(t), \quad t \in [s_{i-1}, t_i],\ i \in N^+, \\ y(t_i^+) = B_i(t_i)y(t_i^-), \quad i \in N^+, \\ y(t) = B_i(t)y(t_i^-), \quad t \in (t_i, s_i],\ i \in N^+, \\ y(s_i^+) = y(s_i^-), \quad i \in N^+, \end{gathered} \tag{1.1}$$

in the Banach space X, where $A(t)$ is a family of closed densely linear unbounded operators on X and $A(t)$ can generate a strongly continuous family of evolution $\{U(t, s), t \geq s \geq 0\}$. The sequence t_i, s_i satisfies $s_{i-1} < t_i < s_i < t_{i+1} < \dots$, $\lim_{i\to\infty} t_i = \infty$, s_i is the connection point and t_i is the pulse point. The symbols $y(t^+) := \lim_{\varepsilon\to 0^+} y(t + \varepsilon)$ and $y(t^-) := \lim_{\varepsilon\to 0^-} y(t + \varepsilon)$ represent the right and left limits of $y(s)$ at $s = t$, respectively. We set $y(t^-) = y(t)$. To help the reader understand (1.1) quickly, we give a sketch map of (1.1), see Figure 1. The black curves are corresponding to the first equation on $[s_{i-1}, t_i]$. The second equation represents the jump at impulsive point t_i. The red curves are corresponding to the third equation on $(t_i, s_i]$, which is the non-instantaneous impulsive action and ends at each connection point s_i. The fourth equation represents that y is continuous at each connection point s_i, which guarantee y can turn to the first differential equation of (1.1) again and again.

We study the structure and existence of (ω, c)-periodic solutions for the linear nonhomogeneous non-instantaneous impulsive differential system,

$$\begin{gathered} y'(t) = A(t)y(t) + h(t), \quad t \in [s_{i-1}, t_i],\ i \in N^+, \\ y(t_i^+) = B_i(t_i)y(t_i^-) + b_i, \quad i \in N^+, \\ y(t) = B_i(t)y(t_i^-) + b_i, \quad t \in (t_i, s_i],\ i \in N^+, \\ y(s_i^+) = y(s_i^-), \quad i \in N^+, \end{gathered} \tag{1.2}$$

where $h : \mathbb{D}_1 \to X$ is continuous, where $\mathbb{D}_1 = \cup_{i=1}^{\infty}[s_{i-1}, t_i]$, $\mathbb{D}_2 = \cup_{i=1}^{\infty}(t_i, s_i]$, and $c_i \in X, i \in N^+$.

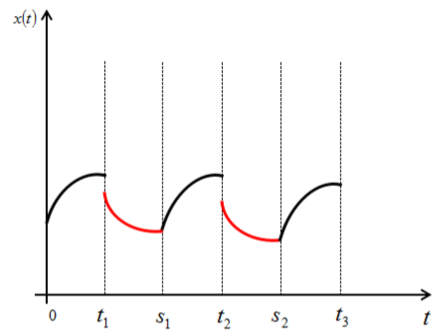

FIGURE 1. Sketch map of (1.1).

We also study the existence, uniqueness and stability of (ω, c)-periodic solutions for the semilinear non-instantaneous impulsive differential system,

$$\begin{gathered}
y'(t) = A(t)y(t) + f(t, y(t)), \quad t \in [s_{i-1}, t_i],\ i \in N^+, \\
y(t_i^+) = B_i(t_i)y(t_i^-) + b_i, \quad i \in N^+, \\
y(t) = B_i(t)y(t_i^-) + b_i, \quad t \in (t_i, s_i],\ i \in N^+, \\
y(s_i^+) = y(s_i^-) = y(s_i), \quad i \in N^+,
\end{gathered} \tag{1.3}$$

where $f : \mathbb{D}_1 \times X \to X$ is continuous.

This work extends the results for time invariant system in [7] to the case of time variable system. We transform the study of systems into the study of Cauchy operators corresponding to systems. Here $A(t)$ and $B_i(t)$ are not required to be commutative when we construct the Cauchy operators corresponding to the system. It is worth mentioning that our situation is more complicated than that in the literature [7]. The existence and uniqueness of (ω, c)-periodic solutions for time-varying unbounded systems are established by fixed point theorems.

To study (ω, c)-periodic solutions, we impose the following assumptions.

- (A1) $A(\cdot)$ is ω-periodic, i.e., for all $t \in \mathbb{D}_1$, $A(t + \omega) = A(t)$, where $\omega > 0$. For each $i \in \mathbb{N}^+$, $B_i(\cdot) \in L_b(X)$, $B(\cdot)$ is invertible, $B_i^{-1}(\cdot) \in L_b(X)$, and $B_i^{-1}(\cdot)$, $B_i(\cdot)$ is ω-periodic, i.e., $B_{i+m}^{-1}(t + \omega) = B_i^{-1}(t)$, $B_{i+m}(t + \omega) = B_i(t)$, for all $t \in \mathbb{D}_2$.
- (A2) The time sequence s_i, t_i are such that $s_{i+m} = s_i + \omega$, $t_{i+m} = t_i + \omega$ for some fixed m, $i \in \mathbb{N}$.
- (A3) $c \notin \sigma(S(\omega, 0))$.
- (A4) $c \in \sigma(S(\omega, 0))$.
- (A5) $h(\cdot)$ is (ω, c)-periodic function, i.e. $h(\cdot + \omega) = ch(\cdot)$ for all $\cdot \in \mathbb{R}^+$ and $b_{i+m} = cb_i$, $i \in \mathbb{N}^+$.
- (A6) For all $t \in \mathbb{R}^+$ and $x \in X$ such that $f(t + \omega, cx) = cf(t, x)$.
- (A7) There is a constant $L_u > 0$ such that $\|f(t, x_1) - f(t, x_2)\| \le L_u\|x_1 - x_2\|$ for all $t \in \mathbb{R}^+$ and $x_1, x_2 \in X$.
- (A8) There are constants $\alpha, \gamma \ge 0$ such that $\|f(t, x)\| \le \alpha + \gamma\|x\|$ for any $t \in \mathbb{R}^+$ and $x \in X$.
- (A9) The resolvent $R(\lambda, A(t))$ is compact, for all $t \ge 0$.
- (A10) (see [15, p.135]) For $t \in [0, \omega]$,

(1) $\{A(t)\}_{t\in[0,\omega]}$ is a stable family of generators of the evolution operator in X with stability indices $M \geq 1, \varpi$.
(2) Let Y be a dense subspace of X and Y is $A(t)$ admissible for each $t \in [0,\omega]$ and the family $\{\tilde{A}(t)\}_{t\in[0,\omega]}$ of parts $\{\tilde{A}(t)\}$ of $A(t)$ in Y, is a stable family in Y with stability constants $\widetilde{M}, \tilde{\varpi}$.
(3) For $t \in [0,\omega], D(A(t)) \supset Y, A(t)$ is a bounded operator from Y into X and $t \to A(t)$ is continuous in the $B(Y,X)$ norm $\|\cdot\|_{Y\to X}$.

(A11) (see [4, p.158]) For $t \in [0,\omega]$ one has
(1) The domain $D(A(t)) = D$ is independent of t and is dense in X.
(2) For $t \geq 0$, the resolvent $R(\lambda, A(t)) = (\lambda I - A(t))^{-1}$ exists for all λ with $\operatorname{Re}\lambda \leq 0$, and there is a constant N independent of λ and t such that
$$R(\lambda, A(t)) \leq N(1 + |\lambda|)^{-1} \quad \text{for } \operatorname{Re}\lambda \leq 0.$$
(3) There exist constants $L > 0$ and $0 < \alpha \leq 1$ such that
$$\|(A(t)A(\theta))A^{-1}(\tau)\| \leq L|t-\theta|^{\alpha}$$
for $t, \theta, \tau \in [0,\omega]$.

2. Preliminaries

Let $L(X)$ denote the space of linear operators in the Banach space X, and $L_b(X)$ denote the space of bounded linear operators in X. Note that $L_b(X)$ is a Banach space with the usual supremum norm. Set $PC(\mathbb{R}^+, X) := \{x : \mathbb{R}^+ \to X | x \in C((t_i, t_{i+1}], X), i = 0, 1, 2, \dots,\}$ and there exist $x(t_i^+), x(t_i^-)$ with $x(t_i^-) = x(t_i)$ with the norm $\|x\|_{PC} := \sup_{t\in\mathbb{R}^+} \|x(t)\|$.

Lemma 2.1 ([4, p.159])**.** *If* (A11) *holds, then the Cauchy problem*
$$y'(t) = A(t)y(t),$$
$$y(s) = y_s \in X, \quad t > s \geq 0.$$
has a unique evolution system $\{U(t,s)|0 \leq s \leq t \leq \omega\}$ in X, and the solution of the Cauchy problem of the linear homogeneous development equation can be written as $y(t) = U(t,s)y(s)$, and satisfies the following properties:
(1) *For all $0 \leq s \leq t \leq \omega, U(t,s) \in L_b(X), U(s,s) = I, s \geq 0$;*
(2) *For all $0 \leq s \leq r \leq t \leq \omega, U(t,r)U(r,s) = U(t,s)$;*
(3) *For all $0 \leq s \leq t \leq a < \infty$, $U(t,s)$ is continuous in the strong operator topology in $L(X)$.*

Lemma 2.2 ([13] or [28, Lemma 2.5])**.** *Assume* (A1), (A9), (A11) *hold. The evolution operator $\{U(t,s)|0 \leq s \leq t \leq \omega\}$ has the following properties:*
(1) *For $0 \leq s \leq t \leq \omega, U(t+\omega, s+\omega) = U(t,s)$.*
(2) *For $0 \leq s \leq t \leq \omega, U(t,s)$ is a compact operator.*

Lemma 2.3 ([15, p.135, Theorem 3.1])**.** *Let $A(t)$, $0 \leq t \leq \omega$, be the infinitesimal generator of a C_0-semigroup $S_t(s), s \geq 0$, on X. If the family $\{A(t)\}_{t\in[0,\omega]}$ satisfies condition* (A10)*, then there exists a unique evolution system $U(t,s)$, $0 \leq s \leq t \leq \omega$, in X satisfying $\|U(t,s)\| \leq M \exp\{\varpi(t-s)\}$, for $0 \leq s \leq t \leq \omega$.*

We consider the homogeneous non-instantaneous impulsive system (1.1) and its corresponding Cauchy problem

$$\begin{gathered}
y'(t) = A(t)y(t), \quad t \in [s_{i-1}, t_i],\ i \in N^+, \\
y(t_i^+) = B_i(t_i)y(t_i^-), \quad i \in N^+, \\
y(t) = B_i(t)y(t_i^-), \quad t \in (t_i, s_i],\ i \in N^+, \\
y(s_i^+) = y(s_i^-) = y(s_i), \quad i \in N^+, \\
y(0) = y_0.
\end{gathered} \tag{2.1}$$

The Cauchy problem (2.1) has a unique classical solution $y \in PC([\mathbb{D}_1 \cup \mathbb{D}_2, \omega]; X)$ and it can be represented by $y(t; s, y_s) = S(t, s)y_s, t \geq 0$ where

$$S(\cdot, \cdot) : \Delta = \{(t, s) \in \mathbb{R}^+ \times \mathbb{D}_1 : t \geq s\} \to X$$

is given by the formula

$$S(t,s) = \begin{cases} U(t,s), \quad \text{if } t, s \in [s_{i-1}, t_i],\ i \in N^+ \\ B_i(t)B_i^{-1}(s), \quad \text{if } t, s \in (t_i, s_i], \\ U(t, s_{k-1}) \prod_{j=i+1}^{k-1} [B_j(s_j)U(t_j, s_{j-1})] B_i(s_i) U(t_i, s) \\ \quad \text{if } s_{i-1} \leq s \leq t_i < \cdots < s_{k-1} \leq t \leq t_k, \\ B_k(t)U(t_k, s_{k-1}) \prod_{j=i+1}^{k-1} [B_j(s_j)U(t_j, s_{j-1})] B_i(s_i) U(t_i, s), \\ \quad \text{if } s_{i-1} \leq s \leq t_i < \cdots < t_k < t \leq s_k, \\ U(t, s_k) \prod_{j=i+1}^{k} [B_j(s_j)U(t_j, \\ \quad \text{if } s_{j-1})] B_i(s_i) B_i^{-1}(s), t_i < s \leq s_i < \cdots < s_k \leq t \leq t_{k+1}, \\ B_k(t)U(t_k, s_{k-1}) \prod_{j=i+1}^{k-1} [B_j(s_j)U(t_j, s_{j-1})] B_i(s_i) B_i^{-1}(s), \\ \quad \text{if } t_i < s \leq s_i < \cdots < t_k < t \leq s_k. \end{cases} \tag{2.2}$$

Note that $B_i(t)S(t_i^-, s) = S(t, s)$ and $S(s, t) := S^{-1}(t, s)$. We use the standard convention $\prod_{j=i+1}^{k-1} = I$ for $i + 1 \geq k - 1$.

Definition 2.4 ([5])**.** A function $g : \mathbb{R} \to X$ is called (ω, c)-periodic if there is a pair (ω, c), where $\omega > 0$ and $c \in \mathbb{R}\backslash\{0\}$ such that $g(t + \omega) = cg(t)$ for all $t \in \mathbb{R}$.

Definition 2.5 ([5])**.** Any solution $y(t; 0, y_0)$ of the non-instantaneous impulsive differential systems (1.1);(1.2);(1.3) is called a (ω, c)-periodic solution if $y(t+\omega; 0, y_0) = cy(t; 0, y_0)$, $t \geq 0$.

Set $\Psi_{\omega,c} := \{y \in PC(\mathbb{R}, X) : cy(\cdot) = y(\cdot + \omega)\}$, i.e. $\Psi_{\omega,c}$ denotes the set of all piecewise continuous and (ω, c)-periodic functions.

Lemma 2.6 ([2] or [11, Lemma 2.2])**.** $y \in \Psi_{\omega,c}$ if and only if $y(\omega) = cy(0)$.

3. Homogeneous linear non-instantaneous impulsive systems

Let $\alpha = \sup_{i \geq 1}(t_i - s_{i-1})$, $\beta = \sup_{i\geq 1} \max_{t \in (t_i, s_i)} \|B_i(t)\|$, and $i(t, s)$ denote the number of impulsive points in (s, t).

Theorem 3.1. *Assume that* (A10) *holds. For any $\tilde{l}, k \in N^+$, and $\tilde{l} \leq k$, $t \in [s_{k-1}, t_k]$, $s \in [s_{\tilde{l}-1}, t_{\tilde{l}}]$, and $s < t$, we obtain*

$$\|S(t, s)\| \leq M^{i(t,s)+1} \beta^{i(t,s)} \exp(\varpi\alpha(i(t, s) + 1)).$$

For any $t \in [t_k, s_k]$ *and* $s \in [s_{\tilde{l}-1}, t_{\tilde{l}}]$, *we obtain*

$$\|S(t,s)\| \le M^{i(t,s)}\beta^{i(t,s)}\exp(\varpi\alpha i(t,s)).$$

Let $K = \max\{\exp(\varpi\alpha), 1\}$ *and* $N = \max\{\beta^m, 1\}$, *then for* $s \in [s_{\tilde{l}-1}, t_{\tilde{l}}]$ *and any* $t \in [0,\omega]$, *we have*

$$\|S(t,s)\| \le KNM^{i(t,s)+1}\exp(\varpi\alpha i(t,s)). \tag{3.1}$$

Proof. For any $t, s \in [s_{\tilde{l}-1}, t_{\tilde{l}}]$ and $s < t$, we have

$$\|S(t,s)\| = \|U(t,s)\| \le M\exp\{\varpi(t-s)\}.$$

For any $s_{\tilde{l}-1} \le s \le t_{\tilde{l}} < \dots < s_{k-1} < t \le t_k$, from (2.2), we have

$$\begin{aligned}
\|S(t,s)\| &= \|U(t,s_{k-1})\prod_{j=\tilde{l}+1}^{k-1}[B_j(s_j)U(t_j,s_{j-1})]B_{\tilde{l}}(s_{\tilde{l}})U(t_{\tilde{l}},s)\| \\
&\le \|U(t,s_{k-1})\|\prod_{j=\tilde{l}+1}^{k-1}[\|B_j(s_j)\|\|U(t_j,s_{j-1})\|]\|B_{\tilde{l}}(s_{\tilde{l}})\|\|U(t_{\tilde{l}},s)\| \\
&\le M^{i(t,s)+1}\beta^{i(t,s)}\exp(\varpi\alpha(i(t,s)+1)).
\end{aligned}$$

For any $s_{\tilde{l}-1} \le s \le t_{\tilde{l}} < \dots < t_k < t \le s_k$, we have

$$\begin{aligned}
\|S(t,s)\| &\le \|B_k(t)U(t_k,s_{k-1})\prod_{j=\tilde{l}+1}^{k-1}[B_j(s_j)U(t_j,s_{j-1})]B_{\tilde{l}}(s_{\tilde{l}})U(t_{\tilde{l}},s)\| \\
&\le \|B_k(t)\|\|U(t_k,s_{k-1})\|\prod_{j=\tilde{l}+1}^{k-1}[\|B_j(s_j)\|\|U(t_j,s_{j-1})\|]\|B_{\tilde{l}}(s_{\tilde{l}})\|\|U(t_{\tilde{l}},s)\| \\
&\le \beta^{i(t,s)}M^{i(t,s)}\exp(\varpi\alpha(i(t,s))).
\end{aligned}$$

The proof is complete. □

Theorem 3.2. *Assume that* (A11) *holds. If* $s \le u \le t, u, s, t \in \mathbb{R}^+$, *then* $S(t,s) = S(t,u)S(u,s)$.

Proof. (a) For all $s, t \in [s_{i-1}, t_i]$, $i \in N^+$, and $s < u < t$, we have

$$S(t,u)S(u,s) = U(t,u)U(u,s) = U(t,s) = S(t,s).$$

(b) For all $s \in [s_{i-1}, t_i]$, $u \in [s_{l-1}, t_l]$, $t \in [s_{k-1}, t_k]$, $i, l, k \in N^+$, and $i < l < k$, we have

$$\begin{aligned}
S(t,u)S(u,s) &= U(t,s_{k-1})\prod_{j=l+1}^{k-1}[B_j(s_j)U(t_j,s_{j-1})]B_l(s_l)U(t_l,u) \\
&\quad \times U(u,s_{l-1})\prod_{j=i+1}^{l-1}[B_j(s_j)U(t_j,s_{j-1})]B_i(s_i)U(t_i,s) \\
&= U(t,s_{k-1})\prod_{j=i+1}^{k-1}[B_j(s_j)U(t_j,s_{j-1})]B_i(s_i)U(t_i,s) \\
&= S(t,s).
\end{aligned}$$

Similarly, the conclusion can be proved when $i \le l < k$, $i < l \le k$.

(c) For all $s \in [s_{i-1}, t_i]$, $u \in [s_{l-1}, t_l]$, $t \in (t_k, s_k]$, $i, l, k \in N^+$, and $i < l < k$, we have

$$\begin{aligned} S(t,u)S(u,s) &= B_k(t)U(t_k, s_{k-1}) \prod_{j=l+1}^{k-1} [B_j(s_j)U(t_j, s_{j-1})] B_l(s_l)U(t_l, u) \\ &\quad \times U(u, s_{l-1}) \prod_{j=i+1}^{l-1} [B_j(s_j)U(t_j, s_{j-1})] B_i(s_i)U(t_i, s) \\ &= B_k(t)U(t_k, s_{k-1}) \prod_{j=i+1}^{k-1} [B_j(s_j)U(t_j, s_{j-1})] B_i(s_i)U(t_i, s) \\ &= S(t,s). \end{aligned}$$

Similarly, the conclusion can be proved when $i \le l \le k$.

(d) For all $s, t \in (t_i, s_i]$, $i \in N^+$, and $s < u < t$, we have

$$S(t,u)S(u,s) = B_i(t)B_i^{-1}(u)B_i(u)B_i^{-1}(s) = B_i(t)B_i^{-1}(s) = S(t,s).$$

(e) For all $s \in (t_i, s_i]$, $u \in [s_{l-1}, t_l]$, $t \in [s_{k-1}, t_k]$, $i, l, k \in N^+$, and $i < l < k$, we have

$$\begin{aligned} S(t,u)S(u,s) &= U(t, s_{k-1}) \prod_{j=l+1}^{k-1} [B_j(s_j)U(t_j, s_{j-1})] B_l(s_l)U(t_l, u) \\ &\quad \times U(u, s_{l-1}) \prod_{j=i+1}^{l-1} [B_j(s_j)U(t_j, s_{j-1})] B_i(s_i)B_i^{-1}(s) \\ &= U(t, s_{k-1}) \prod_{j=i+1}^{k-1} [B_j(s_j)U(t_j, s_{j-1})] B_i(s_i)B_i^{-1}(s) \\ &= S(t,s). \end{aligned}$$

Similarly, the conclusion can be proved when $i < l \le k$.

(f) For all $s \in (t_i, s_i]$, $u \in [s_{l-1}, t_l]$, $t \in (t_k, s_k]$, $i, l, k \in N^+$, and $i < l < k$, we have

$$\begin{aligned} S(t,u)S(u,s) &= B_k(t)U(t_k, s_{k-1}) \prod_{j=l+1}^{k-1} [B_j(s_j)U(t_j, s_{j-1})] B_l(s_l)U(t_l, u) \\ &\quad \times U(u, s_{l-1}) \prod_{j=i+1}^{l-1} [B_j(s_j)U(t_j, s_{j-1})] B_i(s_i)B_i^{-1}(s) \\ &= B_k(t)U(t_k, s_{k-1}) \prod_{j=i+1}^{k-1} [B_j(s_j)U(t_j, s_{j-1})] B_i(s_i)B_i^{-1}(s) \\ &= S(t,s). \end{aligned}$$

Similarly, the conclusion can be proved when $i < l \le k$.

When $u \in (t_l, s_l]$, we can prove $S(t,s) = S(t,u)S(u,s)$ in the same way. The proof is complete. □

Remark 3.3. Assume that (A11) holds. If $s \le t \le u$ or $u \le s \le t$, then $S(t,u)S(u,s) = S(t,s)$.

Proof. If $s \le t \le u$, then

$$S(t,u)S(u,s) = S(t,u)S(u,t)S(t,s) = S(t,u)S^{-1}(t,u)S(t,s) = S(t,s).$$

If $u \le s \le t$, then

$$S(t,u)S(u,s) = S(t,s)S(s,u)S(u,s) = S(t,s)S(s,u)S^{-1}(s,u) = S(t,s). \qquad \square$$

Theorem 3.4. *If* (A1) *and* (A2) *hold, then* $S(t+\omega, s+\omega) = S(t,s)$.

Proof. Clearly, for each $t \in [s_{i-1}, t_i], 0 < s < t$, we have $U(t+\omega, s+\omega) = U(t,s)$ by (A1).

(a) For all $s \in [s_{i-1}, t_i]$, $t \in [s_{k-1}, t_k]$, and $k \in N^+$, we have

$$\begin{aligned}
S(t+\omega, s+\omega) &= U(t+\omega, s_{k-1+m}) \prod_{j=i+1}^{k-1} [B_{j+m}(s_{j+m})U(t_{j+m}, s_{j+m-1})] \\
&\quad \times B_{i+m}(s_{i+m})U(t_{i+m}, s+\omega) \\
&= U(t+\omega, s_{k-1}+\omega) \prod_{j=i+1}^{k-1} [B_{j+m}(s_j+\omega)U(t_j+\omega, s_{j-1}+\omega)] \\
&\quad \times B_{i+m}(s_i+\omega)U(t_i+\omega, s+\omega) \\
&= U(t, s_{k-1}) \prod_{j=i+1}^{k-1} [B_j(s_j)U(t_j, s_{j-1})]B_i(s_i)U(t_i, s) \\
&= S(t,s).
\end{aligned}$$

(b) For any $s \in [s_{i-1}, t_i]$, $t \in (t_k, s_k]$, and $k \in N^+$, we have

$$\begin{aligned}
&S(t+\omega, s+\omega) \\
&= B_{k+m}(t+\omega)U(t_{k+m}, s_{k+m-1}) \\
&\quad \times \prod_{j=i+1}^{k-1} [B_{j+m}(s_{j+m})U(t_{j+m}, s_{j+m-1})]B_{i+m}(s_{i+m})U(t_{i+m}, s) \\
&= B_{k+m}(t+\omega)U(t_k+\omega, s_{k-1}+\omega) \\
&\quad \times \prod_{j=i+1}^{k-1} [B_{j+m}(s_j+\omega)U(t_j+\omega, s_{j-1}+\omega)]B_{i+m}(s_i+\omega)U(t_i+\omega, s) \\
&= B_k(t)U(t_k, s_{k-1}) \prod_{j=i+1}^{k-1} [B_j(s_j)U(t_j, s_{j-1})]B_i(s_i)U(t_i, s) \\
&= S(t,s).
\end{aligned}$$

(c) For all $s \in (t_i, s_{i+1}]$, $t \in [s_{k-1}, t_k]$, and $k \in N^+$, we have

$$\begin{aligned}
&S(t+\omega, s+\omega) \\
&= U(t+\omega, s_{k-1+m}) \prod_{j=i+1}^{k-1} [B_{j+m}(s_{j+m})U(t_{j+m}, s_{j+m-1})] \\
&\quad \times B_{i+m}(s_{i+m})B_{i+m}^{-1}(s+\omega)
\end{aligned}$$

$$
\begin{aligned}
&= U(t+\omega, s_{k-1}+\omega) \prod_{j=i+1}^{k-1} [B_{j+m}(s_j+\omega) U(t_j+\omega, s_{j-1}+\omega)] \\
&\quad \times B_{i+m}(s_i+\omega) B_{i+m}^{-1}(s+\omega) \\
&= U(t, s_{k-1}) \prod_{j=i+1}^{k-1} [B_j(s_j) U(t_j, s_{j-1})] B_i(s_i) B_i^{-1}(s) \\
&= S(t,s).
\end{aligned}
$$

(d) For any $s \in (t_i, s_i]$, $t \in (t_k, s_k]$, and $k \in N^+$, we have

$$
\begin{aligned}
&S(t+\omega, s+\omega) \\
&= B_{k+m}(t+\omega) U(t_{k+m}, s_{k+m-1}) \\
&\quad \times \prod_{j=i+1}^{k-1} [B_{j+m}(s_{j+m}) U(t_{j+m}, s_{j+m-1})] B_{i+m}(s_{i+m}) U(t_{i+m}, s) \\
&= B_{k+m}(t+\omega) U(t_k+\omega, s_{k-1}+\omega) \\
&\quad \times \prod_{j=i+1}^{k-1} [B_{j+m}(s_j+\omega) U(t_j+\omega, s_{j-1}+\omega)] B_{i+m}(s_i+\omega) B_{i+m}^{-1}(s+\omega) \\
&= B_k(t) U(t_k, s_{k-1}) \prod_{j=i+1}^{k-1} [B_j(s_j) U(t_j, s_{j-1})] B_i(s_i) B_i^{-1}(s) \\
&= S(t,s).
\end{aligned}
$$

The proof is complete. □

Theorem 3.5. *Assume that* (A1),(A9), (A11) *hold. Then $S(t,s)$ is compact operator for $0 \le s \le t \le \omega$.*

Proof. Let $s, t \in [s_{i-1}, t_i], i \in N^+$, from Lemma 2.2, we see that $U(t,s)$ is a compact operator.

Let $s_{i-1} \le s \le t_i < \cdots < s_{k-1} \le t \le t_k$, $i \le k, i, k \in N^+$, since $B_j(s_j) \in L_b(X)$, $U(t_j, s_{j-1}) \in L_b(X)$, $\prod_{j=i+1}^{k-1} [B_j(s_j) U(t_j, s_{j-1})] \in L_b(X)$, $B_i(s_i) \in L_b(X)$,$U(t_i, s) \in L_b(X)$, and $U(t, s_{k-1})$ is a compact operator, we see that

$$
U(t, s_{k-1}) \prod_{j=1}^{k-1} [B_j(s_j) U(t_j, s_{j-1})] B_i(s_i) U(t_i, s)
$$

is a compact operator.

Let $s_{i-1} \le s \le t_i < \cdots < t_k < t \le s_k$, $i \le k, i, k \in N^+$, since $B_j(s_j) \in L_b(X)$, $U(t_j, s_{j-1}) \in L_b(X)$ $B_k(t) \in L_b(X)$, $U(t_k, s_{k-1}) \in L_b(X)$, $\prod_{j=i+1}^{k-1} [B_j(s_j) U(t_j, s_{j-1})] \in L_b(X)$, $B_i(s_i) \in L_b(X)$, and $U_k(t, s_{k-1})$ is a compact operator, we see that

$$
B_k(t) U(t_k, s_{k-1}) \prod_{j=i+1}^{k-1} [B_j(s_j) U(t_j, s_{j-1})] B_i(s_i) U(t_i, s)
$$

is a compact operator.

Let $t_i < s \le s_i < \cdots < s_k \le t \le t_{k+1}$, $i \le k \in N^+$, since $B_j(s_j) \in L_b(X)$, $U(t_j, s_{j-1}) \in L_b(X)$, $\prod_{j=i+1}^{k}[B_j(s_j)U(t_j, s_{j-1})] \in L_b(X)$, $B_i(s_i) \in L_b(X)$, $B_i^{-1}(s) \in L_b(X)$, and $U(t, s_k)$ is a compact operator, we see that

$$U(t, s_k) \prod_{j=i+1}^{k} [B_j(s_j)U(t_j, s_{j-1})]B_i(s_i)B_i^{-1}(s)$$

is a compact operator.

Let $t_i < s \le s_i < \cdots < t_k < t \le s_k, i \le k \in N^+$, since $B_j(s_j) \in L_b(X)$, $U(t_j, s_{j-1}) \in L_b(X)$, $B_k(t) \in L_b(X)$, $U(t_k, s_{k-1}) \in L_b(X)$, $\prod_{j=i+1}^{k-1}[B_j(s_j)U(t_j, s_{j-1})] \in L_b(X)$, $B_i(s_i) \in L_b(X)$, $B_i^{-1}(s) \in L_b(X)$, and $U_k(t, s_{k-1})$ is a compact operator, we see that

$$B_k(t)U(t_k, s_{k-1}) \prod_{j=i+1}^{k-1} [B_j(s_j)U(t_j, s_{j-1})]B_i(s_i)B_i^{-1}(s)$$

is a compact operator. In summary, for any $0 \le s \le t \le \omega$, the operator $S(t, s)$ is a compact. □

Lemma 3.6. *Assume that* (A1)—(A3) *hold. Then* (1.1) *has a solution* $y \in \Psi_{\omega,c}$ *if and only if*

$$\big(cI - S(\omega, 0)\big)y_0 = 0.$$

Proof. Note if $y \in PC(\mathbb{R}^+, X)$, then (1.1) can be formulated as

$$y(t; y_0) = S(t, t_0)y_0, \quad t \ge t_0.$$

Using Definition 2.4 we have

$$\begin{aligned} y(t+\omega) = cy(t) &\Longleftrightarrow S(t+\omega, 0)y_0 = cS(t, 0)y_0 \\ &\Longleftrightarrow S(t+\omega, \omega)S(\omega, 0)y_0 = cS(t, 0)y_0 \\ &\Longleftrightarrow S(t, 0)S(\omega, 0)y_0 = cS(t, 0)y_0 \\ &\Longleftrightarrow \big(cI - S(\omega, 0)\big)y_0 = 0. \end{aligned}$$

This proof is complete. □

4. Nonhomogeneous linear non-instantaneous impulsive systems

In this section we consider the existence of (ω, c)-periodic solutions of (1.2).

Theorem 4.1. *Assuming that* $h \in C(\mathbb{R}^+, X)$. *The solution* $y \in PC(\mathbb{D}_1, X)$ *of* (1.2) *with the initial value* y_0 *is given by*

$$\begin{aligned} y(t; 0, y_0) = S(t, 0)y_0 &+ \sum_{j=0}^{i(t,0)-1} \int_{s_j}^{t_{j+1}} S(t, \tau)h(\tau)d\tau \\ &+ \int_{s_{i(t,0)}}^{t} S(t, \tau)h(\tau)d\tau + \sum_{j=1}^{i(t,0)} S(t, s_j)b_j. \end{aligned} \tag{4.1}$$

We set

$$\tilde{h}(t) = \begin{cases} h(t), & t \in \mathbb{D}_1, \\ 0, & t \in \mathbb{D}_2. \end{cases}$$

Then (4.1) *can be rewritten as*

$$y(t;0,y_0)=S(t,0)y_0+\int_0^t S(t,\tau)\tilde{h}(\tau)d\tau+\sum_{j=1}^{i(t,0)}S(t,s_j)b_j. \tag{4.2}$$

Proof. For $t\in[0,t_1]$, the solutions of (1.2) can be represented as

$$y(t;0,y_0)=S(t,0)y_0+\int_0^t S(t,\tau)h(\tau)d\tau,$$

so, $y(t_1^-)=S(t_1^-,0)y_0+\int_0^{t_1}S(t_1^-,\tau)h(\tau)d\tau$.

For $t\in(t_1,s_1]$, we have

$$\begin{aligned}
y(t)&=B_1(t)y(t_1^-)+b_1\\
&=B_1(t)(S(t_1^-,0)y_0+\int_0^{t_1}S(t_1^-,\tau)h(\tau)d\tau)+b_1\\
&=S(t,0)y_0+\int_0^{t_1}S(t,\tau)h(\tau)d\tau+b_1,
\end{aligned}$$

so $y(s_1)=S(s_1,0)y_0+\int_0^{t_1}S(s_1,\tau)h(\tau)d\tau+b_1$.

For $t\in[s_1,t_2]$, by using the variation of parameter method, we have

$$\begin{aligned}
y(t,0,s_1)&=S(t,s_1)y_{s_1}+\int_{s_1}^t S(t,\tau)h(\tau)d\tau\\
&=S(t,s_1)(S(s_1,0)y_0+\int_0^{t_1}S(s_1,\tau)h(\tau)d\tau+b_1)+\int_{s_1}^t S(t,\tau)h(\tau)d\tau\\
&=S(t,0)y_0+\int_0^{t_1}S(t,\tau)h(\tau)d\tau+\int_{s_1}^t S(t,\tau)h(\tau)d\tau+S(t,s_1)b_1.
\end{aligned}$$

Assume that (4.1) holds for $t\in[s_k,t_{k+1}]$, $k\in N^+$. Then we have

$$y(t;0,y_0)=S(t,0)y_0+\sum_{j=0}^{k-1}\int_{s_j}^{t_{j+1}}S(t,\tau)h(\tau)d\tau+\int_{s_k}^t S(t,\tau)h(\tau)d\tau+\sum_{j=1}^k S(t,s_j)b_j,$$

so,

$$\begin{aligned}
y(t_{k+1}^-)&=S(t_{k+1}^-,0)x_0+\sum_{j=0}^{k-1}\int_{s_j}^{t_{j+1}}S(t_{k+1}^-,\tau)h(\tau)d\tau+\int_{s_k}^{t_{k+1}}S(t_{k+1}^-,\tau)h(\tau)d\tau\\
&\quad+\sum_{j=1}^k S(t_{k+1}^-,s_j)b_j\,.
\end{aligned}$$

Then for $t\in(t_{k+1},s_{k+1}]$, we have

$$\begin{aligned}
y(t)&=B_{k+1}(t)y(t_{k+1}^-)+b_{k+1}\\
&=B_{k+1}(t)\Big(S(t_{k+1}^-,0)y_0+\sum_{j=0}^k\int_{s_j}^{t_{j+1}}S(t_{k+1}^-,\tau)h(\tau)d\tau\\
&\quad+\sum_{j=1}^k S(t_{k+1}^-,s_j)b_j\Big)+b_{k+1}
\end{aligned}$$

$$= S(t,0)y_0 + \sum_{j=0}^{k}\int_{s_j}^{t_{j+1}} S(t,\tau)h(\tau)d\tau + \sum_{j=1}^{k} S(t_{k+1},s_j)b_j + b_{k+1}.$$

Thus, for $t \in [s_{k+1}, t_{k+2}]$, we have

$$\begin{aligned}
&y(t,0,s_{k+1})\\
&= S(t,s_{k+1})y_{s_{k+1}} + \int_{s_{k+1}}^{t} S(t,\tau)h(\tau)d\tau\\
&= S(t,s_{k+1})\Big(S(s_{k+1},0)y_0 + \sum_{j=0}^{k}\int_{s_j}^{t_{j+1}} S(s_{k+1},\tau)h(\tau)d\tau\\
&\quad + \sum_{j=1}^{k} S(s_{k+1},s_j)b_j + b_{k+1}\Big) + \int_{s_{k+1}}^{t} S(t,\tau)h(\tau)d\tau\\
&= S(t,0)y_0 + \sum_{j=0}^{k}\int_{s_j}^{t_{j+1}} S(t,\tau)h(\tau)d\tau + \int_{s_{k+1}}^{t} S(t,\tau)h(\tau)d\tau\\
&\quad + \sum_{j=1}^{k} S(t,s_j)b_j + S(t,s_{k+1})b_{k+1}\\
&= S(t,0)y_0 + \sum_{j=0}^{k}\int_{s_j}^{t_{j+1}} S(t,\tau)h(\tau)d\tau + \int_{s_{k+1}}^{t} S(t,\tau)h(\tau)d\tau + \sum_{j=1}^{k+1} S(t,s_j)b_j.
\end{aligned}$$

By mathematical induction, we can complete the proof. □

To study the existence of (ω, c)-periodic solutions of (1.2), we consider two cases:

Case 1: $c \notin \sigma(S(\omega, 0))$.

Lemma 4.2. *Assume that* (A1)–(A3), (A5) *hold. Then the* (ω, c)*-periodic solution* $y \in \Psi = PC([0,\omega], X)$ *of* (1.2) *is*

$$y(t) = \int_0^{\omega} Q(t,\tau)\tilde{h}(\tau)d\tau + \sum_{j=1}^{m} Q(t,s_j)b_j, \quad t \in \mathbb{D}_1,$$

where $Q(\cdot,\cdot)$ *is the Green's function*

$$Q(t,\tau) = \begin{cases} S(t,0)(cI - S(\omega,0))^{-1}S(\omega,\tau) + S(t,\tau), & 0 < \tau < t,\\ S(t,0)(cI - S(\omega,0))^{-1}S(\omega,\tau), & t \le \tau < \omega. \end{cases} \tag{4.3}$$

Proof. From Lemma 2.6 and (4.2), for any solution $y \in \Psi$ of (1.2), we have

$$y(\omega) = S(\omega,0)y_0 + \int_0^{\omega} S(\omega,\tau)\tilde{h}(\tau)d\tau + \sum_{j=1}^{i(\omega,0)} S(\omega,s_j)b_j = cy_0,$$

which is equivalent to

$$y_0 = (cI - S(\omega,0))^{-1}\Big(\int_0^{\omega} S(\omega,\tau)\tilde{h}(\tau)d\tau + \sum_{j=1}^{i(\omega,0)} S(\omega,s_j)b_j\Big),$$

where we used (A3). Therefore, the solution of (1.2) is equivalent to

$$
\begin{aligned}
y(t) &= S(t,0)(cI - S(\omega,0))^{-1}\Big(\int_0^{\omega} S(\omega,\tau)\tilde{h}(\tau)d\tau + \sum_{j=1}^{i(\omega,0)} S(\omega,s_j)b_j\Big) \\
&\quad + \int_0^t S(t,\tau)\tilde{h}(\tau)d\tau + \sum_{j=1}^{i(t,0)} S(t,s_j)b_j \\
&= \int_0^{\omega} S(t,0)(cI - S(\omega,0))^{-1}S(\omega,\tau)\tilde{h}(\tau)d\tau \\
&\quad + \sum_{j=1}^{i(\omega,0)} S(t,0)(cI - S(\omega,0))^{-1}S(\omega,s_j)b_j \\
&\quad + \int_0^t S(t,\tau)\tilde{h}(\tau)d\tau + \sum_{j=1}^{i(t,0)} S(t,s_j)b_j \\
&= \int_0^{t} S(t,0)(cI - S(\omega,0))^{-1}S(\omega,\tau)\tilde{h}(\tau)d\tau \\
&\quad + \int_t^{\omega} S(t,0)(cI - S(\omega,0))^{-1}S(\omega,\tau)\tilde{h}(\tau)d\tau \\
&\quad + \sum_{j=1}^{i(t,0)} S(t,0)(cI - S(\omega,0))^{-1}S(\omega,s_j)b_j \\
&\quad + \sum_{j=i(t,0)+1}^{i(\omega,0)} S(t,0)(cI - S(\omega,0))^{-1}S(\omega,s_j)b_j \\
&\quad + \int_0^t S(t,\tau)\tilde{h}(\tau)d\tau + \sum_{j=1}^{i(t,0)} S(t,s_j)b_j \\
&=: J_1 + J_2,
\end{aligned}
$$

where

$$
\begin{aligned}
J_1 &:= \int_0^{t} S(t,0)(cI - S(\omega,0))^{-1}S(\omega,\tau)\tilde{h}(\tau)d\tau + \int_0^t S(t,\tau)\tilde{h}(\tau)d\tau \\
&\quad + \int_t^{\omega} S(t,0)(cI - S(\omega,0))^{-1}S(\omega,\tau)\tilde{h}(\tau)d\tau,
\end{aligned}
$$

and

$$
\begin{aligned}
J_2 &:= \sum_{j=1}^{i(t,0)} S(t,0)(cI - S(\omega,0))^{-1}S(\omega,s_j)b_j + \sum_{j=1}^{i(t,0)} S(t,s_j)b_j \\
&\quad + \sum_{j=i(t,0)+1}^{i(\omega,0)} S(t,0)(cI - S(\omega,0))^{-1}S(\omega,s_j)b_j.
\end{aligned}
$$

Also we have

$$
J_1 = \int_0^{t} S(t,0)(cI - S(\omega,0))^{-1}S(\omega,\tau)\tilde{h}(\tau)d\tau + \int_0^t S(t,\tau)\tilde{h}(\tau)d\tau
$$

$$
\begin{aligned}
&\quad + \int_t^\omega S(t,0)(cI - S(\omega,0))^{-1} S(\omega,\tau)\tilde{h}(\tau)d\tau \\
&= \int_0^t [S(t,0)(cI - S(\omega,0))^{-1} S(\omega,,\tau) + S(t,\tau)]\tilde{h}(\tau)d\tau \\
&\quad + \int_t^\omega S(t,0)(cI - S(\omega,0))^{-1} S(\omega,\tau)\tilde{h}(\tau)d\tau \\
&= \int_0^\omega Q(t,\tau)\tilde{h}(\tau)d\tau,
\end{aligned}
$$

and

$$
\begin{aligned}
J_2 &= \sum_{j=1}^{i(t,0)} S(t,0)(cI - S(\omega,0))^{-1} S(\omega,s_j)b_j + \sum_{j=1}^{i(t,0)} S(t,s_j)b_j \\
&\quad + \sum_{j=i(t,0)+1}^{i(\omega,0)} S(t,0)(cI - S(\omega,0))^{-1} S(\omega,s_j)b_j \\
&= \sum_{j=1}^{i(t,0)} [S(t,0)(cI - S(\omega,0))^{-1} S(\omega,s_j) + S(t,s_j)]b_j \\
&\quad + \sum_{j=i(t,0)+1}^{i(\omega,0)} S(t,0)(cI - S(\omega,0))^{-1} S(\omega,s_j)b_j \\
&= \sum_{0<s_j<\omega} Q(t,s_j)b_j.
\end{aligned}
$$

This proof is complete. □

Case 2: $c \in \sigma(S(\omega,0))$**.** Suppose X is a Hilbert space. Now we study the existence of (ω,c)-periodic solutions of (2.1) when the operator $(cI - S(\omega,0))^{-1}$ does not exist. We consider the adjoint system to (2.1) as follows

$$
\begin{gathered}
x'(t) = -A^\top(t)x(t), \quad t \in (s_i,t_i],\ i = 0,1,2,\dots, \\
x(t_i^+) = [B_i^\top(t_i)]^{-1} x(t_i^-), \quad i = 1,2,\dots, \\
x(t) = [B_i^\top(t)]^{-1} x(t_i^-), \quad t \in (t_i,s_i],\ i = 1,2,\dots, \\
x(s_i^+) = x(s_i^-), \quad i = 1,2,\dots.
\end{gathered}
\tag{4.4}
$$

Here $A^\top(t), B_i^\top(t)$ is the adjoint operator of $A(t), B_i(t)$, respectively. By assumption (A1), $A^\top(t+\omega) = A^\top(t), B_{i+m}^\top(t+\omega) = B_i^\top(t)$. Let $U^\top(\cdot,\cdot)$ be the adjoint operator of $U(\cdot,\cdot)$ and $U^\top(\cdot,\cdot)$ satisfies some properties similar to $U(\cdot,\cdot)$ because of the convexity of X^*. From the reflexivity of X note $A^\top$ is also an infinitesimal generating element of C_0-semigroup $\{U^\top(\cdot,\cdot)\}$ in X^* and $B_i^\top(t) \in \mathcal{L}_b(X^*)$.

It is easy to obtain that the solution of (4.4) with the initial value $x(0) = x_0$ is given by

$$
x(t) = [S^\top(t,0)]^{-1} x_0 \,. \tag{4.5}
$$

Theorem 4.3. *Assume* (A1), (A2), (A4) *hold. Then the adjoint system* (4.4) *of* (2.1) *has l linearly independent $(\omega, \frac{1}{c})$-periodic solutions for $1 \le l \le n$.*

Proof. From (A4), we know that $(cI - S(\omega,0))^{-1}$ does not exist, so we assume that the operator equation $(cI - S(\omega,0))y_0 = 0$ has l linearly independent solutions for $1 \le l \le n$. This implies that $\dim\ker[cI - S^\top(\omega,0)] = \dim\ker[cI - S(\omega,0)]$.

From (4.5), we have $x(t) = [S^\top(t,0)]^{-1}x_0$. Then

$$\begin{aligned} x(\omega) = \frac{1}{c}x_0 \Longleftrightarrow [S^\top(\omega,0)]^{-1}x_0 = \frac{1}{c}x_0 \Longleftrightarrow \big[cI - S^\top(\omega,0)\big]x_0 = 0 \\ \Longleftrightarrow x_0 \in \ker\big(cI - S^\top(\omega,0)\big) = \ker(cI - S(\omega,0))^\top. \end{aligned} \tag{4.6}$$

Therefore,

$$\dim\ker(cI - S^\top(\omega,0)) = n - \operatorname{rank}(cI - S^\top(\omega,0)) = n - \operatorname{rank}(cI - S(\omega,0)) = l.$$

Hence the adjoint system (4.4) has l linearly independent $(\omega, 1/c)$-periodic solutions. □

Theorem 4.4. *Let y and x be the solutions of* (2.1) *and* (4.4)*, respectively. Then $\langle y(t), x(t)\rangle$ is constant for $t \ge 0$.*

Proof. Let $t \in (s_i, t_{i+1}]$, $i = 0,1,\dots$. Then

$$\begin{aligned} \langle y(t), x(t)\rangle' &= \langle y'(t), x(t)\rangle + \langle y(t), x'(t)\rangle \\ &= \langle A(t)y(t), x(t)\rangle + \langle y(t), -A^\top(t)x(t)\rangle \\ &= \langle y(t), A^\top(t)x(t)\rangle + \langle y(t), -A^\top(t)x(t)\rangle = 0. \end{aligned}$$

Let $t \in (t_i, s_i]$, $i = 1,2,\dots$. Then

$$\begin{aligned} \langle y(t), x(t)\rangle &= \langle B_i(t)y(t_i^-), [B_i^\top(t)]^{-1}x(t_i^-)\rangle \\ &= \langle y(t_i^-), B_i^\top(t)[B_i^\top(t)]^{-1}x(t_i^-)\rangle \\ &= \langle y(t_i^-), x(t_i^-)\rangle. \end{aligned}$$

Let $t = t_i$, $i = 1,2,\dots$, then

$$\begin{aligned} \langle y(t_i^+), x(t_i^+)\rangle &= \langle B_i(t_i)y(t_i^-), [B_i^\top(t_i)]^{-1}x(t_i^-)\rangle \\ &= \langle y(t_i^-), B_i^\top(t_i)[B_i^\top(t_i)]^{-1}x(t_i^-)\rangle \\ &= \langle y(t_i^-), x(t_i^-)\rangle. \end{aligned}$$

Thus, $\langle x(t), y(t)\rangle = \langle x(0), y(0)\rangle$ which is a constant. □

Lemma 4.5. *Assume that* (A1), (A2), (A4) *hold. Then* (1.2) *has l linearly independent (ω, c)-periodic solutions if and only if*

$$\int_0^\omega \langle x(\tau), \tilde h(\tau)\rangle_{X^*,X}\,d\tau + \sum_{k=1}^m \langle x(s_k), b_k\rangle_{X^*,X} = 0 \tag{4.7}$$

Proof. Let y be a (ω, c)-periodic solution of (1.2), and the initial condition $y(0) = y_0$ satisfy

$$(cI - S(\omega,0))y_0 = \int_0^\omega S(\omega,\tau)\tilde h(\tau)d\tau + \sum_{k=1}^{i(\omega,0)} S(\omega, s_k)b_k. \tag{4.8}$$

Let $x(t)$ be a nontrivial $(\omega, 1/c)$-periodic solution of the adjoint system (4.4). Using (4.6), we obtain

$$0 = \big\langle (cI - S(\omega,0))^\top x_0, y_0\big\rangle_{X^*,X} = \big\langle x_0, (cI - S(\omega,0))y_0\big\rangle_{X^*,X}$$

$$
\begin{aligned}
&= \big\langle x_0, \int_0^\omega S(\omega,\tau)\tilde{h}(\tau)d\tau + \sum_{i=1}^m S(\omega,s_k)b_k\big\rangle_{X^*,X} \\
&= \int_0^\omega \langle x_0, S(\omega,\tau)\tilde{h}(\tau)\rangle_{X^*,X} d\tau + \sum_{i=1}^m \big\langle x_0, S(\omega,s_k)b_k\big\rangle_{X^*,X} \\
&= \int_0^\omega \langle S^\top(\omega,\tau)x_0, \tilde{h}(\tau)\rangle_{X^*,X} d\tau + \sum_{i=1}^m \big\langle S^\top(\omega,s_k)x_0, b_k\big\rangle_{X^*,X} \\
&= \int_0^\omega \langle x(\tau), \tilde{h}(\tau)\rangle_{X^*,X} d\tau + \sum_{i=1}^m \langle x(s_k), b_k\rangle_{X^*,X}.
\end{aligned}
$$

This proves the necessity part.

Suppose (4.7) holds. Assume the solution $x(t)$ is a $(\omega,1/c)$-periodic solution of the adjoint system (4.4) with initial condition $x(0)=x_0$. Therefore, the solution is given by $x(\tau)=S^\top(\omega,\tau)x_0$, where a_j is a nonzero constant. Using (4.7), we have

$$
\begin{aligned}
0 &= \int_0^\omega \langle x(\tau), \tilde{h}(\tau)\rangle_{X^*,X} d\tau + \sum_{i=1}^m \langle x(s_k), b_k\rangle_{X^*,X} \\
&= \int_0^\omega \langle S^\top(\omega,\tau)x_0, \tilde{h}(\tau)\rangle_{X^*,X} d\tau + \sum_{k=1}^m \langle S^\top(\omega,s_k)x_0, b_k\rangle_{X^*,X} \\
&= \int_0^\omega \langle x_0, S(\omega,\tau)\tilde{h}(\tau)\rangle_{X^*,X} d\tau + \sum_{k=1}^m \langle x_0, S(\omega,s_k)b_k\rangle_{X^*,X} \\
&= \big\langle x_0, \int_0^\omega S(\omega,\tau)\tilde{h}(\tau)d\tau + \sum_{k=1}^m S(\omega,s_k)b_k\big\rangle_{X^*,X} \\
&= \big\langle x_0, (cI - S(\omega,0))y_0\big\rangle_{X^*,X}
\end{aligned} \tag{4.9}
$$

Note that (4.8) and (4.9) imply that $(cI-S(\omega,0))y_0=0$ is equivalent to $(cI-S(\omega,0))^\top x_0=0$. The system (4.8) has a solution if and only if $\operatorname{rank}(cI-S(\omega,0)) = \operatorname{rank}(cI-S(\omega,0))^\top = n-l$. Thus, the system (1.2) has l linearly independent (ω,c)-periodic solutions. This proof is complete. □

Lemma 4.6. *Assume that* (A10) *holds. Then*

$$
\begin{aligned}
&\sum_{j=1}^m \|Q(t,s_j)b_j\| \\
&\le L_\varpi := \begin{cases} KNM^{m+1}\exp(\varpi\alpha m)(KNM^{(m+1)}\exp(\varpi\alpha m) \\ \times\|(cI-S(\omega,0))^{-1}\|+1)\sum_{j=1}^{m-1}\|b_j\|, & \varpi>0, \\ KNM^{m+1}\sum_{j=1}^m[KNM^{m+1}\|(cI-S(\omega,0))^{-1}\|+1]\|b_j\|, & \varpi\le 0. \end{cases}
\end{aligned}
$$

for any $t\in[0,\omega]$.

Proof. According to (4.3) and Theorem 3.1, we have

$$
\sum_{j=1}^m \|Q(t,s_j)b_j\|
$$

$$
\begin{aligned}
&\leq \sum_{j=1}^{m} \|Q(t,s_j)\| \|b_j\| \\
&= \sum_{j=1}^{i(t,0)-1} \|Q(t,s_j)\| \|b_j\| + \sum_{j=i(t,0)}^{m} \|Q(t,s_j)\| \|b_j\| \\
&\leq \sum_{j=1}^{i(t,0)-1} \|S(t,0)(cI - S(\omega,0))^{-1} S(\omega,s_j) + S(t,s_j)\| \|b_j\| \\
&\quad + \sum_{j=i(t,0)}^{m} \|S(t,0)(cI - S(\omega,0))^{-1} S(\omega,s_j)\| \|b_j\| \\
&\leq \sum_{j=1}^{i(t,0)-1} \Big[\|S(t,0)\| \|(cI - S(\omega,0))^{-1}\| \|S(\omega,s_j)\| + \|S(t,s_j)\|\Big] \|b_j\| \\
&\quad + \sum_{j=i(t,0)}^{m} \|S(t,0)\| \|(cI - S(\omega,0))^{-1}\| \|S(\omega,s_j)\| \|b_j\| \\
&\leq \sum_{j=1}^{i(t,0)-1} [KNM^{i(t,0)+1} \exp(\varpi\alpha i(t,0)) \|(cI - S(\omega,0))^{-1}\| KNM^{i(t,0)+1} \\
&\quad \times \exp(\varpi\alpha i(\omega,s_j)) + KNM^{i(t,0)+1} \exp(\varpi\alpha i(t,s_j))] \|b_j\| \\
&\quad + \sum_{j=i(t,0)}^{m} KNM^{i(t,0)+1} \exp(\varpi\alpha i(t,0)) \|(cI - S(\omega,0))^{-1}\| KNM^{i(t,0)+1} \\
&\quad \times \exp(\varpi\alpha i(\omega,s_j)) \|b_j\| \\
&= \sum_{j=1}^{i(t,0)-1} [K^2N^2M^{2(i(t,0)+1)} \exp(\varpi\alpha(i(t,0) + i(\omega,s_j)) \|(cI - S(\omega,0))^{-1}\| \\
&\quad + KNM^{i(t,0)+1} \exp(\varpi\alpha i(t,s_j))] \|b_j\| \\
&\quad + \sum_{j=i(t,0)}^{m} K^2N^2M^{2(i(t,0)+1)} \exp(\varpi\alpha(i(t,0) + i(\omega,s_j))) \|(cI - S(\omega,0))^{-1}\| \|b_j\|.
\end{aligned}
$$

For $\varpi > 0$,

$$
\begin{aligned}
&\sum_{j=1}^{m} \|Q(t,s_j)\| \|b_j\| \\
&\leq \sum_{j=1}^{i(t,0)-1} [K^2N^2M^{2(i(t,0)+1)} \exp(\varpi\alpha(i(t,0) + i(\omega,s_j)) \|(cI - S(\omega,0))^{-1}\| \\
&\quad + KNM^{i(t,0)+1} \exp(\varpi\alpha i(t,s_j))] \|b_j\| \\
&\quad + \sum_{j=i(t,0)}^{m} K^2N^2M^{2(i(t,0)+1)} \exp(\varpi\alpha(i(t,0) + i(\omega,s_j))) \|(cI - S(\omega,0))^{-1}\| \|b_j\|
\end{aligned}
$$

$$
\begin{aligned}
&\le \sum_{j=1}^{m} K^2N^2M^{2(m+1)}\exp(\varpi\alpha 2m)\|(cI-S(\omega,0))^{-1}\|\|b_j\| \\
&\quad + \sum_{j=1}^{m} KNM^{m+1}\exp(\varpi\alpha m)\|b_j\| \\
&\le KNM^{m+1}\exp(\varpi\alpha m)(KNM^{(m+1)}\exp(\varpi\alpha m)\|(cI-S(\omega,0))^{-1}\|+1)\sum_{j=1}^{m-1}\|b_j\|.
\end{aligned}
$$

For $\varpi \le 0$,

$$
\begin{aligned}
&\sum_{j=1}^{m}\|Q(t,s_j)\|\|b_j\| \\
&\le \sum_{j=1}^{i(t,0)-1}[K^2N^2M^{2(i(t,0)+1)}\exp(\varpi\alpha(i(t,0)+i(\omega,s_j))\|(cI-S(\omega,0))^{-1}\| \\
&\quad + KNM^{i(t,0)+1}\exp(\varpi\alpha i(t,s_j))]\|b_j\| \\
&\quad + \sum_{j=i(t,0)}^{m} K^2N^2M^{2(i(t,0)+1)}\exp(\varpi\alpha(i(t,0)+i(\omega,s_j)))\|(cI-S(\omega,0))^{-1}\|\|b_j\| \\
&\le \sum_{j=1}^{m}[K^2N^2M^{2(m+1)}\|(cI-S(\omega,0))^{-1}\|+KNM^{m+1}]\|b_j\| \\
&\le KNM^{m+1}\sum_{j=1}^{m}[KNM^{m+1}\|(cI-S(\omega,0))^{-1}\|+1]\|b_j\|.
\end{aligned}
$$

The proof is complete. □

Lemma 4.7. *Assume that* (A10) *holds. For* $0<t<\omega$*, we have*

$$
\begin{aligned}
&\int_0^{\omega}\|Q(t,\tau)\|d\tau \\
&\le K_{\varpi} \\
&:= \begin{cases} KNM^{m+1}[KNM^{m+1}\exp(\varpi\alpha m)\|(cI-S(\omega,0))^{-1}\|+1]\exp(\varpi\alpha m)\omega, & \varpi>0, \\ KNM^{m+1}[KNM^{m+1}\exp(\varpi\alpha m)\|(cI-S(\omega,0))^{-1}\|+1]\omega, & \varpi\le 0. \end{cases}
\end{aligned}
$$

Proof. According to (4.3), from (3.1), we have

$$
\begin{aligned}
&\int_0^{\omega}\|Q(t,\tau)\|d\tau \\
&\le \int_0^t\|Q(t,\tau)\|d\tau+\int_t^{\omega}\|Q(t,\tau)\|d\tau \\
&= \int_0^t\|S(t,0)(cI-S(\omega,0))^{-1}S(\omega,\tau)+S(t,\tau)\|d\tau \\
&\quad + \int_t^{\omega}\|S(t,0)(cI-S(\omega,0))^{-1}S(\omega,\tau)\|d\tau \\
&\le \int_0^t[\|S(t,0)\|\|(cI-S(\omega,0))^{-1}\|\|S(\omega,\tau)\|+\|S(t,\tau)\|]d\tau
\end{aligned}
$$

$$
\begin{aligned}
&+\int_t^{\omega}\|S(t,0)\|\|(cI-S(\omega,0))^{-1}\|\|S(\omega,\tau)\|d\tau\\
&\le \int_0^{\omega}\|S(t,0)\|\|(cI-S(\omega,0))^{-1}\|\|S(\omega,\tau)\|d\tau+\int_0^t\|S(t,\tau)\|d\tau\\
&\le \int_0^{\omega}KNM^{i(t,0)+1}\exp(\varpi\alpha i(t,0))\|(cI-S(\omega,0))^{-1}\|KNM^{i(\omega,\tau)+1}\\
&\quad\times\exp(\varpi\alpha i(\omega,\tau))d\tau+\int_0^t KNM^{i(t,\tau)+1}\exp(\varpi\alpha i(t,\tau))d\tau\\
&\le KNM^{m+1}[KNM^{m+1}\exp(\varpi\alpha m)\|(cI-S(\omega,0))^{-1}\|+1]\int_0^{\omega}\exp(\varpi\alpha i(t,\tau))d\tau.
\end{aligned}
$$

For $\varpi > 0$,

$$
\begin{aligned}
&\int_0^{\omega}\|Q(t,\tau)\|d\tau\\
&\le KNM^{m+1}[KNM^{m+1}\exp(\varpi\alpha m)\|(cI-S(\omega,0))^{-1}\|+1]\int_0^{\omega}\exp(\varpi\alpha i(t,\tau))d\tau\\
&\le KNM^{m+1}[KNM^{m+1}\exp(\varpi\alpha m)\|(cI-S(\omega,0))^{-1}\|+1]\exp(\varpi\alpha m)\omega.
\end{aligned}
$$

For $\varpi \le 0$,

$$
\begin{aligned}
&\int_0^{\omega}\|Q(t,\tau)\|d\tau\\
&\le KNM^{m+1}[KNM^{m+1}\exp(\varpi\alpha m)\|(cI-S(\omega,0))^{-1}\|+1]\int_0^{\omega}\exp(\varpi\alpha i(t,\tau))d\tau\\
&\le KNM^{m+1}[KNM^{m+1}\exp(\varpi\alpha m)\|(cI-S(\omega,0))^{-1}\|+1]\omega.
\end{aligned}
$$

The proof is complete. □

5. Nonlinear non-instantaneous impulsive systems

In this section, we apply the Banach fixed point theorem and the Schauder fixed point theorem to establish existence theorems for (ω, c)-periodic solutions of (1.3).

Theorem 5.1. *Assume that* (A1)–(A3), (A6), (A7), (A10) *hold. If* $0 < L_u K_\varpi < 1$, *then* (1.3) *has a unique* (ω, c)*-periodic solution* $y \in \Psi_{\omega,c}$ *satisfying*

$$
\|y\|_{PC} \le \frac{\tilde{f}_0 K_\varpi + L_\varpi}{1 - L_u K_\varpi},
$$

where $\tilde{f}_0 = \max_{t\in[0,\omega]} |f(t,0)|$.

Proof. Consider any $y \in \Psi_{\omega,c}$, i.e., $y(\cdot + \omega) = cy(\cdot)$. From assumption (A6),

$$
f(t+\omega, y(t+\omega)) = f(t+\omega, cy(t)) = cf(t,y), \quad t \in \mathbb{R}^+.
$$

Thus, $f(\cdot, y(\cdot)) \in \Psi_{\omega,c}$.

From Lemma 4.2, our goal is to consider the fixed point problem

$$
y(t) = \int_0^{\omega} Q(t,\tau)\tilde{f}(\tau, y(\tau))d\tau + \sum_{j=1}^{m} Q(t,s_j)b_j.
$$

We consider the operator $P : \Psi \to \Psi$ given by

$$Py(t) = \int_0^\omega Q(t,\tau)\tilde{f}(\tau, y(\tau))d\tau + \sum_{j=1}^m Q(t, s_j)b_j. \tag{5.1}$$

For any $x, y \in \Psi$, we have

$$\begin{aligned}
\|Px(t) - Py(t)\| &\le \int_0^\omega \|Q(t,\tau)\tilde{f}(\tau, x(\tau)) - Q(t,\tau)\tilde{f}(\tau, y(\tau))\|d\tau \\
&\le \int_0^\omega \|Q(t,\tau)\|\|\tilde{f}(\tau, x(\tau)) - \tilde{f}(\tau, y(\tau))\|d\tau \\
&\le L_u\|x - y\|_{PC} \int_0^\omega Q(t,\tau)d\tau \\
&\le L_u K_\varpi \|x - y\|_{PC}.
\end{aligned}$$

This implies that $\|Px - Py\|_{PC} \le L_u K_\varpi \|x - y\|_{PC}$. Since $0 < L_u K_\varpi < 1$, operator P is a contraction mapping. Thus, P has a unique fixed point. Furthermore, using Lemma 4.6, we have

$$\begin{aligned}
\|y(t)\| = \|Py(t)\| &\le \int_0^\omega \|Q(t,\tau)\|\|\tilde{f}(\tau, y(\tau))\|d\tau + \sum_{j=1}^m \|Q(t, s_j)\|\|b_j\| \\
&\le \int_0^\omega \|Q(t,\tau)\|\|\tilde{f}(\tau, y(\tau)) - \tilde{f}(\tau, 0) + \tilde{f}(\tau, 0)\|d\tau + \sum_{j=1}^m \|Q(t, s_j)\|\|b_j\| \\
&\le L_u\|y\|_{PC} \int_0^\omega \|Q(t,\tau)\|d\tau + \int_0^\omega \|Q(t,\tau)\|\|\tilde{f}(\tau, 0)\|d\tau + \sum_{j=1}^m \|Q(t, s_j)\|\|b_j\| \\
&\le L_u K_\varpi \|y\|_{PC} + \|\tilde{f}_0\| K_\varpi + L_\varpi.
\end{aligned}$$

Thus

$$\|y\|_{PC} \le \frac{\tilde{f}_0 K_\varpi + L_\varpi}{1 - L_u K_\varpi}.$$

The proof is complete. □

Theorem 5.2. *Assume that* (A1)–(A3), (A8)–(A11) *hold. If* $0 < \gamma K_\varpi < 1$, *then* (1.3) *has a* (ω, c)*-periodic solution* $y \in \Psi_{\omega,c}$.

Proof. Consider the operator P defined in (5.1) on $B_l := \{y \in \Omega \mid \|y\| \le l\}$, and $l \ge \frac{\alpha K_\varpi + L_\varpi}{1 - \gamma K_\varpi}$.

Step 1. We show that $P(B_l) \subset B_l$. For any $y \in B_l, t \in [0, \omega]$, by Lemmas 4.6 and 4.7 and (A8), we have

$$\begin{aligned}
\|(Py)(t)\| &\le \int_0^\omega \|Q(t,\tau)\|\|\tilde{f}(\tau, y(\tau))\|d\tau + \sum_{j=1}^m \|Q(t, s_j)b_j\| \\
&\le \gamma \int_0^\omega \|Q(t,\tau)\|\|y(\tau)\|d\tau + \alpha \int_0^\omega \|Q(t,\tau)\|d\tau + \sum_{i=1}^m \|Q(t, s_j)b_j\| \\
&\le \gamma K_\varpi \|y\|_{PC} + \alpha K_\varpi + L_\varpi = l,
\end{aligned}$$

which implies that $\|Py\|_{PC} \le l$. Thus $P(B_l) \subset B_l$ for any $y \in B_l$ and $t \in [0, \omega]$.

Step 2. We prove that P is continuous. Let y_n be a Cauchy sequence such that $y_n \to y$ (as $n \to \infty$) in B_l. Since $\tilde{f}(\cdot, y_n(\cdot)) \to \tilde{f}(\cdot, y(\cdot))$ as $y_n \to y$ for any $t \in [0, \omega]$, we obtain

$$\begin{aligned}
\|(Py_n)(t) - (Py)(t)\| &\le \int_0^\omega \|Q(t,\tau)\| \|\tilde{f}_n(\tau, y_n(\tau) - \tilde{f}(\tau, y(\tau))\| d\tau \\
&\le \|\tilde{f}_n - \tilde{f}\|_{PC} \int_0^\omega \|Q(t,\tau)\| d\tau \\
&\le K_\varpi \|\tilde{f}_n - \tilde{f}\|_{PC}.
\end{aligned}$$

Thus, P is continuous.

Step 3. We show that $P(B_l)$ is relatively compact set. Since $P(B_l) \subset B_l$, it is easy to see that $P(B_l)$ is uniformly bounded. Next, we show that P is an equicontinuous operator. For $0 < t_1 < t_2 \le \omega$ and $y \in B_l$, we have

$$\begin{aligned}
&\|(Py)(t_2) - (Py)(t_1)\| \\
&\le \int_0^\omega \|Q(t_2,\tau) - Q(t_1,\tau)\| \|\tilde{f}(\tau, y(\tau))\| d\tau + \sum_{j=1}^m \|Q(t_2, s_j) - Q(t_1, s_j)\| \|b_j\| \\
&\le (\alpha + \beta\|y\|) \int_0^\omega \|Q(t_2,\tau) - Q(t_1,\tau)\| d\tau + \sum_{i=1}^m \|Q(t_2, s_j) - Q(t_1, s_j)\| \|b_j\|.
\end{aligned} \tag{5.2}$$

From (4.3), we obtain

$$\begin{aligned}
&\|Q(t_2,\tau) - Q(t_1,\tau)\| \\
&= \begin{cases} \|S(t_2,0)(cI - S(\omega,0))^{-1}S(\omega,\tau) + S(t_2,\tau) \\ -S(t_1,0)(cI - S(\omega,0))^{-1}S(\omega,\tau) - S(t_1,\tau)\|, & \text{if } 0 < \tau < t_1 < t_2, \\ \|S(t_2,0)(cI - S(\omega,0))^{-1}S(\omega,\tau) \\ -S(t_1,0)(cI - S(\omega,0))^{-1}S(\omega,\tau)\|, & \text{if } t_1 < t_2 < \tau < \omega. \end{cases} \\
&\le \begin{cases} \|(cI - S(\omega,0))^{-1}\| \|S(\omega,\tau)\| \|S(t_2,0) - S(t_1,0))\| \\ +\|S(t_2,\tau) - S(t_1,\tau))\|, & \text{if } 0 < \tau < t_1 < t_2, \\ \|S(t_2,0) - S(t_1,0)\| \|(cI - S(\omega,0))^{-1}\| \|S(\omega,\tau))\|, & \text{if } t_1 < t_2 < \tau < \omega. \end{cases}
\end{aligned} \tag{5.3}$$

Letting $t_1 \to t_2$, from (5.3) and the compactness of $S(\cdot,\cdot)$ we have

$$Q(t_2,\tau) \to Q(t_1,\tau), \text{ as } t_1 \to t_2.$$

Thus, we have $\|(Py)(t_1) - (Py)(t_2)\| \to 0$ as $t_1 \to t_2$. Then P is an equicontinuous operator.

Now consider the approximate operator P_ϵ on B_l as follows

$$\int_0^\omega Q(t-\epsilon,\tau)\tilde{f}(\tau, y(\tau)) d\tau + \sum_{j=1}^m Q(t-\epsilon, s_j) b_j, \ t \in [0,\omega]. \tag{5.4}$$

Consider $K = \{(Py)(t) : t \in [0,\omega]\}$ and

$$K_\epsilon = S(\epsilon, 0)\{(P_\epsilon y)(t) : t \in [0,\omega]\}, \quad 0 < \epsilon < \omega.$$

From Theorem 3.5 and K bounded, K_ϵ is precompact. Next,

$$\begin{aligned}\|(P_\epsilon y)(t)-(Py)(t)\| &\le \int_0^\omega \|Q(t,\tau)-Q(t-\epsilon,\tau)\|\|f(\tau,y(\tau))\|d\tau\\ &\quad+\sum_{i=1}^m \|Q(t,s_i)-Q(t-\epsilon,s_i)\|\|b_j\|\\ &\le (\alpha+\beta l)\int_0^\omega \|Q(t-\epsilon,s)-Q(t,s)\|ds\\ &\quad+\sum_{i=1}^m \|Q(t-\epsilon,s_i)-Q(t,s_i)\|\|b_i\|.\end{aligned} \tag{5.5}$$

Then $\|(P_\epsilon y)(t)-(Py)(t)\|$ tends to zero when $\epsilon\to 0$. Thus K can be approximated to an arbitrary degree of accuracy by a precompact set K_ϵ. Hence K itself is a precompact set in X, that is, B takes a bounded set into a precompact set in X. The Arzelà-Ascoli theorem implies the compactness of B. Thus Schauder's fixed point theorem guarantees the result. The proof is complete. $\square$

Next we present an example. Since we develop our theory mainly for infinite dimensional Banach spaces, we need consider a partial differential equation.

Example 5.3. We consider the problem

$$\begin{gathered}\frac{\partial}{\partial t}y(t,x)=(2+\sin 2t)\frac{\partial^2}{\partial x^2}y(t,x)+a\sin t+b\frac{y^3}{2+y^2},\\ x\in(0,\pi),\quad t\in[s_{i-1},t_i],\quad i\in N^+,\\ y(t_i^+,x)=2y(t_i^-,x),\quad x\in(0,\pi),\\ y(t,x)=(2-\frac{t-t_i}{s_i-t_i})y(t_i^-,x),\quad t\in(t_i,s_i],\ x\in(0,\pi),\\ y(s_i^+,x)=y(s_i^-,x)\,[=y(t_i^-,x)],\quad x\in(0,\pi),\\ y(t,0)=y(t,\pi)=0,\quad t\ge 0,\end{gathered} \tag{5.6}$$

where $a,b\in\mathbb{R}$, $a\neq 0$, $b\neq 0$, $0=t_0=s_0$, $t_i=(2i-1)\pi/2$, $s_i=i\pi$.

Let $B_i(t)=2-\frac{t-t_i}{s_i-t_i}$, $m=1$, $\omega=\pi$. then

$$\begin{aligned}B_{i+m}(t+\omega)=B_{i+1}(t+\pi)&=2-\frac{t+\pi-t_{i+1}}{s_{i+1}-t_{i+1}}=2-\frac{t+\pi-t_i-\pi}{s_i+\pi-t_i-\pi}\\ &=2-\frac{t-t_i}{s_i-t_i}=B_i(t).\end{aligned}$$

Set $X=L^2(0,\pi)$ with a norm $\|y\|=\sqrt{\int_0^\pi y^2(t)}dt$. Define $A(t)y=(2+\sin 2t)\frac{\partial^2}{\partial x^2}y$ for $y\in D(A)=\big\{y\in X:\frac{\partial y}{\partial x},\frac{\partial^2 y}{\partial x^2}\in X, y(0)=y(\pi)=0\big\}$. Then $A(t)$ is the infinitesimal generator of a strongly continuous semigroup $\{S(t,s),t\ge 0\}$ in X. Indeed, we know that the sequence $\big\{\sqrt{2/\pi}\sin kx\}_{k\in N}$ is an orthonormal basis of X. Thus for

$$y_0=\sum_{k\in N}y_{0k}\sqrt{\frac{2}{\pi}}\sin kt\in X,\quad \|y_0\|=\sqrt{\sum_{k\in N}y_{0k}^2},$$

we have

$$\begin{aligned} S(t,s)y_0 &= \sum_{k\in N} \exp\Big(-k^2\Big(2(t-s)+\frac{\cos 2s-\cos 2t}{2}\Big)\Big)y_{0k}\sqrt{\frac{2}{\pi}}\sin kt \\ \Rightarrow \|S(t,s)y_0\| &= \sqrt{\sum_{k\in N}\exp(-k^2(4(t-s)+\cos 2s-\cos 2t))y_{0k}^2} \\ &\le \exp(-(t-s))\|y_0\| \end{aligned}$$

Hence $M=1$ and $\varpi=-1$. Moreover, $\sigma(S(\pi,0))=\{e^{-2\pi k^2}, k\in\mathbb{N}\}$, so $-1\notin\sigma(S(\omega,0))$. Next,

$$(-I-S(\pi,0))^{-1}y_0 = -\sum_{k\in\mathbb{N}}\frac{1}{1+e^{-2\pi k^2}}y_{0k}\sqrt{\frac{2}{\pi}}\sin kx,$$

and then

$$\|(-I-S(\pi,0))^{-1}y_0\| = \Big\|\sum_{k\in\mathbb{N}}\frac{1}{1+e^{-2\pi k^2}}y_{0k}\sqrt{\frac{2}{\pi}}\sin kx\Big\| = \sqrt{\sum_{k\in\mathbb{N}}\frac{1}{(1+e^{-2\pi k^2})^2}y_{0k}^2}$$

and

$$\sup_{k\in\mathbb{N}}\frac{1}{(1+e^{-2\pi k^2})^2}=1,\quad \|y_0\|=\sqrt{\sum_{k\in\mathbb{N}}y_{0k}^2},$$

thus

$$\|(-I-S(\pi,0))^{-1}\| = \sup_{\|y_0\|=1}\|(-I-S(\pi,0))^{-1}y_0\|=1,$$

and $K_\varpi=2\pi$.

On the other hand, we have $f(t,y)=a\sin t+b\frac{y^3}{2+y^2}$, $t\in\mathbb{R}^+$. Then

$$f(t+\pi,-y)=a\sin(t+\pi)+b\frac{(-y)^3}{2+(-y)^2}=-(a\sin t+b\frac{y^3}{2+y^2})=-f(t,y)$$

for $t\in\mathbb{R}^+$, so $c=-1$ and

$$\begin{aligned} \|f(t,y)\| &\le \|a\sin t\|+\Big\|b\frac{y^3}{2+y^2}\Big\| \\ &\le |a|\|\sin t\|+|b|\Big(\int_0^\pi\frac{y^6(t)}{(2+y^2(t))^2}dt\Big)^{1/2} \\ &\le |a|\sqrt{\pi/2}+|b|\Big(\int_0^\pi y^2(t)dt\Big)^{1/2} \\ &\le |a|\sqrt{\pi/2}+|b|\|y\|. \end{aligned}$$

We have $\alpha=|a|\sqrt{\pi/2}$ and $\gamma=|b|$. Then $0<\gamma K_\varpi<1$ reduces to $0<2\pi|b|<1$, which holds for some suitable b: Let $|b|=1/(3\pi)$ and then $0<\gamma K_\varpi=2/3<1$. Thus all the assumptions in Theorem 5.2 are satisfied. Accordingly, system (5.6) has a $(\pi,-1)$-periodic solution.

6. Conclusions

In this paper, time-varying development systems in infinite-dimensional space are studied for the first time. When $A(t)$ and $B_i(t)$ are not commutative, the Cauchy operator of the linear homogeneous system is constructed, and the study of the system is transformed into its corresponding Cauchy operator. Firstly, some properties of Cauchy operator are obtained. A sufficient and necessary condition for the existence of (ω, c)-periodic solutions for linear homogeneous systems is given. The existence of (ω, c)-periodic solutions in critical and noncritical cases for linear inhomogeneous systems is discussed. The existence of periodic solutions for nonlinear systems (ω, c)-periodic solutions is obtained using Banach's fixed point theorem and Schauder's fixed point theorem.

Acknowledgments. This work was partially supported by the National Natural Science Foundation of China (12161015), by the Guizhou Provincial Science and Technology Foundation ([2020]1Y002), by the Training Object of High Level and Innovative Talents of Guizhou Province ((2016)4006), by the Youth Development Project of Guizhou Provincial Education Department ([2021]266), by the Academic seedling cultivation and innovation exploration special cultivation project plan: GZLGXM-17, by the Guizhou Data Driven Modeling Learning and Optimization Innovation Team ([2020]5016), by the Slovak Research and Development Agency under the contract No. APVV-18-0308, and by the Slovak Grant Agency VEGA No. 1/0358/20 and No. 2/0127/20.

References

[1] Abbas, S.; Benchohra, M.; *Uniqueness and Ulam stabilities results for partial fractional differential equations with not instantaneous impulses*, Applied Mathematics and Computation, 2015, 257, 190-198.

[2] Agaoglou, M.; Fečkan, M.; Panagiotidou, A.; *Existence and uniqueness of (ω, c)-periodic solutions of semilinear evolution equations*, International Journal of Dynamical Systems and Differential Equations, 2020, 10, 149–166.

[3] Agarwal, R.; O'Regan, D.; Hristova, S.; *Stability with initial time difference of Caputo fractional differential equations by Lyapunov functions*, Zeitschrift Fur Analysis Und Ihre Anwendungen, 2017, 36, 49–77.

[4] Ahmed, N.; *Semigroup theory with applications to systems and control, vol.246 of Pitman Research notes im mathematics series*, Longman Scientific and Technical, Harlow, UK, 1991.

[5] Alvarez, E.; Gómez, A.; Pinto, M.; *(ω, c)-periodic functions and mild solutions to abstract fractional integro-differential equations*, Electronic Journal of Qualitative Theory of Differential Equations, 2018 (2018), 16, 1-8.

[6] Cooke, C.; Kroll, J.; *The existence of periodic solutions to certain impulsive differential equations*, Computers and Mathematics Applications, 2002, 44, 667–676.

[7] Fečkan, M.; Liu, K.; Wang, J.; *(ω, c)-periodic solutions of non-instantaneous impulsive evolution equations*, Dynamic Systems and Applications, 2020, 29, 3359-3380.

[8] Fečkan, M.; Wang, J.; Zhou, Y.; *Existence of periodic solutions for nonlinear evolution equations with non-instantaneous impulses*, Nonautonomous Dynamical Systems, 2014, 1, 93–101.

[9] Gautam, G.; Dabas, J.; *Mild solutions for class of neutral fractional functional differential equations with not instantaneous impulses*, Applied Mathematics and Computation, 2015, 259, 480–489.

[10] Hernández, E.; O'Regan, D.; *On a new class of abstract impulsive differential equations*, Proceedings of the American Mathematical Society, 2013, 141, 1641–1649.

[11] Li, M.; Wang, J.; Fečkan, M.; *(ω, c)-periodic solutions for impulsive differential systems*, Communications in Mathematical Analysis, 2018, 21, 35–45.

[12] Li, X.; Martin, B.; Wang, C.; *Impulsive differential equations: periodic solutions and applications*, Automatica, 2015, 52, 173–178.

[13] Liu, J.; *Bounded and periodic solutions of finite delay evolution equations*, Nonlinear Analysis: Theory, Methods Applications, 1998, 34, 101–111.

[14] Muslim, M.; Kumar, A.; Fečkan, M.; *Existence, uniqueness and stability of solutions to second order nonlinear differential equations with non-instantaneous impulses*, Journal of King Saud University Science, 2018, 30, 204–213.

[15] Pazy, A.; *Semigroups of linear operators and applications to partial differential equations*, Springer-Verlag, 1983.

[16] Peng, Y.; Xiang, X.; *Second order nonlinear impulsive time-veriant systems with unbounded perturbation and optimal controls*, Journal of Industrial and Management Optimization, 2008, 4, 17–32.

[17] Peng, Y.; Xiang, X.; *Second order nonlinear impulsive evolution equations with time-varying generating operators and optimal controls*, optimization, 2008, 57, 827–840.

[18] Peng, Y.; Xiang, X.; Wei, W.; *Nonlinear impulsive integro-differential equations of mixed type with time-varying generating operators and optimal controls*, Dynamic Systems and Applications, 2007, 16, 481–496.

[19] Peng, Y.; Xiang, X.; Wei, W.; *Second-order nonlinear impulsive integro-differential equations of mixed type with time-varying generating operators and optimal controls*, Computers and Mathematics Applications, 2009, 57, 42–53.

[20] Pierri, M.; Henríquez, H.; Prokczyk, A.; *Global solutions for abstract differential equations with non-instantaneous impulses*, Mediteranean Journal of Mathematics, 2016, 34, 1685–1708.

[21] Pierri, M.; O'Regan, D.; Rolnik, V.; *Existence of solutions for semi-linear abstract differential equations with not instantaneous impulses*, Applied Mathematics and Computation, 2013, 219, 6743–6749.

[22] Trudinger, N.; *Elliptic partial differential equations of second order*, Springer-Verlag, New York, 1977.

[23] Wang, C.; *Almost periodic solutions of impulsive BAM neural networks with variable delays on time scales*, Communications in Nonlinear Science and Numerical Simulation, 2014, 19, 2828–2842.

[24] Wang, J.; *Stability of noninstantaneous impulsive evolution equations*, Applid Mathmatics Letters, 2017, 73, 157–162.

[25] Wang, J.; Fečkan, M. Tian, Y.; *Stability analysis for a general class of non-instantaneous impulsive differential equations*, Mediteranean Journal of Mathematics, 2017, 14, 1–21.

[26] Wang, J.; Ibrahim, A.; O'Regan, D.; *Hilfer type fractional differential switched inclusions with noninstantaneous impulsive and nonlocal conditions*, Nonlinear Analysis-Modelling and Control, 2018, 23, 921–941.

[27] Wang, J.; Ren, L.; Zhou, Y.; *(ω, c)-periodic solutions for time varying impulsive differential equations*, Advances in Difference Equations, 2019, 2019, No.259.

[28] Wang, J.; Xiang, X.; Wei, W.; *Linear impulsive periodic system with time-varying generating operators on Banach space*, Advances in Difference Equations, 2007, 2007, 1–16.

[29] Wang, J.; Xiang, X.; Wei, W.; *Periodic solutions of semilinear impulsive periodic system with time-varying generating operators on Banach space*, Mathematical Problems in Engineering, 2008, 2008, 1–15.

[30] Wang, J.; Xiang, X.; Wei, W.; *Periodic solutions of a class of integro-differential impulsive periodic systems with time-varying generating operators on Banach space*, Electronic Journal of Qualitative Theory of Differential Equations, 2009 (2009), 4, 1–17.

[31] Wang, J.; Xiang, X.; Wei, W.; *Linear impulsive periodic system with time-varying generating operators on Banach space*, Advances in Difference Equations, 2007, 2007, 1–16.

[32] Yang, D.; Wang, J.; *Non-instantaneous impulsive fractional-order implicit differential equations with random effects*, Stochastic Analysis and Applications, 2017, 35, 719–741.

[33] Yang, D.; Wang, J.; O'Regan, D.; *On the orbital Hausdorff dependence of differential equations with non-instantaneous impulses*, Comptes Rendus Mathematique, 2018, 356, 150–171.

[34] Yang, P.; Wang, J.; Fečkan, M.; *Periodic nonautonomous differential equations with non-instantaneous impulsive effects*, Mathematical Methods in the Applied Sciences, 2019, 42, 3700–3720.

Kui Liu
Department of Mathematics, Guizhou University, Guiyang, Guizhou 550025, China.
Science College, Guizhou Institute of Technology, Guiyang, Guizhou 550025, China
Email address: liuk180916@163.com

Michal Fečkan
Department of Mathematical Analysis and Numerical Mathematics, Faculty of Mathematics, Physics and Informatics, Comenius University in Bratislava, Mlynská dolina, 842 48 Bratislava, Slovakia.
Mathematical Institute, Slovak Academy of Sciences, Štefánikova 49, 814 73 Bratislava, Slovakia
Email address: Michal.Feckan@fmph.uniba.sk

Donal O'Regan
School of Mathematics, Statistics and Applied Mathematics, National University of Ireland, Galway, Ireland
Email address: donal.oregan@nuigalway.ie

Jinrong Wang (corresponding author)
Department of Mathematics, Guizhou University, Guiyang, Guizhou 550025, China
Email address: jrwang@gzu.edu.cn