Electronic Journal of Differential Equations, Vol. 2022 (2022), No. 23, pp. 1–30.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu

OPEN MAPPINGS: THE CASE FOR A NEW DIRECTION IN FIXED POINT THEORY

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Abstract. Classical fixed point theorems often begin with the assumption that we have a mapping P of a closed convex set in a Banach space G into itself. It then adds a number of conditions which will ensure that there is at least one fixed point in the set G. We continue two earlier studies in which we now propose to stop the process after we have mapped G not only into itself, but into its interior. We then study what we may deduce from this alone.

1. Introduction

This article addresses the difficulties raised for fixed point theory by a product of terms in quadratic integral equations of the form

$$x(t) = g(t, x(t)) + f(t, x(t)) \int_0^t A(t-s)v(t, s, x(s))ds \quad t \geq 0. \tag{1.1}$$

Because of f, the right-hand-side of (1.1) does not map sets of bounded continuous functions into compact sets. We advance the idea of addressing the difficulties by means of construction of a closed bounded convex nonempty set G which is mapped into its interior by the natural mapping defined by the equation. Among several other properties, this shows that all possible fixed points of the mapping of the whole space reside entirely in G.

This article is motivated by three ideas which we wish to describe in some detail so that the reader can follow the subsequent work. We believe that these ideas establish a general pattern which is quite useful in attacking fixed point problems.

(i) Many fixed point theorems begin with the assumption that there is a mapping P of a closed convex set G in a normed space mapping into G. Typically, nothing is said at this point about the origin of G but it might be assumed that the investigator chooses G for technical convenience in construction and for the fact that any point in G would be a satisfactory fixed point for the problem at hand. Let us say that the points in G are "good points" while there may, indeed, be points outside of G which would not be at all satisfactory fixed points for this project (i.e., not having desired properties, e.t.c.) and we could refer to them as "bad points".

In any event, having chosen G, there are then added technical conditions from which we would then deduce that there is a fixed point in G. Not only can it be

2020 *Mathematics Subject Classification.* 34A08, 45D05, 45G05, 47H09, 47H10.
Key words and phrases. Open mappings; progressive contractions; existence; uniqueness; quadratic integral equations.

Submitted October 15, 2020. Published March 21, 2022.

a struggle to show that there is a fixed point in G, it is still a worry that there might also be a bad fixed point outside of G in the frequent case of non-uniqueness. Perhaps we should work a bit harder on G before we proceed to those additional technical assumptions which are to bring in a fixed point.

This leads us to two earlier studies [7, 8] showing that if we strengthen P to the conclusion that $P : G \to G^o$, the interior of G, and ignore the additional technical assumptions, then we can say that any fixed point will be a good one. Upon reflection the investigator may find that this information is almost as good as the solution of the entire original problem when we realize that we have said that any solution is a good one.

We will return to this statement in the next section, but it is timely to mention that this result would be a counterpart to Schaefer's fixed point theorem which tells us that there is at least one fixed point, but by the very nature of the theorem we have no idea where the fixed point is or how many there might be. In summary, the first idea is to work on G and show that $P : G \to G^o$.

(ii) Here is the second idea. We are concerned with avoiding abstractions and we wish to present all of this in the form of well-known fundamental problems with roots in several kinds of real-world problems. Early and enduring fixed point investigations have centered on integral equations of the form

$$x(t) = g(t, x(t)) + \int_0^t K(t, s)v(t, s, x(s))ds,$$

for their importance in applied mathematics and the fact that the integral term under a wide set of conditions will define a compact map [10, 17, 11], and compactness of the mapping is one of the frequent technical assumptions on P. There is, however, a large set of very important problems discussed by Darwish and Henderson [13] and [14] called quadratic integral equations of the form

$$x(t) = g(x, x(t)) + f(t, x(t)) \int_0^t K(t, s)v(t, s, x(s))ds,$$

with g, f, v satisfying global Lipschitz conditions. The problem now is that the coefficient function f destroys the properties of the integral which no longer can define an equicontinuous map. Darwish and Henderson then employ Darbo's fixed point theorem and measures of non-compactness to obtain a non-unique, but global solution. In addition to the non-uniqueness one may find a long term pattern of such work relying on properties of f and v to force the integral term to tend to zero. See, for example, an early result of Banas and Rezepka [2] and later the paper by Darwish and Henderson [14, p. 76, (h_4)]. We seek fewer conditions and a more elementary approach, as well as a worthy target, to display the proposed method of this paper. That method is progressive contractions. In Section 6 we avoid both the smoothness requirements and conditions driving the integral term to zero.

In sum, our second idea is that although Lipschitz conditions can be very difficult to verify from examination of a physical problem, a Lipschitz relation

$$|g(t, x) - g(t, y)| \le L|x - y|$$

with $y = 0$ yields a mild growth condition on g of the form

$$|g(t, x)| \le |g(t, 0)| + |x|,$$

and we will see that this can replace the Lipschitz condition and take us well on our way to showing $P : G \to G^o$. But it is better than that. Once we have all fixed points in G we could reinstate the Lipschitz condition to hold only in G which can now be regarded as a complete metric space and the same mapping P would map G into G all ready for a contraction. Those conditions reinstated into G can be drastically reduced in case G is bounded and since the whole space is now G. The scarcely noticed opening assumption of all the upcoming fixed point theorems that $P : G \to G$ has become the cornerstone and the uniting property.

(iii) This brings us to the third idea and a fairly recent elementary technique called progressive contractions. Once we obtain the set G containing all possible fixed points then G can be considered as a complete metric space and it is true that $P : G \to G^o$. Now all of our work is taking place in the complete metric space G which has inherited the metric from the original Banach or norm space. If, for example, G is bounded then we could reinstate the Lipschitz conditions asking that they hold only in G which drastically reduces the growth conditions. For example, if G is bounded then $f(x) = x^2$ has become globally Lipschitz. Then we can often use the reduced Lipschitz conditions on f and v, together with progressive contractions, to produce contractions using the properties of K and short intervals of t. Progressive contractions do depend on K having certain properties found widely in both applied mathematics and fractional equations. The work of Darwish and Henderson does involve such a kernel. Finally we parlay this into a unique global solution in G. Obviously, these last two items will only become clear later as we introduce the progressive contractions.

As described, our work here is a two stage problem, with Stage 1 being the construction of G and showing that $P : G \to G^o$. Stage 2 is then the process of showing that there is a fixed point in G. Both are done in Section 6.

It is interesting to note that this work will be a counterpart to the well-known fixed point theorem of Schaefer which states that there is a fixed point, but we have no idea where it is. In our case here, after we complete Stage 1 we do not know if there is a fixed point, but, if there is one, we know it is in G. For reference, here is Schaefer's theorem and some terminology from the book of Smart [22, p. 25] which we follow here.

Definition 1.1. Let P map a set $\mathcal{S}$ into a topological space $\mathcal{X}$. If $P\mathcal{S}$ is contained in a compact subset of $\mathcal{X}$, we say that P is compact.

Theorem 1.2. *Let $\mathcal{B}$ be a normed space, P a continuous mapping of $\mathcal{B} \to \mathcal{B}$ which is compact on each bounded subset $\mathcal{X}$ of $\mathcal{B}$. Then either*

(i) *the equation $x = \lambda Px$ has a solution for $\lambda = 1$, or*
(ii) *the set of all such solutions x, for $0 < \lambda < 1$ (if any), is unbounded.*

This article is organized in nine sections. The first five sections represent an introduction to a method of attacking a wide class of important problems from the real world by means of fixed point theory. The main existence result along with some lemmas are given in Section 6. In Section 7 an asymptotic result concerning the (unique) solution of (1.1) is presented. In Section 8 we give two examples illustrating the results obtained, thus overcoming difficulties found in work on this subject over the last twenty years. The last section is devoted to a comparison between the results of this paper and others obtained by the use of Darbo's theorem.

Regarding context, four distinct parts may be seen in the paper. While they all work together toward the goal of getting a unique solution on $[0, \infty)$ and many of its properties, each part can also stand alone and be read, largely, independently of the other parts not discussed here. Part I covers Sections 1-5 and has focus on a description and properties of the construction of the mapping set, G. Part II is Section 6 through the first half of the proof of Theorem 6.3. It deals with very simple assumptions and at that point of the proof it shows that there is a unique solution on a short interval which resides in G restricted to that interval. In short, it is almost certain that if we can find that set G then solutions are restricted to G and there is, indeed, a unique solution in G on a short interval. Our task now is to extend that solution to $[0,\infty)$. That is where Part III begins and the more demanding conditions. It is a fundamental contribution in that, for equations of this type, it offers a result parallel to the classical result given in Miller [20, pp. 97-98] for the equation without f by showing how to extend that solution to $[0,\infty)$. The conditions are much like those of Miller. Part IV covers Sections 7-9 and offers a number of qualitative properties of this solution and concrete examples. It is of particular interest that, as proved in Section 7, the asymptotic behavior or the solution of (1.1) is related to the behavior of the solution of an algebraic equation. Parts II and IV may be especially suitable as stand alone topics which quickly show the main results without a large investment of time.

2. Three theorems and a guiding example

We have three theorems which generated this study. We begin with Schauder's second theorem [22, p. 25] which is the old line result on which so many results rest.

Theorem 2.1. *Let G be a non-empty convex subset of a normed space $\mathcal{B}$. Let P be a continuous mapping of G into a compact set $\mathcal{K} \subset G$. Then P has a fixed point.*

The next theorem is known as Krasnoselskii's fixed point theorem [22, p. 31] on the sum of two operators.

Theorem 2.2. *Let G be a closed convex non-empty subset of a Banach space $\mathcal{B}$. Suppose that A and B map G into $\mathcal{B}$ and that*

(i) $Ax + By \in G \quad (\forall x, y \in G)$.
(ii) *A is compact and continuous.*
(iii) *B is a contraction mapping.*

Then there exists $y \in G$ such that $Ay + By = y$.

The following is a form of Darbo's fixed point theorem as given in [16].

Theorem 2.3. *Let G be a nonempty, bounded, closed and convex subset of the Banach space $\mathcal{B}$ and let $P : G \to G$ be a contraction with respect to the measure of noncompactness μ. Then P has a fixed point in the set G.*

We see that each of these major theorems begins with the assumption that there is a mapping in a normed or Banach space of a set G into itself. The idea here is to stop at that point and see what can be proved. We conjecture that it is sign and growth conditions which yield G, while conditions to satisfy the fixed point theorem are on the order of technical necessity.

We want to give this paper some perspective so we continually refer back to two interesting papers by Darwish [13], and, by Darwish and Henderson [14], concerning a variety of real world problems modeled by the "quadratic" fractional integral equation

$$x(t) = g(t, x(t)) + \frac{f(t, x(t))}{\Gamma(\beta)} \int_0^t \frac{v(t, s, x(s))}{(t-s)^{1-\beta}} ds, \tag{2.1}$$

under a list of conditions on the functions.

Papers [13] and [14] offer a very impressive list of references of the application of that equation. Their objective is to use Darbo's fixed point to prove the existence of solutions in the space of real functions defined, continuous, and bounded on an unbounded interval. None of these theorems mention how many fixed point exist or assure us that all fixed points are in G. These authors do not separate the problem into stages as we have described, but rather list their final assumptions in four main groups which we will mention later. However, as we examine the problem we would expect to solve the Stage 1 by requiring continuity, sign conditions, and, decay conditions on the functions involved, in order to obtain the mapping set G. As we look at the conditions obtained by the authors, it appears that the first two requirements are technical conditions on the functions which will enable them to invoke Darbo's theorem. On the other hand, we see that the sign and growth conditions are required for the existence of G, and these conditions do not seem to rely on the aforementioned technical assumptions such as contractions and Lipschitz requirements. Our intent is to get G just based on sign and growth conditions, not on the stronger Lipschitz conditions (see Lemma 6.2).

3. Volterra operators, G, and uniqueness

Much can be said immediately about G and uniqueness, if the mapping involves Volterra operators. A definition is found in [12, p. 84]. In order that the operator V (defined in the Banach space $(\mathcal{B}, \|\cdot\|)$ of continuous functions $\phi : [0, T] \to \mathbb{R}$ with the supremum norm) be a Volterra operator, V should satisfy $(Vx)(t) = (Vy)(t)$, for any pair of functions $x, y \in \mathcal{B}$ for which $x(s) = y(s)$ for $0 \le s < t \le T$. Volterra operators are said to be *non-anticipative.* Equation (2.1) is a prime example generating a Volterra operator where

$$\lim_{t \downarrow 0} \int_0^t (t-s)^{\beta-1} ds = 0.$$

When P is the natural operator defined by (1.1) and the integral has the value zero at $t = 0$, then for any function $x \in \mathcal{B}$ it holds $(Px)(0) = g(0, x(0))$. In particular if $\phi \in \mathcal{B}$ is a fixed point of P so that $P\phi = \phi$, then it is true that

$$(P\phi)(0) = \phi(0) = g(0, \phi(0)). \tag{3.1}$$

Now if the equation

$$x = g(0, x),$$

has a unique solution $x_0 \in \mathbb{R}$, as is the case of g being a contraction, then there is a unique starting value x_0 for all solutions ϕ of (1.1) (if any), and any solution ϕ starts at $\phi(0) = x_0$, although there can be many solutions starting at that point, as may be seen for

$$x(t) = \int_0^t e^{-(t-s)} x^{1/3}(s) ds.$$

Thus, if $\phi(0)$ is such that (3.1) holds, then we would construct G satisfying at least the following condition: There is a function ψ residing in the interior of G

with

$$\psi(0) = \phi(0).$$

4. All fixed points reside in G

This section is mainly a repeat of part of a theorem we offered in [7] and [8]. Here, we take G to be a ball, but the proof is quite simple and may be extended to other closed bounded non-empty convex sets.

Theorem 4.1. *Let $T > 0$ and $(\mathcal{B}, \|\cdot\|)$ be the Banach space of continuous functions $\phi : [0,T] \to \Re$ with the supremum norm and let P be a Volterra operator mapping $\mathcal{B} \to \mathcal{B}$ which is continuous. Let $r > 0$ and G be the closed ball of center zero and radius r in $\mathcal{B}$:*

$$G := \{\phi \in \mathcal{B} : \|\phi\| \le r\}.$$

Suppose that $P : G \to G^o$ has the property that if $\phi \in \mathcal{B}$ and if $(P\phi)(0) = \phi(0)$ then $|\phi(0)| < r$. If $\phi \in \mathcal{B}$ is a fixed point of P, then ϕ resides in G^o.

Proof. If the theorem is false, then there is a fixed point ϕ not residing in G. Recall that if ϕ is a fixed point then $|\phi(0)| < r$ and, hence, by the continuity of $P\phi$ there is a $T^* \in (0,T]$ with ϕ in G^o on $[0,T^*)$, but $|\phi(T^*)| = r$. Now ϕ is a fixed point so $(P\phi)(T^*) = \phi(T^*)$ Then for the function

$$\phi_{T^*}(t) = \phi(t) \quad 0 \le t < T^*,$$

and

$$\phi_{T^*}(t) = r \quad T^* \le t \le T,$$

we have ϕ_{T^*} in G so $P\phi_{T^*}$ is in the interior of G yielding the contradiction $r = |(P\phi)(T^*)| = |\phi(T^*)| < r$. □

5. A summary

In almost every problem in applied mathematics of this sort there might be functions x which might satisfy (1.1), but they are of such a nature that we cannot tolerate them in our working model. They are *excluded* and our mapping set G must be constructed to exclude them. Now, suppose we have found G and it contains no excluded points. Suppose also that we have satisfied the other conditions of one of these three theorems mentioned in Section 2. We then have a fixed point, but none of the theorems give uniqueness and there could be a fixed point outside G.

In the case of (1.1), our first consideration (Stage 1) is to check to see that for some $\phi(0)$ solving $\phi(0) = g(0, \phi(0)))$ there is $\psi \in G^o$, the interior of G with $\psi(0) = \phi(0)$. Then, in our second consideration (Stage 2), the entire process depends on being able to show that $P : G \to G^o$.

If we can complete Stage 1, then Stage 2 is less restrictive because it applies only to G. Moreover, then we can be sure that any fixed point will not be one of those which we have excluded. We know that any fixed point resides in G and, hence, it will satisfy the properties common to all functions in G. There is, then, a sense in which we have a unique solution, because it shares these common properties. The conditions on the functions in P now only apply in G, which is now the whole space for this work.

The conditions for Stage 1 will frequently be much simpler than those required for an one step application of one of our three fixed point theorems. Next, we have control over the location of the fixed points and can be sure that our application of the fixed point theorem now will not produce an excluded point.

6. G is the whole space, Stage 2

Equation (1.1) will be the vehicle to show how the ideas in the introduction are implemented. It is in 2 parts, namely (A) and (B).

(A) Lemma 6.2 obtains a closed bounded convex nonempty set G and growth conditions without Lipschitz smoothness on f or v to ensure that the natural mapping P defined by the right-hand-side of (1.1) would map the set G to its interior G^o, i.e., $P : G \to G^o$, without requiring that the integral term converges to zero. At this point we will know that if there is a solution, then this solution resides in G, a result that is useful information even if we proceed no further. In the paragraph containing (1.1) in the Introduction we pointed out that investigators use Darbo's theorem to avoid the difficulty of the integral term being noncompact. But the price of this were several conditions making that term tend to zero. By using progressive contractions we avoid that completely as seen in both, Lemma 6.2, and, Theorem 6.3.

(B) After Lemma 6.2 we introduce some smoothness and show that there is a unique solution in G on any interval $[0, T]$. This is Theorem 6.3 and, again, the integral term need not converge to zero. We could construct such a solution on each of the intervals $[0, 1], \dots, [0, n]$ and parlay them into a unique global solution on $[0, \infty)$ which is bounded because G is. To see this, if ϕ_i is the fixed point on $[0, i]$, let $\Phi_i = \phi_i$ on $[0, i]$ and extend it by $\Phi_i(t) = (\Phi_i)(i)$ on $[i, \infty)$. This sequence then converges uniformly on compact sets to a continuous function which is a fixed point.

To begin, we recall that our equation is

$$x(t) = g(t, x(t)) + f(t, x(t)) \int_0^t K(t, s)v(t, s, x(s))\, ds, \quad t \geq 0, \tag{E}$$

with $g : [0, \infty) \times \mathbb{R} \to \mathbb{R}$, $f : (0, \infty) \times \mathbb{R} \to \mathbb{R}$, $K : \{(t, s) : 0 < s < t\} \to \mathbb{R}$, $v : \{(t, s) : 0 < s < t\} \times \mathbb{R} \to \mathbb{R}$ continuous, and assume that: (A1) There exist bounded continuous functions $\ell_1, g_1 : [0, \infty) \to [0, \infty)$ with

$$g_1^* := \sup_{0\leq t} g_1(t), \quad \ell_1^* := \sup_{0\leq t} \ell_1(t),$$

and, a continuous nondecreasing function $Z_1 : [0, \infty) \to [0, \infty)$ such that

$$|g(t, x)| \leq g_1(t) + \ell_1(t)Z_1(|x|) \quad t \geq 0 \quad x \in \mathbb{R}. \tag{6.1}$$

(A2) There exist continuous functions $m_1, f_1 : (0, \infty) \to [0, \infty)$ and a continuous nondecreasing function $\Psi_1 : [0, \infty) \to [0, \infty)$ such that

$$|f(t, x)| \leq f_1(t) + m_1(t)\Psi_1(|x|), \quad t > 0\ x \in \mathbb{R}. \tag{6.2}$$

(A3) There exist continuous functions $n_1, v_1 : \{(t, s) : 0 < s < t\} \to [0, \infty)$ and a continuous nondecreasing function $\Phi_1 : [0, \infty) \to [0, \infty)$ such that

$$|v(t, s, x)| \leq v_1(t, s) + n_1(t, s)\Phi_1(|x|) \quad t > 0, \quad x \in \mathbb{R}. \tag{6.3}$$

(A4) The functions $\eta_1, \psi_1, \phi_1, \xi_1$ defined by

$$\eta_1(t) = f_1(t) \int_0^t |K(t, s)|v_1(t, s)\, ds \quad t \geq 0,$$

$$\begin{aligned} \psi_1(t) &:= m_1(t)\int_0^t |K(t,s)|v_1(t,s)\,ds \quad t\geq 0,\\ \phi_1(t) &:= f_1(t)\int_0^t |K(t,s)|n_1(t,s)\,ds \quad t\geq 0,\\ \xi_1(t) &:= m_1(t)\int_0^t |K(t,s)|n_1(t,s)\,ds \quad t\geq 0, \end{aligned}$$

are continuous.

Note that the functions f, K, m_1, f_1, n_1, v_1 may not be defined at $t = 0$, yet K, v, n_1, v_1 may not be defined at $t = s$, so singularities at these points are allowed. By (A4) we require that the functions $\eta_1, \phi_1, \psi_1, \xi_1$ are defined and are continuous at $t = 0$, thus we ask that the limits of these functions for $t \to 0+$ are real numbers. Consequently, since we are concerned with solutions continuous at $t = 0$, for the initial condition at $t = 0$ we will always have

$$x(0) = g(0, x(0)) + \lim_{t\to 0+} f(t, x(t))\int_0^t K(t,s)v(t,s,x(s))\,ds \in \mathbb{R}.$$

It should be mentioned that the limits of $\eta_1, \phi_1, \psi_1, \xi_1$ for $t \to 0+$ may not be necessarily zero. Our first lemma concerns the starting point of the solutions (if any) of the equation (1.1) in case that these limits are zero.

Lemma 6.1. *Assume that* (A1)–(A4) *are satisfied. If*

$$\lim_{t\to 0+}\eta_1(t) = 0 = \lim_{t\to 0+}\psi_1(t) = \lim_{t\to 0+}\phi_1(t) = \lim_{t\to 0+}\xi_1(t), \tag{6.4}$$

then for any solution x of the equation (E) we have that $x(0) = x_0$, where x_0 satisfies the equation

$$x_0 = g(0, x_0).$$

Proof. Let x be a solution of (1.1) and consider a $T > 0$. As x is continuous at $t = 0$, by the continuity of g we have that

$$\begin{aligned} x(0) = \lim_{t\to 0+} x(t) &= \lim_{t\to 0+}\Big[g(t,x(t)) + f(t,x(t))\int_0^t K(t,s)v(t,s,x(s))\,ds\Big]\\ &= g(0,x(0)) + \lim_{t\to 0+} f(t,x(t))\int_0^t K(t,s)v(t,s,x(s))\,ds. \end{aligned}$$

By continuity of x there exists an $r_1 > 0$ with $|x(t)| \leq r_1, t \in [0,T]$. Employing (A1)–(A4) we take for $t \in (0,T]$

$$\begin{aligned} &\big|f(t,x(t))\int_0^t K(t,s)v(t,s,x(s))\,ds\big|\\ &\leq |f(t,x(t))|\int_0^t |K(t,s)||v(t,s,x(s))|\,ds\\ &\leq [f_1(t) + m_1(t)\Psi_1(|x(t)|)]\int_0^t |K(t,s)|[v_1(t,s) + n_1(t,s)\Phi_1(|x(t)|)]ds\\ &\leq [f_1(t) + m_1(t)\Psi_1(r_1)]\int_0^t |K(t,s)|[v_1(t,s) + n_1(t,s)\Phi_1(r_1)]ds\\ &\leq f_1(t)\int_0^t |K(t,s)|v_1(t,s)\,ds + \Phi_1(r_1)|f_1(t)|\int_0^t |K(t,s)|n_1(t,s)\,ds \end{aligned}$$

$$
\begin{aligned}
&+ \Psi_1(r_1)m_1(t)\int_0^t |K(t,s)|v_1(t,s)\,ds \\
&+ \Psi_1(r_1)\Phi_1(r_1)m_1(t)\int_0^t |K(t,s)|n_1(t,s)\,ds \\
&= \eta_1(t) + \Phi_1(r_1)\phi_1(t) + \Psi_1(r_1)\psi_1(t) + \Psi_1(r_1)\Phi_1(r_1)\xi_1(t),
\end{aligned}
$$

from which by (6.4) we have

$$
\lim_{t\to 0+} \big|f(t,x(t))\int_0^t K(t,s)v(t,s,x(s))\,ds\big| = 0,
$$

thus $x(0) = g(0,x_0) = x_0$. □

The next lemma concerns part (A). It gives conditions yielding the existence of a subset G of the Banach space $BC(I)$ of bounded continuous functions on $I = [0,\infty)$ equipped with the usual sup-norm, such that the mapping $\mathcal{T} : BC(I) \to C(I)$ with

$$
\mathcal{T}x(t) := g(t,x(t)) + f(t,x(t))\int_0^t K(t,s)v(t,s,x(s))\,ds \quad t \ge 0,
$$

maps this set into its interior. For a positive $r > 0$ we set

$$
B_r := \{x \in BC(I) : \|x\| \le r\}.
$$

Lemma 6.2. *Assume that* (A1)–(A4) *hold. Moreover, assume that*

(A5) *the functions $\eta_1, \psi_1, \phi_1, \xi_1$ are bounded on $[0,\infty)$ with bounds, respectively, $\eta_1^*, \psi_1^*, \phi_1^*, \xi_1^*$, and, there exists an $r_0 > 0$ such that*

$$
g_1^* + \ell_1^* Z_1(r_0) + \eta_1^* + \Phi_1(r_0)\phi_1^* + \Psi_1(r_0)\psi_1^* + \Psi_1(r_0)\Phi_1(r_0)\xi_1^* < r_0. \tag{6.5}
$$

Then $\mathcal{T}(B_{r_0}) \subset B_{r_0}^o$.

Proof. Let (A1)–(A5) hold and $r_0 > 0$ satisfy (6.5). In view of (A1)–(A4), for $x \in B_{r_0}$ we have, for $t \ge 0$,

$$
\begin{aligned}
|\mathcal{T}x(t)| &\le |g(t,x(t))| + |f(t,x(t))|\int_0^t |K(t,s)||v(t,s,x(s))|\,ds \\
&\le g_1(t) + \ell_1(t)Z_1(|x(t)|) \\
&\quad + [f_1(t) + m_1(t)\Psi_1(|x(t)|)]\int_0^t |K(t,s)|[v_1(t,s) + n_1(t,s)\Phi_1(|x(s)|)]ds \\
&\le g_1^* + \ell_1^* Z_1(\|x\|) + \\
&\quad + [f_1(t) + m_1(t)\Psi_1(\|x\|)]\int_0^t |K(t,s)|[v_1(t,s) + n_1(t,s)\Phi_1(\|x\|)]ds \\
&\le g_1^* + \ell_1^* Z_1(r_0) \\
&\quad + [f_1(t) + m_1(t)\Psi_1(r_0)]\int_0^t |K(t,s)|[v_1(t,s) + n_1(t,s)\Phi_1(r_0)]ds \\
&= g_1^* + \ell_1^* Z_1(r_0) + f_1(t)\int_0^t |K(t,s)|v_1(t,s)\,ds \\
&\quad + \Phi_1(r_0)f_1(t)\int_0^t |K(t,s)|n_1(t,s)\,ds \\
&\quad + \Psi_1(r_0)m_1(t)\int_0^t |K(t,s)|v_1(t,s)\,ds
\end{aligned}
$$

$$\begin{aligned}
&+ \Psi_1(r_0)\Phi_1(r_0)m_1(t)\int_0^t |K(t,s)|n_1(t,s)\,ds\\
&= g_1^* + \ell_1^* Z_1(r_0)\\
&\quad + \eta_1(t) + \Phi_1(r_0)\phi_1(t) + \Psi_1(r_0)\psi_1(t) + \Psi_1(r_0)\Phi_1(r_0)\xi_1(t),
\end{aligned}$$

and so, for $x \in B_{r_0}$ we have for $t \geq 0$

$$|\mathcal{T}x(t)| \leq g_1^* + \ell_1^* Z_1(r_0) + \eta_1^* + \Phi_1(r_0)\phi_1^* + \Psi_1(r_0)\psi_1^* + \Psi_1(r_0)\Phi_1(r_0)\xi_1^*, \tag{6.6}$$

from which, in view of (6.5), it follows that $\|\mathcal{T}x\| < r_0 \Longrightarrow \mathcal{T}x \in B_{r_0}^o$. □

Now we cite the smoothness conditions mentioned in (B), above. With r_0 given in (6.5), for the functions g, f and v we assume the following:

(H1) There exists a continuous function $\ell_2 : [0,\infty) \to [0,\infty)$ *such that*

$$|g(t,x) - g(t,y)| \leq \ell_2(t)|x-y|, \quad |x|, |y| \leq r_0, t \geq 0, \tag{6.7}$$

and which is bounded with

$$\ell_2^* := \sup_{0\leq t} \ell_2(t) < 1.$$

(H2) There exists a continuous function $m_2 : (0,\infty) \to [0,\infty)$ *with*

$$|f(t,x) - f(t,y)| \leq m_2(t)|x-y|, \quad |x|, |y| \leq r_0, \quad t > 0, \tag{6.8}$$

and such that the function

$$\psi_2(t) := m_2(t)\int_0^t |K(t,s)|\,|v(t,s,0)|\,ds, \quad t \geq 0.$$

tends to zero for $t \to 0$

H3 There exists a continuous function $n_2(t,s) : \{(t,s) : 0 < s < t\} \to [0,\infty)$ with

$$|v(t,s,x) - v(t,s,y)| \leq n_2(t,s)|x-y| \quad |x|, |y| \leq r_0, \quad 0 < s < t, \tag{6.9}$$

and such that the functions

$$\phi_2(t) := |f(t,0)|\int_0^t |K(t,s)|n_2(t,s)\,ds, \quad t \geq 0,$$

$$\xi_2(t) := m_2(t)\int_0^t |K(t,s)|n_2(t,s)\,ds, \quad t \geq 0,$$

tend to zero for $t \to 0$.

In view of (H1)–(H3) we consider a $T_0 > 0$ such that

$$c_0 := \ell_2^* + \sup_{0\leq t\leq T_0}\{\phi_2(t) + 2r_0\xi_2(t) + \psi_2(t)\} < 1, \tag{6.10}$$

with r_0 satisfying (6.5).

Theorem 6.3. *Let* $T > 0$, (A1)–(A5) *hold, and,* r_0 *be a positive number satisfying* (6.5). *Suppose that* (H1)–(H3) *and* (6.4) *are satisfied and let* T_0 *be defined by* (6.10). *Furthermore, assume that:*

(H4) *There exists* $\gamma \in (1 - \ell_2^*)$ *and* $\delta > 0$ *with*

$$m_2(t+h)\int_0^t |K(t+h,s)|[n_2(t+h,s)r_0 + |v(t+h,s,0)|]ds \leq \gamma \tag{6.11}$$

for all $h \in [0,\delta]$ *and* $t \in [T_0, T]$.

(H5) It holds

$$\lim_{h\to 0}\int_0^h |K(t+h,t+s)|[n_2(t+h,t+s)+|v(t+h,t+s,0)|]ds=0,$$

uniformly for all $t\in[T_0,T]$.

Then equation (1.1) has a unique solution on $[0,T]$*. This solution starts from the unique solution* $x_0\in(-r_0,r_0)$ *of the equation* $g(0,x_0)=x_0$ *and is bounded by* r_0*. When* T *may be arbitrarily chosen, then* x_0 *can be extended to the whole real line.*

Proof. Let $T>0$ be an arbitrary positive number. We denote by $B^0:=BC([0,T_0])$ the Banach space of continuous functions defined on the interval $[0,T_0]$ equipped with the sup-norm $\|\cdot\|_0$, and set $B^0_{r_0}:=\{x\in B^0:\|x\|\le r_0\}$. Let the mapping $\mathcal{T}_0:B^0_{r_0}\to B^0$ be defined by

$$\mathcal{T}_0x(t):=g(t,x(t))+f(t,x(t))\int_0^t K(t,s)v(t,s,x(s))\,ds,\ t\in[0,T_0],$$

and note that by Lemma (6.2) it holds $\mathcal{T}_0(B^0_{r_0})\subset(B^0_{r_0})$. Now, for any $x,y\in B^0_{r_0}$ and $t\in[0,T_0]$ we have

$$\begin{aligned}
|\mathcal{T}_0x(t)-\mathcal{T}_0y(t)| &\le |g(t,x(t))-g(t,y(t))|+|f(t,x(t))\int_0^t K(t,s)v(t,s,x(s))\,ds\\
&\quad -f(t,y(t))\int_0^t K(t,s)v(t,s,y(s))\,ds|\\
&\le \ell_2(t)|x(t)-y(t)|\\
&\quad +|f(t,x(t))|\int_0^t |K(t,s)||v(t,s,x(s))-v(t,s,y(s))|\,ds\\
&\quad +|f(t,y(t))-f(t,x(t))|\int_0^t |K(t,s)||v(t,s,y(s))|\,ds\\
&\le \ell_2(t)|x(t)-y(t)|\\
&\quad +[|f(t,0)|+m_2(t)\|x\|]\int_0^t |K(t,s)|n_2(t,s)|x(s)-y(s)|\,ds\\
&\quad +m_2(t)|x(t)-y(t)|\int_0^t |K(t,s)|[n_2(t,s)\|x\|+|v(t,s,0)|]ds,
\end{aligned}$$

or,

$$\begin{aligned}
&|\mathcal{T}_0x(t)-\mathcal{T}_0y(t)|\\
&\le \ell_2^*\|x-y\|+[|f(t,0)|+m_2(t)r_0]\int_0^t |K(t,s)|n_2(t,s)\,ds\|x-y\|\\
&\quad +m_2(t)\int_0^t |K(t,s)|[n_2(t,s)r_0+|v(t,s,0)|]ds\|x-y\|,
\end{aligned}$$

from which, in view of the definitions of T_0 and c_0 in (6.10), we take

$$\|\mathcal{T}_0x-\mathcal{T}_0y\|\le c_0\|x-y\|_0\quad x,y\in B^0_{r_0},$$

i.e., $\mathcal{T}_0$ is a contraction in $B^0_{r_0}$. By Banach's fixed point theorem it follows that $\mathcal{T}_0$ has a unique fixed point in $B^0_{r_0}$, thus (1.1) has a unique solution x^0 in $B^0_{r_0}$.

If $T = T_0$, then our assertion has been proven. So now we assume that $T > T_0$. By (H5) we consider a positive number $\tau > 0$ with

$$0 < \tau \le \min\{\delta, T - T_0\}, \quad \frac{T - T_0}{\tau} := n \in \mathbb{N},$$

and such that for all $h \in [0, \tau]$ and all $t \in [T_0, T]$ it holds

$$\begin{aligned} &\int_0^h |K(t+h, t+s)|[n_2(t+h, t+s) + |v(t+h, t+s, 0)|]\, ds \\ &< \frac{1 - \gamma - \ell_2^*}{2(f_T + 2m_{2,T})}, \end{aligned} \tag{6.12}$$

where, in view of the continuity of f and m_2, we have set

$$f_T := \sup_{[T_0,T]\times[-r_0,r_0]} |f(t,u)|, \quad m_{2,T} := \sup_{[T_0,T]\times[-r_0,r_0]} |m_2(t,u)|.$$

Let

$$T_i := T_0 + i\tau \quad i = 1, \dots, n.$$

Since a solution of (1.1) on the interval $[0, T_0]$ has been established, our strategy now is to show that this solution can be extended, successively, on the inervals $[T_{i-1}, T_i]$, $i = 1, \dots, n$, thus obtaining a solution to (1.1) on the whole interval $[0, T]$.

In view of the above, we consider the set B^1 of continuous functions $x : [0, T_1] \to \mathbb{R}$, with $x(t) = x^0(t)$, $t \in [0, T_0]$, where x^0 is the (already established, unique) solution on $[0, T_0]$ of (1.1) i.e., we set

$$B^1 := \{x \in C([0, T_1]) : x(t) = x^0(t), t \in [0, T_0]\},$$

and note that this is a complete metric space when equipped with the sup-norm

$$\|\cdot\|_1 := \sup_{t\in[0,T_1]} |x(t)|.$$

For the number $r_0 > 0$ established in (6.5) we let

$$B^1_{r_0} := \{x \in B^1 : \|x\|_1 \le r_0\},$$

and consider the mapping $\mathcal{T}_1 : B^1_{r_0} \to B^1$ by

$$\mathcal{T}_1 x(t) := g(t, x(t)) + f(t, x(t)) \int_0^t K(t,s) v(t, s, x(s))\, ds \quad t \in [0, T_1]. \tag{6.13}$$

As by Lemma (6.2) we have that $\mathcal{T}_1(B^1_{r_0}) \subseteq B^1_{r_0}$, in order to prove existence of a solution to (1.1) on $[0, T_1]$ it suffices to prove that $\mathcal{T}_1$ is a contraction in $B^1_{r_0}$. For $y_1, y_2 \in B^1_{r_0}$ we note that $y_1(t) = y_2(t) = x^0(t)$, $t \in [0, T_0]$, so

$$|\mathcal{T}_1 y_1(t) - \mathcal{T}_1 y_2(t)| = 0 \quad t \in [0, T_0].$$

Now, instead of considering $t \in [T_0, T_1]$, we set $T_0 + t$ in (6.13) with $t \in [0, T]$, thus transferring (6.13) to

$$\begin{aligned} \mathcal{T}_1 x(T_0 + t) &:= g(T_0 + t, x(T_0 + t)) \\ &\quad + f(T_0 + t, x(T_0 + t)) \int_0^{T_0+t} K(T_0 + t, s) v(T_0 + t, s, x(s))\, ds, \end{aligned}$$

for $t \in [0, \tau]$. Finally, letting $z(t) = x(T_0 + t) \quad t \in [0, \tau]$, we have

$$\mathcal{T}_1 z(t) = g(T_0 + t, z(t))$$

$$
\begin{aligned}
&\quad + f(T_0+t, z(t)) \int_0^{T_0+t} K(T_0+t, s) v(T_0+t, s, z(s-T_0))\, ds \\
&= g(T_0+t, z(t)) \\
&\quad + f(T_0+t, z(t)) \int_0^{T_0} K(T_0+t, s) v(T_0+t, s, z(s-T_0))\, ds \\
&\quad + f(T_0+t, z(t)) \int_{T_0}^{T_0+t} K(T_0+t, s) v(T_0+t, s, z(s-T_0))\, ds,
\end{aligned}
$$

and

$$
\begin{aligned}
\mathcal{T}_1 z(t) := g(T_0+t, z(t)) + f(T_0+t, z(t)) \int_0^{T_0} K(T_0+t, s) v(T_0+t, s, x^0(s))\, ds \\
+ f(T_0+t, z(t)) \int_0^{t} K(T_0+t, s+T_0) v(T_0+t, s+T_0, z(s))\, ds
\end{aligned}
$$

for $t \in [0, \tau]$. Thus, for the y_1, y_2 considered, by setting $z_1(t) = y_1(T_0+t), z_2(t) = y_2(T_0+t)$ we see that

$$
\mathcal{T}_1 y_1(t) - \mathcal{T}_1 y_2(t) \quad t \in [T_0, T_1],
$$

is equivalent to

$$
\mathcal{T}_1 z_1(t) - \mathcal{T}_1 z_2(t) \quad t \in [0, \tau := T_1, T_0].
$$

It follows that all we have to prove is that $\mathcal{T}_1$ is a contraction on the ball of radius r_0 in the space of continuous functions on $[0, \tau]$ (with the sup-norm), i.e., that there exists a $\gamma_0 \in (0,1)$ such that for $z_1, z_2 \in C([0,\tau])$ with $\|z_1\|, \|z_2\| \le r_0$ it holds

$$
|\mathcal{T}_1 z_1(t) - \mathcal{T}_1 z_2(t)| \le \gamma_0 \|z_1 - z_2\| \quad t \in [0, \tau]. \tag{6.14}
$$

To this end, for $t \in [0, \tau]$ we have

$$
\begin{aligned}
&|\mathcal{T}_1 z_1(t) - \mathcal{T}_1 z_2(t)| \\
&\le |g(T_0+t, z_1(t)) - g(T_0+t, z_2(t))| + \Big| f(T_0+t, z_1(t)) \\
&\quad - f(T_0+t, z_2(t)) \Big| \int_0^{T_0} |K(T_0+t, s)|\, |v(T_0+t, s, x^0(s))|\, ds \\
&\quad + \Big| f(T_0+t, z_1(t)) \int_0^{t} K(T_0+t, s+T_0) v(T_0+t, s+T_0, z_1(s))\, ds \\
&\quad - f(T_0+t, z_2(t)) \int_0^{t} K(T_0+t, s+T_0) v(T_0+t, s+T_0, z_2(s))\, ds \Big|.
\end{aligned} \tag{6.15}
$$

By (H1) for $t \in [0, \tau]$ we take

$$
\begin{aligned}
&|g(T_0+t, z_1(t)) - g(T_0+t, z_2(t))| \\
&\le \ell_2(T_0+t) |z_1(t) - z_2(t)| \le \ell_2^* \|z_1 - z_2\|,
\end{aligned} \tag{6.16}
$$

while by (H4) we obtain

$$
\begin{aligned}
&|f(T_0+t,z_1(t))-f(T_0+t,z_2(t))\\
&\times \Big|\int_0^{T_0}|K(T_0+t,s)||v(T_0+t,s,x^0(s))|\,ds\\
&\le \|z_1-z_2\|m_2(T_0+t)\\
&\quad\times\int_0^{T_0}|K(T_0+t,s)|[n_2(T_0+t,s)r_0+|v(T_0+t,s,0)|]ds\\
&\le \gamma\|z_1-z_2\|.
\end{aligned}
\tag{6.17}
$$

Let

$$
\begin{aligned}
D(t;z_1,z_2):=\Big|&f(T_0+t,z_1(t))\int_0^t K(T_0+t,s+T_0)v(T_0+t,s+T_0,z_1(s))\,ds\\
&-f(T_0+t,z_2(t))\int_0^t K(T_0+t,s+T_0)v(T_0+t,s+T_0,z_2(s))\,ds\Big|.
\end{aligned}
$$

Employing (H2) and (H3) for $t\in[0,\tau]$ we have

$$
\begin{aligned}
&D(t;z_1,z_2)\\
&\le |f(T_0+t,z_1(t))|\int_0^t|K(T_0+t,s+T_0)|\\
&\quad\times|v(T_0+t,s+T_0,z_1(s))-v(T_0+t,s+T_0,z_2(s))|\,ds\\
&\quad+|f(T_0+t,z_1(t))-f(T_0+t,z_2(t))|\\
&\quad\times\int_0^t|K(T_0+t,s+T_0)||v(T_0+t,s+T_0,z_2(s))|\,ds\\
&\le f_T\|z_1-z_2\|\int_0^t|K(T_0+t,s+T_0)|n_2(T_0+t,s+T_0)\,ds+\|z_1-z_2\|\\
&\quad\times m_2(T_0+t)\int_0^t|K(T_0+t,s+T_0)|[n_2(T_0+t,s+T_0)+|v(T_0+t,s+T_0,0)|]ds\\
&=\|z_1-z_2\|\{f_T\int_0^t|K(T_0+t,s+T_0)|n_2(T_0+t,s+T_0)\,ds\\
&\quad+m_{2,T}\int_0^t|K(T_0+t,s+T_0)|[n_2(T_0+t,s+T_0)+|v(T_0+t,s+T_0,0)|]ds\}.
\end{aligned}
$$

In view of the definition of τ for $u=T_0$, from (6.12) we take

$$
\begin{aligned}
&D(t;z_1,z_2)\\
&\le\|z_1-z_2\|\int_0^t|K(T_0+t,s+T_0)|\\
&\quad\times[(f_T+m_{2,T})n_2(T_0+t,s+T_0)+m_{2,T}|v(T_0+t,s+T_0,0)|]ds\\
&\le(f_T+2m_{2,T})\|z_1-z_2\|\\
&\quad\times\int_0^t|K(T_0+t,s+T_0)|[n_2(T_0+t,s+T_0)+|v(T_0+t,s+T_0,0)|]ds,
\end{aligned}
$$

and

$$
D(t;z_1,z_2)\le\frac{1-\gamma-\ell_2^*}{2}\|z_1-z_2\|\quad t\in[0,\tau]. \tag{6.18}
$$

Thus, by (6.15), (6.16), (6.17) and (6.18) we have

$$|\mathcal{T}_1 z_1(t) - \mathcal{T}_1 z_2(t)| \le \Big(\ell_2^* + \gamma + \frac{1-\gamma-\ell_2^*}{2}\Big)\|z_1 - z_2\| \quad t \in [0,\tau],$$

from which it follows that

$$\|\mathcal{T}_1 z_1 - \mathcal{T}_1 z_2\| \le \gamma_0 \|z_1 - z_2\|,$$

with

$$\gamma_0 := \ell_2^* + \gamma + \frac{1-\gamma-\ell_2^*}{2} \le \frac{1+\gamma+\ell_2^*}{2} < \frac{1+1-\ell_2^*+\ell_2^*}{2} = 1,$$

i.e., (6.14) is proved. We conclude that there exists a unique solution x^1 to equation (1.1) on the interval $[0,T_1]$.

If $T = T_1$ then our assertion has been proved. If $T_1 < T$, we employ the solution x^1 and transfer (1.1) to the equation

$$\begin{aligned} \mathcal{T}_1 x(T_1+t) &:= g(T_1+t, x(T_1+t)) \\ &\quad + f(T_1+t, x(T_1+t)) \int_0^{T_1+t} K(T_1+t,s) v(T_1+t,s,x(s))\, ds, \end{aligned}$$

with $t \in [0,\tau]$, thus, by setting $z(t) = x(T_1+t)$, $t \in [0,\tau]$, we deal with the equation

$$\begin{aligned} \mathcal{T}_1 z(t) &:= g(T_1+t, z(t)) \\ &\quad + f(T_1+t, z(t)) \int_0^{T_1} K(T_1+t,s) v(T_1+t,s,x^1(s))\, ds \\ &\quad + f(T_1+t, z(t)) \int_0^{t} K(T_1+t,s+T_1) v(T_1+t,s+T_1,z(s))\, ds, \end{aligned}$$

with $t \in [0,\tau]$. Considering the set of continuous functions $x : [0,T_2] \to \mathbb{R}$, with $x(t) = x^1(t)$, $t \in [0,T_1]$, and following the argumentation used to prove existence on $[0,T_1]$ we may prove that x^1 is extended to a (unique) solution x^2 on $[0,T_2]$. It is now apparent that the procedure is finalized in a finite number of steps leading to a unique solution to (1.1) on the interval $[0,T]$. $\square$

Though condition (H4) seems rather peculiar, it can be replaced by a condition which is simpler to verify.

Lemma 6.4. *If there exists a* $\gamma \in (1-\ell_2^*)$ *with*

$$\sup_{t>0} m_2(t) \int_0^t |K(t,s)|[n_2(t,s) r_0 + |v(t,s,0)|] ds \le \gamma, \tag{6.19}$$

then (6.11) *is satisfied.*

Proof. For given $T, T_0 > 0$ and any $h > 0$, for $t \in [T_0, T]$, we have

$$\begin{aligned} & m_2(t+h) \int_0^t |K(t+h,s)|[n_2(t+h,s) r_0 + |v(t+h,s,0)|] ds \\ & \le m_2(t+h) \int_0^t |K(t+h,s)|[n_2(t+h,s) r_0 + |v(t+h,s,0)|] ds \\ & \quad + m_2(t+h) \int_t^{t+h} |K(t+h,s)|[n_2(t+h,s) r_0 + |v(t+h,s,0)|] ds \\ & = m_2(t+h) \int_0^{t+h} |K(t+h,s)|[n_2(t+h,s) r_0 + |v(t+h,s,0)|] ds \end{aligned}$$

$$\leq \sup_{t>0} m_2(t) \int_0^t |K(t,s)|[n_2(t,s)r_0 + |v(t,s,0)|]ds \leq \gamma,$$

which proves the assertion. □

Remark 6.5. If the kernel K is either continuous on $\{0 \leq s \leq t, t \geq 0\}$, or of fractional type, and $n_2(t,s)$ and $v(t,s,0)$ are continuous on $\{0 \leq s \leq t, t \geq 0\}$, then (H5) is satisfied. Indeed, for a fractional kernel it holds

$$\begin{aligned}
&\lim_{h\to 0} \int_0^h |K(t+h,t+s)|[n_2(t+h,t+s) + |v(t+h,t+s,0)|]ds \\
&= (N_2 + V) \lim_{h\to 0} \int_0^h (h-s)^{\beta-1} ds = (N_2 + V) \lim_{h\to 0} \frac{h^\beta}{\beta} = 0,
\end{aligned}$$

(with N_2 and V being bounds of n_2, v), the limit being independent of $t \in [T_0, T],$, i.e., (H5) is satisfied. The case of continous kernel is straightforward.

It is straightforward that if the functions f and v satisfy Lipschitz conditions on $[0,\infty) \times \mathbb{R}$, then (A1)–(A4) are satisfied with

$$\begin{gathered}
g_1(t) = g(t,0) \quad f_1(t) = f(t,0) \quad v_1(t,s) = v(t,s,0), \\
\ell_1(t) = \ell_2(t), \quad Z_1(x) = \Phi_1(x) = \Psi_1(x) = x.
\end{gathered}$$

7. An asymptotic result

The next lemma concerns the behavior of the operator $\mathcal{A} : B_p \to C(I)$ defined by the part of (1.1) containing the integral, namely,

$$(\mathcal{A}x)(t) := f(t,x(t)) \int_0^t K(t,s)v(t,s,x(s))\, ds \quad t \geq 0.$$

It turns out that, if the functions $\xi_1, \eta_1, \psi_1, \phi_1$ tend to zero at infinity, then $(\mathcal{A}x)(t) \to 0$ as $t \to \infty$, for any element $x \in BC(I)$. The lemma will be useful in studying the asymptotic behavior of solutions to (1.1).

Lemma 7.1. *Assume that* (A1)–(A4) *hold. If*

$$\xi_1(t), \eta_1(t), \psi_1(t), \phi_1(t) \to 0 \quad \text{as } t \to \infty, \tag{7.1}$$

then $(\mathcal{A}x)(t) \to 0$ as $t \to \infty$ for any $x \in BC(I)$. In particular, the set $\mathcal{A}(B_p)$ is equiconvergent to zero at infinity for any $p > 0$.

Proof. Let $y \in BC(I)$ with $\|y\| = p > 0$. In view of (A1)–(A4), for any $x \in B_p := \{x \in BC(I) : \|x\| \leq p\}$ we have

$$\begin{aligned}
&|\mathcal{A}x(t)| \\
&\leq \big|f(t,x(t)) \int_0^t K(t,s)v(t,s,x(s))\, ds\big| \\
&\leq |f(t,x(t))| \int_0^t |K(t,s)||v(t,s,x(s))|\, ds \\
&\leq [f_1(t) + m_1(t)\Psi_1(|x(t)|)] \int_0^t |K(t,s)|[v_1(t,s) + n_1(t,s)\Phi_1(|x(t)|)]ds \\
&\leq [f_1(t) + m_1(t)\Psi_1(p)] \int_0^t |K(t,s)|[v_1(t,s) + n_1(t,s)\Phi_1(p)]ds
\end{aligned}$$

$$
\begin{aligned}
&\le f_1(t)\int_0^t |K(t,s)|v_1(t,s)\,ds + \Phi_1(p)f_1(t)\int_0^t |K(t,s)|n_1(t,s)\,ds\\
&\quad + \Psi_1(p)m_1(t)\int_0^t |K(t,s)|v_1(t,s)\,ds\\
&\quad + \Psi_1(p)\Phi_1(p)m_1(t)\int_0^t |K(t,s)|n_1(t,s)\,ds,
\end{aligned}
$$

i.e., for any $x \in B_p$ and all $t \ge 0$ it holds

$$
|\mathcal{A}x(t)| \le \eta_1(t) + \Phi_1(p)\phi_1(t) + \Psi_1(p)\psi_1(t) + \Psi_1(p)\Phi_1(p)\xi_1(t), \tag{7.2}
$$

from which by (7.1) it follows that all functions in $\mathcal{A}(B_p)$ tend to zero uniformly as $t \to \infty$. Thus, for a given $\varepsilon > 0$ there exist a $T_{p,\varepsilon} > 0$ such that

$$
|\mathcal{A}x(t)| < \varepsilon \quad x \in B_p \ t > T_{p,\varepsilon}, \tag{7.3}
$$

in other words, the set $\mathcal{A}(B_p)$ is equiconvergent to zero at infinity. □

We now want to employ the above result in order to study the asymptotic behavior of solutions to equation (1.1). So we assume that the assumptions of Theorem 6.3 as well as those of Lemma 7.1 are satisfied. Let $\varepsilon > 0$ be arbitrary. In view of (7.1) for $p = r_0$, we may find a $T_\varepsilon > 0$ such that for any function x in B_{r_0} we have

$$
|x(t) - g(t,x(t))| = |\mathcal{A}x(t)| < \varepsilon(1-\ell_2^*), \quad t \ge T_\varepsilon.
$$

In particular, for the unique solution $\widetilde{x}$ of the (1.1) yielded by Theorem 6.3 it holds

$$
|\mathcal{A}\widetilde{x}(t)| < \varepsilon(1-\ell_2^*), \quad t \ge T_\varepsilon. \tag{7.4}
$$

For an arbitrary (but fixed) $t_0 \ge T_\varepsilon$, the function

$$
\widetilde{g}(u) := g(t_0,u) \quad |u| \le r_0,
$$

satisfies

$$
|\widetilde{g}(u_1) - \widetilde{g}(u_2)| = |g(t_0,u_1) - g(t_0,u_2)| \le \ell_2^*|u_1-u_2| \quad |u_1|,|u_2| \le r_0,
$$

hence it is a contraction with a unique fixed point $u_{t_0} \in [-r_0,r_0]$, so

$$
u_{t_0} = g(t,u_{t_0}).
$$

We claim that $u_{t_0} \in (-r_0,r_0)$. Indeed, for the constant function $x_0(t) = u_{t_0}$, $t \ge 0$, we have $x \in B_{r_0}$ and so, as in the part of the proof leading to (6.6) we have

$$
|\mathcal{T}x_0(t)| \le |g(t,x_0(t))| + |f(t,x_0(t))|\int_0^t |K(t,s)||v(t,s,x_0(s))|\,ds < r_0.
$$

Since x_0 is arbitrary, we set

$$
y(t) = u_t \quad t \ge 0,
$$

and see that the function $y(t)$, $t \ge 0$, satisfies

$$
y(t) = g(t,y(t)), \quad t \ge 0.
$$

Then for $t \ge T_\varepsilon$, in view of (7.4) we have

$$
\begin{aligned}
&|\widetilde{x}(t) - y(t)|\\
&= \Big|g(t,\widetilde{x}(t)) + f(t,\widetilde{x}(t))\int_0^t K(t,s)v(t,s,\widetilde{x}(s))\,ds - g(t,y(t))\Big|\\
&\le |g(t,\widetilde{x}(t)) - g(t,y(t))| + \Big|f(t,\widetilde{x}(t))\int_0^t K(t,s)v(t,s,\widetilde{x}(s))\,ds\Big|
\end{aligned}
$$

$$\begin{aligned}&\le \ell_2(t)|\widetilde{x}(t)-y(t)|+|\mathcal{A}\widetilde{x}(t)|\\&\le \ell_2(t)|\widetilde{x}(t)-y(t)|+\varepsilon(1-\ell_2^*),\end{aligned}$$

and so

$$|\widetilde{x}(t)-y(t)|\le \frac{\varepsilon(1-\ell_2^*)}{1-\ell_2(t)}\le \varepsilon \quad t\ge T_\varepsilon.$$

By the above arguments we can formulate the following result.

Theorem 7.2. *Let the assumptions of Theorem 6.3 hold and assume that* (7.1) *is fulfilled. Then for the unique solution* $\widetilde{x}$ *of* (1.1) *we have*

$$\widetilde{x}(t)\to y(t)$$

where y *is the unique solution of the (algebraic) equation*

$$y(t)=g(t,y(t)) \quad t\ge 0. \tag{7.5}$$

8. An application

In this section we use our results to study solutions to a highly nonlinear fractional equation, namely,

$$x(t)=\ell(t)[x^n(t)+p(t)]+d(t)x^{k_1}(t)\int_0^t (t-s)^{\beta_1-1}w(t,s)x^{k_2}(s)\,ds,\ t\ge 0, \tag{8.1}$$

where $n,k_1,k_2\in\mathbb{N}$, $\beta_1\in(0,1)$, $\ell,p:[0,\infty)\to\mathbb{R}$, $d:(0,\infty)\to\mathbb{R}$ are continuous, and, $w:\{0<s\le t, t>0\}\to\mathbb{R}$ is continuous and satisfies

$$|w(t,s)|\le \frac{s^{\beta_2-1}}{c(t)} \quad 0<s\le t,$$

with $\beta_2\in(0,1)$ and $c:(0,\infty)\to(0,\infty)$, continuous. We assume that

$$\ell_1^*+g_1^*+k_1k_2\frac{\Gamma(\beta_1)\Gamma(\beta_2)}{\Gamma(\beta_1+\beta_2)}\sup_{0<t}\frac{|d(t)|}{c(t)}t^{\beta_1+\beta_2-1}<1, \tag{8.2}$$

$$n\ell_1^*+k_1k_2\frac{\Gamma(\beta_1)\Gamma(\beta_2)}{\Gamma(\beta_1+\beta_2)}\sup_{0<t}\frac{|d(t)|}{c(t)}t^{\beta_1+\beta_2-1}<1, \tag{8.3}$$

$$\lim_{t\to 0+}\frac{|d(t)|}{c(t)}t^{\beta_1+\beta_2-1}=0=\lim_{t\to+\infty}\frac{|d(t)|}{c(t)}t^{\beta_1+\beta_2-1}, \tag{8.4}$$

where

$$\ell_1^*=\sup_{0\le t}|\ell(t)|, \quad g_1^*:=\sup_{0\le t}|\ell(t)p(t)|.$$

In terms of (1.1), here we have

$$\begin{gathered}g(t,x)=\ell(t)[x^n+p(t)], \quad f(t,x)=d(t)x^{k_1},\\ v(t,s,x)=w(t,s)x^{k_2}, \quad K(t,s)=(t-s)^{\beta_1-1}.\end{gathered}$$

To apply our main result, Theorem 6.3, our first concern is to show that (A1)–(A5) and (6.4)–(6.5) are satisfied. In particular, these conditions imply that Lemmas 6.1 and 6.2 do hold. Then we verify (H1)–(H5) thus concluding that all the assumptions of Theorem 6.3 are fulfilled. It should be noticed that in all these considerations we let T be arbitrary. Finally, we show that (7.1) is valid so we can use Theorem 7.2 to deduce the asymptotic behavior of the unique solution to equation (8.1).

Firstly, we notice that:

(i) For the function $g(t,x)$ we have
$$|g(t,x)| = |\ell(t)[x^n + p(t)]| \le |\ell(t)p(t)| + |\ell(t)||x|^n \quad t \ge 0, \quad x \in \mathbb{R},$$
so (A1) is satisfied with $\ell_1(t) = |\ell(t)|$, $g_1(t) := |\ell(t)p(t)|$ for $t \ge 0$, and $Z_1(x) := x^n$ for $x \ge 0$.

(ii) $f(t,x) = d(t)x^{k_1}$, so (A2) is satisfied with $f_1(t) = 0$ for $t \ge 0$ $m_1(t) = |d(t)$, for $t > 0$, $\Psi_1(x) = x^{k_1}$ for $x \ge 0$.

(iii) $v(t,s,x) = w(t,s)x^{k_2}$, so (A3) is satisfied with $v_1(t,s) = 0$, $n_1(t,s) = |w(t,s)$ for $0 < s < t$, $\Phi_1(x) = x^{k_2}$ for $x \ge 0$.

Furthermore, we find that
$$\begin{gathered}
\eta_1(t) := f_1(t) \int_0^t |K(t,s)|\,|v(t,s,0)|\,ds = 0, \quad t > 0, \\
\psi_1(t) := m_1(t) \int_0^t |K(t,s)| v_1(t,s)\,ds = 0, \quad t > 0, \\
\phi_1(t) := f_1(t) \int_0^t |K(t,s)| n_1(t,s)\,ds = 0, \quad t > 0,
\end{gathered}$$
from which it immediately follows that
$$\eta_1^* = \psi_1^* = \phi_1^* = 0. \tag{8.5}$$
Also,
$$\begin{aligned}
\xi_1(t) &:= m_1(t) \int_0^t |K(t,s)| n_1(t,s)\,ds \\
&= |d(t)| \int_0^t (t-s)^{\beta_1 - 1} |w(t,s)|\,ds \\
&\le \frac{|d(t)|}{c(t)} \int_0^t (t-s)^{\beta_1 - 1} s^{\beta_2 - 1} ds \\
&= \frac{\Gamma(\beta_1)\Gamma(\beta_2)}{\Gamma(\beta_1+\beta_2)} \frac{|d(t)|}{c(t)} t^{\beta_1+\beta_2-1},
\end{aligned}$$
so
$$0 \le \xi_1(t) \le \frac{\Gamma(\beta_1)\Gamma(\beta_2)}{\Gamma(\beta_1+\beta_2)} \frac{|d(t)|}{c(t)} t^{\beta_1+\beta_2-1}, \quad t > 0. \tag{8.6}$$
In view of (8.2), from inequality (8.6) we have
$$\xi_1^* := \sup_{0 \le t} \xi_1(t) \le \sup_{0 \le t} \frac{\Gamma(\beta_1)\Gamma(\beta_2)}{\Gamma(\beta_1+\beta_2)} \frac{|d(t)|}{c(t)} t^{\beta_1+\beta_2-1} < 1. \tag{8.7}$$
By (8.4) and setting $\xi_1(0) = 0$ we have that ξ_1 is continuous on $[0,\infty)$ and (A4) is satisfied.

From their definition, the functions η_1, ψ_1 and ϕ_1 are bounded and, by (8.7), so does ξ_1. As it concerns inequality (6.5), namely,
$$g_1^* + \ell_1^* Z_1(r_0) + \eta_1^* + \Phi_1(r_0)\phi_1^* + \Psi_1(r_0)\psi_1^* + \Psi_1(r_0)\Phi_1(v)\xi_1^* < r_0,$$
we see that by (8.5) and definitions of Z_1, Φ_1 and Ψ_1 it reduces to
$$g_1^* + \ell_1^* r_0^n + r_0^{k_1} r_0^{k_2} \xi_1^* < r_0,$$
which for $r_0 = 1$ becomes
$$g_1^* + \ell_1^* + \xi_1^* < 1, \tag{8.8}$$

this holding by (8.2). We conclude that (A5) is satisfied.

Finally, in view of (8.4), (8.5) and (8.6), we have that (6.4) holds. In turn, Lemma 6.1 yields that any solution of (8.1) starts from a (real) root x_0 of the equation

$$x_0 = g(0, x_0) = \ell(0)[x_0^n + p(0)].$$

Our next step is to verify (H1)–(H5). As inequality (6.5) is satisfied with $r_0 = 1$, we may restrict ourselves to considering only x, y with $|x|, |y| \le 1$.

For $t \ge 0$ and some $\vartheta \in [-1, 1]$, we have

$$\begin{aligned} |g(t,x) - g(t,x)| &\le |\ell(t)||x^n - y^n| \le |\ell(t)|n|x-y||\vartheta|^n \\ &\le n\ell(t)|x-y|, \end{aligned}$$

so (H1) is satisfied with $\ell_2(t) = n|\ell(t)|$, since, by (8.3), it holds

$$\ell_2^* := \sup_{0 \le t} \ell_2(t) = \sup_{0 \le t} n|\ell(t)| = n\ell_1^* < 1.$$

Also, for some $\xi \in [0, 1]$,

$$|f(t,x) - f(t,y)| = |d(t)x^{k_1} - d(t)y^{k_1}| = |d(t)|k_1|\xi|^{k_1}|x-y| \le k_1|d(t)||x-y|,$$

so f satisfies (6.8) with $m_2(t) = k_1|d(t)|$, for $t \ge 0$, while

$$\psi_2(t) := m_2(t) \int_0^t |K(t,s)||v(t,s,0)|\,ds = 0, \tag{8.9}$$

is continuous with $\lim_{t\to 0+} \psi_2(t) = 0$, and (H2) is satisfied.

In a similar way for $|x|, |y| \le 1$, $0 < s < t$, we have

$$|v(t,s,x) - v(t,s,y)| = |w(t,s)x^{k_2} - w(t,s)y^{k_2}| \le k_2|w(t,s)||x-y|,$$

or

$$|v(t,s,x) - v(t,s,y)| \le n_2(t,s)|x-y| \quad |x|, |y| \le 1, 0 < s < t,$$

with $n_2(t,s) = k_2|w(t,s)|$. Since $f(t,0) = 0$, we have

$$\phi_2(t) := |f(t,0)| \int_0^t |K(t,s)|n_2(t,s)\,ds = 0, \quad t \ge 0,$$

while

$$\begin{aligned} \xi_2(t) &:= m_2(t) \int_0^t |K(t,s)|n_2(t,s)\,ds \\ &= k_1|d(t)| \int_0^t (t-s)^{\beta_1 - 1} k_2|w(t,s)|\,ds \\ &\le k_1k_2 \frac{|d(t)|}{c(t)} \int_0^t (t-s)^{\beta_1-1} s^{\beta_2-1} ds, \end{aligned}$$

i.e.,

$$0 \le \xi_2(t) \le k_1k_2 \frac{\Gamma(\beta_1)\Gamma(\beta_2)}{\Gamma(\beta_1+\beta_2)} \frac{|d(t)|}{c(t)} t^{\beta_1+\beta_2-1}. \tag{8.10}$$

By (8.4) and (8.10) we find that

$$\lim_{t\to 0+} \xi_2(t) = \lim_{t\to 0+} \frac{|d(t)|}{c(t)} t^{\beta_1+\beta_2-1} = 0,$$

so for $\xi_2(0) = 0$ we have that ϕ_2 and ξ_2 are continuous for all $t \ge 0$ and [H3] is satisfied.

Now we are concerned with (H4). Since $r_0 = 1$, $v(t,s,0) = 0$ and $n_2(t,s) = k_2|w(t,s)|$, we have to show that there exists $\gamma \in (0, 1-\ell_2^*)$, and $\delta > 0$ such that inequality (6.11) holds, i.e. that

$$m_2(t+h)\int_0^t |K(t+h,s)|[n_2(t+h,s)r_0 + |v(t+h,s,0)|]ds \le \gamma,$$

for all $h \in [0,\delta]$ and $t \in [T_0,T]$.

Taking into consideration (8.2) we find that

$$\begin{aligned}
&m_2(t+h)\int_0^t |K(t+h,s)|[n_2(t+h,s)r_0 + |v(t+h,s,0)|]ds \\
&\le k_1|d(t+h)|\int_0^t (t+h-s)^{\beta_1-1}k_2|w(t+h,s)|r_0 ds \\
&\le k_1|d(t+h)|\int_0^t (t+h-s)^{\beta_1-1}k_2\frac{s^{\beta_2-1}}{c(t+h)}ds \\
&\le k_1k_2\frac{|d(t+h)|}{c(t+h)}\int_0^{t+h} (t+h-s)^{\beta_1-1}s^{\beta_2-1}ds \\
&= k_1k_2\frac{|d(t+h)|}{c(t+h)}\frac{\Gamma(\beta_1)\Gamma(\beta_2)}{\Gamma(\beta_1+\beta_2)}(t+h)^{\beta_1+\beta_2-1} \\
&\le k_1k_2\frac{\Gamma(\beta_1)\Gamma(\beta_2)}{\Gamma(\beta_1+\beta_2)}\sup_{t>0}\frac{|d(t)|}{c(t)}t^{\beta_1+\beta_2-1} \\
&< 1-\ell_1^* < 1,
\end{aligned}$$

so, by setting

$$\gamma := k_1k_2\frac{\Gamma(\beta_1)\Gamma(\beta_2)}{\Gamma(\beta_1+\beta_2)}\sup_{t>0}\frac{|d(t)|}{c(t)}t^{\beta_1+\beta_2-1},$$

assumption (H4) is satisfied for all $t \in [T_0,T]$ and any $\delta > 0$.

Finally, as $\beta_2 \in (0,1)$, $v(t,s,0) = 0$, $n_2(t,s) = k_2|w(t,s)|$ and noting that by continuity of c we have $\inf\limits_{u\in[T_0,2T]} c(u) > 0$, for $h \in [0,T]$ and $t \in [T_0,T]$ we take

$$\begin{aligned}
0 &\le \int_0^h (h-s)^{\beta_1-1}[n_2(t+h,t+s) + |v(t+h,t+s,0)|]ds \\
&\le k_2\int_0^h (h-s)^{\beta_1-1}\frac{(t+s)^{\beta_2-1}}{c(t+h)}ds \\
&\le \frac{k_2}{c(t+h)}\int_0^h (h-s)^{\beta_1-1}t^{\beta_2-1}ds \\
&\le \frac{k_2}{\inf_{u\in[T_0,2T]} c(u)T_0^{1-\beta_2}}\lim_{h\to 0}\frac{h^{\beta_1}}{\beta_1} \longrightarrow 0,
\end{aligned}$$

uniformly for all $t \in [T_0,T]$ as $h \to 0^+$, so (H_5) holds.

Consequently, as all the assumptions of Theorem 6.3 are satisfied (with $T > 0$ arbitrary), we conclude that there exists a unique solution $\widetilde{x}$ to equation (8.1) defined on $[0,\infty)$.

Furthermore, noting that, by definition, we have

$$\eta_1(t) = \psi_1(t) = \phi_1(t) = 0, \quad t > 0,$$

while, by (8.6) and (8.4) we take

$$\lim_{t\to\infty} \xi_1(t) = 0,$$

we see that (7.1) is satisfied, so Theorem 7.2 applies. It follows that for the solution $\widetilde{x}$ of equation (8.1) we have

$$\lim_{t\to\infty} |\widetilde{x}(t) - u(t)| = 0,$$

where $u(t)$ is the unique (real) solution of the equation

$$g(t, u(t)) = u(t), \quad \text{for } t \ge 0.$$

In other words, the solution $\widetilde{x}$ is asymptotic at $+\infty$ to the unique real solution u (with $\|u\| \le 1$) of the (algebraic) equation

$$u(t) = \ell(t)[u^n(t) + p(t)] \quad t \ge 0.$$

From the above discussion we have the following result.

Corollary 8.1. *If* (8.2)–(8.4) *are satisfied, then* (8.1) *has a unique solution* $\widetilde{x}$ *defined for all* $t \ge 0$. *Solution* $\widetilde{x}$ *starts from the (unique, real) solution of the equation* $y_0 = \ell(0)[y_0^n + p(0)]$ *with* $|y_0| \le 1$ *and is bounded by* 1. *Moreover,*

$$\widetilde{x}(t) \to y(t),$$

where y is the unique real solution of the (algebraic) equation

$$y(t) = \ell(t)[y^n(t) + p(t)],\ t \ge 0.$$

Note that, if $p(t) = 0$, $t \ge 0$ or $\ell(t) = 0$, $t \ge 0$ then, due to (8.3) we have $\sup |\ell(t)| < 1/n$, so the unique solution of the above algebraic equation is the trivial solution. To illustrate the asymptotic result in Corollary 8.1, we consider equation (8.1) with $n = 1$, and $n = 2$.

Example 8.2. When $n = 1$, i.e., when g is linear in x, equation (8.1) becomes

$$x(t) = \ell(t)[x(t) + p(t)] + d(t)x^{k_1}(t) \int_0^t (t-s)^{\beta_1 - 1} w(t,s) x^{k_2}(s) \tag{8.11}$$

for $t \ge 0$. In this case, condition (8.3) is implied by (8.2), so from Corollary 8.1 we see that (8.11) has a unique solution $\widetilde{x}$ defined for all $t \ge 0$ and which starts from $y(0) = \frac{\ell(0)p(0)}{1-\ell(0)}$, is bounded by 1. If, in addition, (8.4) holds, then

$$\widetilde{x}(t) \to \frac{\ell(t)p(t)}{1-\ell(t)}, \quad \text{as } t \to \infty.$$

Example 8.3. When $n = 2$, equation (8.1) is

$$x(t) = \ell(t)[x^2(t) + p(t)] + d(t)x^{k_1}(t) \int_0^t (t-s)^{\beta_1 - 1} w(t,s) x^{k_2}(s)\, ds, \tag{8.12}$$

for $t \ge 0$. For a fixed, arbitrary $t \ge 0$, equation (7.5) becomes

$$\ell(t)y^2(t) - y(t) + \ell(t)p(t) = 0 \quad t \ge 0. \tag{8.13}$$

Observe that (8.2) gives

$$|\ell(t)| + |\ell(t)p(t)| \le \sup_{0\le t} |\ell(t)| + \sup_{0\le t} |\ell(t)p(t)| < 1,$$

so

$$4\ell^2(t)|p(t)| \le [|\ell(t)| + |\ell(t)p(t)|]^2 < 1,$$

while from (8.3) we have

$$\ell(t) \le \sup_{0\le t} |\ell(t)| < \frac{1}{2} \Longrightarrow \frac{1}{2|\ell(t)|} \ge 1. \tag{8.14}$$

It follows that the quadratic equation (8.13) has two real solutions y_1 and y_2, namely

$$y_1(t) = \frac{1-\sqrt{1-4\ell^2(t)p(t)}}{2\ell(t)}, \quad y_2(t) = \frac{1+\sqrt{1-4\ell^2(t)p(t)}}{2\ell(t)},$$

for all $t \ge 0$ for which $\ell(t) \ne 0$. As by (8.14) we have

$$|y_2(t)| = \frac{1+\sqrt{1-4\ell^2(t)p(t)}}{2|\ell(t)|} \ge 1 + \sqrt{1-4\ell^2(t)p(t)} > 1,$$

we conclude that y_1 is the unique solution of the quadratic equation (8.13) bounded by 1. Note that y_1 may be written as

$$y_1(t) = \frac{2\ell(t)p(t)}{1+\sqrt{1-4\ell^2(t)p(t)}},$$

and this expression is valid for all $t \ge 0$ regardless of $\ell(t)$ being zero or not.

We have the following result:

If (8.2)-(8.4) are satisfied, then (8.12) has a unique solution $\widetilde{x}$ defined for all $t \ge 0$. This solution starts from $\frac{2\ell(0)p(0)}{1+\sqrt{1-4\ell^2(0)p(0)}}$, is bounded by 1 and

$$\widetilde{x}(t) \to \frac{2\ell(t)p(t)}{1+\sqrt{1-4\ell^2(t)p(t)}} \quad \text{as } t \to \infty.$$

In our last example we focus on singularities in the functions f, K, v.

Example 8.4. Consider the equation

$$x(t) = \frac{2t+1}{10t^2+7}[x(t)+b] + \frac{kx^2(t)}{\sqrt[3]{t}(t^2+1)} \int_0^t (t-s)^{-1/4} \frac{1}{\sqrt[5]{ts}} x^3(s)\, ds, \tag{8.15}$$

for $t \ge 0$ which is (8.1) with $n = 1$, $k_1 = 2$, $k_2 = 3$,

$$\ell(t) = \frac{2t+1}{10t^2+7}, \quad p(t) = b, \quad d(t) = \frac{k}{\sqrt[3]{t}(t^2+1)},$$

$$K(t,s) = (t-s)^{\frac{3}{4}-1}, \quad w(t,s) = \frac{s^{\frac{4}{5}-1}}{t^{\frac{1}{5}}}, \, c(t) = t^{\frac{1}{5}},$$

so $\beta_1 = 3/4, \beta_2 = 4/5$. We find that

$$\ell_1^* = \ell_2^* = \sup_{0\le t} |\ell(t)| = \frac{1}{7} \quad g_1^* := \sup_{0\le t} |\ell(t)p(t)| = \frac{|b|}{7},$$

$$\frac{|d(t)|}{c(t)} t^{\beta_1+\beta_2-1} = \frac{|k|}{\sqrt[3]{t}(t^2+1)t^{\frac{1}{5}}} t^{\frac{3}{4}+\frac{4}{5}-1} = \frac{|k|}{t^2+1} t^{\frac{3}{4}+\frac{4}{5}-1-\frac{1}{5}-\frac{1}{3}} = |k| \frac{t^{\frac{1}{60}}}{t^2+1}$$

so

$$\lim_{t\to 0+} \frac{|d(t)|}{c(t)} t^{\beta_1+\beta_2-1} = 0 = \lim_{t\to\infty} \frac{|d(t)|}{c(t)} t^{\beta_1+\beta_2-1}.$$

Therefore, (8.4) is satisfied. Furthermore, for $b \in (-6, 6)$ and sufficiently small values of $|k|$, we have

$$k_1 k_2 \frac{\Gamma(\beta_1)\Gamma(\beta_2)}{\Gamma(\beta_1+\beta_2)} \frac{|d(t)|}{c(t)} t^{\beta_1+\beta_2-1} = 6|k| \frac{\Gamma(\frac{3}{4})\Gamma(\frac{4}{5})}{\Gamma(\frac{3}{4}+\frac{4}{5})} \frac{t^{\frac{1}{60}}}{t^2+1}$$

$$< 1 - \ell_2^* - g_1^* = 1 - \frac{1+|b|}{7} < 1$$

thus

$$\ell_2^* + g_1^* + k_1 k_2 \frac{\Gamma(\beta_1)\Gamma(\beta_2)}{\Gamma(\beta_1+\beta_2)} \frac{|d(t)|}{c(t)} t^{\beta_1+\beta_2-1} < 1,$$

i.e., (8.2) is also satisfied. In view of the result in Example 8.2, we have that for $|b| < 6$ and sufficiently small $|k|$, equation (8.15) has a unique solution $\widetilde{x}$ defined for all $t \geq 0$. Solution $\widetilde{x}$ starts from $\widetilde{x}(0) = \frac{b}{6}$, is bounded by 1 and satisfies

$$\widetilde{x}(t) \to \frac{p(t)\ell(t)}{1-\ell(t)} = \frac{b\frac{2t+1}{10t^2+7}}{1-\frac{2t+1}{10t^2+7}} = b\frac{2t+1}{10t^2-2t+6} \quad \text{as } t \to \infty.$$

In particular, $\lim_{t\to\infty} \widetilde{x}(t) = 0$.

9. Discussion

As already mentioned, intending to apply a fixed point theorem to obtain existence of fixed points of an operator $\mathcal{T} : B \to B$, the main idea in this paper is that when a set $G \subseteq B$ with $\mathcal{T}(G) \subseteq G^o$ is spotted, then one might essentially reduce or simplify some of the requirements on the elements of the mapping $\mathcal{T}$ (which, in our case, is defined by the right-hand-side of the equation (1.1) to hold only on the set G rather on the (larger) set B . In this paper we employed this idea along with the method of progressive contractions to prove existence and uniqueness of a solution to a quadratic Volterra equation. This section is devoted to discussing our results in relation to some results obtained by the use of Darbo's theorem. We use the equation which has been the motive for our work as the field for this discussion. It is of particular interest that our asymptotic result might still be useful in cases where Theorem 6.3 cannot be applied. To be more specific, let us take a look at the equation

$$x(t) = \frac{\alpha t}{t+1} + \beta \sin\Big(\frac{t^2+x^2(t)}{t^2+1}\Big) \int_0^t \frac{ste^{-t} + \frac{s}{t^2+1}x^2(s)}{1+s^2+t^2} ds \quad t \geq 0,$$

appearing as equation (4.1) in the example of the recent work by Banas and Dubiel [1, p.12], and write it as

$$x(t) = \frac{\alpha t}{t+1} + \frac{\beta}{t^2+1} \sin\Big(\frac{t^2+x^2(t)}{t^2+1}\Big) \int_0^t s\frac{(t^2+1)te^{-t} + x^2(s)}{1+s^2+t^2} ds \quad t \geq 0. \tag{9.1}$$

It is assumed that $\alpha \in (0,1)$, $\beta > 0$ and $r > 0$ are such that

$$\alpha + \frac{\pi}{4}(2\beta r^2 + \beta \sin 1)\Big(\frac{4}{e^2} + \frac{1}{2}r^2\Big) \leq r.$$

This inequality is satisfied with $r = 1$ whenever

$$\frac{\pi}{4}\beta(2+\sin 1)\Big(\frac{4}{e^2} + \frac{1}{2}\Big) \leq 1 - \alpha. \tag{9.2}$$

Under these assumptions the authors conclude that (9.1) has a solution in the space $BC(\mathbb{R}^+)$ which converges to a finite limit at infinity.

Intending to apply the results in this paper to (9.1), in terms of our equation (1.1), here we take

$$K(t,s) := \frac{s}{1+s^2+t^2} \quad v(t,s,x) := (t^2+1)te^{-t} + x^2(s),$$

$$g(t,x)=\frac{\alpha t}{t+1}, t\geq 0 \quad f(t,x):=\frac{\beta}{t^2+1}\sin\Big(\frac{t^2+x^2(t)}{t^2+1}\Big).$$

Firstly, we check conditions (A1)–(A5). Clearly, as g does not depend on x we can take

$$\ell_1(t)=0=Z_1(x)\quad g_1(t)=\frac{\alpha t}{t+1},$$

and see that (A1) is satisfied. Note that $g_1^*=\alpha$ and $\ell_1^*=0$.

Next, we have

$$|f(t,x)|\leq\frac{\beta}{t^2+1}\big|\sin\big(\frac{t^2+x^2(t)}{t^2+1}\big)\big|\leq\frac{\beta}{t^2+1}\cdot 1,$$

so by taking

$$f_1(t)=\frac{\beta}{t^2+1},\quad m_1(t)=0=\Psi_1(x)=0,$$

we have that (A2) is fulfilled. As $m_1(t)=0$, by definition it follows that

$$\psi_1(t)=0=\xi_1(t),\quad \psi_1^*=\xi_1^*=0. \tag{9.3}$$

Also

$$|v(t,s,x)|=(t^2+1)te^{-t}+x^2(s),$$

and taking

$$v_1(t,s)=(t^2+1)te^{-t},\quad n_1(t,s)=1,\quad \Phi_1(x)=x^2,$$

we see that (A3) is verified.

As for $t\geq 0$ we have

$$\begin{aligned}\eta_1(t)&=f_1(t)\int_0^t|K(t,s)|v_1(t,s)\,ds\\&=\frac{\beta}{t^2+1}(t^2+1)te^{-t}\int_0^t\frac{s}{1+s^2+t^2}ds\\&=\beta te^{-t}\ln\Big(\frac{1+2t^2}{1+t^2}\Big)\\&\leq\frac{\beta\ln(2)}{2e}:=\eta_1^*,\end{aligned}$$

and

$$\begin{aligned}\phi_1(t)&=f_1(t)\int_0^t|K(t,s)|n_1(t,s)\,ds\\&=\frac{\beta}{t^2+1}\int_0^t\frac{s}{1+s^2+t^2}\cdot 1ds\\&=\frac{\beta}{t^2+1}\ln\Big(\frac{1+2t^2}{1+t^2}\Big)\\&\leq\frac{\beta\ln(2)}{2}:=\phi_1^*,\end{aligned}$$

we see that the functions ψ_1,η_1,ϕ_1 and ξ_1 are continuous and (A4) holds. We note that

$$\phi_1^*:=\frac{\beta\ln(2)}{2},\quad \eta_1^*:=\frac{\beta\ln(2)}{2e},\quad \lim_{t\to\infty}\phi_1(t)=0=\lim_{t\to\infty}\eta_1(t). \tag{9.4}$$

In view of the above, the left-hand side of inequality (6.5) becomes

$$g_1^*+\ell_1^*Z_1(r_0)+\eta_1^*+\Phi_1(r_0)\phi_1^*+\Psi_1(r_0)\psi_1^*+\Psi_1(r_0)\Phi_1(r_0)\xi_1^*$$

$$= \alpha + 0 + \frac{\beta \ln(2)}{2e} + r_0^2 \frac{\beta \ln(2)}{2} + 0 + 0 \le r_0,$$

and so (A5) requires the existence of a positive number r_0 with

$$\alpha + \frac{\beta \ln(2)}{2e} + r_0^2 \frac{\beta \ln(2)}{2} \le r_0.$$

For the last inequality to be satisfied with $r_0 = 1$, it suffices to have

$$\frac{\beta \ln(2)}{2} \big(1 + \frac{1}{e}\big) \le 1 - \alpha. \tag{9.5}$$

Comparing this inequality with (9.2) one may see that (9.5) leaves more "space" for the constant β than (9.2) .

Now we are concerned with conditions (H1)–(H5). Firstly we note that g can be regarded as a contraction in x with constant any number in $(0, 1)$, thus (H1) is automatically satisfied with $\ell_2 = 0$. Furthermore, for x, y with $|x|, |y| \le 1$ we take

$$\begin{aligned} |f(t,x) - f(t,y)| &= \frac{\beta}{t^2+1} \Big| \sin\Big(\frac{t^2 + x^2(t)}{t^2+1}\Big) - \sin\Big(\frac{t^2 + y^2(t)}{t^2+1}\Big) \Big| \\ &\le \frac{2\beta}{t^2+1} |x - y|, \end{aligned}$$

i.e., f satisfies (6.8) with $m_2(t) = \frac{2\beta}{t^2+1}$. Note that

$$f(t, 0) = \frac{\beta}{t^2+1} \sin(\frac{t^2}{t^2+1}),$$

and

$$\lim_{t\to 0} f(t, 0) = 0. \tag{9.6}$$

Now, by $v(t, s, 0) = (t^2 + 1)te^{-t}$, we find that

$$\begin{aligned} \psi_2(t) &:= m_2(t) \int_0^t |K(t,s)||v(t,s,0)|\, ds \\ &= \frac{2\beta(t^2+1)te^{-t}}{t^2+1} \int_0^t \frac{s}{1+s^2+t^2} ds \\ &= \beta te^{-t} \ln(\frac{1+2t^2}{1+t^2}) = 2\eta_1(t), \end{aligned}$$

and

$$\lim_{t\to 0} \psi_2(t) = 0, \tag{9.7}$$

so (H2) is satisfied

Next, for x, y with $|x|, |y| \le 1$ we have

$$|v(t, s, x) - v(t, s, y)| = |x^2 - y^2| \le 2|x - y|,$$

that is v satisfies (6.9) with $n_2(t, s) = 2$. Also,

$$\begin{aligned} \phi_2(t) &:= |f(t,0)| \int_0^t |K(t,s)| n_2(t,s)\, ds \\ &= \frac{\beta}{t^2+1} \sin\Big(\frac{t^2}{t^2+1}\Big) \int_0^t \frac{2s}{1+s^2+t^2} ds \\ &= \frac{2\beta}{2(t^2+1)} \sin\Big(\frac{t^2}{t^2+1}\Big) \ln\Big(\frac{1+2t^2}{1+t^2}\Big), \end{aligned}$$

so

$$\lim_{t\to 0} \phi_2(t) = 0. \tag{9.8}$$

Finally,

$$\begin{aligned}
\xi_2(t) &:= m_2(t) \int_0^t |K(t,s)| n_2(t,s)\, ds \\
&= \frac{2\beta}{t^2+1} \int_0^t \frac{2s}{1+s^2+t^2} ds \\
&= \frac{2\beta}{t^2+1} \ln(\frac{1+2t^2}{1+t^2}) = 2\phi_1(t),
\end{aligned}$$

so

$$\lim_{t\to 0} \xi_2(t) = 0, \tag{9.9}$$

and

$$\xi_2^* \le 2\phi_1 = 2\frac{\beta \ln 2}{2} = \beta \ln 2,$$

thus condition [H3] holds.

To check (H4) and (H5) we firstly verify (6.19) with $r_0 = 1$. Note that because $\ell_2^* = 0$, it suffices that $\gamma \in (0,1)$. For $t \ge 0$ we have

$$\begin{aligned}
& m_2(t) \int_0^t |K(t,s)| [n_2(t,s) + |v(t,s,0)|] ds \\
&= \frac{2\beta}{t^2+1} \int_0^t \frac{s}{1+s^2+t^2} [2 + (t^2+1)te^{-t}] ds \\
&= \frac{\beta[2 + (t^2+1)te^{-t}]}{t^2+1} \int_0^t \frac{2s}{1+s^2+t^2} ds \\
&= \big[\frac{2}{t^2+1} + te^{-t}\big] \beta \ln\Big(\frac{1+2t^2}{1+t^2}\Big) \\
&= \frac{2\beta}{t^2+1} \ln\Big(\frac{1+2t^2}{1+t^2}\Big) + \beta t e^{-t} \ln(\frac{1+2t^2}{1+t^2}) \\
&= 2\phi_1(t) + 2\eta_1(t) \\
&\le 2\phi_1^* + 2\eta_1^* = \beta \ln 2 + \frac{\beta \ln(2)}{e},
\end{aligned}$$

so

$$m_2(t) \int_0^t |K(t,s)| [n_2(t,s) + |v(t,s,0)|] ds \le \beta \ln(2)(1 + \frac{1}{e}),$$

which implies that it suffices to take

$$\gamma_0 := \beta \ln 2 \cdot (1 + \frac{1}{e}) < 1.$$

It follows that for sufficiently small values of $\beta > 0$ condition (H4) is satisfied because of Lemma 6.4, while (H5) is fulfilled by Remark 6.5, the kernel K and the functions $n_2(t,s) = 2$ and $v(t,s,0) = (t^2+1)te^{-t}$ being continuous on $\{0 \le s \le t, t \ge 0\}$.

Therefore, all conditions of Theorem 6.3 are fulfilled and we can infer that (9.1) has a unique solution x starting from zero and being bounded by one. Furthermore, by (9.3) and (9.4) we see that condition (7.1) is satisfied, and hence, from Theorem 7.2 we have that the unique solution of (9.1) tends asymptotically to the unique solution of the algebraic equation $u(t) = g(t, u(t)) = \frac{at}{t+1}$, so we may conclude that the solution x of (9.1) tends asymptotically to $u(t) = \frac{at}{t+1}$ as $t \to \infty$.

A major advantage of the method proposed here is that it yields existence and uniqueness of a global solution with the cost that both functions f and v satisfy Lipschitz conditions in a bounded domain of the variable x . However, when v is not locally Lipschitz (with respect to x), then Theorem 6.3 cannot be applied. But the result in [1] may be still valid, the cost now being rather expected: loss of uniqueness. To illustrate this, take a look at the equation

$$x(t) = \frac{\alpha t}{t+1} + \beta \sin\Big(\frac{t^2 + x^2(t)}{t^2+1}\Big) \int_0^t \frac{ste^{-t} + \frac{s}{t^2+1}|x(s)|^p}{1+s^2+t^2} ds \quad t \geq 0, \tag{9.10}$$

with $p \in (0, 1)$, which comes from (9.1) with only modification $|x|^p$ in place of x^2. The changes are that, now, we have $|v(t, s, x)| = (t^2 + 1)te^{-t} + |x(s)|^p$ and $\Phi(x) = |x|^p$. As the function v is no more Lipschitz in x, thus Theorem 6.3 cannot be applied, however, the existence result in [1] can yield existence of solutions in B_1. Note that in view of continuity we immediately see that for any solution x it holds $x(0) = 0$, a fact that is also implied by Lemma 6.1, as one may easily observe that (A1)-(a_4) are still valid. Moreover, writing the second summand in the right-hand-side of (9.10) as

$$\frac{\beta}{t^2+1} \sin\Big(\frac{t^2 + x^2(t)}{t^2+1}\Big) \int_0^t s\frac{(t^2+1)te^{-t} + |x(s)|^p}{1+s^2+t^2} ds \quad t \geq 0,$$

it is not difficult to see that for any bounded function x it holds

$$\begin{aligned}
0 &\leq \frac{\beta}{t^2+1}\Big|\sin\Big(\frac{t^2 + x^2(t)}{t^2+1}\Big)\Big| \int_0^t s\frac{(t^2+1)te^{-t} + |x(s)|^p}{1+s^2+t^2} ds \\
&\leq \frac{\beta te^{-t}}{t^2+1}\int_0^t s ds + \frac{\beta\|x\|^p}{t^2+1}\int_0^t \frac{s}{1+s^2+t^2} ds \\
&\leq \frac{\beta t^3 e^{-t}}{2(t^2+1)} + \frac{\beta\|x\|^p}{t^2+1}\int_0^t \frac{s}{1+t^2} ds \\
&\leq \beta te^{-t} + \frac{\beta\|x\|^p t^2}{2(t^2+1)^2},
\end{aligned}$$

and so

$$\lim_{t\to\infty} \frac{\beta}{t^2+1} \sin\Big(\frac{t^2 + x^2(t)}{t^2+1}\Big) \int_0^t s\frac{(t^2+1)te^{-t} + |x(s)|^p}{1+s^2+t^2} ds = 0.$$

Consequently, all solutions of (9.10) tend asymptotically to the function $g(t) := \frac{\alpha t}{t^2+1}$ as $t \to \infty$.

Remarks analogous to the above can be made regarding the result by Darwish and Henderson in [14]. Comments concerning solutions of the equation (4.1) in [14, p. 83] may be found in [9].

We now put together all the pieces of the work here. Under the set of assumptions posed, Theorem 6.3 tells us of a unique solution in the ball B_{r_0}, say x. Do there exist any other bounded solutions? By Lemma 6.1 we see that any other bounded solution has to start from the specific x_0. Then, in view of (6.5) in (A5), from Theorem 4.1 we have that any other possible bounded solution should belong to B_{r_0}, and Theorem 6.3 yields there can be no other bounded solutions. What about unbounded solutions? The answer is negative thanks to Theorem 6.3, again. For if an unbounded solution existed, then it would had to start at the specific x_0, and, by continuity, be bounded on the set $[0, T]$ for any $T > 0$. But there is only one solution defined on $[0, T]$ and this is x. We may conclude that if the conditions of Theorem 6.3 are satisfied, then equation (1.1) has no other solutions than the one yielded by this theorem. Furthermore, if condition (7.1) is satisfied, then the asymptotic behavior of the solution is described in Theorem 7.2.

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