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BOUNDS FOR THE SPECTRAL RADIUS OF POSITIVE OPERATORS

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Abstract. We use partially ordered spaces to obtain bounds for the spectral radius of positive linear operators, and to improve the corresponding fixed point theorems.

1. Introduction

In [9], methods of partially ordered spaces are used for obtaining bounds for the spectral radius of positive operators. In Theorem 2.2 below we present a result which generalizes those in [9], by weakening the hypothesis that the operator under consideration is bounded by some u_0 with "u_0 is a quasi-interior element of the cone". In Theorem 2.5 we improve a variant of the well known contraction mapping principle given by Krasnosel'skii and Zabreiko in [9] by requiring only estimates for the difference of the values of the operator on comparable elements. Also, we use the previous results to prove that the operator given in Theorem 2.5 satisfies all the conditions of the converse to the Banach contraction theorem.

2. Main results

Let $(E, \|\cdot\|_E)$ be a real Banach space and P be a nonempty closed convex set in E. P is called a cone if it satisfies the following two conditions:

(i) if $x \in P$ and $\lambda \geq 0$, then $\lambda x \in P$;
(ii) if $x \in P$ and $-x \in P$, then $x = \theta$, where θ is the zero element in E.

A cone P is said to be reproducing if $E = P - P$, i.e., every element $x \in E$ can be represented in the form $x = u - v$ where $u, v \in P$. The cone P defines a linear ordering in E by

$$x \leq y \quad \text{if and only if} \quad y - x \in P.$$

The cone P is said to be normal if there exists a constant $K > 0$ such that

$$\theta \leq x \leq y \Longrightarrow \|x\| \leq K\|y\|, \quad \forall x, y \in P.$$

For $L : E \to E$, a bounded linear operator, we define its spectral radius by

$$\sigma(L) = \lim_{n\to+\infty} \|L^n\|^{1/n}.$$

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The linear operator L is called positive if it transforms the cone P into itself. It is not hard to see that it follows from $x \le y$ for arbitrary elements $x, y \in E$ that $Lx \le Ly$.

Let u_0 be a non-zero element of P. Following Krasnosel'skii [8], we say that the positive operator L is u_0-bounded above if there exists a natural number m such that for every non-zero element $x \in P$ there exists a positive number $\beta(x)$ such that

$$L^m x \le \beta(x) u_0.$$

Arguing as in [2, Lemma 1.4.2] we have the following result.

Theorem 2.1. *Let P be a normal reproducing cone. If the positive linear operator L is u_0-bounded above then there exist constants $\tau > 0$ and $m > 0$ such that for any $x \in E$ there exist an element $y \in P$ and a constant $\beta(x) > 0$ such that*

$$x \le y, \quad L^m y \le \beta(x) u_0, \quad \|y\| \le \tau \|x\|, \quad \|\beta(x) u_0\| \le \tau \|x\|.$$

Proof. Since P is a reproducing cone, then for $x \in E$ there exist $y, z \in P$ such that $x = y - z$. From the u_0-boundedness above of L there exists an integer $m > 0$ such that $L^m y \le \beta(x) u_0$ for some constant $\beta(x) > 0$. From which it follows that $E = \cup_{n=1}^{\infty} E_n$, where

$$\begin{aligned} E_n = \big\{ x \in E : &\text{there are } y \in P \text{ and } \beta(x) > 0 \text{ such that } x \le y, \\ &L^m y \le \beta(x) u_0,\ \|y\| \le n\|x\|,\ \|\beta(x) u_0\| \le n\|x\| \big\}, \end{aligned}$$

for $n = 1, 2, 3, \dots$. By the Baire-Hausdorff's Theorem (that says nonempty complete metric spaces are second Baire sets) there exist positive integer $n_1, x_0 \in E$ and $R > r > 0$ satisfying

$$B_0 = \{x \in E : r < \|x - x_0\| < R\} \subset \overline{E_{n_1}}.$$

Let $-x_0 = y_0 - z_0$ where $y_0, z_0 \in P$. Take positive constant β_0 and positive integer n_2 satisfying $L^m y_0 \le \beta_0 u_0$, $\|y_0\| \le n_2 \|x_0\|$, and $\|\beta_0 u_0\| \le n_2 \|x_0\|$.

Let $B = \{x \in E : r < \|x\| < R\}$, and choose some integer n_3 satisfying

$$n_3 > n_1 + \frac{1}{r}(n_1 + n_2)\|x_0\|.$$

Now we prove that $B \subset \overline{E}_{n_3}$. Indeed, for any $x \in B$, we have $y = x_0 + x \in B_0$, then there exists a sequence $\{x_i\} \subset E_{n_1}$ such that $x_i \to y$ as $i \to \infty$. Clearly, we can assume that $x_i \in B_0$ for $i = 1, 2, 3, \dots$. Take elements $y_i \in P$ and constants $\beta_i > 0$ such that $x_i \le y_i$, $L^m y_i \le \beta_i u_0$, $\|y_i\| \le n_1 \|x_i\|$, and $\|\beta_i u_0\| \le n_1 \|x_i\|$, then we obtain $x_i - x_0 \le y_i + y_0$ and

$$\begin{aligned} \|y_i + y_0\| &\le n_1 \|x_i\| + n_2 \|x_0\| \\ &\le (n_1 + n_2)\|x_0\| + n_1 \|x_i - x_0\| \\ &\le \Big[(n_1 + n_2)\frac{\|x_0\|}{r} + n_1\Big] \|x_i - x_0\| \\ &\le n_3 \|x_i - x_0\|. \end{aligned}$$

On the other hand we have $L^m(y_i + y_0) = L^m y_i + L^m y_0 \le (\beta_i + \beta_0) u_0$ and

$$\|(\beta_i + \beta_0) u_0\| \le n_1 \|x_i\| + n_2 \|x_0\| \le n_3 \|x_i - x_0\|,$$

from which it follows that $x_i - x_0 \in E_{n_3}$ for $n = 1, 2, 3, \dots$. From the fact that $x_i - x_0 \to y - x_0$ as $i \to \infty$ we obtain $x \in \overline{E}_{n_3}$. Therefore $B \subset \overline{E}_{n_3}$. Clearly, from $x \in \overline{E}_{n_3}$, we can easily prove that $tx \in \overline{E}_{n_3}$, for all $t \ge 0$. Consequently, $E = \overline{E}_{n_3}$.

Finally, we show that $E = E_{3n_3}$. Taking $x \in E$ such that $x \neq \theta$, then there exists $x_1 \in E_{n_3}$ satisfying

$$\|x - x_1\| < \frac{1}{2}\|x\|.$$

Since $x_1 \in E_{n_3}$, there exist $y_1 \in P$ and $\beta_1 > 0$ such that

$$x_1 \leq y_1, \quad L^m y_1 \leq \beta_1 u_0, \quad \|y_1\| \leq n_3\|x_1\|, \quad \|\beta_1 u_0\| \leq n_3\|x_1\|.$$

Similarly, there exist $x_2 \in E_{n_3}$, $y_2 \in P$, and $\beta_2 > 0$ such that

$$\|x - x_1 - x_2\| < \frac{1}{2^2}\|x\|, \quad x_2 \leq y_2,$$
$$L^m y_2 \leq \beta_2 u_0, \quad \|y_2\| \leq n_3\|x_2\|, \quad \|\beta_2 u_0\| \leq n_3\|x_2\|.$$

Inductively, we find sequences $\{x_k\} \subset E_{n_3}$, $\{y_k\} \subset P$, and $\{\beta_k\} > 0$ satisfying

$$\|x - x_1 - x_2 - \cdots - x_k\| < \frac{1}{2^k}\|x\|, \quad x_k \leq y_k,$$
$$L^m y_k \leq \beta_k u_0, \quad \|y_k\| \leq n_3\|x_k\|, \quad \|\beta_k u_0\| \leq n_3\|x_k\|$$

for $k = 1, 2, 3 \dots$. Clearly, $x = \sum_{k=1}^{\infty} x_k$ and

$$\|x_k\| \leq \|x - \sum_{i=1}^{k-1} x_i\| + \|x - \sum_{i=1}^{k} x_i\| < \frac{3\|x\|}{2^k}, \quad k = 1, 2, \dots .$$

From which it follows that

$$\sum_{k=1}^{\infty} \|y_k\| \leq n_3 \sum_{k=1}^{\infty} \|x_k\| \leq 3n_3\|x\| < \infty,$$
$$\sum_{k=1}^{\infty} \|\beta_k u_0\| \leq n_3 \sum_{k=1}^{\infty} \|x_k\| \leq 3n_3\|x\| < \infty.$$

Consequently the series $\sum_{k=1}^{\infty} y_k$ and $\sum_{k=1}^{\infty} \beta_k$ converge to and element $y \in P$ and a constant $\beta > 0$, respectively. Clearly

$$x = \sum_{k=1}^{\infty} x_k \leq \sum_{k=1}^{\infty} y_k = y, \quad \|y\| \leq \sum_{k=1}^{\infty} \|y_k\| \leq 3n_3\|x\|.$$

On the other hand by using that a linear operator which maps a reproducing cone into a normal cone must be continuous (see [6]), L^m is continuous. Then

$$L^m y = L^m\Big(\sum_{k=1}^{\infty} y_k\Big) = \sum_{k=1}^{\infty} L^m y_k \leq \sum_{k=1}^{\infty} \beta_k u_0 = \beta u_0,$$
$$\|\beta u_0\| \leq \sum_{k=1}^{\infty} \|\beta_k u_0\| \leq 3n_3\|x\|.$$

Therefore, $x \in E_{3n_3}$, which implies that $E = E_{3n_3}$. □

Our main result reads as follows.

Theorem 2.2. *Let (E, P) be an ordered Banach space with normal, reproducing cone P. Let $L : E \to E$ be u_0-bounded above positive linear operator. If there exist a positive integer n and a $\lambda \geq 0$ such that $L^n u_0 \leq \lambda u_0$, then*

$$r(L) \leq \lambda^{1/n}$$

Proof. First, suppose that $\lambda > 0$. By Theorem 2.1 there exist positive constants τ and m such that for every $x \in E$ there exist $y \in P$, $\beta(x) > 0$ such that $x \leq y$, $L^m y \leq \beta(x)u_0$, $\|y\| \leq \tau\|x\|$, $\|\beta(x)u_0\| \leq \tau\|x\|$.

Taking $z = y - x \in P$, then by Theorem 2.1 we can find an element $\omega \in P$ and a constant $\beta(z) > 0$ such that $z \leq \omega$, $L^m\omega \leq \beta(z)u_0$, $\|\omega\| \leq \tau\|z\|$, and $\|\beta(z)u_0\| \leq \tau\|z\|$. If $u = y + z$, then $-u \leq x \leq u$ and

$$L^m u = L^m y + L^m z \leq \beta(x)u_0 + \beta(z)u_0.$$

Since

$$\begin{aligned}
\|\beta(x)u_0 + \beta(z)u_0\| &\leq \|\beta(x)u_0\| + \|\beta(z)u_0\| \\
&\leq \tau\|x\| + \tau\|z\| \\
&\leq \tau\|x\| + \tau(\|y\| + \|x\|) \\
&\leq \tau\|x\| + \tau(\tau\|x\| + \|x\|) \\
&\leq \tau(\tau + 2)\|x\|,
\end{aligned}$$

there is a constant $\sigma > 0$ such that for any $x \in E$ there exist $u(x) \in P$ and $\beta'(x) > 0$ such that

$$-u(x) \leq x \leq u(x), \quad L^m u(x) \leq \beta'(x)u_0, \quad \|\beta'(x)u_0\| \leq \sigma\|x\|.$$

Therefore, there is a constant β ($\beta > \frac{\sigma}{\|u_0\|}$) such that for every $x \in E$ satisfying $\|x\| \leq 1$, we can find an element $u(x) \in P$ such that

$$-u(x) \leq x \leq u(x) \quad \text{and} \quad L^m u(x) \leq \beta u_0.$$

Hence

$$-\beta u_0 \leq -L^m u(x) \leq L^m x \leq L^m u(x) \leq \beta u_0,$$

and inductively, for any $p \geq 1$,

$$-\beta\lambda^p u_0 \leq -L^{m+np}u(x) \leq L^{m+np}x \leq L^{m+np}u(x) \leq \beta\lambda^p u_0,$$

from which it follows that

$$0 \leq L^{m+np}x + \beta\lambda^p u_0 \leq 2\beta\lambda^p u_0.$$

Since P is normal we obtain

$$\|L^{m+np}x + \beta\lambda^p u_0\| \leq 2K\beta\lambda^p\|u_0\|,$$

where K is the normal constant of P. Then for every $x \in E$, satisfying $\|x\| \leq 1$ and $p \geq 1$ we have

$$\|L^{m+np}x\| \leq (2K + 1)\beta\lambda^p\|u_0\|.$$

From which it follows that

$$\|L^{m+np}\| \leq (2K + 1)\beta\lambda^p\|u_0\|.$$

Then

$$\|L^{m+np}\|^{\frac{1}{m+np}} \leq (2K + 1)^{\frac{1}{m+np}}\beta^{\frac{1}{m+np}}\lambda^{\frac{p}{m+np}}\|u_0\|^{\frac{1}{m+np}}.$$

Letting $p \to \infty$, for all $\epsilon > 0$ there exists N, such that for all $p \geq N$, we have

$$r(L) \leq \|L^{m+np}\|^{\frac{1}{m+np}} \leq \lambda^{\frac{1}{n}} + \epsilon.$$

By letting $\epsilon \to 0$ we have

$$r(L) \leq \lambda^{\frac{1}{n}}.$$

Now, if $\lambda = 0$ then for any $\epsilon > 0$ we have $L^n u_0 \leq \epsilon^n u_0$ and from what has already been proved we have $r(L) \leq \epsilon$. This completes the proof. □

Remark 2.3. Theorem 2.2 generalizes Theorem 2 in Zabreiko, Krasnosel'skii and Stetsenko [9], where the element u_0 is supposed to be a "quasi-interior element of the cone", a more restrictive condition.

Remark 2.4. If we assume that P is a solid cone and u_0 is a quasi interior element of P (as in [9]), then it follows from [2, Theorem 1.4.1] that u_0 belongs to the interior of the cone P.

As a consequence of Theorem 2.2, by using a result by Krasnosel'skii and Zabreiko [6, Theorem 49.3, p. 320] (see also [3, Theorem 3.1.14]), we have the following result.

Theorem 2.5. *Let (E, P) be an ordered Banach space with normal reproducing cone P and $A : E \to E$ be an operator. Suppose that there exists an u_0-bounded above linear positive operator $L : E \to E$ such that*

$$-L(x-y) \leq A(x) - A(y) \leq L(x-y), \quad x, y \in E,\ x \geq y. \tag{2.1}$$

If $L^n u_0 \leq \lambda u_0$ for some $\lambda \in [0, 1)$, an integer n and an element u_0 of $P \setminus \{0\}$ then A has a unique fixed point x^* in E and for any $x_0 \in E$, if $x_n = Ax_{n-1}(n = 1, 2, 3, \dots.)$, then $x_n \to x^*$ as $n \to \infty$.

Proof. It follows from Theorem 2.2 that $r(L) < 1$. □

It should be remarked above that [6, Theorem 49.3, p. 320] remains valid if we replace condition (2.1) by the condition

$$-L_1(x-y) \leq A(x) - A(y) \leq L_2(x-y), \quad x, y \in E,\ x \geq y,$$

where L_1 and L_2 are positive linear operators with $\sigma(L_1 + L_2) < 1$. Here, we observe that even if A is a linear operator the latter condition cannot be replaced by the inequalities $\sigma(L_1) < 1$ and $\sigma(L_2) < 1$ (See the remark after the proof of [6, Theorem 49.3, p. 320]). By using Theorem 2.5 we can solve this problem if we impose new conditions concerning every operators L_1 and L_2 as follows.

Corollary 2.6. *Let (E, P) be an ordered Banach space with normal reproducing cone P and $A : E \to E$ be an operator. Suppose that there exist linear positive operators $L_1, L_2 : E \to E$ such that*

$$-L_1(x-y) \leq A(x) - A(y) \leq L_2(x-y), \quad x, y \in E,\ x \geq y. \tag{2.2}$$

Assume that $L_1 + L_2$ is u_0-bounded above, then if $L_1 u_0 \leq \lambda_1 u_0$ and $L_2 u_0 \leq \lambda_2 u_0$ for some $\lambda_1, \lambda_2 \geq 0$ satisfying $0 \leq \lambda_1 + \lambda_2 < 1$, then A has a unique fixed point x^* in E.

Proof. It is easily seen that

$$-(L_1 + L_2)(x-y) \leq A(x) - A(y) \leq (L_1 + L_2)(x-y), \quad x, y \in E,\ x \geq y.$$

Then all conditions of Theorem 2.5 are satisfied. □

Remark 2.7. If A satisfies the conditions of any of the theorems above, then for each $y \in$, the equation $x = Ax + y$ has a unique solution.

Next, we use the following converse to the Banach contraction theorem [5].

Theorem 2.8. *Let X be a metrizable topological space and let its topology be generated by the metric ρ. Then for each $\lambda \in (0, 1)$ there exists a metric ρ_λ on X, complete if ρ is complete, such that f is a ρ_λ- contraction if and only if*

(i) *for some $\xi \in X$, $f(\xi) = \xi$;*

(ii) *$f^n(x) \to \xi$ as $n \to \infty$ for all $x \in X$;*
(iii) *there exists an open neighborhood U of ξ such that $f^n(U) \to \{\xi\}$, which implies that given any neighborhood V of ξ there exists an integer $n(V) > 0$ such that $f^n(U) \subset V$ for all $n \geq n(V)$.*

As a consequence of the previous results we obtain the following statement.

Theorem 2.9. *Let (E, P) be an ordered Banach space with normal reproducing cone P and $A : E \to E$ be an operator. Suppose that there exists an u_0-bounded above linear positive operator $L : E \to E$ such that*

$$-L(x-y) \leq A(x) - A(y) \leq L(x-y), \quad x, y \in E,\ x \geq y. \tag{2.3}$$

If $Lu_0 \leq \lambda u_0$ for some $\lambda \in [0,1)$, then for every $\mu \in (0,1)$ there exists a metric d_μ such that A is a $d_\mu-$contraction. This means that for $x, y \in E$,

$$d_\mu(A(x), A(y)) \leq \lambda d_\mu(x, y).$$

Proof. We shall prove that A satisfies all the hypotheses of Theorem 2.8. Indeed, the hypotheses (i) and (ii) can easly derived from Theorem 2.5. Then it remains for us to prove (iii).

Take $U = \{x : \|x - \xi\| < 1\}$ where ξ is the unique fixed point of A. Then by using the same arguments as in the proof of Theorem 2.2, there exist constants $\beta > 0$ and $m > 0$ such that for every $x \in U$ an element $u \in P$ can be found such that $-u \leq x - \xi \leq u$ and $L^m u \leq \beta u_0$. It follows from the inequalities

$$x \geq \frac{1}{2}(x + \xi - u), \quad \xi \geq \frac{1}{2}(x + \xi - u)$$

and from (2.3) that

$$-L\Big(\frac{x-\xi+u}{2}\Big) \leq A(x) - A\Big(\frac{x+\xi-u}{2}\Big) \leq L\Big(\frac{x-\xi+u}{2}\Big), \tag{2.4}$$

$$-L\Big(\frac{\xi-x+u}{2}\Big) \leq \xi - A\Big(\frac{x+\xi-u}{2}\Big) \leq L\Big(\frac{\xi-x+u}{2}\Big). \tag{2.5}$$

By subtracting (2.5) from (2.4), we have

$$-Lu \leq A(x) - \xi \leq Lu.$$

By repeating this argument $m+n$ times for any integer n, we obtain the inequality

$$-L^{m+n}u \leq A^{m+n}(x) - \xi \leq L^{m+n}u.$$

By using the hypothesis of the theorem we obtain

$$-\beta\lambda^n u_0 \leq -L^{m+n}u \leq A^{m+n}(x) - \xi \leq L^{m+n}u \leq \beta\lambda^n u_0,$$

from which it follows that

$$\|A^{m+n}(x) - \xi\| \leq (2K+1)\beta\lambda^n\|u_0\|. \tag{2.6}$$

Since $\lambda < 1$, for any neighborhood V of ξ we can choose $n(V)$ so large that for all $n \geq n(V)$,

$$\{x : \|x - \xi\| < (2K+1)\beta\lambda^n\|u_0\|\} \subset V,$$

which implies by (2.6) that $A^{m+n}(U) \subset V$. Consequently for every $n \geq n(V) + m$ we have $A^n(U) \subset V$. Thus (iii) is proved. This completes the proof. $\square$

Conclusion. This article generalizes and improves well-known results by Krasnosel'skii, Zabreiko and Stetsenko, and other authors. Note that the present results can be used for generalizing other results in the literature.

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