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EXISTENCE OF SOLUTION TO CRITICAL KIRCHHOFF-TYPE EQUATION WITH DIPOLE-TYPE POTENTIAL

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Abstract. Dipole-type potential arises in the area of nonrelativistic molecular physics. In this paper, we establish the existence and nonexistence of solution to critical Kirchhoff-type equation with dipole-type potential.

1. Introduction

We consider the Kirchhoff-type equation

$$-\Big(1+b\int_{\mathbb{R}^N}|\nabla u|^2\mathrm{d}x\Big)\Delta u-\mu\frac{\Phi(x/|x|)}{|x|^2}u=|u|^{2^*-2}u,\quad x\in\mathbb{R}^N, \tag{1.1}$$

where $N\geqslant 3$, $b\geqslant 0$ and $2^*=\frac{2N}{N-2}$ is the Sobolev critical exponent. The function Φ and the parameter μ satisfy the following condition:

(A1) $0\leq\Phi\in L^p(\mathbb{S}^{N-1})$, $p\geq\frac{(N-2)^2}{2(N-1)}+1$, and $\mu\in(0,\Lambda_\Phi)$, where

$$\Lambda_\Phi:=\frac{(N-2)^2}{4}|\mathbb{S}^{N-1}|^{1/p}\|\Phi\|^{-1}_{L^p(\mathbb{S}^{N-1})}.$$

On the other hand the Laplace operator with dipole-type potential is

$$\mathcal{L}_\Phi:=-\Delta-\mu\frac{\Phi(x/|x|)}{|x|^2},\quad x\in\mathbb{R}^N,$$

where $N\geq 3$. This kind of operator arises in the area of nonrelativistic molecular physics. Specifically, the Schrödinger equation for the wave function of an electron interacting with a polar molecule can be written as

$$H=-\frac{\hbar}{2m}\Delta+e\frac{x\cdot\mathbf{D}}{|x|^3}-E,$$

where $\mathbf{D}$ is the dipole moment of the molecule, e and m denote the charge and the mass of the electron, see [19]. The operator with different kinds of singular potentials have been largely studied, see [7, 8, 9, 10, 23, 26, 28] and references therein.

On the other hand, equation (1.1) is related to the stationary analogue of equation

$$\rho\frac{\partial^2u}{\partial t^2}-\Big(\frac{P_0}{h}+\frac{E}{2L}\int_0^L|\frac{\partial u}{\partial x}|\mathrm{d}x\Big)\frac{\partial^2u}{\partial x^2}=0,$$

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which was proposed by Kirchhoff in [18] as an extension of the classical D'Alembert wave equation for free vibrations of elastic strings. The existence of solution of Kirchhoff-type equation with Laplacian was explored in [3, 25], and with fractional Laplacian was investigated in [21].

Liu-Liao-Tang [20] studied equation (1.1) with $\Phi = 0$:

$$-\Big(a + b\int_{\mathbb{R}^N} |\nabla u|^2 \mathrm{d}x\Big)\Delta u = |u|^{2^*-2}u, \quad x \in \mathbb{R}^N. \tag{1.2}$$

By using the minimizing of best constant

$$S := \inf_{u\in D^{1,2}(\mathbb{R}^N)\setminus\{0\}} \frac{\|u\|^2_{D^{1,2}(\mathbb{R}^N)}}{(\int_{\mathbb{R}^N} |u|^{2^*}\mathrm{d}x)^{2/2^*}}$$

as follows

$$U_{\varepsilon,y} = [N(N-2)]^{\frac{N-2}{4}} \frac{\varepsilon^{\frac{N-2}{2}}}{(\varepsilon^2 + |x-y|^2)},$$

they established the existence and nonexistence of solutions for equation (1.2) with respect to parameters N, a and b. The existence of solution of equation (1.2) with p-Laplacian was presented in [17, 22].

For Φ =Constant, Fiscella-Pucci [11] established the Concentration Compactness Principle with Hardy potential, and then they established the existence of solutions for Kirchhoff-type equations involving Hardy potential and different critical nonlinearities. For more recent work, we refer to [1, 12, 13].

The case where the potential Φ is a constant was discussed in [11, 17, 20, 22]. Therefore, it is natural to ask whether equation (1.1) admits a solution for Φ nonconstant. To the best of our knowledge, there is no result on this problem.

If $b = 0$, equation (1.1) becomes

$$-\Delta u - \mu\frac{\Phi(x/|x|)}{|x|^2}u = |u|^{2^*-2}u, \quad x \in \mathbb{R}^N. \tag{1.3}$$

We study the following minimizing problem:

$$S_\Phi := \inf_{u\in D^{1,2}_{\rm rad}(\mathbb{R}^N)\setminus\{0\}} \frac{\|u\|^2_\Phi}{\big(\int_{\mathbb{R}^N} |u|^{2^*}\mathrm{d}x\big)^{2/2^*}}.$$

Extremals for S_Φ are solutions of the Euler-Lagrange equation (1.3). The following is our first result.

Theorem 1.1. *Assume that $N \geq 3$ and* (A1) *hold. Then equation* (1.3) *has a radially symmetric solution $\bar{v} \in D^{1,2}_{\rm rad}(\mathbb{R}^N)$, and infinitely many nonradial solutions $\bar{v}_k$ such that $\int_{\mathbb{R}^N} |\bar{v}_k|^{2^*}\mathrm{d}x \to \infty$ as $k \to \infty$.*

Remark 1.2. Note that the Sobolev embedding $D^{1,2}(\mathbb{R}^N) \hookrightarrow L^{2^*}(\mathbb{R}^N)$ is not compact. Hence, it is hard to show that the minimizing sequence of S_Φ has a convergence subsequence. We investigate this problem by two different methods. In the first method, we obtain a radially symmetric solution. In the second method, we obtain infinitely many nonradial solutions.

For $b > 0$ and $N = 3 \Leftrightarrow 2^* > 4$, we have

Theorem 1.3. *Assume that $N = 3$, $b > 0$ and condition* (A1) *holds. Then* (1.1) *has a radially symmetric ground state solution $v \in D^{1,2}_{\rm rad}(\mathbb{R}^N)$. Moreover, if $\mu \in (0, 4\Lambda_\Phi/(2^*)^2)$, then $v \in L^{2^*\cdot\frac{2^*}{2}}(\mathbb{R}^N)$.*

When $N \geq 4 \Leftrightarrow 2^* \leq 4$, equation (1.1) is more complicated.

Theorem 1.4. *Assume that $N \geq 4$, $b > 0$ and condition* (A1) *holds. Then the following statements are true.*

(1) *For $N = 4$ and $b \geq S^{-2}$, equation* (1.1) *has no nontrivial solution.*
(2) *For $N > 4$ and $b > \frac{2^*-2}{2}\big(\frac{\Lambda_\Phi-\mu}{\Lambda_\Phi}\big)^{\frac{4-2^*}{2^*-2}}\big(\frac{4-2^*}{2}\big)^{\frac{4-2^*}{2^*-2}}S^{-\frac{2^*}{2^*-2}}$, equation* (1.1) *has no nontrivial solution, where Λ_Φ and μ are defined in condition* (A1).
(3) *For $N \geq 4$, there exists $b_0 > 0$ small enough such that for all $b \in (0, b_0)$, equation* (1.1) *has a radially symmetric.*

We summarize of Theorems 1.1–1.4 as follows:

$$b = 0, N \geq 3 \begin{cases} \text{a radially symmetric solution,} \\ \text{infinitely many nonradial solutions,} \end{cases}$$

$$b > 0 \begin{cases} N = 3, \ \text{a radially symmetric ground state solution,} \\ N = 4, b \geq S^{-2}, \ \text{no nontrivial solution,} \\ N \geq 5 \begin{cases} b > \frac{2^*-2}{2}\big(\frac{\Lambda_\Phi-\mu}{\Lambda_\Phi}\big)^{\frac{4-2^*}{2^*-2}}\big(\frac{4-2^*}{2}\big)^{\frac{4-2^*}{2^*-2}}S^{-\frac{2^*}{2^*-2}}, \ \text{no nontrivial solution,} \\ b \in (0, b_0), \ \text{a radially symmetric solution.} \end{cases} \end{cases}$$

This article is organized as follows. In Section 2, we present notation. In Sections 3-5, we give the proofs of Theorems 1.1–1.4, respectively.

2. Preliminaries

The space $D^{1,2}(\mathbb{R}^N)$ is the completion of $C_0^\infty(\mathbb{R}^N)$ with respect to the semi-norm

$$\|u\|^2_{D^{1,2}(\mathbb{R}^N)} := \int_{\mathbb{R}^N} |\nabla u|^2 \mathrm{d}x.$$

We denote by $D^{1,2}_{\rm rad}(\mathbb{R}^N)$ the space of radial functions in $D^{1,2}(\mathbb{R}^N)$. We define the best constant

$$S := \inf_{u \in D^{1,2}(\mathbb{R}^N)\setminus\{0\}} \frac{\|u\|^2_{D^{1,2}(\mathbb{R}^N)}}{(\int_{\mathbb{R}^N} |u|^{2^*} \mathrm{d}x)^{2/2^*}}.$$

We know that S can be attained in $\mathbb{R}^N$, see [5].

For all $u \in D^{1,2}(\mathbb{R}^N)$, we have the Hardy inequality, see [14],

$$\frac{(N-2)^2}{4} \int_{\mathbb{R}^N} \frac{|u|^2}{|x|^2} \mathrm{d}x \leq \int_{\mathbb{R}^N} |\nabla u|^2 \mathrm{d}x.$$

We introduce the measure $\mathrm{d}\vartheta$ induced by Lebesgues measure on the unit sphere $\mathbb{S}^{N-1} \subset \mathbb{R}^N$. We denote by $\|\cdot\|_{L^q(\mathbb{S}^{N-1})}$ the quantity

$$\|\Phi\|^q_{L^q(\mathbb{S}^{N-1})} = \int_{\mathbb{S}^{N-1}} |\Phi(\vartheta)|^q \mathrm{d}\vartheta.$$

Lemma 2.1 ([15])**.** *Let $N \geq 3$, $0 \leq \Phi \in L^p(\mathbb{S}^{N-1})$ and $p \geq \frac{(N-2)^2}{2(N-1)} + 1$. Then*

$$\int_{\mathbb{R}^N} |\nabla u|^2 \mathrm{d}x \geq \Lambda_\Phi \int_{\mathbb{R}^N} \frac{\Phi(x/|x|)|u|^2}{|x|^2} \mathrm{d}x,$$

where $u \in D^{1,2}(\mathbb{R}^N)$ and $\Lambda_\Phi := \frac{(N-2)^2}{4}|\mathbb{S}^{N-1}|^{1/p}\|\Phi\|^{-1}_{L^p(\mathbb{S}^{N-1})}$.

By using Lemma 2.1 and $\mu \in (0, \Lambda_\Phi)$,

$$\|u\|_\Phi^2 =: \int_{\mathbb{R}^N} |\nabla u|^2 \mathrm{d}x - \mu \int_{\mathbb{R}^N} \frac{\Phi(x/|x|)|u|^2}{|x|^2} \mathrm{d}x$$

is an equivalent norm in $D^{1,2}(\mathbb{R}^N)$.

A measurable function $u : \mathbb{R}^N \to \mathbb{R}$ belongs to the Morrey space $\|u\|_{\mathcal{L}^{q,\varpi}(\mathbb{R}^N)}$ with $q \in [1, \infty)$ and $\varpi \in (0, N]$ if and only if

$$\|u\|^q_{\mathcal{L}^{q,\varpi}(\mathbb{R}^N)} = \sup_{R>0, x\in\mathbb{R}^N} R^{\varpi-3} \int_{B(x,R)} |u(y)|^q \mathrm{d}y < \infty.$$

Lemma 2.2 ([24]). *For $N \geq 3$, there exists $C > 0$ such that for ι and ϑ satisfying $\frac{2}{2^*} \leq \iota < 1$, $1 \leq \vartheta < 2^*$, we have*

$$\Big(\int_{\mathbb{R}^N} |u|^{2^*} \mathrm{d}x\Big)^{1/2^*} \leq C\|u\|^\iota_{D^{1,2}(\mathbb{R}^N)} \|u\|^{1-\iota}_{\mathcal{L}^{\vartheta, \frac{\vartheta(N-2)}{2}}(\mathbb{R}^N)},$$

for any $u \in D^{1,2}(\mathbb{R}^N)$.

Equation (1.1) is variational and its solutions are the critical points of the functional defined in $D^{1,2}(\mathbb{R}^N)$ by

$$I_b(u) = \frac{1}{2}\|u\|^2_{D^{1,2}(\mathbb{R}^N)} - \frac{\mu}{2}\int_{\mathbb{R}^N} \frac{\Phi(x/|x|)|u|^2}{|x|^2}\mathrm{d}x + \frac{b}{4}\|u\|^4_{D^{1,2}(\mathbb{R}^N)} - \frac{1}{2^*}\int_{\mathbb{R}^N} |u|^{2^*}\mathrm{d}x.$$

It is easy to see that the functional $I_b \in C^1(D^{1,2}(\mathbb{R}^N), \mathbb{R})$. It is easy to see that if $u \in D^{1,2}(\mathbb{R}^N)$ is a critical point of I_b, i.e.,

$$\begin{aligned} 0 &= \langle I_b'(u), \varphi\rangle \\ &= \Big(1 + b\|u\|^2_{D^{1,2}(\mathbb{R}^N)}\Big)\int_{\mathbb{R}^N} \nabla u \nabla\varphi \mathrm{d}x \\ &\quad - \mu\int_{\mathbb{R}^N} \Phi\Big(\frac{x}{|x|}\Big)\frac{u\varphi}{|x|^2}\mathrm{d}x - \int_{\mathbb{R}^N} |u|^{2^*-2}u\varphi\mathrm{d}x, \end{aligned}$$

for all $\varphi \in D^{1,2}(\mathbb{R}^N)$.

3. Proof of Theorem 1.1

We separate the proof of Theorem 1.1 into two parts: (i) radially symmetric solution; (ii) nonradial solution.

Proof of Theorem 1.1. (radially symmetric solution).
Step 1. Note that $\mu \in (0, \Lambda_\Phi)$. Applying Lemma 2.2 with $\vartheta = 2$, we obtain

$$\Big(\int_{\mathbb{R}^N} |u|^{2^*}\mathrm{d}x\Big)^{1/2^*} \leq C\|u\|_\Phi^{2\iota}\|u\|^{2(1-\iota)}_{\mathcal{L}^{2,N-2}(\mathbb{R}^N)}, \tag{3.1}$$

for $u \in D^{1,2}(\mathbb{R}^N)$. Let $\{u_n\} \subset D^{1,2}_{\rm rad}(\mathbb{R}^N)$ be a minimizing sequence of S_Φ, that is

$$\|u_n\|^2_\Phi \to S_\Phi \quad \text{as } n \to \infty,$$

and

$$\int_{\mathbb{R}^N} |u_n|^{2^*}\mathrm{d}x = 1.$$

According to (3.1), there exists $C > 0$ such that for any n it holds

$$\|u_n\|_{\mathcal{L}^{2,N-2}(\mathbb{R}^N)} \geq C > 0.$$

On the other hand, we note that $\{u_n\}$ is bounded in $D^{1,2}_{\rm rad}(\mathbb{R}^N)$ and

$$D^{1,2}_{\rm rad}(\mathbb{R}^N) \hookrightarrow L^{2^*}(\mathbb{R}^N) \hookrightarrow \mathcal{L}^{2,N-2}(\mathbb{R}^N).$$

Then

$$\|u_n\|_{\mathcal{L}^{2,N-2}(\mathbb{R}^N)} \le C,$$

Hence, there exists $C_0 > 0$ such that for any n it holds

$$C_0 \le \|u_n\|_{\mathcal{L}^{2,N-2}(\mathbb{R}^N)} \le C_0^{-1}.$$

From above inequality, we deduce that for any $n \in \mathbb{N}$ there exist $\sigma_n > 0$ and $x_n \in \mathbb{R}^N$ such that

$$\frac{1}{\sigma_n^2}\int_{B(x_n,\sigma_n)} |u_n(y)|^2 \mathrm{d}y \ge \|u_n\|^2_{\mathcal{L}^{2,N-2}(\mathbb{R}^N)} - \frac{C}{2n} \ge C_1 > 0.$$

Let $v_n(x) = \sigma_n^{\frac{N-2}{2}} u_n(\sigma_n x)$. By scaling invariance, we have

$$\|v_n\|_\Phi^2 \to S_\Phi, \quad \text{as } n \to \infty,$$
$$\int_{\mathbb{R}^N} |v_n|^{2^*} \mathrm{d}x = 1,$$

and

$$\int_{B(\frac{x_n}{\sigma_n},1)} |v_n(y)|^2 \mathrm{d}y = \frac{1}{\sigma_n^2}\int_{B(x_n,\sigma_n)} |u_n(y)|^2 \mathrm{d}y \ge C_1 > 0. \tag{3.2}$$

Hence, we assume that

$$v_n \rightharpoonup v \text{ in } D^{1,2}_{\rm rad}(\mathbb{R}^N), \quad v_n \to v \text{ a.e. in } \mathbb{R}^N, \quad v_n \to v \text{in } L^q_{\rm loc}(\mathbb{R}^N)$$

for all $q \in [2, 2^*)$.

Step 2. We show that $\{\frac{x_n}{\sigma_n}\}$ is bounded. Suppose on the contrary that $\frac{x_n}{\sigma_n} \to \infty$ as $n \to \infty$. By the boundedness of $\{u_n\}$ in $D^{1,2}_{\rm rad}(\mathbb{R}^N)$, we have $\|v_n\|_{D^{1,2}(\mathbb{R}^N)} = \|u_n\|_{D^{1,2}(\mathbb{R}^N)} \le C$. It follows from the uniform decay estimates of radial functions that

$$|v_n(x)| \le \frac{C}{|x|^{\frac{N-2}{2}}}\|v_n\|_{D^{1,2}(\mathbb{R}^N)} \le \frac{C}{|x|^{\frac{N-2}{2}}}, \quad \text{a.e. } \mathbb{R}^N.$$

For $\sqrt{\frac{C_1}{|B(0,1)|}} > \varepsilon > 0$, there exists $M > 0$ for any $n > M$ it holds

$$|v_n(x)| \le \frac{C_3}{|\frac{x_n}{\sigma_n} - 1|^{\frac{N-2}{2}}} \le \varepsilon, \quad x \in B^c(0, |\frac{x_n}{\sigma_n} - 1|).$$

Note that $B(\frac{x_n}{\sigma_n}, 1) \subset B^c(0, |\frac{x_n}{\sigma_n} - 1|)$. Then

$$\int_{B(\frac{x_n}{\sigma_n},1)} |v_n(y)|^2 \mathrm{d}y \le \varepsilon^2 \int_{B(\frac{x_n}{\sigma_n},1)} \mathrm{d}y = \varepsilon^2 |B(\frac{x_n}{\sigma_n}, 1)| = \varepsilon^2|B(0,1)| < C_1.$$

This contradicts (3.2). Hence, $\{\frac{x_n}{\sigma_n}\}$ is bounded. There exists $R > 0$ such that

$$\int_{B(0,R)} |v_n(y)|^2 \mathrm{d}y \ge \int_{B(\frac{x_n}{\sigma_n},1)} |v_n(y)|^2 \mathrm{d}y \ge C_1 > 0.$$

Since the embedding $D^{1,2}_{\rm rad}(\mathbb{R}^N) \hookrightarrow L^r_{\rm loc}(\mathbb{R}^N)$, $r \in [2, 2^*)$ is compact, we deduce that $v \not\equiv 0$.

Step 3. Set

$$h(t) = t^{2^*}, \ t \ge 0.$$

It is easy to see that $h(t)$ is a convex function. By $h(0) = 0$ and $l \in [0,1]$, we know

$$h(lt) = h(lt + (1-l)\cdot 0) \le lh(t) + (1-l)h(0) = lh(t).$$

For $t_1, t_2 \in [0,\infty)$, applying last inequality, we obtain

$$\begin{aligned} h(t_1) + h(t_2) &= h\Big((t_1+t_2)\frac{t_1}{t_1+t_2}\Big) + h\Big((t_1+t_2)\frac{t_2}{t_1+t_2}\Big) \\ &\le \frac{t_1}{t_1+t_2} h\big(t_1+t_2\big) + \frac{t_2}{t_1+t_2} h(t_1+t_2) \\ &= h(t_1+t_2). \end{aligned}$$

Step 4. We claim that $v_n \to v$ strongly in $D^{1,2}(\mathbb{R}^N)$. It follows from Brézis-Lieb type lemma [2] that

$$\|v\|_\Phi^2 + \lim_{n\to\infty} \|v_n - v\|_\Phi^2 = \lim_{n\to\infty} \|v_n\|_\Phi^2 = S_{\Phi,\alpha},$$

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} |v_n|^{2^*} \mathrm{d}x = \lim_{n\to\infty} \int_{\mathbb{R}^N} |v_n - v|^{2^*} \mathrm{d}x + \int_{\mathbb{R}^N} |v|^{2^*} \mathrm{d}x.$$

Therefore,

$$\begin{aligned} 1 &= \lim_{n\to\infty} \int_{\mathbb{R}^N} |v_n|^{2^*} \mathrm{d}x \\ &= \lim_{n\to\infty} \int_{\mathbb{R}^N} |v_n - v|^{2^*} \mathrm{d}x + \int_{\mathbb{R}^N} |v|^{2^*} \mathrm{d}x \\ &\le S_\Phi^{-\frac{2^*}{2}} \lim_{n\to\infty} \|v_n - v\|_\Phi^{2^*} + S_\Phi^{-\frac{2^*}{2}} \|v\|_\Phi^{2^*} \\ &\le S_\Phi^{-\frac{2^*}{2}} \Big(\lim_{n\to\infty} \|v_n - v\|_\Phi + \|v\|_\Phi\Big)^{2^*} = 1. \end{aligned}$$

Therefore, all the inequalities above have to be equalities. We know that

$$\lim_{n\to\infty} \|v_n - v\|_\Phi^{2^*} + \|v\|_\Phi^{2^*} = \Big(\lim_{n\to\infty} \|v_n - v\|_\Phi + \|v\|_\Phi\Big)^{2^*}.$$

This further gives: either $\lim_{n\to\infty} \|v_n - v\|_\Phi = 0$ or $\|v\|_\Phi = 0$.

From $v \not\equiv 0$, so we have $\|v\|_\Phi \neq 0$. Then

$$\lim_{n\to\infty} \|v_n - v\|_\Phi = 0.$$

We can choose $v \ge 0$.

There exists $C > 0$ such that $\bar{v} = Cv$ satisfies

$$-\Delta \bar{v} - \mu \frac{\Phi(x/|x|)}{|x|^2} \bar{v} = |\bar{v}|^{2^*-1}, \quad x \in \mathbb{R}^N.$$

The proof is complete. □

To study the nonradial solution of equation (1.1), we need the following result.

Lemma 3.1 ([6]). *Let X be a closed subspace of $H^1(\mathbb{S}^{N-1})$. Suppose that the embedding $X \subset L^q(\mathbb{S}^{N-1})$ is compact. Then the restriction of function K on X, $K|_X$ satisfies the Palais-Smale condition. Furthermore, if X is infinite dimensional, then $K|_X$ has a sequence of critical points ϕ_k in X, such that $\int_{\mathbb{S}^{N-1}} |\phi_k|^q \mathrm{d}\vartheta \to \infty$ as $k \to \infty$.*

Proof of Theorem 1.1. (nonradial solutions). It is easy to see that

$$u(x) = |x|^{\frac{2-N}{2}}\phi\Big(\frac{x}{|x|}\Big) \tag{3.3}$$

solves equation (1.3), if and only if ϕ is a solution of the equation

$$-\Delta_\vartheta\phi + \frac{(N-2)^2}{4}\phi - \mu\Phi\phi = |\phi|^{2^*-2}\phi, \quad \text{in } \mathbb{S}^{N-1}. \tag{3.4}$$

The energy functional of equation (3.4) is

$$\begin{aligned} K(\phi) = \frac{1}{2}\int_{\mathbb{S}^{N-1}} |\nabla\phi|^2 \mathrm{d}\vartheta + \frac{(N-2)^2}{8}\int_{\mathbb{S}^{N-1}} |\phi|^2 \mathrm{d}\vartheta - \frac{\mu}{2}\int_{\mathbb{S}^{N-1}} \Phi|\phi|^2 \mathrm{d}\vartheta \\ - \frac{1}{2^*}\int_{\mathbb{S}^{N-1}} |\phi|^{2^*} \mathrm{d}\vartheta \end{aligned}$$

and

$$\begin{aligned} \langle K'(\phi), \varphi\rangle = \int_{\mathbb{S}^{N-1}} \nabla\phi\nabla\varphi \mathrm{d}\vartheta + \frac{(N-2)^2}{4}\int_{\mathbb{S}^{N-1}} \phi\varphi \mathrm{d}\vartheta - \mu\int_{\mathbb{S}^{N-1}} \Phi\phi\varphi \mathrm{d}\vartheta \\ - \int_{\mathbb{S}^{N-1}} |\phi|^{2^*-2}\phi\varphi \mathrm{d}\vartheta. \end{aligned}$$

Suppose that $G = O(k) \times O(m) \subset O(N)$, where $k + m = N$, then $H^1_G(\mathbb{S}^{N-1})$ is an infinite dimensional closed subspace of $H^1(\mathbb{S}^{N-1})$, and $H^1_G(\mathbb{S}^{N-1})$ is compactly embedded in $L^q(\mathbb{S}^{N-1})$ for every $q \in [1, \frac{2(N-1)}{N-3})$, see [6].

Since $2^* \in [1, \frac{2(N-1)}{N-3})$, so we have that $H^1_G(\mathbb{S}^{N-1})$ is compactly embedded in $L^{2^*}(\mathbb{S}^{N-1})$. Applying Lemma 3.1 with $X = H^1_G(\mathbb{S}^{N-1})$ and $q = 2^*$, then we have that $K|_{H^1_G(\mathbb{S}^{N-1})}$ has a sequence of critical points ϕ_k in $H^1_G(\mathbb{S}^{N-1})$, such that $\int_{\mathbb{S}^{N-1}} |\phi_k|^{2^*}\mathrm{d}\vartheta \to \infty$ as $k \to \infty$.

According to (3.3), we know that $\bar{v}_k(x) = |x|^{\frac{2-N}{2}}\phi_k(\frac{x}{|x|})$ are solutions of equation (1.3), and $\int_{\mathbb{R}^N} |\bar{v}_k|^{2^*}\mathrm{d}x = \int_{\mathbb{S}^{N-1}} |\phi_k|^{2^*}\mathrm{d}\vartheta \to \infty$ as $k \to \infty$. □

4. Proof of Theorem 1.3

Define

$$J_b = I_b|_{D^{1,2}_{\mathrm{rad}}(\mathbb{R}^N)}, \quad c = \inf_{\Upsilon\in\Gamma}\max_{t\in[0,1]} J_b(\Upsilon(t)),$$

where

$$\Gamma = \{\Upsilon \in C([0,1], D^{1,2}_{\mathrm{rad}}(\mathbb{R}^N)) | \Upsilon(0) = 0, J_b(\Upsilon(1)) < 0\}.$$

It is easy to see that J_b possesses the mountain pass geometry, there exists $\{u_n\} \subset D^{1,2}_{\mathrm{rad}}(\mathbb{R}^N)$ such that

$$J_b(u_n) \to c > 0 \quad \text{and} \quad J_b'(u_n) \to 0 \quad \text{as } n \to \infty.$$

And $\{u_n\}$ is uniformly bounded in $D^{1,2}_{\mathrm{rad}}(\mathbb{R}^N)$.

The Nehari manifold on $D^{1,2}_{\mathrm{rad}}(\mathbb{R}^N)$ is defined by

$$\mathcal{N}_b = \{u \in D^{1,2}_{\mathrm{rad}}(\mathbb{R}^N) | \langle J_b'(u), u\rangle = 0,\ u \neq 0\},$$

and

$$\bar{\bar{c}} = \inf_{u\in\mathcal{N}_b} J_b(u) \quad \text{and } \bar{c} = \inf_{u\in D^{1,2}_{\mathrm{rad}}(\mathbb{R}^N)}\max_{t\geq 0} J_b(tu).$$

With minor change the proof of [27, Theorem 4.2], we can show that

$$\bar{\bar{c}} = \bar{c} = c.$$

Lemma 4.1. *Assume the assumptions in Theorem 1.3 hold. Then for each $u \in D^{1,2}_{\rm rad}(\mathbb{R}^N) \setminus \{0\}$, there exists a unique $t_u > 0$ such that $t_u u \in \mathcal{N}_b$. Moreover, $J_b(t_u u) = \max_{t\geq 0} J_b(tu)$.*

Proof. For each $u \in D^{1,2}_{\rm rad}(\mathbb{R}^N) \setminus \{0\}$, and $t \in (0,\infty)$, we set

$$f_1(t) = J_b(tu) = \frac{t^2}{2}\|u\|_\Phi^2 + \frac{bt^4}{4}\|u\|^4_{D^{1,2}(\mathbb{R}^N)} - \frac{t^{2^*}}{2^*}\int_{\mathbb{R}^N} |u|^{2^*}\mathrm{d}x,$$

$$f_1'(t) = t\|u\|_\Phi^2 + bt^3\|u\|^4_{D^{1,2}(\mathbb{R}^N)} - t^{2^*-1}\int_{\mathbb{R}^N} |u|^{2^*}\mathrm{d}x.$$

This implies that $f_1'(\cdot) = 0$ if and only if

$$t^{2-2^*}\|u\|_\Phi^2 + bt^{4-2^*}\|u\|^4_{D^{1,2}(\mathbb{R}^N)} = \int_{\mathbb{R}^N} |u|^{2^*}\mathrm{d}x.$$

Set

$$f_2(t) = t^{2-2^*}\|u\|_\Phi^2 + bt^{4-2^*}\|u\|^4_{D^{1,2}(\mathbb{R}^N)}.$$

We know that $\lim_{t\to 0} f_2(t) = \infty$, $\lim_{t\to\infty} f_2(t) = 0$ and $f_2(\cdot)$ is strictly decreasing on $(0,\infty)$. Then there exists a unique $0 < t_u < \infty$ such that

$$f_2(t) \begin{cases} < \int_{\mathbb{R}^N} |u|^{2^*}\mathrm{d}x, & t_u < t < \infty, \\ = \int_{\mathbb{R}^N} |u|^{2^*}\mathrm{d}x, & t = t_u, \\ > \int_{\mathbb{R}^N} |u|^{2^*}\mathrm{d}x, & 0 < t < t_u. \end{cases}$$

This is showing that $t_u u \in \mathcal{N}_b$. Moreover,

$$f_1'(t) \begin{cases} < 0, & t_u < t < \infty, \\ = 0, & t = t_u, \\ > 0, & 0 < t < t_u. \end{cases}$$

This shows that $f_1(\cdot)$ admits a unique critical point t_u on $(0,\infty)$ such that $f_1(\cdot)$ takes the maximum at t_u.

To prove the uniqueness of t_u, let us assume that $0 < \bar{t} < \bar{\bar{t}}$ satisfy $f_1'(\bar{t}) = f_1'(\bar{\bar{t}}) = 0$. We obtain

$$\int_{\mathbb{R}^N} |u|^{2^*}\mathrm{d}x = f_2(\bar{t}) = f_2(\bar{\bar{t}}).$$

Since $0 < \bar{t} < \bar{\bar{t}}$, the above equality leads to the contradiction: $u = 0$. Hence, for each $u \in D^{1,2}_{\rm rad}(\mathbb{R}^N) \setminus \{0\}$, there exists a unique $t_u > 0$ such that $t_u u \in \mathcal{N}_b$. □

Lemma 4.2. *Assume that the assumptions in Theorem 1.3 hold. Let $\{u_n\}$ be a $(PS)_c$ sequence of J_b at $c > 0$. Then up to a subsequence, $u_n \rightharpoonup u$ in $D^{1,2}_{\rm rad}(\mathbb{R}^N)$ with $u \not\equiv 0$ being a weak solution of equation* (1.1).

Proof. It is easy to see that $\{u_n\}$ is uniformly bounded in $D^{1,2}_{\rm rad}(\mathbb{R}^N)$. In order to see that u is a weak solution of J_b, we recall

$$u_n \rightharpoonup u \text{ in } D^{1,2}_{\rm rad}(\mathbb{R}^N), \quad u_n \to u \text{ a.e. in } \mathbb{R}^N, \quad u_n \to u \text{ in } L^r_{\rm loc}(\mathbb{R}^N)$$

for all $r \in [2, 2^*)$. Moreover, there exists $A \in \mathbb{R}$, such that

$$\lim_{n\to\infty} \|u_n\|^2_{D^{1,2}(\mathbb{R}^N)} = A. \tag{4.1}$$

Then by Fatou's lemma,

$$\|u\|^2_{D^{1,2}(\mathbb{R}^N)} \le A.$$

We claim that $\|u\|^2_{D^{1,2}(\mathbb{R}^N)} = A$. To obtain a contradiction, we assume that $\|u\|^2_{D^{1,2}(\mathbb{R}^N)} < A$. Since $u_n \rightharpoonup u$ weakly in $D^{1,2}_{\rm rad}(\mathbb{R}^N)$, we know that for each $\varphi \in D^{1,2}_{\rm rad}(\mathbb{R}^N)$

$$\begin{aligned} &\lim_{n\to\infty} \int_{\mathbb{R}^N} \nabla u_n \nabla \varphi \mathrm{d}x - \mu \int_{\mathbb{R}^N} \Phi\Big(\frac{x}{|x|}\Big) \frac{u_n \varphi}{|x|^2} \mathrm{d}x \\ &= \int_{\mathbb{R}^N} \nabla u \nabla \varphi \mathrm{d}x - \mu \int_{\mathbb{R}^N} \Phi\Big(\frac{x}{|x|}\Big) \frac{u\varphi}{|x|^2} \mathrm{d}x \end{aligned} \tag{4.2}$$

and

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} |u_n|^{2^*-2} u_n \varphi \mathrm{d}x = \int_{\mathbb{R}^N} |u|^{2^*-2} u \varphi \mathrm{d}x. \tag{4.3}$$

From $\lim_{n\to\infty} \langle J_b'(u_n), \varphi\rangle = 0$, we have

$$\begin{aligned} 0 = &\lim_{n\to\infty} (1 + b\|u_n\|^2_{D^{1,2}(\mathbb{R}^N)}) \int_{\mathbb{R}^N} \nabla u_n \nabla \varphi \mathrm{d}x - \mu \int_{\mathbb{R}^N} \Phi\Big(\frac{x}{|x|}\Big) \frac{u_n \varphi}{|x|^2} \mathrm{d}x \\ &- \int_{\mathbb{R}^N} |u_n|^{2^*-2} u_n \varphi \mathrm{d}x. \end{aligned}$$

Applying (4.1), we obtain

$$0 = (1 + bA) \int_{\mathbb{R}^N} \nabla u_n \nabla \varphi \mathrm{d}x - \mu \int_{\mathbb{R}^N} \Phi\Big(\frac{x}{|x|}\Big) \frac{u_n \varphi}{|x|^2} \mathrm{d}x - \int_{\mathbb{R}^N} |u_n|^{2^*-2} u_n \varphi \mathrm{d}x.$$

By using (4.2), (4.3) and $\|u\|^2_{D^{1,2}(\mathbb{R}^N)} < A$, we know that

$$\langle J_b'(u), u\rangle < 0. \tag{4.4}$$

On the other hand, we have

$$\langle J_b'(tu), tu\rangle = f_1'(t)t = t^2 \|u\|^2_\Phi + bt^4 \|u\|^4_{D^{1,2}(\mathbb{R}^N)} - t^{2^*} \int_{\mathbb{R}^N} |u|^{2^*} \mathrm{d}x, \tag{4.5}$$

Applying Lemma 4.1, there exists a unique $t_0 > 0$ satisfying $f_1'(t_0) = 0$, which implies that

$$\langle J_b'(t_0 u), t_0 u\rangle = f_1'(t_0) t_0 = 0 \tag{4.6}$$

Now, we show that $t_0 < 1$. Combining (4.4) and (4.5), we know that $f_1'(1) < 0$. Taking $t_\varepsilon > 0$ small enough in (4.5), we know $f_1'(t_\varepsilon)t_\varepsilon > 0$, which implies $f_1'(t_\varepsilon) > 0$. According to Intermediate value theorem, there exists $t_1 \in (t_\varepsilon, 1)$ such that $f_1'(t_1) = 0$. By using the uniqueness of t_0, we have

$$t_0 = t_1 \in (t_\varepsilon, 1) \tag{4.7}$$

From (4.5)-(4.7), we obtain

$$\begin{aligned} c &= J_b(t_0 u) \\ &= J_b(t_0 u) - \frac{1}{4} \langle J_b'(t_0 u), t_0 u\rangle \\ &= \frac{t_0^2}{4} \|u\|^2_\Phi + \big(\frac{1}{4} - \frac{1}{2^*}\big) t_0^{2^*} \int_{\mathbb{R}^N} |u|^{2^*} \mathrm{d}x \\ &< \frac{1}{4} \|u\|^2_\Phi + \big(\frac{1}{4} - \frac{1}{2^*}\big) \int_{\mathbb{R}^N} |u|^{2^*} \mathrm{d}x \end{aligned}$$

$$\begin{aligned} &\le \frac{1}{4}\lim_{n\to\infty}\|u_n\|_\Phi^2 + \big(\frac{1}{4}-\frac{1}{2^*}\big)\lim_{n\to\infty}\int_{\mathbb{R}^N}|u_n|^{2^*}\mathrm{d}x \\ &= \lim_{n\to\infty} J_b(u_n) - \frac{1}{4}\lim_{n\to\infty}\langle J_b'(u_n),u_n\rangle = c \end{aligned}$$

which is a contradiction. Then

$$\lim_{n\to\infty}\|u_n\|^2_{D^{1,2}(\mathbb{R}^N)} = A = \|u\|^2_{D^{1,2}(\mathbb{R}^N)}.$$

Thus for any $\varphi \in D^{1,2}(\mathbb{R}^N)$, we obtain

$$\lim_{n\to\infty}\langle J_b'(u_n),\varphi\rangle = 0 = \langle J_b'(u),\varphi\rangle.$$

The proof is complete. □

The following result implies the non-vanishing of $(PS)_c$ sequence.

Lemma 4.3. *Assume that all the assumptions descripted in Theorem 1.3 hold. Let $\{u_n\}$ be a $(PS)_c$ sequence of J_b at $c>0$. Then*

$$\lim_{n\to\infty}\int_{\mathbb{R}^N}|u_n|^{2^*}\mathrm{d}x > 0.$$

Proof. It is easy to see that $\{u_n\}$ is uniformly bounded in $D^{1,2}_{\rm rad}(\mathbb{R}^N)$. Then there exists a constant $0<C<\infty$ such that $\|u_n\|_\Phi \le C$.

Suppose on the contrary that

$$\lim_{n\to\infty}\int_{\mathbb{R}^N}|u_n|^{2^*}\mathrm{d}x = 0. \tag{4.8}$$

According to (4.8) and the definition of $(PS)_c$ sequence, we obtain

$$c+o(1) = \frac{1}{2}\|u_n\|_\Phi^2 + \frac{b}{4}\|u_n\|^4_{D^{1,2}(\mathbb{R}^N)} \quad \text{and} \quad o(1) = \|u_n\|_\Phi^2 + b\|u_n\|^4_{D^{1,2}(\mathbb{R}^N)}.$$

This implies $c+o(1) = -\frac{1}{4}\|u_n\|_\Phi^2$, which contradicts $0<c$. □

Proof of Theorem 1.3. (i) Note that $\{u_n\}$ is a bounded sequence of J_b at level c in $D^{1,2}_{\rm rad}(\mathbb{R}^N)$. Up to a subsequence, we assume

$$u_n \rightharpoonup u \text{ in } D^{1,2}_{\rm rad}(\mathbb{R}^N), \quad u_n \to u\text{a.e. in } \mathbb{R}^N, \quad u_n \to u \text{ in } L^r_{\rm loc}(\mathbb{R}^N)$$

for all $r \in [2,2^*)$. Let $v_n(x) = \sigma_n^{\frac{N-2}{2}} u_n(\sigma_n x)$. We assume that

$$v_n \rightharpoonup v \text{ in } D^{1,2}_{\rm rad}(\mathbb{R}^N), \quad v_n \to v \text{ a.e. in } \mathbb{R}^N, \quad v_n \to v \text{ in } L^q_{\rm loc}(\mathbb{R}^N)$$

for all $q \in [2,2^*)$. From Lemma 4.3, we have

$$\lim_{n\to\infty}\int_{\mathbb{R}^N}|u_n|^{2^*}\mathrm{d}x > 0.$$

Similar to the proof of Theorem 1.1 Steps 1 and 2, we deduce that $v \not\equiv 0$. From Lemma 4.2, we know $v \in \mathcal{N}_b$. We show that $v_n \to v$ strongly in $D^{1,2}_{\rm rad}(\mathbb{R}^N)$. Applying

Brézis-Lieb lemma [2], we obtain

$$\begin{aligned} c &= \lim_{n\to\infty} J_b(v_n) - \lim_{n\to\infty} \frac{1}{2^*}\langle J_b'(v_n), v_n\rangle \\ &= \lim_{n\to\infty} \big(\frac{1}{2}-\frac{1}{2^*}\big)\|v_n\|_\Phi^2 + \lim_{n\to\infty} \big(\frac{1}{4}-\frac{1}{2^*}\big)\|v_n\|^4_{D^{1,2}(\mathbb{R}^N)} \\ &\geq \big(\frac{1}{2}-\frac{1}{2^*}\big)\|v\|_\Phi^2 + \big(\frac{1}{4}-\frac{1}{2^*}\big)\|v\|^4_{D^{1,2}(\mathbb{R}^N)} \\ &= J_b(v) \geq c. \end{aligned} \tag{4.9}$$

Thus, the inequalities above have to be equalities. We know that

$$\lim_{n\to\infty} \|v_n\|_\Phi^2 = \|v\|_\Phi^2.$$

By Brézis-Lieb lemma again, we have

$$\lim_{n\to\infty} \|v_n\|_\Phi^2 - \lim_{n\to\infty} \|v_n - v\|_\Phi^2 = \|v\|_\Phi^2,$$

which implies

$$\lim_{n\to\infty} \|v_n - v\|_\Phi^2 = 0.$$

Using (4.9) again, we know that $J_b(v) = c$. This implies that v attains the minimum of J_b at c. Moreover, we can choose $v \geq 0$. The principle of symmetric criticality implies that the critical point of J_b is also a critical point of I_b.

(ii) For each $L > 1$, define

$$v_L(x) = \begin{cases} v(x) & \text{if } v(x) \leq L, \\ L & \text{if } v(x) > L. \end{cases}$$

For $\beta = 2^*/2 > 1$. Set $\phi = v v_L^{2(\beta-1)}$. It is easy to see that $\phi \in D^{1,2}_{\rm rad}(\mathbb{R}^N)$.

We know that v is a nonnegative solution of equation (1.1). Then

$$\Big(1 + b\|v\|^2_{D^{1,2}(\mathbb{R}^N)}\Big)\int_{\mathbb{R}^N} \nabla v \nabla \varphi \mathrm{d}x - \mu \int_{\mathbb{R}^N} \Phi\Big(\frac{x}{|x|}\Big)\frac{v\varphi}{|x|^2}\mathrm{d}x = \int_{\mathbb{R}^N} |v|^{2^*-2} v\varphi \mathrm{d}x.$$

Plugging ϕ into above equation, we obtain

$$\Big(1 + b\|v\|^2_{D^{1,2}(\mathbb{R}^N)}\Big)\int_{\mathbb{R}^N} \nabla v \nabla \phi \mathrm{d}x - \mu \int_{\mathbb{R}^N} \Phi\Big(\frac{x}{|x|}\Big)\frac{v\phi}{|x|^2}\mathrm{d}x = \int_{\mathbb{R}^N} |v|^{2^*-2} v\phi \mathrm{d}x.$$

A direct calculation yields

$$\int_{\mathbb{R}^N} \nabla v \nabla \phi \mathrm{d}x \geq \int_{\mathbb{R}^N} v_L^{2(\beta-1)} |\nabla v|^2 \mathrm{d}x. \tag{4.10}$$

Notice that

$$|\nabla(v v_L^{\beta-1})|^2 = v_L^{2(\beta-1)}|\nabla v|^2 + (\beta-1)^2 v^2 v_L^{2(\beta-2)}|\nabla v_L|^2 + 2(\beta-1) v v_L^{2\beta-3}\nabla v \nabla v_L.$$

Then one has

$$\begin{gathered} \int_{\mathbb{R}^N} v^2 v_L^{2(\mu-2)} |\nabla v_L|^2 \mathrm{d}x \leq \int_{\mathbb{R}^N} v_L^{2(\mu-1)}|\nabla v|^2 \mathrm{d}x, \\ \int_{\mathbb{R}^N} v v_L^{2\mu-3}\nabla v \nabla v_L \mathrm{d}x \leq \int_{\mathbb{R}^N} v_L^{2(\mu-1)}|\nabla v|^2 \mathrm{d}x. \end{gathered}$$

Therefore,

$$\int_{\mathbb{R}^N} |\nabla(v v_L^{\beta-1})|^2 \mathrm{d}x \leq \beta^2 \int_{\mathbb{R}^N} v_L^{2(\beta-1)}|\nabla v|^2 \mathrm{d}x. \tag{4.11}$$

It follows from (4.10) and (4.11) that

$$\frac{1}{\beta^2}\int_{\mathbb{R}^N}|\nabla(vv_L^{\beta-1})|^2\mathrm{d}x \le \int_{\mathbb{R}^N}\nabla v\nabla\phi\mathrm{d}x.$$

Hence,

$$\begin{aligned}\int_{\mathbb{R}^N}|v|^{2^*-2}|vv_L^{\beta-1}|^2\mathrm{d}x &= \Big(1+b\|v\|^2_{D^{1,2}(\mathbb{R}^N)}\Big)\int_{\mathbb{R}^N}\nabla v\nabla\phi\mathrm{d}x-\mu\int_{\mathbb{R}^N}\Phi\Big(\frac{x}{|x|}\Big)\frac{v\phi}{|x|^2}\mathrm{d}x\\ &\ge \int_{\mathbb{R}^N}\nabla v\nabla\phi\mathrm{d}x-\mu\int_{\mathbb{R}^N}\Phi\Big(\frac{x}{|x|}\Big)\frac{v\phi}{|x|^2}\mathrm{d}x\\ &\ge \frac{1}{\beta^2}\int_{\mathbb{R}^N}|\nabla(vv_L^{\beta-1})|^2\mathrm{d}x-\mu\int_{\mathbb{R}^N}\Phi\Big(\frac{x}{|x|}\Big)\frac{|vv_L^{\beta-1}|^2}{|x|^2}\mathrm{d}x\\ &\ge \Big(\frac{1}{\beta^2}-\frac{\mu}{\Lambda_\Phi}\Big)\|vv_L^{\beta-1}\|^2_\Phi.\end{aligned}$$

Then, combining above inequality and Moser iteration technique, we deduce that $v\in L^{2^*\cdot\frac{2^*}{2}}(\mathbb{R}^N)$. □

5. Proof of Theorem 1.4

5.1. **Perturbation equation.** In this subsection, we look equation (1.1) as a perturbation of (1.3). The energy functional of equation (1.3) is

$$I_0(u)=\frac{1}{2}\|u\|^2_\Phi-\frac{1}{2^*}\int_{\mathbb{R}^N}|u|^{2^*}\mathrm{d}x.$$

Set

$$J_0=I_0|_{D^{1,2}_{\mathrm{rad}}(\mathbb{R}^N)},$$

and define

$$c_0=\inf_{\Upsilon\in\Gamma_0}\max_{t\in[0,1]}J_0(\Upsilon(t)),$$

where $\Gamma_0=\{\Upsilon\in C([0,1],D^{1,2}_{\mathrm{rad}}(\mathbb{R}^N))|\Upsilon(0)=0,J_0(\Upsilon(1))<0\}$. The Nehari manifold is

$$\mathcal{N}_0=\{u\in D^{1,2}_{\mathrm{rad}}(\mathbb{R}^N)|\langle J_0'(u),u\rangle=0,\ u\neq 0\},$$

and

$$\bar{c}_0=\inf_{u\in D^{1,2}_{\mathrm{rad}}(\mathbb{R}^N)}\max_{t\ge 0}J_0(tu)\quad\text{and}\quad \bar{\bar{c}}_0=\inf_{u\in\mathcal{N}_0}J_0(u).$$

We can show that $c_0=\bar{c}_0=\bar{\bar{c}}_0$.

Lemma 5.1. *Assume that the assumptions in Theorem 1.4 hold. Then the energy functional J_0 satisfies the following properties*

- (M1) *There exist $\rho,\iota>0$ such that if $\|u\|_{D^{1,2}(\mathbb{R}^N)}=\rho$, then $J_0(u)\ge\iota$, and $e_0\in D^{1,2}_{\mathrm{rad}}(\mathbb{R}^N)$ exists such that $\|e_0\|_{D^{1,2}(\mathbb{R}^N)}>\rho$ and $J_0(e_0)<0$.*
- (M2) *There exists $v_0\not\equiv 0$ such that $J_0(v_0)=c_0:=\min_{\Upsilon\in\Gamma_0}\max_{t\in[0,1]}J_0(\Upsilon(t))$, where $\Gamma_0=\{\Upsilon\in C([0,1],D^{1,2}_{\mathrm{rad}}(\mathbb{R}^N))|\Upsilon(0)=0,J_0(\Upsilon(1))<0\}$.*
- (M3) $c_0=\inf\{J_0(u)|\|J_0'(u)\|_{D^{-1,2}(\mathbb{R}^N)}=0,u\in D^{1,2}_{\mathrm{rad}}(\mathbb{R}^N)\setminus\{0\}\}$.
- (M4) *There exists a path $\Upsilon_0(t)\in\Gamma_0$ passing through v_0 at $t=t_0$ and satisfying*
$$J_0(v_0)>J_0(\Upsilon_0(t))\quad\textit{for all } t\neq t_0.$$
- (M5) *The set $\mathcal{S}:=\{u\in D^{1,2}_{\mathrm{rad}}(\mathbb{R}^N)|\|J_0'(u)\|_{D^{-1,2}(\mathbb{R}^N)}=0,J_0(u)=c_0\}$ is compact in $D^{1,2}_{\mathrm{rad}}(\mathbb{R}^N)$ with the strong topology up to dilations in $\mathbb{R}^N$.*

Proof. As in Theorem 1.3, we have (M1)–(M4).

(M5) Note that J_0 is invariant by dilations. It follows from Theorem 1.3 that the weak convergence of the dilated subsequence can be upgraded into strong convergence. This further implies that the set $\mathcal{S}$ is compact in $D^{1,2}_{\rm rad}(\mathbb{R}^N)$ with the topology up to dilations in $\mathbb{R}^N$. □

5.2. **Perturbation method.** We define a modified mountain pass level of J_b

$$c_b := \min_{\Upsilon\in\Gamma_M}\max_{t\in[0,1]} J_b(\Upsilon(t)),$$

where

$$\Gamma_M = \{\Upsilon\in\Gamma_0 : \sup_{t\in[0,1]}\|\Upsilon(t)\|_{D^{1,2}(\mathbb{R}^N)} \le M\} \quad \text{with}$$
$$M = 2\{\sup_{u\in\mathcal{S}}\|u\|_{D^{1,2}(\mathbb{R}^N)},\ \sup_{t\in[0,1]}\|\Upsilon(t)\|_{D^{1,2}(\mathbb{R}^N)}\} \quad \text{fixed.}$$

By the choice of M, $\Upsilon_0\in\Gamma_M$, we have $c_0 = \min_{\Upsilon\in\Gamma_M}\max_{t\in[0,1]} J_0(\Upsilon(t))$. becasue $\Gamma_M \subsetneqq \Gamma_0$, the standard mountain pass theorem becomes unavailable.

Lemma 5.2. *Let $b>0$. Then $\lim_{b\to 0} c_b = c_0$.*

Proof. For $b>0$, it is easy to obtain $c_b \ge c_0$. We take $e_0 = Tv_0$ in (M_1), where $T > (2^*/2)^{\frac{1}{2^*-1}}$. Then $\Upsilon_0(t)\in C([0,1], D^{1,2}_{\rm rad}(\mathbb{R}^N))$ defined as

$$\Upsilon_0(t) = te_0 = tTv_0,$$

and $t_0 = \frac{1}{T}$ in (M4). We know that

$$\lim_{b\to 0} c_b = \lim_{b\to 0} J_b(\Upsilon_0(t)) \le J_0(\Upsilon_0(t)) + \lim_{\lambda\to 0}\frac{b}{4}\|\Upsilon_0(t)\|^4_{D^{1,2}(\mathbb{R}^N)} = J_0(v_0) = c_0.$$

□

For any $d>0$, and any subset A of $D^{1,2}_{\rm rad}(\mathbb{R}^N)$, we set

$$A^d := \bigcup_{u\in A} B_d(u),$$

where $B_d(u) := \{v\in D^{1,2}_{\rm rad}(\mathbb{R}^N) | \|u-v\|_{D^{1,2}(\mathbb{R}^N)} \le d\}$.

Lemma 5.3. *Let $d>0$ and $\{u_j\}\subset\mathcal{S}^d$. Then there exists $\{\sigma_j\}$ such that*

$$\|\bar u_j\|_{D^{1,2}(\mathbb{R}^N)} = \|u_j\|_{D^{1,2}(\mathbb{R}^N)}$$

where $\bar u_j(x) = \sigma_j^{\frac{N-2}{2}} u_j(\sigma_j x)$. Up to a subsequence, $\bar u_j \rightharpoonup \bar u \in \mathcal{S}^{2d}$.

Proof. Let $\{u_j\}\subset\mathcal{S}^d$. From $\mathcal{S}^d$ and Lemma 5.1 (M5), there exists $w_j\in\mathcal{S}$ such that

$$\|u_j - w_j\|_{D^{1,2}(\mathbb{R}^N)} \le d.$$

From (M5), there exists $\{\sigma_j\}$ such that $\bar w_j\in\mathcal{S}$,where $\bar w_j(x) = \sigma_j^{\frac{N-2}{2}} w_j(\sigma_j x)$. It is easy to prove that $\bar w_j \to \bar w\in\mathcal{S}$. And

$$\|\bar u_j\|_{D^{1,2}(\mathbb{R}^N)} = \|u_j\|_{D^{1,2}(\mathbb{R}^N)}, \quad \|\bar u_j - \bar w_j\|_{D^{1,2}(\mathbb{R}^N)} = \|u_j - w_j\|_{D^{1,2}(\mathbb{R}^N)} \le d.$$

For j large enough, we have

$$\begin{aligned}\|\bar u_j - \bar w\|_{D^{1,2}(\mathbb{R}^N)} &= \|\bar u_j - \bar w_j + \bar w_j - \bar w\|_{D^{1,2}(\mathbb{R}^N)}\\ &\le \|\bar u_j - \bar w_j\|_{D^{1,2}(\mathbb{R}^N)} + \|\bar w_j - \bar w\|_{D^{1,2}(\mathbb{R}^N)} \le 2d.\end{aligned}$$

This shows that $\{\bar{u}_j\}$ is bounded. Up to a subsequence, we assume that $\bar{u}_j \rightharpoonup \bar{u}$ in $D^{1,2}_{\rm rad}(\mathbb{R}^N)$. Note that $B_{2d}(\bar{w})$ is weakly closed in $D^{1,2}_{\rm rad}(\mathbb{R}^N)$. We obtain $\bar{u} \in B_{2d}(\bar{w}) \subset \mathcal{S}^{2d}$. $\square$

Lemma 5.4. *Let $d_1 := \frac{1}{2}\sqrt{\frac{2\cdot 2^*}{2^*-2}c_0}$ and $d \in (0, d_1)$. Suppose that there exist sequences $b_j > 0$, $b_j \to 0$, and $\{u_j\} \subset \mathcal{S}^d$ satisfying*

$$\lim_{j\to\infty} J_{b_j}(u_j) \le c_0 \quad \text{and} \quad \lim_{j\to\infty} \|J'_{b_j}(u_j)\|_{D^{-1,2}(\mathbb{R}^N)} = 0.$$

Then there exists a sequence $\{\sigma_j\}$ such that $\|\bar{u}_j\|_{D^{1,2}(\mathbb{R}^N)} = \|u_j\|_{D^{1,2}(\mathbb{R}^N)}$, where $\bar{u}_j(x) = \sigma_j^{\frac{N-2}{2}} u_j(\sigma_j x)$. Up to a subsequence, $\{\bar{u}_j\}$ converges to $\bar{u} \in \mathcal{S}$.

Proof. Let $\lim_{j\to\infty} \|J'_{b_j}(u_j)\|_{D^{-1,2}(\mathbb{R}^N)} = 0$ and $\{u_j\}$ be bounded. From Lemma 5.3, up to a subsequence, $\bar{u}_j \rightharpoonup \bar{u} \in \mathcal{S}^{2d}$. From d_1, we know that $\bar{u} \not\equiv 0$.

Let $\bar{u}_j(x) = \sigma_j^{\frac{N-2}{2}} u_j(\sigma_j x)$. We have

$$\lim_{j\to\infty} J_{b_j}(\bar{u}_j) = \lim_{j\to\infty} J_{b_j}(u_j) \le c_0.$$

For all $\varphi \in D^{1,2}_{\rm rad}(\mathbb{R}^N)$, we obtain

$$\begin{aligned}
&|\langle J'_{b_j}(\bar{u}_j), \varphi\rangle| \\
&= |\langle J'_{b_j}(u_j), \bar{\varphi}\rangle| \\
&\le \|J'_{b_j}(u_j)\|_{D^{-1,2}(\mathbb{R}^N)} \|\bar{\varphi}\|_{D^{1,2}(\mathbb{R}^N)} \\
&= o(1)\|\bar{\varphi}\|_{D^{1,2}(\mathbb{R}^N)},
\end{aligned}$$

where $\bar{\varphi} = \sigma_j^{-\frac{N-2}{2}} \varphi(x/\sigma_j)$. Note that $\|\bar{\varphi}\|_{D^{1,2}(\mathbb{R}^N)} = \|\varphi\|_{D^{1,2}(\mathbb{R}^N)}$. We know that

$$\|J'_{b_j}(\bar{u}_j)\|_{D^{-1,2}(\mathbb{R}^N)} \to 0 \quad \text{as } j \to \infty,$$

which further implies

$$\langle J'_0(\bar{u}), \varphi\rangle = \lim_{j\to\infty} \langle J'_{b_j}(\bar{u}_j), \varphi\rangle - \frac{b_j}{4}\|\bar{u}_j\|^4_{D^{1,2}(\mathbb{R}^N)} = 0.$$

This shows that $\|J'_0(\bar{u})\|_{D^{-1,2}(\mathbb{R}^N)} = 0$.

It follows from $\bar{u}_j \in \mathcal{S}^{2d}$ that

$$\begin{aligned}
\lim_{j\to\infty} \langle J'_0(\bar{u}_j), \varphi\rangle &= \lim_{j\to\infty} \langle J'_{b_j}(\bar{u}_j), \varphi\rangle - \lim_{j\to\infty} b_j \|\bar{u}_j\|^2_{D^{1,2}(\mathbb{R}^N)} \int_{\mathbb{R}^N} \nabla \bar{u}_j(x) \nabla \varphi(x) \mathrm{d}x \\
&= o(1)\|\varphi\|_{D^{1,2}(\mathbb{R}^N)}.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
c_0 &\ge \lim_{j\to\infty} J_{b_j}(\bar{u}_j) \\
&= \lim_{j\to\infty} J_0(\bar{u}_j) + \lim_{j\to\infty} \frac{b_j}{4}\|\bar{u}_j\|^4_{D^{1,2}(\mathbb{R}^N)} \\
&= \lim_{j\to\infty} J_0(\bar{u}_j).
\end{aligned} \tag{5.1}$$

So $\{\bar{u}_j\}$ is a $(PS)_m$ sequence for J_0 with $m := \lim_{j\to\infty} J_0(\bar{u}_j)$. Up to a subsequence, $\bar{u}_j \rightharpoonup \bar{u}$ and

$$J_0(\bar{u}) = \frac{1}{2}\|\bar{u}\|^2_\Phi - \frac{1}{2^*}\int_{\mathbb{R}^N} |\bar{u}|^{2^*} \mathrm{d}x$$

$$
\begin{aligned}
&= \big(\frac{1}{2}-\frac{1}{2^*}\big)\|\bar u\|_\Phi^2 \\
&\le \big(\frac{1}{2}-\frac{1}{2^*}\big)\liminf_{j\to\infty}\|\bar u_j\|_\Phi^2 \\
&= \liminf_{j\to\infty}\Big(J_0(\bar u_j)-\frac{1}{2^*}\langle J_0'(\bar u_j),\bar u_j\rangle\Big) = m.
\end{aligned}
$$

It follows from (M3) that $m \ge J_0(\bar u) \ge c_0$. From (5.1), one has $m = J_0(\bar u) = c_0$, which implies $\bar u \in \mathcal{S}$. □

Set

$$
m_b := \max_{t\in[0,1]} J_b(\Upsilon_0(t)). \tag{5.2}
$$

Then $c_b \le m_b$. It is easy to see that $\lim_{b\to 0} m_b \le c_0$. From this inequality and Lemmas 5.2 and 5.4 , one has

$$
\lim_{b\to 0} c_b = \lim_{b\to 0} m_b = c_0.
$$

We define

$$
J_b^{m_b} = \{u \in D^{1,2}_{\rm rad}(\mathbb{R}^N) | J_b(u) \le m_b\}.
$$

Proposition 5.5. *Let $d_2, d_3 > 0$ satisfying $d_3 < d_2 < d_1$. Then there exist $\iota > 0$ and $\tilde b > 0$ depending on d_2, d_3 such that for $b \in (0, \tilde b)$, it holds*

$$
\|J_b'(u)\|_{D^{-1,2}(\mathbb{R}^N)} \ge \iota, \quad u \in J_b^{m_b} \cap (\mathcal{S}^{d_2}\backslash \mathcal{S}^{d_3}).
$$

Proof. Suppose on the contrary that $d_2, d_3 > 0$ satisfying $d_3 < d_2 < d_1$, there exist sequences $\{b_j\}$ with $\lim_{j\to\infty} b_j = 0$, and $\{u_j\} \in J_{b_j}^{m_{b_j}} \cap (\mathcal{S}^{d_2} \setminus \mathcal{S}^{d_3})$ such that

$$
\lim_{j\to\infty} J_{b_j}(u_j) \le c_0 \quad \text{and} \quad \lim_{j\to\infty} \|J_{b_j}'(u_j)\|_{D^{-1,2}(\mathbb{R}^N)} = 0.
$$

From (M5), there exists sequence $\{\sigma_j\}$ such that

$$
\begin{gathered}
\{\bar u_j\} \in J_{b_j}^{m_{b_j}} \cap (\mathcal{S}^{d_2} \setminus \mathcal{S}^{d_3}), \quad \lim_{j\to\infty} J_{b_j}(\bar u_j) \le c_0, \\
\lim_{j\to\infty} \|J_{b_j}'(\bar u_j)\|_{D^{-1,2}(\mathbb{R}^N)} = 0,
\end{gathered}
$$

where $\bar u_j(x) = \sigma_j^{\frac{N-2}{2}} u_j(\sigma_j x)$. Hence, we can apply Lemma 5.4 and the existence of $\bar u \in \mathcal{S}$ such that $\bar u_j \to \bar u$ in $D^{1,2}_{\rm rad}(\mathbb{R}^N)$. As a consequence, $\operatorname{dist}(\bar u_j, \mathcal{S}) \to 0$ as $j \to \infty$. This is a contradiction with $\bar u_j \notin \mathcal{S}^{d_3}$. □

Proposition 5.6. *For any $d > 0$, there exists $\delta > 0$ such that if $b > 0$ small enough, then*

$$
J_b(\Upsilon_0(t)) \ge c_b - \delta \textit{ implies } \Upsilon_0(t) \in \mathcal{S}^d, \quad t \in [0,1].
$$

The proof of the above proposition follows by repeating the proof of [16, Propositions 4].

Proposition 5.7. *For any $d \in (0, d_1)$, there exist $b_0 > 0$ and a sequence $\{u_j\} \subset J_b^{m_b} \cap \mathcal{S}^d$ such that $\|J_b'(u_j)\|_{D^{-1,2}(\mathbb{R}^N)} \to 0$ as $j \to \infty$, for all $b \in (0, b_0)$.*

The proof of the above proposition follows from a discussion in [4, Propositions 5.3], by Propositions 5.5 and 5.6.

Proof of Theorem 1.4. (i) Suppose on the contrary that $u \in D^{1,2}(\mathbb{R}^N)\backslash\{0\}$ is a solution of (1.1). It follows from $2^* = 4$ and $b \geq S^{-2}$ that

$$\begin{aligned}
\langle I_b'(u), u\rangle &= \|u\|^2_{D^{1,2}(\mathbb{R}^N)} - \mu\int_{\mathbb{R}^N}\frac{\Phi(x/|x|)|u|^2}{|x|^2}\mathrm{d}x + b\|u\|^4_{D^{1,2}(\mathbb{R}^N)} - \int_{\mathbb{R}^N}|u|^{2^*}\mathrm{d}x \\
&\geq \|u\|^2_{D^{1,2}(\mathbb{R}^N)} - \mu\int_{\mathbb{R}^N}\frac{\Phi(x/|x|)|u|^2}{|x|^2}\mathrm{d}x + b\|u\|^4_{D^{1,2}(\mathbb{R}^N)} - S^{-2}\|u\|^4_{D^{1,2}(\mathbb{R}^N)} \\
&\geq \|u\|^2_{D^{1,2}(\mathbb{R}^N)} - \mu\int_{\mathbb{R}^N}\frac{\Phi(x/|x|)|u|^2}{|x|^2}\mathrm{d}x > 0.
\end{aligned}$$

This is a contradiction.

(ii) Suppose on the contrary that $u \in D^{1,2}(\mathbb{R}^N)\backslash\{0\}$ is a solution of (1.1). Applying Young's inequality and

$$b > \frac{2^*-2}{2}\Big(\frac{\Lambda_\Phi - \mu}{\Lambda_\Phi}\Big)^{\frac{4-2^*}{2^*-2}}\Big(\frac{4-2^*}{2}\Big)^{\frac{4-2^*}{2^*-2}} S^{-\frac{2^*}{2^*-2}},$$

we have

$$\begin{aligned}
&\Big(1-\frac{\mu}{\Lambda_\Phi}\Big)\|u\|^2_{D^{1,2}(\mathbb{R}^N)} + b\|u\|^4_{D^{1,2}(\mathbb{R}^N)} \\
&\leq \|u\|^2_{D^{1,2}(\mathbb{R}^N)} - \mu\int_{\mathbb{R}^N}\frac{\Phi(x/|x|)|u|^2}{|x|^2}\mathrm{d}x + b\|u\|^4_{D^{1,2}(\mathbb{R}^N)} \\
&= \int_{\mathbb{R}^N}|u|^{2^*}\mathrm{d}x \\
&\leq S^{-\frac{2^*}{2}}\|u\|^{2^*}_{D^{1,2}(\mathbb{R}^N)} \\
&= \Big[S^{-\frac{2^*}{2}}\Big(\frac{2b}{2^*-2}\Big)^{\frac{2-2^*}{2}}\|u\|^{4-2^*}_{D^{1,2}(\mathbb{R}^N)}\Big]\Big[\Big(\frac{2b}{2^*-2}\Big)^{\frac{2^*-2}{2}}\|u\|^{2(2^*-2)}_{D^{1,2}(\mathbb{R}^N)}\Big] \\
&\leq \frac{4-2^*}{2}\Big[S^{-\frac{2^*}{2}}\Big(\frac{2b}{2^*-2}\Big)^{\frac{2-2^*}{2}}\|u\|^{4-2^*}_{D^{1,2}(\mathbb{R}^N)}\Big]^{\frac{2}{4-2^*}} \\
&\quad + \frac{2^*-2}{2}\Big[\Big(\frac{2b}{2^*-2}\Big)^{\frac{2^*-2}{2}}\|u\|^{2(2^*-2)}_{D^{1,2}(\mathbb{R}^N)}\Big]^{\frac{2}{2^*-2}} \\
&= \frac{4-2^*}{2}S^{-\frac{2^*}{4-2^*}}\Big(\frac{2^*-2}{2b}\Big)^{\frac{2^*-2}{4-2^*}}\|u\|^2_{D^{1,2}(\mathbb{R}^N)} + b\|u\|^4_{D^{1,2}(\mathbb{R}^N)}.
\end{aligned}$$

which is a contradiction.

(iii) Taking $d \in (0, d_1)$, by Proposition 5.7, there exists $b_0 > 0$ such that for all $\lambda \in (0, b_0)$, there exists a Palais-Smale sequence $\{u_j\} \subset \mathcal{S}^{d/2}$. By applying (M5), there exists sequence $\{\sigma_j\}$ such that $\{\bar{u}_j\} \subset \mathcal{S}^{d/2}$ where $\bar{u}_j(x) = \sigma_j^{\frac{3-2s}{2}}u_j(\sigma_j x)$. Clearly, $\{\bar{u}_j\}$ is bounded in $D^{1,2}_{\rm rad}(\mathbb{R}^N)$. Then by Lemma 5.4, up to a subsequence, there exists $\bar{u} \in \mathcal{S}^{\frac{d}{2}\cdot 2} = \mathcal{S}^d$ such that $\bar{u}_j \rightharpoonup \bar{u}$. Then we obtain $\|J_b'(\bar{u})\|_{D^{-1,2}(\mathbb{R}^N)} = 0$. It follows from $d \in (0, d_1)$ that $\bar{u} \not\equiv 0$. Hence $\bar{u}$ is a nontrivial critical point of J_b. The principle of symmetric criticality implies that the critical point of J_b is also a critical point of I_b. □

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