Electronic Journal of Differential Equations, Vol. 2022 (2022), No. 38, pp. 1–30.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu

GLOBAL SOLUTIONS AND BLOW-UP FOR A KIRCHHOFF-TYPE PROBLEM ON A GEODESIC BALL OF THE POINCARÉ BALL MODEL

HANG DING, JUN ZHOU

Abstract. This article concerns a Kirchhoff-type parabolic problem on a geodesic ball of hyperbolic space. Firstly, we obtain conditions for finite time blow-up, and for the existence of global solutions for $J(u_0) \le d$, where $J(u_0)$ denotes the initial energy and d denotes the depth of the potential well. Secondly, we estimate the upper and lower bounds of the blow-up time. In addition, we derive the growth rate of the blow-up solution and the decay rate of the global solution. Thirdly, we establish a new finite time blow-up condition which is independent of d and prove that the solution can blow up in finite time with arbitrary high initial energy, by using this blow-up condition. Finally, we present some equivalent conditions for the solution existing globally or blowing up in finite time.

1. Introduction

In this article, we consider the Kirchhoff-type parabolic problem

$$\begin{gathered} u_t - \Big(a + b\int_{B_R} |\nabla_H u(\sigma)|^2\,d\mu\Big)\Delta_H u = \xi|u|^{q-1}u, \quad \sigma \in B_R,\ t>0,\\ u(\sigma,t) = 0, \quad \sigma\in\partial B_R,\ t>0,\\ u(\sigma,0)=u_0(\sigma), \quad \sigma\in B_R, \end{gathered} \tag{1.1}$$

where Δ_H is the Laplace-Beltrami operator on the Poincaré ball model $\mathbb{B}^3$, which is a model of the hyperbolic space $\mathbb{H}^3$, $B_R \subset \mathbb{B}^3$ denotes a geodesic ball centered in zero with radius R, the initial value $u_0 \in H_0^1(B_R)$, and the parameters a, b, ξ and q satisfy

$$a \ge 0, \quad b>0, \quad \xi>0, \quad 3<q<5. \tag{1.2}$$

We first recall the definitions of $\mathbb{B}^3$, Δ_H, B_R, $H_0^1(B_R)$ and ∇_H, which can be found in [2, 30].

(1) The Poincaré ball is

$$\mathbb{B}^3 := \{\sigma=(x_1,x_2,x_3)\in\mathbb{R}^3 : |\sigma|<1\}$$

2020 *Mathematics Subject Classification.* 35K55, 35B40, 35B44.
Key words and phrases. Parabolic problem of Kirchhoff type; hyperbolic space; poincaré ball model; global solution; blow-up.

Submitted July 18, 2021 Published May 13, 2022.

endowed with the Riemannian metric

$$g_{ij} := \frac{4}{(1-|\sigma|^2)^2}\delta_{ij} \quad (\sigma \in \mathbb{B}^3;\ i,j = 1,2,3),$$

where δ_{ij} and $|\cdot|$ denote the usual Kronecker delta and the Euclidean distance, respectively.

(2) For $i, j = 1, 2, 3$, we define

$$g^{ij} := (g_{ij})^{-1} \quad \text{and} \quad g := \det(g_{ij}).$$

In this setting the operator Δ_H is locally defined by

$$\Delta_H := \frac{1}{\sqrt{g}} \sum_{i=1}^{3} \frac{\partial}{\partial x_i}\Big(\sqrt{g}\sum_{j=1}^{3} g^{ij}\frac{\partial}{\partial x_j}\Big).$$

As usual, let

$$d\mu := \sqrt{g}dx = \frac{8}{(1-|\sigma|^2)^3}dx$$

be the Riemannian volume element in $\mathbb{B}^3$, where dx is the standard Lebesgue measure in the Euclidean space $\mathbb{R}^3$. Therefore, if (note that $|\sigma| < 1$)

$$d_H(\sigma, 0) := 2\int_0^{|\sigma|} \frac{1}{1-t^2}dt = \log(\frac{1+|\sigma|}{1-|\sigma|})$$

denotes the geodesic distance of $\sigma \in \mathbb{B}^3$ from the origin, a direct calculation ensures that the operator Δ_H has the more convenient form

$$\Delta_H = \frac{1}{4}(1-|\sigma|^2)^2 \sum_{i=1}^{3} \frac{\partial^2}{\partial x_i^2} + \frac{1}{2}(1-|\sigma|^2)\sum_{i=1}^{3} x_i \frac{\partial}{\partial x_i}.$$

(3) The geodesic ball B_R and its surface ∂B_R are defined by

$$B_R := \{\sigma \in \mathbb{B}^3 : d_H(\sigma, 0) < R\}, \quad \partial B_R := \{\sigma \in \mathbb{B}^3 : d_H(\sigma, 0) = R\},$$

where

$$d_H(\sigma_1, \sigma_2) := \cosh^{-1}\Big(1 + \frac{2|\sigma_2 - \sigma_1|^2}{(1-|\sigma_1|^2)(1-|\sigma_2|^2)}\Big), \quad \forall \sigma_1, \sigma_2 \in \mathbb{B}^3$$

denotes the hyperbolic distance in the Poincaré ball model $\mathbb{B}^3$.

(4) For a geodesic ball $B_R \subset \mathbb{B}^3$, we denote by $H_0^1(B_R)$ the completion of $C_0^\infty(B_R)$ with respect to the Hilbertian norm

$$\|u\| := \Big(\int_{B_R} |\nabla_H u(\sigma)|^2\, d\mu\Big)^{1/2}, \tag{1.3}$$

where

$$\nabla_H := \Big(\frac{1-|\sigma|^2}{2}\Big)^2 \nabla,$$

denotes the hyperbolic gradient. Then we have

$$\int_{B_R} (\Delta_H u)\phi\, d\mu = \int_{B_R} (\nabla_H u)(\nabla_H \phi)\, d\mu, \quad \forall u, \phi \in C_0^\infty(B_R). \tag{1.4}$$

By a denseness argument, $H_0^1(B_R)$ denotes the Sobolev space of the functions $u \in L^2(B_R)$ such that $\nabla_H u$ exists in the sense of distributions and $|\nabla_H u|$ is in $L^2(B_R)$, endowed with the natural norm (1.3).

(5) From standard theory, we know that the embedding $H_0^1(B_R) \hookrightarrow L^\nu(B_R)$ is continuous for any $\nu \in [1,6]$, while it is compact whenever $\nu \in [1,6)$. Therefore, there is a positive constant C_ν such that

$$\|u\|_{L^\nu(B_R)} \leq C_\nu \|u\| \quad \text{for all } u \in H_0^1(B_R) \text{ and } \nu \in [1,6]. \tag{1.5}$$

Below we introduce the research history of problem (1.1). Kirchhoff in 1883 proposed the model

$$\rho \frac{\partial^2 u}{\partial t^2} - \Big(\frac{P_0}{h} + \frac{E}{2L}\int_0^L |\frac{\partial u(x)}{\partial x}|^2 dx\Big)\frac{\partial^2 u}{\partial x^2} = 0,$$

which was as a generalization of the well-known D'Alembert wave equation for free vibrations of elastic strings. The above parameters have the following definitions: ρ denotes the mass density, P_0 denotes the initial tension, h denotes the area of the cross-section, E denotes the Young modulus of the material and L denotes the length of the string.

Recently, Xiang et al. [34] considered the Kirchhoff-type parabolic problem involving the fractional Laplacian,

$$\begin{gathered} u_t + M([u]_s^2)\mathcal{L}_K u = |u|^{p-2}u, \quad \text{in } \Omega \times (0,+\infty), \\ u(x,0) = u_0(x), \quad \text{in } \Omega, \\ u(x,t) = 0, \quad \text{in } (\mathbb{R}^n \setminus \Omega) \times (0,+\infty). \end{gathered} \tag{1.6}$$

Firstly, by using the classical Galerkin method, the authors showed the local existence of solutions. Secondly, they obtained the finite time blow-up of solutions with negative initial energy. Finally, they also estimated the upper and lower bounds of the blow-up time by some differential inequality techniques. For more recent references on Kirchhoff-type problems, we refer to [6, 7, 8, 12, 13, 14, 26, 27, 28, 32, 33, 35, 36, 37, 39, 41].

It is worth mentioning that the potential well method was introduced by Sattinger in [31] to study the global existence of solutions to the nonlinear hyperbolic equations. From then on, many researchers applied this method to study the nonlinear evolution equations, see [5, 9, 10, 18, 21, 22, 23, 24, 25, 29, 38, 43]. Especially, Payne and Sattinger [29] investigated the existence and finite time blow-up of solutions to the initial boundary value problem of semilinear parabolic equations and semilinear hyperbolic equations. Ikehata and Suzuki [18] studied the stable and unstable sets for the parabolic equations and hyperbolic equations. Liu et al. [22, 23, 24, 25, 38] treated the existence of the global solution for the double dispersion equations, semilinear wave equations and parabolic equations.

In recent years, there has been a lot of work on evolution/steady-state problems in the hyperbolic space, especially in the Poincaré ball model. The research contents include existence, uniqueness, multiplicity, global existence and blow-up, see [1, 2, 4, 15, 16, 17, 30] and references therein. In particular, the reference [2] dealt with the steady-state problem corresponding to problem (1.1), i.e.,

$$\begin{gathered} -\Big(a + b\int_{B_R} |\nabla_H u(\sigma)|^2\, d\mu\Big)\Delta_H u = \xi f(u), \quad \text{in } B_R, \\ u(\sigma,t) = 0, \quad \text{on } \partial B_R. \end{gathered} \tag{1.7}$$

By using the topological and variational methods, the existence and multiplicity of the weak solution to the above problem were studied.

Motivated by the above work, in this article, we consider the evolution problem corresponding to (1.7) with $f(u) = |u|^{q-1}u$, i.e., problem (1.1). By using the potential well theory, we obtain

(1) conditions for the existence of global solutions and for finite time blow-up;
(2) growth rate of the blow-up solution and the decay rate of the global solution;
(3) upper and lower bounds of the blow-up time;
(4) necessary and sufficient conditions for the solution existing globally or blowing up in finite time.

The remaining parts of this article are organized as follows. In Section 2, we give the main results of this paper. In Section 3, we introduce some important lemmas. In Section 4, we prove the main results.

2. Main results

To introduce the main results, we first introduce some notation. Throughout this paper, the norm of the space $L^\varrho(B_R)$ for $1 \le \varrho \le +\infty$ is denoted by $\|\cdot\|_\varrho$. Namely, for any $v \in L^\varrho(B_R)$,

$$\|v\|_\varrho = \begin{cases} \big(\int_{B_R} |v(\sigma)|^\varrho \,d\mu\big)^{1/\varrho}, & \text{if } 1 \le \varrho < +\infty; \\ \operatorname{ess\,sup}_{\sigma\in B_R} |v(\sigma)|, & \text{if } \varrho = +\infty. \end{cases}$$

Moreover, the inner product of the Hilbert space $L^2(B_R)$ is defined by

$$(u, v) := \int_{B_R} uv \,d\mu, \quad \forall u, v \in L^2(B_R).$$

Secondly, the energy functional J and the Nehari functional I are defined by

$$J(u) := \frac{1}{2}\Big(a\|u\|^2 + \frac{b}{2}\|u\|^4\Big) - \frac{\xi}{q+1}\|u\|_{q+1}^{q+1}, \tag{2.1}$$

$$I(u) := \langle J'(u), u\rangle = a\|u\|^2 + b\|u\|^4 - \xi\|u\|_{q+1}^{q+1}, \tag{2.2}$$

where $\langle\cdot,\cdot\rangle$ denotes the dual product between $H^{-1}(B_R)$ and $H_0^1(B_R)$. By $3 < q < 5$ and (1.5), we know that J and I are well defined in $H_0^1(B_R)$. Moreover, from (1.4), we see that the critical points of J are weak solutions of the steady-state problem corresponding to (1.1) (see [2]).

Obviously, from (2.1) and (2.2), one has

$$J(u) = \frac{(q-1)a}{2(q+1)}\|u\|^2 + \frac{(q-3)b}{4(q+1)}\|u\|^4 + \frac{1}{q+1}I(u). \tag{2.3}$$

The depth of the potential well is defined by

$$d := \inf_{u\in N} J(u), \tag{2.4}$$

where N denotes the Nehari manifold and

$$N := \{u \in H_0^1(B_R) \setminus \{0\} : I(u) = 0\}. \tag{2.5}$$

From Lemma 3.3, we see that d is a positive constant and

$$d \ge M := \frac{2ar_0^2(q-1) + br_0^4(q-3)}{4(q+1)}, \tag{2.6}$$

where $r_0 > 0$ is the constant given in Lemma 3.2.

In addition, we set

$$N_+ := \{u \in H_0^1(B_R) : I(u) > 0\}, \tag{2.7}$$

$$N_- := \{u \in H_0^1(B_R) : I(u) < 0\}. \tag{2.8}$$

Thirdly, we define the potential well W and the outer space of the potential well V as follows:

$$W := \{u \in H_0^1(B_R) : J(u) < d, I(u) > 0\} \cup \{0\}, \tag{2.9}$$

$$V := \{u \in H_0^1(B_R) : J(u) < d, I(u) < 0\}. \tag{2.10}$$

To introduce the main results, we need the following three definitions.

Definition 2.1. Let $u_0 \in H_0^1(B_R)$ and $T > 0$. A function $u = u(t)$ in the space $L^\infty(0, T; H_0^1(B_R))$ with $u_t \in L^2(0, T; L^2(B_R))$ is said to be a weak solution of (1.1), if

$$\begin{aligned} &\int_{B_R} u_t \phi \, d\mu + \Big(a + b \int_{B_R} |\nabla_H u(\sigma)|^2 \, d\mu\Big) \int_{B_R} \nabla_H u \nabla_H \phi \, d\mu \\ &= \xi \int_{B_R} |u|^{q-1} u \phi \, d\mu, \end{aligned} \tag{2.11}$$

for any $\phi \in H_0^1(B_R)$. In addition, the following energy inequality holds

$$J(u(t)) + \int_0^t \|u_\tau\|_2^2 \, d\tau \le J(u_0) \tag{2.12}$$

for a.e. $t \in (0, T)$.

Definition 2.2. Assume $u = u(t)$ is a weak solution of (1.1), then the maximal existence time T of u is defined by:

1. (1) If there is a $t_0 \in (0, +\infty)$ such that u exists for $t \in [0, t_0)$, but does not exist at $t = t_0$, then the maximal existence time $T = t_0$;
2. (2) If u exists for all $t \in [0, +\infty)$, then the maximal existence time $T = +\infty$.

Definition 2.3. *Assume $u = u(t)$ is a weak solution of* (1.1)*. If the maximal existence time $T < +\infty$ and*

$$\lim_{t \to T^-} \int_0^t \|u\|_2^2 \, d\tau = +\infty, \tag{2.13}$$

then we say that u blows up in finite time.

Now, we introduce the main results of the present paper. Firstly, we give the existence of global solutions.

Theorem 2.4. *Assume* (1.2) *holds and $u_0 \in H_0^1(B_R)$. If $J(u_0) < d$ and $I(u_0) > 0$, then* (1.1) *admits a global weak solution $u(t) \in L^\infty(0, +\infty; H_0^1(B_R))$ with $u_t \in L^2(0, +\infty; L^2(B_R))$ and $u(t) \in W$ for all $t \in [0, +\infty)$. In addition, if the weak solution is bounded, then it is unique. Furthermore, if $J(u_0) < d_0$, then*

$$\|u\|_2^2 \le \frac{\|u_0\|_2^2}{Dt\|u_0\|_2^2 + 1},$$

where

$$D := 2\lambda_1^2 \Big[b - \xi C_{q+1}^{q+1} \Big(\frac{4(q+1)J(u_0)}{(q-3)b}\Big)^{\frac{q-3}{4}}\Big] > 0, \quad d_0 := \frac{(q-3)b}{4(q+1)} \Big(\frac{b}{\xi C_{q+1}^{q+1}}\Big)^{\frac{4}{q-3}}.$$

Here, C_{q+1} is defined in (1.5)*, and $\lambda_1 > 0$ is the first eigenvalue of the eigenvalue problem*

$$\begin{aligned} -\Delta_H u = \lambda u, &\quad \text{in } B_R; \\ u = 0, &\quad \text{on } \partial B_R, \end{aligned} \tag{2.14}$$

which can be characterized as

$$\lambda_1 = \inf_{u \in H_0^1(B_R)\setminus\{0\}} \frac{\|u\|^2}{\|u\|_2^2}. \tag{2.15}$$

Remark 2.5. Note that $d_0 \le d$. Indeed, for any $u \in N$, it follows from (2.3) and Lemma 3.2(3) that

$$\begin{aligned} J(u) &= \frac{(q-1)a}{2(q+1)}\|u\|^2 + \frac{(q-3)b}{4(q+1)}\|u\|^4 \\ &\ge \frac{(q-3)b}{4(q+1)}\|u\|^4 \\ &\ge \frac{(q-3)b}{4(q+1)} r_0^4 \\ &= \frac{(q-3)b}{4(q+1)} \big(\frac{b}{\xi C_{q+1}^{q+1}}\big)^{\frac{4}{q-3}} = d_0. \end{aligned}$$

From Theorem 2.4, we can obtain the following corollary.

Corollary 2.6. *Assume* (1.2) *holds and* $u_0 \in H_0^1(B_R)$. *If* $J(u_0) \le d$ *and* $I(u_0) \ge 0$, *then* (1.1) *admits a global weak solution* $u(t) \in L^\infty(0, +\infty; H_0^1(B_R))$ *with* $u_t \in L^2(0, +\infty; L^2(B_R))$ *and* $u(t) \in \overline{W}$ *for all* $t \in [0, +\infty)$.

Next, we introduce a result about finite time blow-up.

Theorem 2.7. *Assume* (1.2) *holds and* $u_0 \in H_0^1(B_R)$. *Let* $u = u(t)$ *be a weak solution of* (1.1). *If* $J(u_0) \le d$ *and* $I(u_0) < 0$, *then* $u(t)$ *blows up at some finite time* T. *Furthermore,*

(1) *if* $J(u_0) < d$, *then* T *can be estimated by*

$$T \le \frac{4q\|u_0\|_2^2}{(q+1)(q-1)^2(d - J(u_0))};$$

(2) *if* $3 < q < 11/3$, *then*

$$T > \frac{\|u_0\|_2^{2-2\gamma}}{2\widehat{C}(\gamma - 1)} \quad \text{and} \quad \|u\|_2 > \big(2\widehat{C}(T-t)(\gamma-1)\big)^{\frac{1}{2(1-\gamma)}},$$

where

$$\gamma = \frac{10-2q}{11-3q} > 1, \quad \widehat{C} = \Big(\frac{\xi \widetilde{C}^{q+1}}{b^{\frac{3q-3}{8}}}\Big)^{\frac{8}{11-3q}}.$$

Here, $\widetilde{C}$ *is the best constant in the inequality*

$$\|u\|_{q+1} \le \widetilde{C}\|u\|^{1-\theta}\|u\|_2^\theta, \tag{2.16}$$

and

$$\theta = \frac{5-q}{2(q+1)} \in (0,1). \tag{2.17}$$

Remark 2.8. The constant $\widetilde{C}$ in (2.16) is well-defined. In fact, by (1.5), we have

$$\|u\|_6 \le C_6\|u\|.$$

Since $3 < q < 5$, by using the interpolation inequality (see [3, 11]), we obtain

$$\|u\|_{q+1} \le \|u\|_6^{1-\theta}\|u\|_2^\theta,$$

where θ is given in (2.17). Combining the above two inequalities, we obtain

$$\|u\|_{q+1} \leq C_6^{1-\theta} \|u\|^{1-\theta} \|u\|_2^{\theta}.$$

So, $\widetilde{C}$ is well-defined and $\widetilde{C} \leq C_6^{1-\theta}$.

The next theorem gives the growth rate of blow-up solutions.

Theorem 2.9. *Assume* (1.2) *holds and* $u_0 \in H_0^1(B_R)$. *Let* $u = u(t)$ *be a weak solution of* (1.1). *If* $J(u_0) \leq M$ *and* $I(u_0) < 0$, *then for any* $\varepsilon \in (0, 1/3]$, *there is a* $t_\varepsilon \in (0, T)$ *such that* u *satisfies*

$$\|u\|_2^2 \geq C_\varepsilon \Big(t^{\frac{(q+1)\varepsilon}{2}} - t^{\frac{(q+1)\varepsilon}{2} - 1} t_\varepsilon \Big)^{\frac{2}{2-(q+1)\varepsilon}}$$

for all $t \in [t_\varepsilon, T)$, *where*

$$C_\varepsilon := \Big[\Big(1 - \frac{(q+1)\varepsilon}{2} \Big) F^{-\frac{(q+1)\varepsilon}{2}}(t_\varepsilon) F'(t_\varepsilon) \Big]^{\frac{2}{2-(q+1)\varepsilon}}, \quad F(t) := \int_0^t \|u\|_2^2 \, d\tau.$$

Next, we give a new blow-up condition which is independent of d.

Theorem 2.10. *Assume* (1.2) *holds and* $u_0 \in H_0^1(B_R)$. *Let* $u = u(t)$ *be a weak solution of* (1.1). *If*

$$J(u_0) < \frac{(q-1)a\lambda_1}{2(q+1)} \|u_0\|_2^2 + \frac{(q-3)b\lambda_1^2}{4(q+1)} \|u_0\|_2^4, \tag{2.18}$$

then $u(t)$ *blows up at some finite time* T. *Furthermore,* T *can be estimated by*

$$T \leq \frac{16q\|u_0\|_2^2}{(q-1)^2[2(q-1)a\lambda_1\|u_0\|_2^2 + (q-3)b\lambda_1^2\|u_0\|_2^4 - 4(q+1)J(u_0)]}.$$

In addition,

$$\|u\|_2^2 \geq \frac{2(q+1)}{S} J(u_0) + \Big(\|u_0\|_2^2 - \frac{2(q+1)}{S} J(u_0) \Big) e^{St},$$

where λ_1 *is defined in* (2.15) *and* $S = \frac{2(q-1)a\lambda_1 + (q-3)b\lambda_1^2\|u_0\|_2^2}{2}$.

Next, we give a finite time blow-up result with arbitrary high initial energy.

Theorem 2.11. *For any constant* $P > d$, *there is a function* $u_P \in H_0^1(B_R)$, *which satisfies* $J(u_P) = P$ *and* (2.18). *Then the weak solution* u *of problem* (1.1) *with the initial value* u_P *blows up in finite time.*

Next, we give a result related to the asymptotic behavior of the energy functional.

Theorem 2.12. *Let* $u = u(t)$ *be a weak solution of* (1.1) *and* T *be the maximal existence time of* u. *If* $J(u_0) \leq d$, $I(u_0) < 0$ *or* (2.18) *holds, then*

$$\lim_{t \to T} J(u(t)) = -\infty. \tag{2.19}$$

The next theorem is about some equivalent conditions for the solution blowing up in finite time or existing globally.

Theorem 2.13. *Let* $u = u(t)$ *be a weak solution of* (1.1) *and* T *be the maximum existence time of* u,

1. *if* $J(u_0) < d$ *and* $u_0 \in H_0^1(B_R) \setminus \{0\}$, *then it holds*
   - (a) $I(u_0) < 0 \Leftrightarrow T < +\infty \Leftrightarrow$ *there is a* $t_0 \in [0, T)$ *such that* $J(u(t_0)) < 0$;
   - (b) $I(u_0) > 0 \Leftrightarrow T = +\infty \Leftrightarrow J(u(t)) > 0$ *for all* $t \in [0, T)$;

(2) *if* $J(u_0) = d$ *and* $u_0 \in H_0^1(B_R) \setminus \{N \cup \{0\}\}$, *then it holds*
   (a) $I(u_0) < 0 \Leftrightarrow T < +\infty \Leftrightarrow$ *there is a* $t_0 \in [0, T)$ *such that* $J(u(t_0)) < 0$;
   (b) $I(u_0) > 0 \Leftrightarrow T = +\infty \Leftrightarrow J(u(t)) > 0$ *for all* $t \in [0, T)$,

where N *is defined in* (2.5).

3. Preliminaries

Lemma 3.1 (see [19, 20]). *Assume that* $0 < T \le +\infty$ *and* $\rho(t) \in C^2[0, T)$ *is a nonnegative function satisfying*

$$\rho''(t)\rho(t) - (1+\gamma)(\rho'(t))^2 \ge 0,$$

where γ *is a positive constant. If* $\rho(0) > 0$ *and* $\rho'(0) > 0$, *then* $T \le \frac{\rho(0)}{\gamma\rho'(0)} < +\infty$ *and* $\rho(t) \to +\infty$ *as* $t \to T$.

Lemma 3.2. *Let* $u \in H_0^1(B_R)$ *and* (1.2) *hold,*

(1) *if* $0 < \|u\| < r_0$, *then* $I(u) > 0$;
(2) *if* $I(u) < 0$, *then* $\|u\| > r_0$;
(3) *if* $I(u) = 0$, *then* $\|u\| \ge r_0$ *or* $\|u\| = 0$,

where $r_0 = \big(\frac{b}{\xi C_{q+1}^{q+1}}\big)^{\frac{1}{q-3}} > 0$.

Proof. (1) It follows from $0 < \|u\| < r_0$ and (1.5) that

$$\xi\|u\|_{q+1}^{q+1} \le \xi C_{q+1}^{q+1}\|u\|^{q+1} = \xi C_{q+1}^{q+1}\|u\|^{q-3}\|u\|^4 < b\|u\|^4 \le a\|u\|^2 + b\|u\|^4,$$

which, together with the definition of $I(u)$, implies $I(u) > 0$.

(2) Because $I(u) < 0$, we infer that $\|u\| \ne 0$. By (1.5), one has

$$a\|u\|^2 + b\|u\|^4 < \xi\|u\|_{q+1}^{q+1} \le \xi C_{q+1}^{q+1}\|u\|^{q+1},$$

which yields

$$\xi C_{q+1}^{q+1}\|u\|^{q-1} > a + b\|u\|^2 \ge b\|u\|^2,$$

this gives $\|u\| > r_0$.

(3) If $\|u\| = 0$, then we obtain $I(u) = 0$. If $I(u) = 0$ and $\|u\| \ne 0$, then we obtain from (1.5) that

$$a\|u\|^2 + b\|u\|^4 = \xi\|u\|_{q+1}^{q+1} \le \xi C_{q+1}^{q+1}\|u\|^{q+1},$$

which implies

$$\xi C_{q+1}^{q+1}\|u\|^{q-1} \ge a + b\|u\|^2 \ge b\|u\|^2,$$

this gives $\|u\| \ge r_0$. □

Lemma 3.3. *Let* (1.2) *hold. Then*

$$d \ge \frac{2ar_0^2(q-1) + br_0^4(q-3)}{4(q+1)}, \tag{3.1}$$

where d *and* r_0 *are defined in* (2.4) *and Lemma 3.2, respectively.*

Proof. For all $u \in N$, we have $I(u) = 0$ and $u \in H_0^1(B_R) \setminus \{0\}$. Then from (2.3) and Lemma 3.2(3) we obtain

$$\begin{aligned} J(u) &= \frac{(q-1)a}{2(q+1)}\|u\|^2 + \frac{(q-3)b}{4(q+1)}\|u\|^4 + \frac{1}{q+1}I(u) \\ &= \frac{(q-1)a}{2(q+1)}\|u\|^2 + \frac{(q-3)b}{4(q+1)}\|u\|^4 \end{aligned}$$

$$\geq \frac{2ar_0^2(q-1)+br_0^4(q-3)}{4(q+1)},$$

which implies (3.1). □

Lemma 3.4. *Let* (1.2) *hold. If* $u \in H_0^1(B_R)$ *and* $I(u)<0$*, then there is a* $r^* \in (0,1)$ *such that* $I(r^*u)=0$.

Proof. We divide the proof into two cases.

Case 1: $a=0$**.** For $r>0$, we set $\phi(r) := \xi r^{q-3}\|u\|_{q+1}^{q+1}$, then it is clear that

$$I(ru) = br^4\|u\|^4 - \xi r^{q+1}\|u\|_{q+1}^{q+1} = r^4(b\|u\|^4 - \phi(r)). \tag{3.2}$$

It follows from $I(u)<0$, (3.2) and Lemma 3.2(2) that

$$\phi(1) > b\|u\|^4 > br_0^4 > 0. \tag{3.3}$$

Furthermore, according to the definition of $\phi(r)$, we reach

$$\lim_{r\to 0^+}\phi(r)=0,$$

which, together with (3.3), implies that there is a $r^*\in(0,1)$ such that $\phi(r^*)=b\|u\|^4$ and $I(r^*u)=0$.

Case 2: $a>0$**.** For $r>0$, we set $\phi(r) := \xi r^{q-1}\|u\|_{q+1}^{q+1} - br^2\|u\|^4$, then we have

$$I(ru) = ar^2\|u\|^2 + br^4\|u\|^4 - \xi r^{q+1}\|u\|_{q+1}^{q+1} = r^2(a\|u\|^2 - \phi(r)). \tag{3.4}$$

It follows from $I(u)<0$, (3.4) and Lemma 3.2(2) that

$$\phi(1) > a\|u\|^2 > ar_0^2 > 0. \tag{3.5}$$

Furthermore, from the definition of $\phi(r)$, we have

$$\lim_{r\to 0^+}\phi(r)=0,$$

which, together with (3.5), implies that there is a $r^*\in(0,1)$ such that $\phi(r^*)=a\|u\|^2$ and $I(r^*u)=0$. □

Lemma 3.5. *Let* (1.2) *hold. If* $u\in H_0^1(B_R)$ *and* $I(u)<0$*, then*

$$I(u) < (q+1)(J(u)-d). \tag{3.6}$$

Proof. By Lemma 3.4, we see that there is a $r^*\in(0,1)$ such that $I(r^*u)=0$. Let

$$f(r) := (q+1)J(ru) - I(ru), \quad r>0,$$

then we have

$$f(r) = \frac{a(q-1)}{2}r^2\|u\|^2 + \frac{b(q-3)}{4}r^4\|u\|^4.$$

It follows from Lemma 3.2(2) that

$$f'(r) = a(q-1)r\|u\|^2 + b(q-3)r^3\|u\|^4 \geq b(q-3)r^3\|u\|^4 > b(q-3)r^3r_0^4 > 0,$$

which implies that $f(r)$ is strictly increasing for $r>0$. Then we obtain from $0<r^*<1$ that $f(1)>f(r^*)$, i.e.,

$$(q+1)J(u) - I(u) > (q+1)J(r^*u) - I(r^*u) = (q+1)J(r^*u) \geq (q+1)d,$$

which means (3.6). □

Lemma 3.6. *Assume* (1.2) *holds and* $u_0 \in H_0^1(B_R)$. *Let* $u = u(t)$ *be a weak solution of* (1.1). *Then*

$$\frac{1}{2}\frac{d}{dt}\|u\|_2^2 = -I(u), \quad \forall t \in [0,T), \tag{3.7}$$

where T *is the maximum existence time of* u.

Proof. Let $\phi = u(t)$ in (2.11), one has

$$\int_{B_R} u_t u \, d\mu + \Big(a + b\int_{B_R} |\nabla_H u(\sigma)|^2 \, d\mu\Big) \int_{B_R} |\nabla_H u|^2 \, d\mu = \xi \int_{B_R} |u|^{q+1} \, d\mu,$$

i.e.,

$$\frac{1}{2}\frac{d}{dt}\|u\|_2^2 = -a\|u\|^2 - b\|u\|^4 + \xi\|u\|_{q+1}^{q+1},$$

which, along with the definition of $I(u)$, yields (3.7). □

Lemma 3.7. *Let* $u = u(t)$ *be a weak solution of* (1.1) *and* T *be the maximum existence time of* u. *If* $J(u_0) \le d$, *then the sets* N_- *and* N_+ *are both invariant for* $u(t)$, *namely, if* $u_0 \in N_-$ *(resp.* $u_0 \in N_+$*), then* $u(t) \in N_-$ *(resp.* $u(t) \in N_+$*) for all* $t \in [0,T)$.

Proof. Because the proof of the invariance of N_+ and N_- is similar, we only show the invariance of N_-. We divide the proof into two cases.

Case 1: $J(u_0) < d$**.** By contradiction, if not, then there must exist a $t_0 \in (0,T)$ such that $I(u(t)) < 0$ for $t \in [0,t_0)$ and $I(u(t_0)) = 0$. Then it follows from Lemma 3.2(2) that $\|u\| > r_0 > 0$ for $t \in [0,t_0)$, which means $\|u(t_0)\| \ge r_0 > 0$. Thus, we infer that $u(t_0) \in N$ and $J(u(t_0)) \ge d$, which contradicts that $J(u(t_0)) \le J(u_0) < d$ (see (2.12)).

Case 2: $J(u_0) = d$**.** By contradiction, if not, then there must exist a $t_1 \in (0,T)$ such that $I(u(t)) < 0$ for $t \in [0,t_1)$ and $I(u(t_1)) = 0$. Then it follows from Lemma 3.2(2) that $\|u\| > r_0 > 0$ for $t \in [0,t_1)$, which implies $u(t_1) \ne 0$. Consequently, we infer that $u(t_1) \in N$ and $J(u(t_1)) \ge d$. Furthermore, by Lemma 3.6, we know $(u_t, u) = -I(u(t)) > 0$ for $t \in [0,t_1)$, which means $\int_0^{t_1} \|u_\tau\|_2^2 \, d\tau > 0$. Hence, we obtain from (2.12) that

$$J(u(t_1)) \le J(u_0) - \int_0^{t_1} \|u_\tau\|_2^2 \, d\tau < d,$$

which contradicts that $J(u(t_1)) \ge d$. □

4. Proofs of main results

Proof of Theorem 2.4. We divide the proof into three steps.

Step 1: Existence of the global weak solution. Let ω_j, $j = 1,2,\dots$ be the eigenfunction of the Laplace-Beltrami operator subject to the Dirichlet boundary condition

$$\begin{gathered} -\Delta_H \omega_j = \lambda_j \omega_j, \quad \sigma \in B_R, \\ \omega_j = 0, \quad \sigma \in \partial B_R. \end{gathered}$$

Furthermore, we normalize ω_j such that $\|\omega_j\|_2 = 1$. Then we see that $\{\omega_j\}_{j=1}^{\infty}$ is a basis of $H_0^1(B_R)$. Constructing the following approximate solution $u_m(t)$ of (1.1),

$$u_m = \sum_{j=1}^{m} g_{jm}(t)\omega_j(\sigma), \quad m = 1, 2 \dots \tag{4.1}$$

which satisfies

$$\begin{aligned} &\int_{B_R} u_{mt}\omega_j \, d\mu + \Big(a + b\int_{B_R} |\nabla u_m(\sigma)|^2 \, d\mu\Big) \int_{B_R} \nabla_H u_m \nabla_H \omega_j \, d\mu \\ &= \xi \int_{B_R} |u_m|^{q-1} u_m \omega_j \, d\mu, \end{aligned} \tag{4.2}$$
$$(u_m(0), \omega_j) = \zeta_{jm}$$

for $j = 1, 2, \dots, m$, where the constant ζ_{jm} satisfies

$$u_m(0) = \sum_{j=1}^{m} \zeta_{jm}\omega_j(\sigma) \to u_0 \quad \text{in } H_0^1(B_R) \text{ as } m \to +\infty. \tag{4.3}$$

According to the standard theory of ODEs, we infer that there is a $T > 0$ depending only on ζ_{jm} $(j = 1, 2, \dots, m)$ such that $g_{jm} \in C^1[0, T]$ and $g_{jm}(0) = \zeta_{jm}$. Hence, $u_m \in C^1([0, T]; H_0^1(B_R))$.

Multiplying the first equation of (4.2) by $g'_{jm}(t)$ and summing for $j = 1, 2, \dots, m$, then integrating with respect to time from 0 to t, one has

$$J(u_m(t)) + \int_0^t \|u_{m\tau}\|_2^2 \, d\tau = J(u_m(0)), \quad t \in [0, T].$$

It follows from (4.3) and $g_{jm}(0) = \zeta_{jm}$ that

$$\lim_{m\to+\infty} J(u_m(0)) = J(u_0) < d, \quad \lim_{m\to+\infty} I(u_m(0)) = I(u_0) > 0.$$

We conclude that $I(u_m(0)) > 0$ and

$$J(u_m(t)) + \int_0^t \|u_{m\tau}\|_2^2 \, d\tau = J(u_m(0)) < d, \quad t \in [0, T] \tag{4.4}$$

for sufficiently large m, which implies $u_m(0) \in W$.

Now, for any $t \in [0, T]$ and sufficiently large m, we show that $u_m(t) \in W$. In fact, if not, we infer that there is a $t_0 \in (0, T]$ and a sufficiently large m such that $u_m(t_0) \in \partial W$, which implies $u_m(t_0) \in H_0^1(B_R) \setminus \{0\}$ and $I(u_m(t_0)) = 0$ or $J(u_m(t_0)) = d$. According to (4.4), we know that $J(u_m(t_0)) = d$ is impossible. Hence, we obtain $u_m(t_0) \in N$, then we infer that $J(u_m(t_0)) \geq d$, a contradiction. Therefore, for any $t \in [0, T]$ and sufficiently large m, we have $u_m(t) \in W$.

According to $I(u_m(t)) > 0$, (4.4), and

$$J(u_m(t)) = \frac{(q-1)a}{2(q+1)}\|u_m\|^2 + \frac{(q-3)b}{4(q+1)}\|u_m\|^4 + \frac{1}{q+1} I(u_m(t)),$$

we readily obtain

$$\frac{(q-1)a}{2(q+1)}\|u_m\|^2 + \frac{(q-3)b}{4(q+1)}\|u_m\|^4 + \int_0^t \|u_{m\tau}\|_2^2 \, d\tau < d$$

for any $t \in [0, T]$ and sufficiently large m. Then

$$\int_0^t \|u_{m\tau}\|_2^2 \, d\tau < d, \quad \forall t \in [0, T], \tag{4.5}$$

$$\|u_m\|^4 < \frac{4(q+1)d}{(q-3)b}, \quad \forall t \in [0, T]. \tag{4.6}$$

Thus, $T = +\infty$ and $u_m(t) \in W$ for $t \in [0, +\infty)$. It follows from (1.5) and (4.6) that

$$\begin{aligned} \int_{B_R} \big||u_m|^{q-1}u_m\big|^{\frac{q+1}{q}}\, d\mu &= \int_{B_R} |u_m|^{q+1}\, d\mu \le C_{q+1}^{q+1}\|u_m\|^{q+1} \\ &< C_d := \Big(\frac{4(q+1)dC_{q+1}^4}{(q-3)b}\Big)^{\frac{q+1}{4}}, \quad \forall t \in [0, +\infty). \end{aligned} \tag{4.7}$$

It follows from (4.5), (4.6), and (4.7) that there exists a function $u = u(t) \in L^\infty(0, +\infty; H_0^1(B_R))$ with $u_t \in L^2(0, +\infty; L^2(B_R))$ and a subsequence of $\{u_m\}_{m=1}^\infty$ (still denoted by $\{u_m\}_{m=1}^\infty$) such that for each $\widetilde{T} > 0$, as $m \to +\infty$,

$$u_{mt} \rightharpoonup u_t \quad \text{weakly in } L^2(0, \widetilde{T}; L^2(B_R)), \tag{4.8}$$

$$u_m \rightharpoonup u \quad \text{weakly star in } L^\infty(0, \widetilde{T}; H_0^1(B_R)), \tag{4.9}$$

$$u_m \rightharpoonup u \quad \text{weakly in } L^2(0, \widetilde{T}; H_0^1(B_R)), \tag{4.10}$$

$$|u_m|^{q-1}u_m \rightharpoonup |u|^{q-1}u \quad \text{weakly star in } L^\infty(0, \widetilde{T}; L^{\frac{q+1}{q}}(B_R)), \tag{4.11}$$

$$|u_m|^{q-1}u_m \rightharpoonup |u|^{q-1}u \quad \text{weakly in } L^2(0, \widetilde{T}; L^{\frac{q+1}{q}}(B_R)). \tag{4.12}$$

In addition, it is clear from $3 < q < 5$ that $H_0^1(B_R) \hookrightarrow L^{q+1}(B_R)$ compactly. Then from [42], we conclude that

$$\big\{u : u \in L^2\big(0, \widetilde{T}; H_0^1(B_R)\big), u_t \in L^2(0, \widetilde{T}; L^2(B_R))\big\} \hookrightarrow L^2(0, \widetilde{T}; L^{q+1}(B_R))$$

compactly. Consequently,

$$u_m \to u \quad \text{strongly in } L^2(0, \widetilde{T}; L^{q+1}(B_R)). \tag{4.13}$$

Now, we choose a function $h \in C^1([0, \widetilde{T}]; H_0^1(B_R))$ and fix a integer $s > 0$ such that

$$h = \sum_{j=1}^{s} f_j(t)\omega_j(\sigma), \tag{4.14}$$

where $\{f_j(t)\}_{j=1}^s$ are arbitrary given C^1 functions. Taking $m \ge s$ in the first equation of (4.2), and multiplying it by $f_j(t)$, summing for $j = 1, 2, \dots, s$, then integrating with respect to t from 0 to $\widetilde{T}$, one has

$$\begin{aligned} &\int_0^{\widetilde{T}} \int_{B_R} u_{mt} h\, d\mu\, dt + \int_0^{\widetilde{T}} \Big(a + b\int_{B_R} |\nabla_H u_m(\sigma)|^2\, d\mu\Big) \int_{B_R} \nabla_H u_m \nabla_H h\, d\mu\, dt \\ &= \xi \int_0^{\widetilde{T}} \int_{B_R} |u_m|^{q-1}u_m h\, d\mu\, dt. \end{aligned}$$

Taking $m \to +\infty$ in the above equality, we obtain from (4.8), (4.10) and (4.12) that

$$\begin{aligned} &\int_0^{\widetilde{T}} \int_{B_R} u_t h\, d\mu\, dt + \int_0^{\widetilde{T}} \Big(a + b\int_{B_R} |\nabla_H u(\sigma)|^2\, d\mu\Big) \int_{B_R} \nabla_H u \nabla_H h\, d\mu\, dt \\ &= \xi \int_0^{\widetilde{T}} \int_{B_R} |u|^{q-1}u h\, d\mu\, dt. \end{aligned}$$

Because the set of functions h is dense in $L^2(0,\widetilde{T};H_0^1(B_R))$, we know the above equality holds for all $h\in L^2(0,\widetilde{T};H_0^1(B_R))$. From the arbitrariness of $\widetilde{T}>0$, we infer that

$$\int_{B_R} u_t\phi\,d\mu+\Big(a+b\int_{B_R}|\nabla_H u(\sigma)|^2\,d\mu\Big)\int_{B_R}\nabla_H u\nabla_H\phi\,d\mu=\xi\int_{B_R}|u|^{q-1}u\phi\,d\mu$$

holds for any $\phi\in H_0^1(B_R)$ and a.e. $t\in(0,+\infty)$.

According to (4.8) and (4.10), we know that $u_m(0)\rightharpoonup u(0)$ weakly in $L^2(B_R)$. Then it follows from (4.1), (4.3) and $g_{jm}(0)=\zeta_{jm}$ that $u(0)=u_0\in H_0^1(B_R)$.

Now, we show that (2.12) holds for a.e. $t\in(0,+\infty)$. Indeed, for a.e. $t\in(0,+\infty)$, we select $\widetilde{T}>t$. Then we obtain from (4.13) that $u_m(t)\to u(t)$ strongly in $L^{q+1}(B_R)$. Thus, it follows from (4.7) and (4.12) that, as $m\to+\infty$,

$$\begin{aligned}
&\Big|\int_{B_R}|u_m|^{q+1}\,d\mu-\int_{B_R}|u|^{q+1}\,d\mu\Big|\\
&\le\Big|\int_{B_R}u(u_m|u_m|^{q-1}-u|u|^{q-1})\,d\mu\Big|+\Big|\int_{B_R}(u_m-u)u_m|u_m|^{q-1}\,d\mu\Big|\\
&\le\Big|\int_{B_R}u(u_m|u_m|^{q-1}-u|u|^{q-1})\,d\mu\Big|+\sqrt[\frac{q+1}{q}]{C_d}\|u_m-u\|_{q+1}\to0,
\end{aligned}\tag{4.15}$$

which, along with (4.1), (4.3), (4.4), (4.8), (4.10), (4.13), and $g_{jm}(0)=\xi_{jm}$, yields

$$\begin{aligned}
&\int_0^t\|u_\tau\|_2^2\,d\tau+\frac{a}{2}\|u\|^2+\frac{b}{4}\|u\|^4\\
&\le\liminf_{m\to+\infty}\int_0^t\|u_{m\tau}\|_2^2\,d\tau+\frac{a}{2}\liminf_{m\to\infty}\|u_m\|^2+\frac{b}{4}\liminf_{m\to+\infty}\|u_m\|^4\\
&\le\liminf_{m\to+\infty}\Big(\int_0^t\|u_{m\tau}\|_2^2\,d\tau+\frac{a}{2}\|u_m\|^2+\frac{b}{4}\|u_m\|^4\Big)\\
&=\liminf_{m\to+\infty}\Big(\int_0^t\|u_{m\tau}\|_2^2\,d\tau+J(u_m)+\frac{\xi}{q+1}\|u_m\|_{q+1}^{q+1}\Big)\\
&=\lim_{m\to+\infty}(\frac{\xi}{q+1}\|u_m\|_{q+1}^{q+1}+J(u_m(0)))\\
&=\frac{\xi}{q+1}\|u\|_{q+1}^{q+1}+J(u_0),
\end{aligned}$$

which means (2.12) holds for a.e. $t\in(0,+\infty)$. In addition, similar to the proof of $u_m(t)\in W$, we can show that $u(t)\in W$ for all $t\in[0,+\infty)$.

Step 2: Uniqueness of the bounded global weak solution. Let u and v be two bounded weak solutions of problem (1.1), then

$$\begin{gathered}
(u_t,\phi)+a(\nabla_H u,\nabla_H\phi)+b\|\nabla_H u\|_2^2(\nabla_H u,\nabla_H\phi)=\xi(|u|^{q-1}u,\phi),\\
(v_t,\phi)+a(\nabla_H v,\nabla_H\phi)+b\|\nabla_H v\|_2^2(\nabla_H v,\nabla_H\phi)=\xi(|v|^{q-1}v,\phi)
\end{gathered}$$

for any $\phi \in H_0^1(B_R)$. Subtracting the above two equalities, then letting $\phi = u-v \in H_0^1(B_R)$ and integrating over $(0,t)$ for any $t>0$, one has

$$\begin{aligned} &\int_0^t \int_{B_R} (u-v)_\tau (u-v) + a|\nabla_H(u-v)|^2 \\ &+ (b\|\nabla_H u\|_2^2 \nabla_H u - b\|\nabla_H v\|_2^2 \nabla_H v)\nabla_H(u-v)\, d\mu\, d\tau \\ &= \xi \int_0^t \int_{B_R} (|u|^{q-1}u - |v|^{q-1}v)(u-v)\, d\mu\, d\tau. \end{aligned} \tag{4.16}$$

By means of the Cauchy-Schwarz inequality, we conclude that

$$\begin{aligned} &a\|\nabla_H(u-v)\|_2^2 + b\int_{B_R} (\|\nabla_H u\|_2^2 \nabla_H u - \|\nabla_H v\|_2^2 \nabla_H v)(\nabla_H u - \nabla_H v)\, d\mu \\ &\geq b\Big(\|\nabla_H u\|_2^4 - \|\nabla_H u\|_2^2 \int_{B_R} \nabla_H u \nabla_H v\, d\mu\Big) \\ &\quad - b\Big(\|\nabla_H v\|_2^2 \int_{B_R} \nabla_H u \nabla_H v\, d\mu - \|\nabla_H v\|_2^4\Big) \\ &\geq b\Big(\|\nabla_H u\|_2^4 - \|\nabla_H u\|_2^2 \cdot \frac{\|\nabla_H u\|_2^2 + \|\nabla_H v\|_2^2}{2}\Big) \\ &\quad - b\Big(\|\nabla_H v\|_2^2 \cdot \frac{\|\nabla_H u\|_2^2 + \|\nabla_H v\|_2^2}{2} - \|\nabla_H v\|_2^4\Big) \\ &= \frac{b}{2}(\|\nabla_H u\|_2^2 - \|\nabla_H v\|_2^2)^2 \geq 0. \end{aligned} \tag{4.17}$$

It follows from (4.16) and (4.17) that

$$\xi \int_0^t \int_{B_R} (|u|^{q-1}u - |v|^{q-1}v)(u-v)\, d\mu\, d\tau \geq \int_0^t \int_{B_R} (u-v)_\tau (u-v)\, d\mu\, d\tau.$$

Since $(u-v)(\sigma,0)=0$ and $3<q<5$, we obtain from the boundedness of u and v that

$$\|\phi\|_2^2 \leq C\int_0^t \|\phi\|_2^2\, d\tau,$$

where the positive constant C depends only on q, ξ and the bounds of u, v. Then by Gronwall's inequality, we have $\|\phi\|_2^2 = 0$. Hence, we obtain $\phi = 0$ a.e. in $B_R \times (0,+\infty)$ and the proof of this step is complete.

Step 3: Decay estimates. Because $d_0 \leq d$, it follows from $J(u_0) < d_0$, $I(u_0) > 0$ and step 1 that $I(u) \geq 0$ for $t \in [0,+\infty)$. By (2.3) and (2.12), we obtain

$$J(u_0) \geq J(u) = \frac{(q-1)a}{2(q+1)}\|u\|^2 + \frac{(q-3)b}{4(q+1)}\|u\|^4 + \frac{1}{q+1}I(u) \geq \frac{(q-3)b}{4(q+1)}\|u\|^4. \tag{4.18}$$

Then it is clear from (1.5) that

$$\|u\|_{q+1} \leq C_{q+1}\|u\| \leq C_{q+1}\Big(\frac{4(q+1)J(u_0)}{(q-3)b}\Big)^{1/4}. \tag{4.19}$$

From (1.5) and (4.19), we have

$$\begin{aligned} \|u\|_{q+1}^{q+1} &= \|u\|_{q+1}^{q-3}\|u\|_{q+1}^4 \leq C_{q+1}^4 \|u\|_{q+1}^{q-3}\|u\|^4 \\ &\leq C_{q+1}^{q+1}\big(\frac{4(q+1)J(u_0)}{(q-3)b}\big)^{\frac{q-3}{4}}\|u\|^4. \end{aligned} \tag{4.20}$$

Moreover, from Lemma 3.6 we obtain

$$\frac{d}{dt}\|u\|_2^2 = -2I(u) = 2\xi\|u\|_{q+1}^{q+1} - 2a\|u\|^2 - 2b\|u\|^4. \tag{4.21}$$

Then by (2.15) and (4.20), we obtain

$$\begin{aligned}
\frac{d}{dt}\|u\|_2^2 &\le -2b\|u\|^4 + 2\xi\|u\|_{q+1}^{q+1} \\
&\le -2b\|u\|^4 + 2\xi C_{q+1}^{q+1}\Big(\frac{4(q+1)J(u_0)}{(q-3)b}\Big)^{\frac{q-3}{4}}\|u\|^4 \\
&= -2\Big[b - \xi C_{q+1}^{q+1}\Big(\frac{4(q+1)J(u_0)}{(q-3)b}\Big)^{\frac{q-3}{4}}\Big]\|u\|^4 \\
&\le -2\lambda_1^2\Big[b - \xi C_{q+1}^{q+1}\Big(\frac{4(q+1)J(u_0)}{(q-3)b}\Big)^{\frac{q-3}{4}}\Big]\|u\|_2^4,
\end{aligned}$$

which implies

$$\|u\|_2^2 \le \frac{\|u_0\|_2^2}{Dt\|u_0\|_2^2 + 1},$$

where

$$D = 2\lambda_1^2\Big[b - \xi C_{q+1}^{q+1}\Big(\frac{4(q+1)J(u_0)}{(q-3)b}\Big)^{\frac{q-3}{4}}\Big] > 0.$$

□

Proof of Corollary 2.6. If $u_0 = 0$, then the proof is complete because problem (1.1) admits a global weak solution $u(t) \equiv 0$. Hence, in the following, we assume that $u_0 \in H_0^1(B_R) \setminus \{0\}$, and then the proof is divided into three cases:

Case 1: $J(u_0) < d$ and $I(u_0) = 0$. Because $u_0 \in N$, then we infer that $J(u_0) \ge d$, which is a contradiction and this case cannot happen.

Case 2: $J(u_0) < d$ and $I(u_0) > 0$. By Theorem 2.4, we know that problem (1.1) admits a global weak solution.

Case 3: $J(u_0) = d$ and $I(u_0) \ge 0$. In this case, we let $h_m = 1 - \frac{1}{m}$ $(m = 2, 3, \dots)$. Considering the problem

$$\begin{gathered}
u_t - \Big(a + b\int_{B_R} |\nabla_H u(\sigma)|^2\, d\mu\Big)\Delta_H u = \xi|u|^{q-1}u, \quad \sigma \in B_R, t > 0, \\
u(\sigma, t) = 0, \quad \sigma \in \partial B_R, t > 0, \\
u(\sigma, 0) = u_{0m}(\sigma) := h_m u_0, \quad \sigma \in B_R.
\end{gathered} \tag{4.22}$$

It follows from $u_0 \in H_0^1(B_R) \setminus \{0\}$, $h_m \in (0, 1)$ and $I(u_0) \ge 0$ that

$$\begin{aligned}
I(u_{0m}) &= ah_m^2\|u_0\|^2 + bh_m^4\|u_0\|^4 - \xi h_m^{q+1}\|u_0\|_{q+1}^{q+1} \\
&= h_m^2\big(a\|u_0\|^2 + bh_m^2\|u_0\|^4 - \xi h_m^{q-1}\|u_0\|_{q+1}^{q+1}\big) > 0.
\end{aligned}$$

Furthermore, from the definition of $J(u)$, we obtain

$$\begin{aligned}
\frac{d}{dh_m}J(h_m u) &= ah_m\|u\|^2 + bh_m^3\|u\|^4 - \xi h_m^q\|u\|_{q+1}^{q+1} \\
&= \frac{1}{h_m}(ah_m^2\|u\|^2 + bh_m^4\|u\|^4 - \xi h_m^{q+1}\|u\|_{q+1}^{q+1}) \\
&= \frac{1}{h_m}I(h_m u).
\end{aligned}$$

Then

$$\frac{d}{dh_m}J(h_m u_0) = \frac{1}{h_m}I(h_m u_0) = \frac{1}{h_m}I(u_{0m}) > 0, \tag{4.23}$$

which means

$$J(u_{0m}) = J(h_m u_0) < J(u_0) = d. \tag{4.24}$$

By Theorem 2.4, for each m, we see that problem (4.22) admits a global weak solution $u_m(t) \in L^\infty(0,+\infty; H_0^1(B_R))$ with $u_{mt} \in L^2(0,+\infty; L^2(B_R))$, which satisfies $u_m(t) \in W$ for all $t \in [0,+\infty)$ and

$$\begin{aligned} &\int_{B_R} u_{mt}\phi \, d\mu + \Big(a + b\int_{B_R} |\nabla_H u_m(\sigma)|^2 \, d\mu\Big) \int_{B_R} \nabla_H u_m \nabla_H \phi \, d\mu \\ &= \xi \int_{B_R} |u_m|^{q-1} u_m \phi \, d\mu \end{aligned} \tag{4.25}$$

for a.e. $t > 0$ and any $\phi \in H_0^1(B_R)$. In addition,

$$J(u_m(t)) + \int_0^t \|u_{m\tau}\|_2^2 \, d\tau = J(u_{0m}) < d. \tag{4.26}$$

Since

$$J(u_m(t)) = \frac{(q-1)a}{2(q+1)}\|u_m\|^2 + \frac{(q-3)b}{4(q+1)}\|u_m\|^4 + \frac{1}{q+1}I(u_m(t)),$$

it follows from (4.26) that

$$\frac{(q-1)a}{2(q+1)}\|u_m\|^2 + \frac{(q-3)b}{4(q+1)}\|u_m\|^4 + \int_0^t \|u_{m\tau}\|_2^2 \, d\tau < d.$$

Then the remainder of the proof is similar to that in the proof of Theorem 2.4. □

Proof of Theorem 2.7. We divide the proof into the following three steps.

Step 1: Blow-up in finite time. We divide this part of the proof into two cases.
Case 1: $J(u_0) < d$. Let $u = u(t)$, $t \in [0, T)$, be a weak solution of (1.1) and T be the maximum existence time of u. Since $J(u_0) < d$ and $I(u_0) < 0$, then it follows from (2.12) and Lemma 3.7 that $u(t) \in V$. Next, we show that $u(t)$ blows up at some finite time T. By contradiction, we assume that the weak solution u exists globally and $T = +\infty$. Set

$$Q(t) := \int_0^t \|u\|_2^2 \, d\tau, \quad t \in [0, T).$$

Then we obtain from Lemma 3.6 that

$$Q'(t) = \|u\|_2^2, \quad Q''(t) = 2(u, u_t) = -2I(u). \tag{4.27}$$

It follows from (2.3) and (2.12) that

$$\frac{(q-1)a}{2(q+1)}\|u\|^2 + \frac{(q-3)b}{4(q+1)}\|u\|^4 + \frac{1}{q+1}I(u) + \int_0^t \|u_\tau\|_2^2 \, d\tau \le J(u_0),$$

which implies

$$-2I(u) \ge (q-1)a\|u\|^2 + \frac{(q-3)b}{2}\|u\|^4 - 2(q+1)J(u_0) + 2(q+1)\int_0^t \|u_\tau\|_2^2 \, d\tau,$$

then by (1.5), we reach

$$
\begin{aligned}
&Q''(t)\\
&\geq (q-1)a\|u\|^2+\frac{(q-3)b}{2}\|u\|^4-2(q+1)J(u_0)+2(q+1)\int_0^t\|u_\tau\|_2^2\,d\tau\\
&\geq \frac{(q-3)b}{2}\|u\|^4-2(q+1)J(u_0)+2(q+1)\int_0^t\|u_\tau\|_2^2\,d\tau\\
&\geq \frac{(q-3)b}{2C_2^4}\|u\|_2^4-2(q+1)J(u_0)+2(q+1)\int_0^t\|u_\tau\|_2^2\,d\tau\\
&= \frac{(q-3)b}{2C_2^4}(Q'(t))^2-2(q+1)J(u_0)+2(q+1)\int_0^t\|u_\tau\|_2^2\,d\tau.
\end{aligned}
\tag{4.28}
$$

Furthermore, we have

$$
\begin{aligned}
\Big(\int_0^t(u_\tau,u)\,d\tau\Big)^2&=\Big(\frac{1}{2}\int_0^t\frac{d}{d\tau}\|u\|_2^2\,d\tau\Big)^2\\
&=\frac{1}{4}(\|u\|_2^2-\|u_0\|_2^2)^2\\
&=\frac{1}{4}(\|u\|_2^4-2\|u_0\|_2^2\|u\|_2^2+\|u_0\|_2^4)\\
&=\frac{1}{4}((Q'(t))^2-2\|u_0\|_2^2Q'(t)+\|u_0\|_2^4).
\end{aligned}
\tag{4.29}
$$

Then it follows from (4.28), (4.29), and Schwarz inequality that

$$
\begin{aligned}
&Q(t)Q''(t)-\frac{q+1}{2}(Q'(t))^2\\
&\geq \frac{(q-3)b}{2C_2^4}Q(t)(Q'(t))^2-(q+1)\|u_0\|_2^2Q'(t)-2(q+1)J(u_0)Q(t)+\frac{q+1}{2}\|u_0\|_2^4\\
&\quad +2(q+1)\int_0^t\|u_\tau\|_2^2\,d\tau\int_0^t\|u\|_2^2\,d\tau-2(q+1)\Big(\int_0^t(u_\tau,u)\,d\tau\Big)^2\\
&\geq \frac{(q-3)b}{2C_2^4}Q(t)(Q'(t))^2-(q+1)\|u_0\|_2^2Q'(t)-2(q+1)J(u_0)Q(t).
\end{aligned}
\tag{4.30}
$$

Furthermore, from $Q''(t)=-2I(u)>0$, we deduce that $Q'(t)\geq Q'(0)=\|u_0\|_2^2>0$. Then it is clear from (4.30) that

$$
\begin{aligned}
&Q(t)Q''(t)-\frac{q+1}{2}(Q'(t))^2\\
&\geq \frac{(q-3)b\|u_0\|_2^2}{2C_2^4}Q(t)Q'(t)-(q+1)\|u_0\|_2^2Q'(t)-2(q+1)J(u_0)Q(t).
\end{aligned}
\tag{4.31}
$$

From (2.12), (4.27), and Lemma 3.5, we know that, for all $t\in[0,+\infty)$,

$$
Q''(t)=-2I(u)>2(q+1)(d-J(u))\geq 2(q+1)(d-J(u_0)):=C_1>0. \tag{4.32}
$$

Then for all $t\in[0,+\infty)$, one has

$$
Q'(t)\geq Q'(0)+C_1t=\|u_0\|_2^2+C_1t>C_1t,\quad Q(t)>Q(0)+\frac{C_1}{2}t^2=\frac{C_1}{2}t^2.
$$

Hence,

$$
\lim_{t\to+\infty}Q(t)=+\infty,\quad \lim_{t\to+\infty}Q'(t)=+\infty.
$$

Then there is a $t_0 \geq 0$ such that

$$\frac{(q-3)b\|u_0\|_2^2}{4C_2^4}Q'(t) > 2(q+1)J(u_0), \quad t_0 \leq t < +\infty,$$
$$\frac{(q-3)b\|u_0\|_2^2}{4C_2^4}Q(t) > (q+1)\|u_0\|_2^2, \quad t_0 \leq t < +\infty,$$

which, together with (4.31), implies, for $t \in [t_0, +\infty)$,

$$\begin{aligned} Q(t)Q''(t) - \frac{q+1}{2}(Q'(t))^2 &\geq (\frac{(q-3)b\|u_0\|_2^2}{4C_2^4}Q'(t) - 2(q+1)J(u_0))Q(t) \\ &\quad + (\frac{(q-3)b\|u_0\|_2^2}{4C_2^4}Q(t) - (q+1)\|u_0\|_2^2)Q'(t) > 0. \end{aligned}$$

It follows from Lemma 3.1 that the maximum existence time T_1 of $Q(t)$ satisfies $T_1 < +\infty$ and $\lim_{t\to T_1} Q(t) = +\infty$, which is a contradiction.

Case 2: $J(u_0) = d$. According to Lemma 3.7, we know $I(u(t)) < 0$ for all $t \geq 0$, then we infer from Lemma 3.6 that $(u_t, u) = -I(u(t)) > 0$ for $t \geq 0$, which implies $\|u_t\|_2^2 > 0$ for $t \geq 0$. Thus, it follows from (2.12) that there is a $t_1 > 0$ such that

$$J(u(t_1)) \leq J(u_0) - \int_0^{t_1} \|u_\tau\|_2^2 \, d\tau < d.$$

Taking t_1 as the initial time, then the remainder of the proof is similar to case 1.

Step 2: Upper bound estimate of the blow-up time. Let $u = u(t)$, $t \in [0, T)$, be a weak solution of (1.1) and T be the maximum existence time of u. Since $J(u_0) < d$ and $I(u_0) < 0$, then it follows from Step 1 that $T < +\infty$. Furthermore, by Lemma 3.7, we know that $I(u) < 0$ for $t \in [0, T)$. We set

$$a(t) := \Big(\int_0^t \|u\|_2^2 \, d\tau\Big)^{1/2}, \quad b(t) := \Big(\int_0^t \|u_\tau\|_2^2 \, d\tau\Big)^{1/2}, \quad \forall t \in [0, T).$$

We define

$$H(t) := \eta(t+\kappa)^2 + a^2(t) + (T-t)\|u_0\|_2^2, \ \forall t \in [0, T), \tag{4.33}$$

where $\kappa, \eta > 0$ are two constants which will be specified later. It follows from Lemma 3.6 and $I(u) < 0$ that

$$H'(t) = 2\eta(t+\kappa) + \|u\|_2^2 - \|u_0\|_2^2 \geq 2\eta(t+\kappa) > 0, \quad t \in [0, T). \tag{4.34}$$

Then we have

$$H(t) \geq H(0) = \eta\kappa^2 + T\|u_0\|_2^2 > 0, \quad t \in [0, T). \tag{4.35}$$

In addition, from Lemma 3.5, Lemma 3.6, and (2.12), we conclude that

$$\begin{aligned} H''(t) &= 2\eta - 2I(u) > 2\eta + 2(q+1)(d - J(u)) \\ &\geq 2\eta + 2(q+1)(d - J(u_0)) + 2(q+1)b^2(t), \quad t \in [0, T). \end{aligned} \tag{4.36}$$

Using the Cauchy-Schwarz inequality, one has

$$\begin{aligned} a(t)b(t) &= \Big(\int_0^t \|u\|_2^2 \, d\tau\Big)^{1/2}\Big(\int_0^t \|u_\tau\|_2^2 \, d\tau\Big)^{1/2} \\ &\geq \int_0^t \|u\|_2\|u_\tau\|_2 \, d\tau \geq \int_0^t (u, u_\tau) \, d\tau \\ &= \frac{1}{2}\int_0^t \frac{d}{d\tau}\|u\|_2^2 \, d\tau, \quad t \in [0, T), \end{aligned}$$

which, together with the definition of $H(t)$, implies

$$
\begin{aligned}
&(H(t)-(T-t)\|u_0\|_2^2)(b^2(t)+\eta)\\
&=(a^2(t)+\eta(t+\kappa)^2)(b^2(t)+\eta)\\
&=a^2(t)b^2(t)+\eta a^2(t)+\eta(t+\kappa)^2b^2(t)+\eta^2(t+\kappa)^2\\
&\geq a^2(t)b^2(t)+2\eta a(t)b(t)(t+\kappa)+\eta^2(t+\kappa)^2\\
&\geq \Big[\eta(t+\kappa)+\frac{1}{2}\int_0^t\frac{d}{d\tau}\|u\|_2^2\,d\tau\Big]^2,\quad t\in[0,T).
\end{aligned}\tag{4.37}
$$

According to (4.34) and (4.37), we have

$$
4H(t)(b^2(t)+\eta)\geq 4\Big(\eta(t+\kappa)+\frac{1}{2}\int_0^t\frac{d}{d\tau}\|u\|_2^2ds\Big)^2=(H'(t))^2,\quad t\in[0,T).\tag{4.38}
$$

Then it follows from (4.35), (4.36) and (4.38) that

$$
\begin{aligned}
&H(t)H''(t)-\frac{q+1}{2}(H'(t))^2\\
&>H(t)\left[2\eta+2(q+1)(d-J(u_0))+2(q+1)b^2(t)-2(q+1)b^2(t)-2(q+1)\eta\right]\\
&=H(t)\left[2(q+1)(d-J(u_0))-2q\eta\right],\quad t\in[0,T).
\end{aligned}
$$

Choosing η small enough such that

$$
0<\eta\leq\frac{(q+1)(d-J(u_0))}{q},\tag{4.39}
$$

we obtain

$$
H(t)H''(t)-\frac{q+1}{2}(H'(t))^2\geq 0,\quad t\in[0,T).
$$

By Lemma 3.1, one has

$$
T\leq\frac{H(0)}{(\frac{q+1}{2}-1)H'(0)}=\frac{1}{q-1}(\kappa+\frac{\|u_0\|_2^2}{\eta\kappa}T).\tag{4.40}
$$

Choosing κ large enough such that

$$
\kappa>\frac{\|u_0\|_2^2}{(q-1)\eta}.\tag{4.41}
$$

Then by (4.40), we obtain

$$
T\leq\frac{\eta\kappa^2}{(q-1)\eta\kappa-\|u_0\|_2^2}.\tag{4.42}
$$

Consequently,

$$
T\leq\inf_{(\delta,\kappa)\in\Psi}g(\delta,\kappa),\tag{4.43}
$$

where $\delta:=\kappa\eta$,

$$
g(\delta,\kappa):=\frac{\delta\kappa}{(q-1)\delta-\|u_0\|_2^2},\quad \Psi:=\Big\{(\delta,\kappa):\delta>\frac{\|u_0\|_2^2}{q-1},\kappa\geq\frac{q\delta}{(q+1)(d-J(u_0))}\Big\}.
$$

Because $g(\delta,\kappa)$ is increasing with respect to κ, we infer that

$$
T\leq\inf_{\delta>\frac{\|u_0\|_2^2}{q-1}}g\Big(\delta,\frac{q\delta}{(q+1)(d-J(u_0))}\Big)
$$

$$
\begin{aligned}
&= \inf_{\delta > \frac{\|u_0\|_2^2}{q-1}} \frac{q\delta^2}{(q+1)(d-J(u_0))((q-1)\delta - \|u_0\|_2^2)} \\
&= \frac{q\delta^2}{(q+1)(d-J(u_0))((q-1)\delta - \|u_0\|_2^2)}\Big|_{\delta=\frac{2\|u_0\|_2^2}{q-1}} \\
&= \frac{4q\|u_0\|_2^2}{(q+1)(q-1)^2(d-J(u_0))}.
\end{aligned}
$$

Step 3: Lower bound estimate of the blow-up time. By Lemma 3.7, we see that $I(u) < 0$ for $t \in [0, T)$. Now, we estimate the lower bound of the blow-up time T and the blow-up rate. Set $L(t) = \frac{1}{2}\|u\|_2^2$, then it is obvious that

$$
L(T) = +\infty. \tag{4.44}
$$

It follows from Lemma 3.6 and the definition of $I(u)$ that

$$
\frac{1}{2}\frac{d}{dt}\|u\|_2^2 = -I(u) = \xi\|u\|_{q+1}^{q+1} - a\|u\|^2 - b\|u\|^4. \tag{4.45}
$$

Then according to (2.16) and $I(u) < 0$, we conclude that

$$
\begin{aligned}
\|u\|_{q+1}^{q+1} &\le \widetilde{C}^{q+1}\|u\|^{(1-\theta)(q+1)}\|u\|_2^{\theta(q+1)} \\
&= \frac{\widetilde{C}^{q+1}}{b^{\frac{(1-\theta)(q+1)}{4}}}(b\|u\|^4)^{\frac{(1-\theta)(q+1)}{4}}\|u\|_2^{\theta(q+1)} \\
&\le \frac{\widetilde{C}^{q+1}}{b^{\frac{(1-\theta)(q+1)}{4}}}(a\|u\|^2 + b\|u\|^4)^{\frac{(1-\theta)(q+1)}{4}}\|u\|_2^{\theta(q+1)} \\
&< \frac{\xi^{\frac{(1-\theta)(q+1)}{4}}\widetilde{C}^{q+1}}{b^{\frac{(1-\theta)(q+1)}{4}}}(\|u\|_{q+1}^{q+1})^{\frac{(1-\theta)(q+1)}{4}}(\|u\|_2^2)^{\frac{\theta(q+1)}{2}}.
\end{aligned} \tag{4.46}
$$

From $\theta = \frac{5-q}{2(q+1)}$ and $3 < q < \frac{11}{3}$, we have

$$
\frac{(1-\theta)(q+1)}{4} = \frac{3q-3}{8} < 1.
$$

Hence, from (4.46) we obtain

$$
\|u\|_{q+1}^{q+1} < \Big(\frac{\xi^{\frac{(1-\theta)(q+1)}{4}}\widetilde{C}^{q+1}}{b^{\frac{(1-\theta)(q+1)}{4}}}\Big)^{\frac{4}{4-(1-\theta)(q+1)}}(\|u\|_2^2)^{\frac{2\theta(q+1)}{4-(1-\theta)(q+1)}}. \tag{4.47}
$$

By a simply calculation, one has

$$
\gamma := \frac{2\theta(q+1)}{4-(1-\theta)(q+1)} = \frac{10-2q}{11-3q} > 1.
$$

Then it follows from (4.45) and (4.47) that

$$
L'(t) = \xi\|u\|_{q+1}^{q+1} - a\|u\|^2 - b\|u\|^4 \le \xi\|u\|_{q+1}^{q+1} < \widehat{C}(\|u\|_2^2)^\gamma = 2^\gamma\widehat{C}(L(t))^\gamma, \tag{4.48}
$$

where

$$
\widehat{C} = \Big(\frac{\xi\widetilde{C}^{q+1}}{b^{\frac{(1-\theta)(q+1)}{4}}}\Big)^{\frac{4}{4-(1-\theta)(q+1)}} = \Big(\frac{\xi\widetilde{C}^{q+1}}{b^{\frac{3q-3}{8}}}\Big)^{\frac{8}{11-3q}}.
$$

Next, we claim that $L(t) > 0$ for $t \in [0, T)$. Indeed, if not, then there is a $t_1 \geq 0$ such that $\|u(t_1)\|_2^2 = 0$, which conflicts with (4.47). Thus, we obtain from (4.48) that

$$\frac{L'(t)}{(L(t))^\gamma} < 2^\gamma \widehat{C}. \tag{4.49}$$

Integrating (4.49) from 0 to t yields

$$(L(0))^{1-\gamma} - (L(t))^{1-\gamma} < 2^\gamma \widehat{C}(\gamma - 1)t,$$

letting $t \to T$ in the above inequality, we obtain from (4.44) that

$$T > \frac{(L(0))^{1-\gamma}}{2^\gamma \widehat{C}(\gamma - 1)} = \frac{\|u_0\|_2^{2-2\gamma}}{2\widehat{C}(\gamma - 1)}.$$

Integrating (4.49) from t to T, it follows from (4.44) that

$$L(t) > \Big(2^\gamma \widehat{C}(T - t)(\gamma - 1)\Big)^{\frac{1}{1-\gamma}},$$

i.e.,

$$\|u\|_2 > \Big(2\widehat{C}(T - t)(\gamma - 1)\Big)^{\frac{1}{2(1-\gamma)}}.$$

□

Proof of Theorem 2.9. Let $u = u(t)$, $t \in [0, T)$, be a weak solution of (1.1) and T be the maximum existence time of u. Since $J(u_0) \leq M$ and $I(u_0) < 0$, then we obtain $I(u) < 0$ for $t \in [0, T)$ from Lemma 3.7. Set

$$F(t) := \int_0^t \|u\|_2^2 \, d\tau, \quad t \in [0, T),$$

then from Lemma 3.6 we obtain

$$F'(t) = \|u\|_2^2, \quad F''(t) = -2I(u) > 0. \tag{4.50}$$

From (2.3), (2.6), (2.12), (4.50), and Lemma 3.2 it follows that

$$\begin{aligned}
&F''(t) \\
&= (q-1)a\|u\|^2 + \frac{(q-3)b}{2}\|u\|^4 - 2(q+1)J(u) \\
&\geq (q-1)a\|u\|^2 + \frac{(q-3)b}{2}\|u\|^4 - 2(q+1)J(u_0) + 2(q+1)\int_0^t \|u_\tau\|_2^2 \, d\tau \\
&\geq 2(q+1)\int_0^t \|u_\tau\|_2^2 \, d\tau + (q-1)ar_0^2 + \frac{(q-3)b}{2} r_0^4 - 2(q+1)J(u_0) \\
&= 2(q+1)\int_0^t \|u_\tau\|_2^2 \, d\tau + 2(q+1)(M - J(u_0)).
\end{aligned} \tag{4.51}$$

Moreover, from

$$\begin{aligned}
4\Big(\int_0^t (u, u_\tau) \, d\tau\Big)^2 &= (\int_0^t \frac{d}{d\tau}\|u\|_2^2 \, d\tau)^2 \\
&= (F'(t) - F'(0))^2 \\
&= (F'(t))^2 - 2F'(t)F'(0) + (F'(0))^2,
\end{aligned}$$

we readily obtain

$$(F'(t))^2 = 2\|u_0\|_2^2 F'(t) - \|u_0\|_2^4 + 4\Big(\int_0^t (u, u_\tau)\, d\tau\Big)^2, \tag{4.52}$$

which, along with (4.51) and Cauchy-Schwarz inequality, yields

$$\begin{aligned}
&F(t)F''(t) - \frac{q+1}{2}(F'(t))^2 \\
&\geq 2(q+1)(M - J(u_0))F(t) - (q+1)\|u_0\|_2^2 F'(t) + \frac{q+1}{2}\|u_0\|_2^4 \\
&\quad + 2(q+1)\int_0^t \|u_\tau\|_2^2\, d\tau \int_0^t \|u\|_2^2\, d\tau - 2(q+1)\Big(\int_0^t (u, u_\tau)\, d\tau\Big)^2 \\
&\geq 2(q+1)(M - J(u_0))F(t) - (q+1)\|u_0\|_2^2 F'(t) \\
&\geq -(q+1)\|u_0\|_2^2 F'(t).
\end{aligned}$$

Then it is clear that, for any $\varepsilon \in (0, 1/3]$,

$$F(t)F''(t) - \frac{(q+1)\varepsilon}{2}(F'(t))^2 \geq \frac{(q+1)(1-\varepsilon)}{2}(F'(t))^2 - (q+1)\|u_0\|_2^2 F'(t). \tag{4.53}$$

By Theorem 2.7, we see that $u(t)$ blows up in finite time, then

$$\lim_{t \to T^-} F'(t) = \lim_{t \to T^-} \|u\|_2^2 = +\infty.$$

Therefore, from (4.53) that there exists a $t_\varepsilon \in (0, T)$ such that

$$F(t)F''(t) - \frac{(q+1)\varepsilon}{2}(F'(t))^2 > 0, \quad t \in [t_\varepsilon, T). \tag{4.54}$$

Since

$$\Big(F^{1-\frac{(q+1)\varepsilon}{2}}(t)\Big)' = \Big(1 - \frac{(q+1)\varepsilon}{2}\Big)F^{-\frac{(q+1)\varepsilon}{2}}(t)F'(t),$$

then we obtain from (4.54) that

$$\begin{aligned}
\Big(F^{1-\frac{(q+1)\varepsilon}{2}}(t)\Big)'' &= \Big(1 - \frac{(q+1)\varepsilon}{2}\Big)F^{-\frac{(q+1)\varepsilon}{2}-1}(t) \\
&\quad \times \big[F(t)F''(t) - \frac{(q+1)\varepsilon}{2}(F'(t))^2\big] > 0, \quad t \in [t_\varepsilon, T).
\end{aligned}$$

Because $2 - (q+1)\varepsilon \geq 2 - \frac{q+1}{3} > 0$ and $F(t_\varepsilon) > 0$, we conclude that

$$\begin{aligned}
F(t) &= \Big[\int_{t_\varepsilon}^t (F^{1-\frac{(q+1)\varepsilon}{2}}(\tau))'\, d\tau + F^{1-\frac{(q+1)\varepsilon}{2}}(t_\varepsilon)\Big]^{\frac{2}{2-(q+1)\varepsilon}} \\
&\geq \Big[(t - t_\varepsilon)(F^{1-\frac{(q+1)\varepsilon}{2}}(\tau))'\big|_{\tau = t_\varepsilon} + F^{1-\frac{(q+1)\varepsilon}{2}}(t_\varepsilon)\Big]^{\frac{2}{2-(q+1)\varepsilon}} \\
&\geq \Big[\big(1 - \frac{(q+1)\varepsilon}{2}\big)(t - t_\varepsilon)F^{-\frac{(q+1)\varepsilon}{2}}(t_\varepsilon)F'(t_\varepsilon)\Big]^{\frac{2}{2-(q+1)\varepsilon}} \\
&= C_\varepsilon (t - t_\varepsilon)^{\frac{2}{2-(q+1)\varepsilon}}, \quad t \in [t_\varepsilon, T),
\end{aligned} \tag{4.55}$$

where

$$C_\varepsilon := \Big[\Big(1 - \frac{(q+1)\varepsilon}{2}\Big)F^{-\frac{(q+1)\varepsilon}{2}}(t_\varepsilon)F'(t_\varepsilon)\Big]^{\frac{2}{2-(q+1)\varepsilon}}.$$

Furthermore, it follows from $F''(t) > 0$ for $t \in [0, T)$ that $tF'(t) \geq \int_0^t F'(\tau)\, d\tau$, i.e.,

$$t\|u\|_2^2 \geq F(t), \quad t \in [0, T). \tag{4.56}$$

From (4.55) and (4.56), we reach, for all $t \in [t_\varepsilon, T)$ and any $\varepsilon \in (0, 1/3]$,

$$\|u\|_2^2 \geq C_\varepsilon \Big(t^{\frac{(q+1)\varepsilon}{2}} - t^{\frac{(q+1)\varepsilon}{2}-1} t_\varepsilon\Big)^{\frac{2}{2-(q+1)\varepsilon}}.$$

□

Proof of Theorem 2.10. We divide the proof into three steps.

Step 1: Blow-up in finite time. Let $u = u(t)$, $t \in [0, T)$, be a weak solution of (1.1) and T be the maximum existence time of u. According to (2.3) and (2.18), we obtain

$$\begin{aligned} I(u_0) &= (q+1)J(u_0) - \frac{(q-1)a}{2}\|u_0\|^2 - \frac{(q-3)b}{4}\|u_0\|^4 \\ &\leq (q+1)J(u_0) - \frac{(q-1)a\lambda_1}{2}\|u_0\|_2^2 - \frac{(q-3)b\lambda_1^2}{4}\|u_0\|_2^4 < 0. \end{aligned}$$

Now, we show that $I(u(t)) < 0$ for $t \in [0, T)$. Indeed, if not, then there exists a $t_0 \in (0, T)$ such that $I(u(t)) < 0$ for $t \in [0, t_0)$ and $I(u(t_0)) = 0$. It follows from Lemma 3.6 that $\|u\|_2^2$ is strictly increasing for $t \in [0, t_0)$. Thus,

$$\begin{aligned} J(u_0) &< \frac{(q-1)a\lambda_1}{2(q+1)}\|u_0\|_2^2 + \frac{(q-3)b\lambda_1^2}{4(q+1)}\|u_0\|_2^4 \\ &< \frac{(q-1)a\lambda_1}{2(q+1)}\|u(t_0)\|_2^2 + \frac{(q-3)b\lambda_1^2}{4(q+1)}\|u(t_0)\|_2^4. \end{aligned} \tag{4.57}$$

Furthermore, it follows from (2.3) and (2.12) that

$$\begin{aligned} J(u_0) &\geq J(u(t_0)) \\ &\geq \frac{(q-1)a}{2(q+1)}\|u(t_0)\|^2 + \frac{(q-3)b}{4(q+1)}\|u(t_0)\|^4 \\ &\geq \frac{(q-1)a\lambda_1}{2(q+1)}\|u(t_0)\|_2^2 + \frac{(q-3)b\lambda_1^2}{4(q+1)}\|u(t_0)\|_2^4, \end{aligned}$$

which conflicts with (4.57). Hence, we obtain $I(u(t)) < 0$ for all $t \in [0, T)$.

Next, we show that $u(t)$ blows up in finite time. By contradiction, we choose $\widetilde{T} = \frac{(4q\|u_0\|_2^2+1)^2+1}{\varrho(q-1)^2}$, where $\varrho := (q-1)a\lambda_1\|u_0\|_2^2 + \frac{(q-3)b\lambda_1^2}{2}\|u_0\|_2^4 - 2(q+1)J(u_0) > 0$, and we assume that $u(t)$ exists globally on $[0, \widetilde{T}]$. Set

$$M(t) := \omega(t+\delta)^2 + \int_0^t \|u\|_2^2 \, d\tau + (\widetilde{T} - t)\|u_0\|_2^2, \quad t \in [0, \widetilde{T}],$$

where $\omega > 0$ and $\delta > 0$ are two constants which will be specified later.

It follows from Lemma 3.6 and $I(u) < 0$ for all $t \in [0, T)$ that $\|u\|_2^2$ is strictly increasing for $t \in [0, T)$. Then we have

$$M'(t) = 2\omega(t+\delta) + \|u\|_2^2 - \|u_0\|_2^2 \geq 2\omega(t+\delta) > 0, \quad t \in [0, \widetilde{T}]$$

and

$$\begin{aligned} M''(t) &= 2\omega - 2I(u) \\ &= 2\omega + (q-1)a\|u\|^2 + \frac{(q-3)b}{2}\|u\|^4 - 2(q+1)J(u) \\ &\geq 2\omega + (q-1)a\lambda_1\|u\|_2^2 + \frac{(q-3)b\lambda_1^2}{2}\|u\|_2^4 \\ &\quad - 2(q+1)J(u_0) + 2(q+1)\int_0^t \|u_\tau\|_2^2\,d\tau \\ &\geq 2\omega + (q-1)a\lambda_1\|u_0\|_2^2 + \frac{(q-3)b\lambda_1^2}{2}\|u_0\|_2^4 \\ &\quad - 2(q+1)J(u_0) + 2(q+1)\int_0^t \|u_\tau\|_2^2\,d\tau > 0, \quad t \in [0, \widetilde{T}]. \end{aligned} \tag{4.58}$$

We define

$$\alpha(t) := \Big(\int_0^t \|u\|_2^2\,d\tau\Big)^{1/2}, \quad \beta(t) := \Big(\int_0^t \|u_\tau\|_2^2\,d\tau\Big)^{1/2}.$$

By Schwarz inequality and Hölder's inequality, we obtain

$$\begin{aligned} &\Big[\omega(t+\delta)^2 + \int_0^t \|u\|_2^2\,d\tau\Big]\Big[\omega + \int_0^t \|u_\tau\|_2^2\,d\tau\Big] - \Big[\omega(t+\delta) + \frac{1}{2}(\|u\|_2^2 - \|u_0\|_2^2)\Big]^2 \\ &= [\omega(t+\delta)^2 + \alpha^2(t)][\omega + \beta^2(t)] - \Big[\omega(t+\delta) + \frac{1}{2}\int_0^t \frac{d}{d\tau}\|u\|_2^2\,d\tau\Big]^2 \\ &\geq [\omega(t+\delta)^2 + \alpha^2(t)][\omega + \beta^2(t)] - \Big[\omega(t+\delta) + \int_0^t \|u\|_2\|u_\tau\|_2\,d\tau\Big]^2 \\ &\geq [\omega(t+\delta)^2 + \alpha^2(t)][\omega + \beta^2(t)] - [\omega(t+\delta) + \alpha(t)\beta(t)]^2 \\ &= [\sqrt{\omega}\alpha(t)]^2 - 2\omega(t+\delta)\alpha(t)\beta(t) + [\sqrt{\omega}(t+\delta)\beta(t)]^2 \\ &= [\sqrt{\omega}\alpha(t) - \sqrt{\omega}(t+\delta)\beta(t)]^2 \geq 0. \end{aligned}$$

Then it is clear that

$$\begin{aligned} -(M'(t))^2 &= -4\Big(\omega(t+\delta) + \frac{1}{2}(\|u\|_2^2 - \|u_0\|_2^2)\Big)^2 \\ &= -4\Big(M(t) - (\widetilde{T}-t)\|u_0\|_2^2\Big)\Big(\omega + \int_0^t \|u_\tau\|_2^2\,d\tau\Big) \\ &\quad + 4\Big(\omega(t+\delta)^2 + \int_0^t \|u\|_2^2\,d\tau\Big)\Big(\omega + \int_0^t \|u_\tau\|_2^2\,d\tau\Big) \\ &\quad - 4\Big(\omega(t+\delta) + \frac{1}{2}(\|u\|_2^2 - \|u_0\|_2^2)\Big)^2 \\ &\geq -4M(t)\Big(\omega + \int_0^t \|u_\tau\|_2^2\,d\tau\Big). \end{aligned}$$

It follows from (4.58) and the above inequality that

$$\begin{aligned} &M(t)M''(t) - \frac{q+1}{2}(M'(t))^2 \\ &\geq M(t)\Big(M''(t) - 2(q+1)\Big(\omega + \int_0^t \|u_\tau\|_2^2\,d\tau\Big)\Big) \end{aligned}$$

$$\geq M(t)\Big((q-1)a\lambda_1\|u_0\|_2^2+\frac{(q-3)b\lambda_1^2}{2}\|u_0\|_2^4-2(q+1)J(u_0)-2q\omega\Big).$$

Taking $\omega=\varrho/(4q)$, we obtain

$$M(t)M''(t)-\frac{q+1}{2}(M'(t))^2\geq 0.$$

It follows from Lemma 3.1 that

$$T\leq\frac{2M(0)}{(q-1)M'(0)}=\frac{\delta}{q-1}+\frac{\|u_0\|_2^2}{\omega\delta(q-1)}\widetilde{T},\quad \lim_{t\to T}M(t)=+\infty.$$

Taking $\delta=\frac{4q\|u_0\|_2^2+1}{\varrho(q-1)}$, we obtain $T<\widetilde{T}$, a contradiction. Therefore, $u(t)$ blows up in finite time.

Step 2: Upper bound estimate of the blow-up time. For $T_1\in(0,T)$, we set

$$B(t):=\omega(t+\delta)^2+\int_0^t\|u\|_2^2\,d\tau+(T-t)\|u_0\|_2^2,\quad t\in[0,T_1],$$

where $\omega,\delta>0$ are two constants to be determined later. Similar to Step 1, we infer that

$$\begin{aligned}&B(t)B''(t)-\frac{q+1}{2}(B'(t))^2\\&\geq B(t)\Big((q-1)a\lambda_1\|u_0\|_2^2+\frac{(q-3)b\lambda_1^2}{2}\|u_0\|_2^4-2(q+1)J(u_0)-2q\omega\Big).\end{aligned}$$

Taking ω small enough such that

$$0<\omega\leq\frac{\varrho}{2q},\tag{4.59}$$

we have

$$B(t)B''(t)-\frac{q+1}{2}(B'(t))^2\geq 0.$$

It follows from Lemma 3.1 that

$$T_1\leq\frac{2B(0)}{(q-1)B'(0)}=\frac{\delta}{q-1}+\frac{\|u_0\|_2^2}{\omega\delta(q-1)}T,\quad \forall T_1\in[0,T).$$

Letting $T_1\to T$, we obtain

$$T\leq\frac{\delta}{q-1}+\frac{\|u_0\|_2^2}{\omega\delta(q-1)}T.\tag{4.60}$$

Taking δ large enough such that

$$\delta>\frac{\|u_0\|_2^2}{(q-1)\omega},\tag{4.61}$$

then it is clear from (4.60) that

$$T\leq\frac{\omega\delta^2}{\omega\delta(q-1)-\|u_0\|_2^2}.$$

According to (4.59) and (4.61), we define

$$\begin{aligned}\Theta&:=\Big\{(\omega,\delta):0<\omega\leq\frac{\varrho}{2q},\delta>\frac{\|u_0\|_2^2}{(q-1)\omega}\Big\}\\&=\Big\{(\omega,\delta):\frac{\|u_0\|_2^2}{(q-1)\delta}<\omega\leq\frac{\varrho}{2q},\delta>\frac{2q\|u_0\|_2^2}{(q-1)\varrho}\Big\},\end{aligned}$$

then

$$T \leq \inf_{(\omega,\delta)\in\Theta} \frac{\omega\delta^2}{\omega\delta(q-1) - \|u_0\|_2^2}.$$

Let $\mu = \omega\delta$ and

$$\chi(\delta,\mu) := \frac{\mu\delta}{\mu(q-1) - \|u_0\|_2^2}.$$

Obviously, $\chi(\delta,\mu)$ is decreasing with respect to μ. Hence,

$$\begin{aligned} T &\leq \inf_{\delta > \frac{2q\|u_0\|_2^2}{(q-1)\varrho}} \chi\Big(\delta, \frac{\varrho\delta}{2q}\Big) \\ &= \inf_{\delta > \frac{2q\|u_0\|_2^2}{(q-1)\varrho}} \frac{\varrho\delta^2}{\varrho\delta(q-1) - 2q\|u_0\|_2^2} \\ &= \frac{\varrho\delta^2}{\varrho\delta(q-1) - 2q\|u_0\|_2^2}\Big|_{\delta = \frac{4q\|u_0\|_2^2}{(q-1)\varrho}} \\ &= \frac{8q\|u_0\|_2^2}{(q-1)^2\varrho}. \end{aligned}$$

Then it follows from the definition of ϱ that

$$T \leq \frac{16q\|u_0\|_2^2}{(q-1)^2[2(q-1)a\lambda_1\|u_0\|_2^2 + (q-3)b\lambda_1^2\|u_0\|_2^4 - 4(q+1)J(u_0)]}.$$

Step 3: Growth estimates.

By Step 1, we know that $I(u) < 0$ for all $t \in [0,T)$, then it follows from Lemma 3.6 that $\|u\|_2^2$ is strictly increasing for $t \in [0,T)$. Furthermore, we obtain from Lemma 3.6 and (2.3) that

$$\begin{aligned} &\frac{d}{dt}\Big(\|u\|_2^2 - \frac{4(q+1)}{2(q-1)a\lambda_1 + (q-3)b\lambda_1^2\|u_0\|_2^2} J(u_0)\Big) \\ &= -2I(u) = (q-1)a\|u\|^2 + \frac{(q-3)b}{2}\|u\|^4 - 2(q+1)J(u), \end{aligned}$$

which, together with (2.12) and (2.15), implies

$$\begin{aligned} &\frac{d}{dt}\Big(\|u\|_2^2 - \frac{4(q+1)}{2(q-1)a\lambda_1 + (q-3)b\lambda_1^2\|u_0\|_2^2} J(u_0)\Big) \\ &\geq (q-1)a\lambda_1\|u\|_2^2 + \frac{(q-3)b\lambda_1^2}{2}\|u\|_2^4 - 2(q+1)J(u_0) \\ &\geq (q-1)a\lambda_1\|u\|_2^2 + \frac{(q-3)b\lambda_1^2\|u_0\|_2^2}{2}\|u\|_2^2 - 2(q+1)J(u_0) \\ &= \frac{2(q-1)a\lambda_1 + (q-3)b\lambda_1^2\|u_0\|_2^2}{2}\Big(\|u\|_2^2 - \frac{4(q+1)}{2(q-1)a\lambda_1 + (q-3)b\lambda_1^2\|u_0\|_2^2} J(u_0)\Big), \end{aligned}$$

this gives

$$\|u\|_2^2 \geq \frac{2(q+1)}{S} J(u_0) + \Big(\|u_0\|_2^2 - \frac{2(q+1)}{S} J(u_0)\Big) e^{St},$$

where $S = \frac{2(q-1)a\lambda_1 + (q-3)b\lambda_1^2\|u_0\|_2^2}{2}$. □

Proof of Theorem 2.11. For all $P > d$, let B_{R_1} and B_{R_2} be two arbitrary disjoint open subsets of B_R. Furthermore, let $\psi \in H_0^1(B_{R_1}) \setminus \{0\}$ be an arbitrary function such that $I(\psi) > 0$. Next, we show that there must exist some sufficiently large $\varsigma > 0$ such that

$$\frac{(q-1)a\lambda_1}{2(q+1)}\|\varsigma\psi\|_2^2 + \frac{(q-3)b\lambda_1^2}{4(q+1)}\|\varsigma\psi\|_2^4 > P, \quad J(\varsigma\psi) \le 0. \tag{4.62}$$

Indeed, when $\varsigma > (\frac{4P(q+1)}{(q-3)b\lambda_1^2\|\psi\|_2^4})^{1/4}$, we have

$$\frac{(q-1)a\lambda_1}{2(q+1)}\|\varsigma\psi\|_2^2 + \frac{(q-3)b\lambda_1^2}{4(q+1)}\|\varsigma\psi\|_2^4 \ge \frac{(q-3)b\lambda_1^2}{4(q+1)}\|\varsigma\psi\|_2^4 > P.$$

On the other hand, from the definition of $J(u)$ we have

$$\begin{aligned} J(\varsigma\psi) &= \frac{a\varsigma^2}{2}\|\psi\|^2 + \frac{b\varsigma^4}{4}\|\psi\|^4 - \frac{\xi\varsigma^{q+1}}{q+1}\|\psi\|_{q+1}^{q+1} \\ &= \varsigma^2\Big[\frac{a}{2}\|\psi\|^2 + \varsigma^2\Big(\frac{b}{4}\|\psi\|^4 - \frac{\xi\varsigma^{q-3}}{q+1}\|\psi\|_{q+1}^{q+1}\Big)\Big]. \end{aligned} \tag{4.63}$$

Then we infer from (4.63) that there must exist some sufficiently large ς such that $J(\varsigma\psi) \le 0$. Therefore, (4.62) holds for some sufficiently large $\varsigma > 0$. For such a ς, we pick a function $\varphi \in H_0^1(B_{R_2})$ such that $J(\varphi) = P - J(\varsigma\psi)$. Then for $u_P = \varsigma\psi + \varphi$, we obtain

$$\begin{gathered} J(u_P) = J(\varsigma\psi) + J(\varphi) = P, \\ \frac{(q-1)a\lambda_1}{2(q+1)}\|u_P\|_2^2 + \frac{(q-3)b\lambda_1^2}{4(q+1)}\|u_P\|_2^4 \ge \frac{(q-1)a\lambda_1}{2(q+1)}\|\varsigma\psi\|_2^2 + \frac{(q-3)b\lambda_1^2}{4(q+1)}\|\varsigma\psi\|_2^4 \\ > J(u_P). \end{gathered}$$

Taking u_P as the initial time, then by Theorem 2.10, we see that the weak solution u blows up in finite time. □

Proof of Theorem 2.12. Let $u = u(t)$, $t \in [0, T)$, be a weak solution of (1.1) and T be the maximum existence time of u. Since $J(u_0) \le d$, $I(u_0) < 0$ or (2.18) holds, then it follows from Theorem 2.7 and Theorem 2.10 that

$$\lim_{t\to T}\|u\|_2 = +\infty. \tag{4.64}$$

Furthermore, by Lemma 3.7 and the proof of Theorem 2.10, we see that $I(u) < 0$ for all $t \in [0, T)$. Then we infer from Lemma 3.6 that $\|u\|_2^2$ is strictly increasing for $t \in [0, T)$.

In addition, from [40, Proposition 3.3] we obtain

$$\|u\|_2 - \|u_0\|_2 \le \|u(t) - u_0\|_2 = \Big\|\int_0^t u_\tau\,d\tau\Big\|_2 \le \int_0^t \|u_\tau\|_2\,d\tau,$$

which, together with $\|u\|_2^2$ begin strictly increasing for $t \in [0, T)$, implies

$$\Big(\int_0^t \|u_\tau\|_2\,d\tau\Big)^2 \ge (\|u\|_2 - \|u_0\|_2)^2.$$

It follows from Hölder's inequality that

$$\int_0^t \|u_\tau\|_2^2\,d\tau \ge \frac{1}{t}(\int_0^t \|u_\tau\|_2\,d\tau)^2 \ge \frac{1}{t}(\|u\|_2 - \|u_0\|_2)^2.$$

According to (2.12), we know that

$$J(u) \leq J(u_0) - \int_0^t \|u_\tau\|_2^2 \, d\tau \leq J(u_0) - \frac{1}{t}(\|u\|_2 - \|u_0\|_2)^2.$$

Then by (4.64), we readily obtain $\lim_{t\to T} J(u(t)) = -\infty$. □

Proof of Theorem 2.13. Let $u = u(t)$, $t \in [0,T)$, be a weak solution of (1.1) and T be the maximum existence time of u. We divide the proof into the following two cases.

Case 1: $J(u_0) < d$ **and** $u_0 \in H_0^1(B_R) \setminus \{0\}$. Firstly, we prove $I(u_0) \neq 0$. If $I(u_0) = 0$, then it follows from $u_0 \neq 0$ and the definition of d that $J(u_0) \geq d$, a contradiction.

(1) If $I(u_0) < 0$, then we infer from $J(u_0) < d$ and Theorem 2.7 that u blows up in finite time, so $T < +\infty$. Next, we show that if $T < +\infty$, then $I(u_0) < 0$. In fact, if not, then we have $I(u_0) > 0$, which, together with $J(u_0) < d$ and Theorem 2.4, implies $T = +\infty$, a contradiction. In addition, if $I(u_0) < 0$, then it follows from $J(u_0) < d$ and Theorem 2.12 that there is a $t_0 \in [0,T)$ such that $J(u(t_0)) < 0$. Next, we prove that

there is a $t_0 \in [0,T)$ such that $J(u(t_0)) < 0$ implies $I(u_0) < 0$.

In fact, it follows from $J(u(t_0)) < 0$ and (2.3) that $I(u(t_0)) < 0$. Hence, one can choose t_0 as the initial time, and it follows from Theorem 2.7 that u blows up in finite time. According to Theorem 2.4, we see that $I(u_0) > 0$ is impossible, so we obtain $I(u_0) < 0$.

(2) If $I(u_0) > 0$, then we obtain $T = +\infty$ from $J(u_0) < d$ and Theorem 2.4. Next, we show that if $T = +\infty$, then $I(u_0) > 0$. In fact, if $I(u_0) < 0$, then it follows from (1) that $T < +\infty$, a contradiction. In addition, if $I(u_0) > 0$, then we obtain from $J(u_0) < d$ and Lemma 3.7 that $I(u) > 0$ for $t \in [0, +\infty)$. Thus, it follows from (2.3) that $J(u) > 0$ for all $t \in [0, +\infty)$. Next, we prove that

$$J(u(t)) > 0 \text{ for all } t \in [0,T) \Rightarrow I(u_0) > 0.$$

In fact, if $I(u_0) < 0$, then it follows from $J(u_0) < d$ and Theorem 2.12 that $\lim_{t\to T} J(u) = -\infty$. Hence, we infer that there is a t_0 such that $J(u(t_0)) < 0$, a contradiction.

Case 2: $J(u_0) = d$ **and** $u_0 \in H_0^1(B_R) \setminus \{N \cup \{0\}\}$. Because $u_0 \in H_0^1(B_R) \setminus \{N \cup \{0\}\}$, we know that $I(u_0) \neq 0$.

(3) If $I(u_0) < 0$, then we obtain from $J(u_0) = d$ and Theorem 2.7 that u blows up in finite time, so $T < +\infty$. Next, we show that if $T < +\infty$, then $I(u_0) < 0$. In fact, if not, then we have $I(u_0) > 0$, which, together with $J(u_0) = d$ and Corollary 2.6, implies $T = +\infty$, a contradiction. In addition, if $I(u_0) < 0$, then it follows from $J(u_0) = d$ and Theorem 2.12 that there is a $t_0 \in [0,T)$ such that $J(u(t_0)) < 0$. Next, we prove that

there is a $t_0 \in [0,T)$ such that $J(u(t_0)) < 0 \Rightarrow I(u_0) < 0$.

In fact, it follows from $J(u(t_0)) < 0$ and (2.3) that $I(u(t_0)) < 0$. Hence, we can choose t_0 as the initial time, and it follows from Theorem 2.7 that u blows up in finite time. From Corollary 2.6, we see that $I(u_0) > 0$ is impossible, so we obtain $I(u_0) < 0$.

(4) If $I(u_0) > 0$, then we obtain $T = +\infty$ from $J(u_0) = d$ and Corollary 2.6. Next, we show that if $T = +\infty$, then $I(u_0) > 0$. In fact, if $I(u_0) < 0$, then it follows

from (3) that $T < +\infty$, a contradiction. In addition, if $I(u_0) > 0$, then it is clear from $J(u_0) = d$ and Lemma 3.7 that $I(u) > 0$ for all $t \in [0, +\infty)$. Therefore, it follows from (2.3) that $J(u) > 0$ for all $t \in [0, +\infty)$. Next, we prove that

$$J(u(t)) > 0 \text{ for all } t \in [0, T) \Rightarrow I(u_0) > 0.$$

In fact, if $I(u_0) < 0$, then by $J(u_0) = d$ and Theorem 2.12, one has $\lim_{t\to T} J(u(t)) = -\infty$. Consequently, we infer that there is a t_0 such that $J(u(t_0)) < 0$, a contradiction. □

References

[1] C. Bandle, M. A. Pozio, A. Tesei; The Fujita exponent for the Cauchy problem in the hyperbolic space. *J. Differential Equations*, **251** (2011), 2143–2163.

[2] G. Molica Bisci; Kirchhoff-type problems on a geodesic ball of the hyperbolic space. *Nonlinear Analysis*, **186** (2019), 55–73.

[3] H. Brezis; *Functional analysis, Sobolev spaces and partial differential equations*. Universitext. Springer, New York, 2011.

[4] E. B. Davies; *Heat kernels and spectral theory*, volume 92 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, 1989.

[5] H. Ding, R. H. Wang, J. Zhou; Infinite time blow-up of solutions to a class of wave equations with weak and strong damping terms and logarithmic nonlinearity. *Stud. Appl. Math.*, **147** (2021), 914–934.

[6] H. Ding, J. Zhou; Global existence and blow-up of solutions to a nonlocal Kirchhoff diffusion problem. *Nonlinearity*, **33** (2020), 6099–6133.

[7] H. Ding, J. Zhou; Local existence, global existence and blow-up of solutions to a nonlocal Kirchhoff diffusion problem. *Nonlinearity*, **33** (2020), 1046–1063.

[8] H. Ding, J. Zhou; Global existence and blow-up for a parabolic problem of Kirchhoff type with logarithmic nonlinearity. *Appl. Math. Optim.*, **83** (2021), 1651–1707.

[9] H. Ding, J. Zhou; Infinite time blow-up of solutions for a class of logarithmic wave equations with arbitrary high initial energy. *Appl. Math. Optim.*, **84** (2021), S1331–S1343.

[10] H. Ding, J. Zhou; Well-posedness of solutions for the dissipative Boussinesq equation with logarithmic nonlinearity. *Nonlinear Anal. Real World Appl.*, **67** (2022), Paper No. 103587.

[11] L. C. Evans; *Partial differential equations*, volume 19 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, second edition, 2010.

[12] A. Fiscella; A fractional Kirchhoff problem involving a singular term and a critical nonlinearity. *Adv. Nonlinear Anal.*, **8** (2019), 645–660.

[13] A. Fiscella, P. Pucci; p-fractional Kirchhoff equations involving critical nonlinearities. *Nonlinear Anal. Real World Appl.*, **35** (2017), 350–378.

[14] A. Fiscella, E. Valdinoci; A critical Kirchhoff type problem involving a nonlocal operator. *Nonlinear Anal.*, **94** (2014), 156–170.

[15] A. Grigor'yan; Analytic and geometric background of recurrence and non-explosion of the Brownian motion on Riemannian manifolds. *Bull. Amer. Math. Soc. (N.S.)*, **36** (1999), 135–249.

[16] A. Grigor'yan; Heat kernels on weighted manifolds and applications. In *The ubiquitous heat kernel*, volume 398 of *Contemp. Math.*, pages 93–191. Amer. Math. Soc., Providence, RI, 2006.

[17] A. Grigor'yan, M. Noguchi; The heat kernel on hyperbolic space. *Bull. London Math. Soc.*, **30** (1998), 643–650.

[18] R. Ikehata, T. Suzuki; Stable and unstable sets for evolution equations of parabolic and hyperbolic type. *Hiroshima Math. J.*, **26** (1996), 475–491.

[19] H. A. Levine; Instability and nonexistence of global solutions to nonlinear wave equations of the form $Pu_{tt} = -Au + F(u)$. *Trans. Amer. Math. Soc.*, **192** (1974), 1–21.

[20] H. A. Levine; Some additional remarks on the nonexistence of global solutions to nonlinear wave equations. *SIAM J. Math. Anal.*, **5** (1974), 138–146.

[21] W. Liu, J. Yu; Global existence and uniform decay of solutions for a coupled system of nonlinear viscoelastic wave equations with not necessarily differentiable relaxation functions. *Stud. Appl. Math.*, **127** (2011), 315–344.

[22] Y. C. Liu; On potential wells and vacuum isolating of solutions for semilinear wave equations. *J. Differential Equations*, **192** (2003), 155–169.

[23] Y. C. Liu, R. Z. Xu; Potential well method for Cauchy problem of generalized double dispersion equations. *J. Math. Anal. Appl.*, **338** (2008), 1169–1187.

[24] Y. C. Liu, J. S. Zhao; Nonlinear parabolic equations with critical initial conditions $J(u_0) = d$ or $I(u_0) = 0$. *Nonlinear Anal.*, **58** (2004), 873–883.

[25] Y. C. Liu, J. S. Zhao; On potential wells and applications to semilinear hyperbolic equations and parabolic equations. *Nonlinear Anal.*, **64** (2006), 2665–2687.

[26] T. L. M. Luna, G. F. Madeira; Parabolic Kirchhoff equations with non-homogeneous flux boundary conditions: well-posedness, regularity and asymptotic behavior. *Nonlinearity*, **34** (2021), 5844–5871.

[27] N. Pan, P. Pucci, B. L. Zhang; Degenerate Kirchhoff-type hyperbolic problems involving the fractional Laplacian. *J. Evol. Equ.*, **18** (2018), 385–409.

[28] N. Pan, B. L. Zhang, J. Cao; Degenerate Kirchhoff-type diffusion problems involving the fractional p-Laplacian. *Nonlinear Anal. Real World Appl.*, **37** (2017), 56–70.

[29] L. E. Payne, D. H. Sattinger; Saddle points and instability of nonlinear hyperbolic equations. *Israel J. Math.*, **22** (1975), 273–303.

[30] F. Punzo; On well-posedness of semilinear parabolic and elliptic problems in the hyperbolic space. *J. Differential Equations*, **251** (2011), 1972–1989.

[31] D. H. Sattinger; On global solution of nonlinear hyperbolic equations. *Arch. Rational Mech. Anal.*, **30** (1968), 148–172.

[32] X. K. Shao; Global existence and blow-up for a Kirchhoff-type hyperbolic problem with logarithmic nonlinearity. *Appl. Math. Optim.*, **84** (2021), 2061–2098.

[33] M. Q. Xiang, G. Molica Bisci, G. H. Tian, B. L. Zhang; Infinitely many solutions for the stationary Kirchhoff problems involving the fractional p-Laplacian. *Nonlinearity*, **29** (2016), 357–374.

[34] M. Q. Xiang, V. D. Rădulescu, B. L. Zhang; Nonlocal Kirchhoff diffusion problems: local existence and blow-up of solutions. *Nonlinearity*, **31** (2018), 3228–3250.

[35] M. Q. Xiang, B. L. Zhang, V. D. Rădulescu; Multiplicity of solutions for a class of quasilinear Kirchhoff system involving the fractional p-Laplacian. *Nonlinearity*, **29** (2016), 3186–3205.

[36] G. Y. Xu, J. Zhou; Qualitative analysis for a degenerate Kirchhoff-type diffusion equation involving the fractional p-Laplacian. *Appl. Math. Optim.*, **84** (2021), S465–S508.

[37] L. P. Xu, H. B. Chen; Ground state solutions for Kirchhoff-type equations with a general nonlinearity in the critical growth. *Adv. Nonlinear Anal.*, **7** (2018), 535–546.

[38] R. Z. Xu, Y. C. Liu; Global existence and nonexistence of solution for Cauchy problem of multidimensional double dispersion equations. *J. Math. Anal. Appl.*, **359** (2009), 739–751.

[39] Y. B. Yang, X. T. Tian, M. N. Zhang, J. H. Shen. Blowup of solutions to degenerate Kirchhoff-type diffusion problems involving the fractional p-Laplacian. *Electron. J. Differential Equations*, **2018** (2018), Paper No. 155.

[40] E. Zeidler; *Nonlinear functional analysis and its applications. I.* Springer-Verlag, New York, 1986. Fixed-point theorems, Translated from the German by Peter R. Wadsack.

[41] B. L. Zhang , A. Fiscella, S.H. Liang; Infinitely many solutions for critical degenerate Kirchhoff type equations involving the fractional p-Laplacian. *Appl. Math. Optim.*, **80** (2019), 63-80.

[42] S. M. Zheng; *Nonlinear evolution equations*, volume 133 of *Chapman & Hall/CRC Monographs and Surveys in Pure and Applied Mathematics*. Chapman & Hall/CRC, Boca Raton, FL, 2004.

[43] J. Zhou; Behavior of solutions to a fourth-order nonlinear parabolic equation with logarithmic nonlinearity. *Appl. Math. Optim.*, **84** (2021), 191–225.

Hang Ding

School of Mathematics and Statistics, Southwest University, Chongqing 400715, China

Email address: `hding0527@163.com`

Jun Zhou (corresponding author)

School of Mathematics and Statistics, Southwest University, Chongqing 400715, China

Email address: `jzhouwm@163.com`