Electronic Journal of Differential Equations, Vol. 2022 (2022), No. 40, pp. 1–16.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu

EXISTENCE OF SOLUTIONS TO NONLOCAL ELLIPTIC PROBLEMS WITH SINGULAR AND COMBINED NONLINEARITIES

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Abstract. We use an approximation scheme together with a variation of the fixed point theorem to show the existence of a positive solution to a nonlocal boundary value problem. This problem has a smooth bounded domain in $\mathbb{R}^N$, a singular term, and combined nonlinearities. We also study the symmetric, monotonicity, and asymptotic behavior of the solutions with respect to a parameter involved in the problem.

1. Introduction

Let $\Omega \subset \mathbb{R}^N$ $(N \geq 2)$ be a smooth bounded domain. We prove the existence of a positive solution to the nonlocal boundary value problem

$$\begin{gathered} -M\Big(\int_\Omega |\nabla u|^2\Big)\Delta u = \lambda(a(x)u^{-\gamma} + u^q) + f(u), \quad x \in \Omega, \\ u = 0, \quad x \in \partial\Omega, \end{gathered} \tag{1.1}$$

where $0 < q < 1$, $0 < \gamma < 1$, $\lambda > 0$ is a parameter, $a \in L^\infty(\Omega)$ with $a \geq 0$, $M\colon \mathbb{R}^+ \to \mathbb{R}^+$ is a continuous and positive function in $[0,1]$ and $f\colon \mathbb{R} \to \mathbb{R}$ is a continuous function satisfying

$$0 \leq tf(t) \leq C|t|^p, \tag{1.2}$$

with $1 < p \leq \frac{N+2}{N-2}$ if $N \geq 3$ or $1 < p$ if $N = 2$. Two typical examples are $M(t) = ct + d$ with $c > 0$ and $d \geq 0$, and $f(u) = u^p$.

By a solution of (1.1) we mean a function $u \in H_0^1(\Omega)$ such that $u > 0$ in Ω and

$$-M\Big(\int_\Omega |\nabla u|^2\Big)\int_\Omega \nabla u \nabla \phi = \lambda\int_\Omega (a(x)u^{-\gamma} + u^q)\phi + \int_\Omega f(u)\phi = 0$$

for all $\phi \in H_0^1(\Omega)$.

Problem (1.1) is called nonlocal because of the presence of the term M, which implies that the equation in (1.1) is no longer a point-wise function. This phenomenon provokes some mathematical difficulties, which makes the study of such problems particularly interesting. This problem has a physical motivation. In fact,

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2020 *Mathematics Subject Classification.* 35J75, 35B40, 35J60, 35B25, 58K55.
Key words and phrases. Nonlocal problem; singular equation; Galerkin method; positive solution; asymptotic behavior.

Submitted October 9, 2021. Published June 27, 2022.

the operator $M(\int_\Omega |\nabla u|^2)\Delta u$ appears in the Kirchhoff equation, which arises in nonlinear vibrations, namely

$$\begin{gathered} u_{tt} - M\Big(\int_\Omega |\nabla u|^2\Big)\Delta u = g(u), \quad x \in \Omega, \\ u = 0, \quad x \in \partial\Omega \times [0,T], \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x). \end{gathered} \tag{1.3}$$

Such a hyperbolic equation is a general version of the Kirchhoff equation

$$\rho\frac{\partial^2 u}{\partial t^2} - \Big(\frac{P_0}{h} + \frac{E}{2L}\int_0^L |\frac{\partial u}{\partial x}|dx\Big)\frac{\partial^2 u}{\partial t^2} = 0 \tag{1.4}$$

presented by Kirchhoff in [20]. This equation extends the classical d'Alembert's wave equation by considering the effects of the changes in the length of the strings during the vibrations. The parameters in equation (1.4) have the following meanings: L is the length of the string, h is the area of cross-section, E is the Young modulus of the material, ρ is the mass density and P_0 is the initial tension.

Equations with singularities have attracted a great attention due to the relationship with models of non-Newtonian fluids, in applications to heat conduction in electrically conducting materials, boundary layer phenomena for viscous fluids, and chemical heterogeneous catalysts (see, e.g., [6, 7, 24, 28] and the references therein). One of the first studies appeared in [10, 12]. There, the authors consider an approximation of the singular equation by a regular problem, where monotonicity methods can be applied and then passing to the limit to obtain the solution of the original equation. Recently, the existence of a solution to the problem (1.1) was investigated, for the cases $f = 0$ or $f(u) = u^p$, $M(t) = ct + d$, and $\Omega \subset \mathbb{R}^3$ is a bounded domain, see [3, 8, 13, 21]. Furthermore, for the cases $a = 0$ and $f(u) = u^p$ where the nonlinearity is convex-concave, the existence of solutions to (1.1) has been extensively researched in [1, 11, 22].

When $M = 1$, equation (1.1) is reduced to the singular semilinear elliptic problem

$$\begin{gathered} -\Delta u = \lambda(a(x)u^{-\gamma} + u^q) + f(u), \quad x \in \Omega, \\ u = 0, \quad x \in \partial\Omega. \end{gathered} \tag{1.5}$$

Many authors have considered problems related with (1.5). In [14], it was studied the existence, nonexistence, and uniqueness of positive solution to the problem $-\Delta u + a(x)g(u) = \mu f(x,u) + \lambda h(x)$ in Ω and $u = 0$ on $\partial\Omega$, where f is a positive function with sublinear growth, $a, h \in C^{0,\alpha}(\overline{\Omega})$ with $a > 0$ and $h > 0$, and g is a singular nonlinearity. The same hypotheses were used to address similar questions by [9] for a different equation $-\Delta u = a(x)g(u) + \lambda f(u)$. In [15], the existence of multiple positive solutions was studied for the singular, critical elliptic problem $-\Delta u = \lambda(u^{-\delta} + u^q + \rho(u))$ in Ω and $u = 0$ on $\partial\Omega$, where $\delta > 0$, $1 < q \le 2^* - 1$ and ρ is a smooth function with subcritical asymptotic behavior at infinite. Moreover, in [16], in addition to studying the existence of a positive solution, the authors investigated the asymptotic behavior of the solutions when the exponent $p \to 1$. Note that in the problem (1.1) we establish the asymptotic behavior of the solutions regarding the parameter λ. Our main results read as follows.

Theorem 1.1. *Suppose that $f\colon \mathbb{R} \to \mathbb{R}$ is a continuous function satisfying* (1.2) *and $M\colon \mathbb{R}^+ \to \mathbb{R}^+$ is a continuous and positive function satisfying* (1.6). *Then* *there exists $\lambda^* > 0$ such that for every $\lambda \in (0, \lambda^*)$, the problem (1.1) has a positive solution $u \in H_0^1(\Omega)$.*

Proposition 1.2. *Suppose that, under the conditions in Theorem 1.1, $a(x) = 0$ and $\Omega = B_\varrho$ is an open ball in $\mathbb{R}^N$ with radius ϱ and center $x = 0$. If $u \in H_0^1(\Omega)$ is a positive solution to (1.1), give by Theorem 1.1, then u is symmetric with respect to the hyperplane $x_1 = 0$ and decreasing in the direction $x_1 > 0$, where $x = (x_1, x') \in B_\varrho$.*

Remark 1.3. If $a(x) = 0$, then we can show using well-known Bootstrap arguments that the positive solutions $u \in H_0^1(\Omega)$ to (1.1) are classical, that is, $u \in C^{2,\alpha}(\overline{\Omega})$ for some $\alpha \in (0, 1)$.

Proposition 1.4. *Suppose that $u_\lambda \in H_0^1(\Omega)$ is a positive solution to (1.1), given by Theorem 1.1, then*

$$\|u_\lambda\|_{H_0^1(\Omega)} \to 0 \quad \text{as } \lambda \to 0^+.$$

Notice that in this article we do not impose any extra hypotheses on M beyond continuity and positivity in $[0, 1]$. In comparison with problems that found in the literature, the novelty in problem (1.1) is that our results hold for a new nonlocal problem, that is, when $M(t) = d - ct$ with $d > c > 0$. Recently, nonsingular problems related to this operator were studied in [25, 27]. The results obtained in Proposition 1.2 are apparently new in the study of nonlocal problems with singularity.

Remark 1.5. If M is a continuous and positive function in the compact set $[0, 1]$, then there exist $m_0 > 0$ and $m_\infty > 0$ such that

$$m_0 \leq M(t) \leq m_\infty \text{ for every } t \in [0, 1]. \tag{1.6}$$

This article is organized as follows: in Section 2 we give some auxiliary results that will be used throughout the paper. We approximate f by a sequence (f_n) of Lipschitz functions. Then, in Section 3 we prove the existence of solutions v_n for an approximate problem (3.1) in finite dimension. In Section 4 we prove Theorem 1.1 where we show that the solutions v_n of (3.1) are bounded and converge to a positive solution of (1.1). Finally, in Section 5, we investigated the symmetry, monotonicity and asymptotic behavior of solutions to the problem (1.1), that is, we prove Propositions 1.2 and 1.4.

2. Auxiliary results

In this section, we present some preliminary results that will be used throughout the paper. Initially, we approximate the function f give in (1.1) by a sequence of Lipschitz functions $f_k : \mathbb{R} \to \mathbb{R}$ defined by

$$f_k(t) = \begin{cases} -k[G(-k - \frac{1}{k}) - G(-k)], & \text{if } t \leq -k \\ -k[G(t - \frac{1}{k}) - G(t)], & \text{if } -k < t \leq -1/k \\ k^2 t[G(-\frac{2}{k}) - G(-\frac{1}{k})], & \text{if } -1/k < t \leq 0 \\ k^2 t[G(\frac{2}{k}) - G(\frac{1}{k})], & \text{if } 0 < t \leq 1/k \\ k[G(t + \frac{1}{k}) - G(t)], & \text{if } 1/k < t \leq k \\ k[G(k + \frac{1}{k}) - G(k)], & \text{if } t > k \end{cases} \tag{2.1}$$

where $G(t) = \int_0^t f(\tau) d\tau$.

The following approximation result was proved in [26] and it uses an explicit expression of the sequence defined in (2.1).

Lemma 2.1. *Let $f\colon \mathbb{R} \to \mathbb{R}$ be a continuous function such that $tf(t) \geq 0$ for every $t \in \mathbb{R}$. Then there exists a sequence $f_k\colon \mathbb{R} \to \mathbb{R}$ of continuous functions satisfying*

(i) *$tf_k(t) \geq 0$ for every $t \in \mathbb{R}$;*
(ii) *$\forall k \in \mathbb{N}$, $\exists c_k > 0$ such that $|f_k(\xi) - f_k(\eta)| \leq c_k|\xi - \eta|$ for every $\xi, \eta \in \mathbb{R}$;*
(iii) *$f_k \to f$ uniformly in bounded subsets of $\mathbb{R}$.*

The sequence (f_k) in Lemma 2.1 has some additional properties that are deduced from (1.2).

Lemma 2.2. *Let $f\colon \mathbb{R} \to \mathbb{R}$ be a continuous function satisfying* (1.2) *for every $t \in \mathbb{R}$. Then the sequence f_k of Lemma 2.1 satisfies*

(i) *$\forall k \in \mathbb{N}$, $0 \leq tf_k(t) \leq C_1|t|^p$ for every $|t| \geq 1/k$;*
(ii) *$\forall k \in \mathbb{N}$, $0 \leq tf_k(t) \leq C_2|t|^2$ for every $|t| \leq 1/k$,*

where C_1 and C_2 are positive constants independent of k.

Proof. The proof consist of four steps and it is basically deduced using the mean value theorem. Everywhere in this proof, the constant C is the one of (1.2).

Step 1. Suppose of $-k \leq t \leq -1/k$. By the mean value theorem, there exists $\eta \in (t - \frac{1}{k}, t)$ such that

$$f_k(t) = -k\big[G\big(t - \frac{1}{k}\big) - G(t)\big] = -kG'(\eta)\big(t - \frac{1}{k} - t\big) = f(\eta)$$

and so, $tf_k(t) = tf(\eta)$. Since $t - \frac{1}{k} < \eta < t < 0$ and $f(\eta) < 0$, we obtain $tf_k(t) \leq \eta f(\eta)$. Therefore,

$$tf_k(t) \leq \eta f(\eta) \leq C|\eta|^p \leq C|t - \frac{1}{k}|^p \leq C\big(|t| + \frac{1}{k}\big)^p \leq C(2|t|)^p \leq C2^p|t|^p.$$

Step 2. Assume $\frac{1}{k} \leq t \leq k$. By the mean value theorem, there exists $\eta \in (t, t + \frac{1}{k})$ such that

$$f_k(t) = k\big[G\big(t + \frac{1}{k}\big) - G(t)\big] = kG'(\eta)\big(t + \frac{1}{k} - t\big) = f(\eta)$$

and thus $f_k(t) = tf(\eta)$. Since $0 < t < \eta < t + \frac{1}{k}$ and $f(\eta) > 0$, we have $tf_k(t) \leq \eta f(\eta)$. Therefore

$$tf_k(t) \leq \eta f(\eta) \leq C|\eta|^p \leq C|t + \frac{1}{k}|^p \leq C(2|t|)^p \leq C2^p|t|^p.$$

Step 3. Suppose that $|t| \geq k$, then

$$f_k(t) = \begin{cases} -k[G(-k - \frac{1}{k}) - G(-k)], & \text{if } t \leq -k \\ k[G(k + \frac{1}{k}) - G(k)], & \text{if } t \geq k. \end{cases}$$

If $t \leq -k$, by the mean value theorem, there exists $\eta \in (-k - \frac{1}{k}, -k)$ such that

$$f_k(t) = k\big[G\big(-k - \frac{1}{k}\big) - G(-k)\big] = kG'(\eta)\big(-k + \frac{1}{k} + k\big) = f(\eta)$$

and $tf_k(t) = tf(\eta)$. Since $-k - \frac{1}{k} < \eta < -k < 0$ and $k < |\eta| < k + \frac{1}{k}$, we conclude that

$$\begin{aligned} tf_k(t) &= \frac{s}{\eta}\eta f(\eta) \le C\frac{|t|}{|\eta|}|\eta|^p \\ &\le C|t|\big(k + \frac{1}{k}\big)^{p-1} \\ &\le C|t|\big(|t| + \frac{1}{k}\big)^{p-1} \le C2^p|t|^p. \end{aligned} \tag{2.2}$$

If $t \ge k$, by the mean value theorem, there exists $\eta \in (k, k + \frac{1}{k})$ such that

$$f_k(t) = k\big[G\big(k, k + \frac{1}{k}\big) - G(k)\big] = kG'(\eta)\big(k + \frac{1}{k} - k\big) = f(\eta).$$

By computations similar to those for (2.2) one has

$$tf_k(t) = \frac{s}{\eta}\eta f(\eta) \le C\frac{|t|}{|\eta|}|\eta|^p \le C2^p|t|^p.$$

Step 4. Suppose that $|t| \le \frac{1}{k}$, then

$$f_k(t) = \begin{cases} k^2t[G(-\frac{2}{k}) - G(-\frac{1}{k})], & \text{if } -\frac{1}{k} \le t \le 0 \\ k^2t[G(\frac{2}{k}) - G(\frac{1}{k})], & \text{if } 0 \le t \le \frac{1}{k}. \end{cases}$$

If $-1/k \le t \le 0$, by the mean value theorem, there exists $\eta \in (-\frac{1}{k}, -\frac{2}{k})$ such that

$$f_k(t) = k^2t\big[G\big(-\frac{2}{k}\big) - G\big(-\frac{1}{k}\big)\big] = k^2tG'(\eta)\big(-\frac{2}{k} + \frac{1}{k})\big) = -ktf(\eta).$$

Therefore,

$$\begin{aligned} tf_k(t) &= -kt^2f(\eta) = -k\frac{t^2}{\eta}\eta f(\eta) \\ &\le k\frac{t^2}{\eta}\eta f(\eta) \le C|t|^2|\eta|^{p-1} \\ &\le Ck|t|^2(\frac{2}{k})^{p-1} \le C2^{p-1}|t|^2 \end{aligned} \tag{2.3}$$

If $0 \le t \le 1/k$, by the mean value theorem, there exists $\eta \in (\frac{1}{k}, \frac{2}{k})$ such that

$$f_k(t) = k\big[G\big(\frac{2}{k}\big) - G\big(\frac{1}{k}\big)\big] = k^2tG'(\eta)\big(\frac{2}{k} - \frac{1}{k}\big) = ktf(\eta).$$

By computations similar to those for (2.3) one obtains

$$tf_k(t) = kt^2f(\eta) = k\frac{t^2}{\eta}\eta f(\eta) \le k\frac{t^2}{\eta}\eta f(\eta) \le C2^{p-1}|t|^2.$$

The proof of lemma follows by talking $C_1 = C2^p$ and $C_2 = C2^{p-1}$. $\square$

The next lemma will be used to show the symmetry and monotonicity of the positive solutions to (1.1).

Lemma 2.3. *Let $\Omega \subset \mathbb{R}^N$ be a bounded open set, convex in the direction of x_1 and symmetric with respect the hyperplane $x_1 = 0$. Let $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ be a positive solution to the problem $-\Delta u = g(u)$ in Ω and $u = 0$ in $\partial\Omega$, where $g\colon \mathbb{R} \to \mathbb{R}$ is a locally Lipschitz function. Then $u(x_1, x') \le u(-x_1, x')$ for every $x = (x_1, x') \in \Omega$ such that $x_1 > 0$. Furthermore, $\frac{\partial u}{\partial x_1} < 0$ for every $x \in \Omega$, $x_1 > 0$. See* [4, Theorem 1.2].

Now, we recall the Hardy-Sobolev inequality, which will play a key role in the proof of our main result:

Lemma 2.4 (Hardy-Sobolev inequality [18]). *If $u \in W_0^{1,p}(\Omega)$ with $1 < p \leq N$, then $\frac{u}{d^\tau} \in L^\sigma(\Omega)$, for $\frac{1}{\sigma} = \frac{1}{p} - \frac{1-\tau}{N}$, $0 < \tau \leq 1$ and*

$$\|\frac{u}{d^\tau}\|_{L^\sigma(\Omega)} \leq C\|\nabla u\|_{L^p(\Omega)},$$

where $d(x) = dist(x, \partial\Omega)$ and $C > 0$ is a constant which does not depend on x.

The following lemma will be used for showing that the solutions of an approximate problem discussed in Section 3 converges to a solution to the problem (1.1).

Lemma 2.5 ([26, Theorem 1.1]). *Let Ω be a bounded open set in $\mathbb{R}^N$, $u_k : \mathbb{R} \to \mathbb{R}$ be a sequence of functions, and $g_k : \mathbb{R} \to \mathbb{R}$ be a sequence of functions such that $g_k(u_k)$ are mensurable in Ω for every $k \in \mathbb{N}$. Assume that $g_k(u_k) \to v$ a.e. in Ω and $\int_\Omega g_k(u_k)u_k \leq C$ for a constant C independent of k. And suppose that for every $B \subset \mathbb{R}$, B bounded, there is a constant C_B depending only on B such that $|g_k(x)| \leq C_B$, for all $x \in B$ and $k \in \mathbb{N}$. Then $v \in L^1(\Omega)$ and $g_k(u_k) \to v$ in $L^1(\Omega)$.*

We conclude this section by presenting a lemma, which is a consequence of Brouwer's Fixed Point Theorem. However, our statement is a subtle (but very useful) generalization by comparing it with the literature.

Lemma 2.6. *Let $F : \mathbb{R}^d \to \mathbb{R}^d$ be a continuous function such that $\langle F(\xi), \xi\rangle \geq 0$ for every $\xi \in \mathbb{R}^d$ with $|\xi| = r$ for some $r > 0$. Then there exists z_0 in the closed ball $\overline{B}_r(0)$ such that $F(z_0) = 0$.*

3. Approximate problem in a finite dimensional space

For each $n \in \mathbb{N}$, consider the sequence (f_n) of Lipschitz functions given by the Lemmas 2.1 and 2.2. We will show the existence of a solution to the approximate problem

$$\begin{gathered} -M\Big(\int_\Omega |\nabla v|^2\Big)\Delta v = \lambda\left(a(x)v^s(v + 1/\sqrt{n})^{-(\gamma+s)} + v^q\right) + f_n(v) + \frac{\phi}{n}, \quad x \in \Omega, \\ u = 0, \quad x \in \partial\Omega, \end{gathered} \tag{3.1}$$

where $0 < \gamma < s < 1$, $0 < q < 1$, $\lambda > 0$ is a parameter, $\phi(x)$ is a positive function such that $\phi \in C^{0,\alpha}(\overline{\Omega})$ for some $\alpha \in (0, 1]$. Notice that for us to show the existence of a solution to the approximate problem (3.1), we will use the Galerkin method together with the fixed point theorem given in Lemma 2.6. The main result in this section is the following.

Lemma 3.1. *For each $n \in \mathbb{N}$, there exists $\lambda^* > 0$ and $n^* \in \mathbb{N}$ such that (3.1) admits a positive solution $v_n \in H_0^1(\Omega)$ for every $\lambda \in (0, \lambda^*)$ and $n \geq n^*$. Furthermore,*

$$\|v_n\|_{H_0^1(\Omega)} \leq r, \quad \forall n \in \mathbb{N},$$

where r does not depend on n.

In the following lemma we prove that nonnegative solutions for the approximate problem (3.1) are in fact regular.

Lemma 3.2. *Let $v \in H_0^1(\Omega)$ be a nonegative solution to (3.1). Then $v \in C^{2,\alpha}(\overline{\Omega})$ for some $\alpha \in (0, 1)$.*

In the next corollary, we will emphasize the significance of considering the sequence (f_n) of Lipchitz functions in the approximate problem (3.1).

Corollary 3.3. *Suppose that $a(x) = 0$ and $\Omega = B_\varrho$. Then, for each $n \in \mathbb{N}$, the solution $v_n \in C^{2,\alpha}(\overline{\Omega})$ to (3.1) satisfies*

$$v_n(x_1, x') \le v_n(-x_1, x'), \tag{3.2}$$

for every $x = (x_1, x') \in B_\varrho$ such that $x_1 > 0$. Furthermore,

$$\frac{\partial u}{\partial x_1} < 0 \quad \text{for every } x \in \Omega,\ x_1 > 0. \tag{3.3}$$

Proof Lemma 3.1. Let $\mathcal{B} = \{w_1, w_2, \dots, w_m, \dots\}$ be an orthonormal basis of $H_0^1(\Omega)$. For each $m \in \mathbb{N}$, we define

$$W_m = [w_1, w_2, \dots, w_m],$$

to be the m-dimensional space generated by $\{w_1, w_2, \dots, w_m\}$. Define the function $F\colon \mathbb{R}^m \to \mathbb{R}^m$ such that

$$F(\xi) = (F_1(\xi), F_2(\xi), \dots, F_m(\xi)),$$

where $\xi = (\xi_1, \xi_2, \dots, \xi_m) \in \mathbb{R}^m$,

$$\begin{aligned} &F_j(\xi)\\ &= M\Big(\int_\Omega |\nabla v|^2\Big)\int_\Omega \nabla v \nabla w_j - \lambda \int_\Omega \Big(a(x)(v_+)^s(v_+ + 1/\sqrt{n})^{-(\gamma+s)} + (v_+)^q\Big) w_j\\ &\quad - \int_\Omega f_n(v_+) w_j - \frac{1}{n}\int_\Omega \phi w_j \end{aligned}$$

for $j = 1, 2, \dots, m$ and $v = \sum_{i=1}^m \xi_i w_i$ belongs to W_m. Therefore,

$$\begin{aligned} &\langle F(\xi), \xi\rangle\\ &= M\Big(\int_\Omega |\nabla v|^2\Big)\int_\Omega |\nabla v|^2 - \lambda \int_\Omega \Big(a(x)(v_+)^{s+1}(v_+ + 1/\sqrt{n})^{-(\gamma+s)} + (v_+)^q\Big)\\ &\quad - \int_\Omega f_n(v_+) v_+ - \frac{1}{n}\int_\Omega \phi v, \end{aligned} \tag{3.4}$$

where $v_+ = \max\{0, v\}$ and $v_- = v_+ - v$.

Notice that F is continuous by Sobolev embedding and dominated convergence theorem. Given $v \in W_m$, we define

$$\Omega_n^+ = \{x \in \Omega : |v| \ge \frac{1}{n}\}, \quad \Omega_n^- = \{x \in \Omega : |v| < \frac{1}{n}\}.$$

Now we rewrite (3.4) as $\langle F(\xi), \xi\rangle = \langle F(\xi), \xi\rangle^+ + \langle F(\xi), \xi\rangle^-$ where

$$\begin{aligned} &\langle F(\xi), \xi\rangle^+\\ &= M\Big(\int_\Omega |\nabla v|^2\Big)\int_{\Omega_n^+} |\nabla v|^2 - \lambda \int_{\Omega_n^+} \Big(a(x)(v_+)^{s+1}(v_+ + 1/\sqrt{n})^{-(\gamma+s)} + (v_+)^q\Big)\\ &\quad - \int_{\Omega_n^+} f_n(v_+) v_+ - \frac{1}{n}\int_{\Omega_n^+} \phi v \end{aligned}$$

and

$$\langle F(\xi), \xi\rangle^-$$

$$= M\Big(\int_\Omega |\nabla v|^2\Big)\int_{\Omega_n^-}|\nabla v|^2 - \lambda\int_{\Omega_n^-}\Big(a(x)(v_+)^{s+1}(v_+ + 1/\sqrt{n})^{-(\gamma+s)} + (v_+)^q\Big)$$
$$- \int_{\Omega_n^-} f_n(v_+)v_+ - \frac{1}{n}\int_{\Omega_n^-}\phi v.$$

Let $K_1 = \|a\|_{L^\infty(\Omega)}$ and $K_2 = \max\{|\phi(x)| : x \in \overline{\Omega}\}$. In the next two steps, we estimate $\langle F(\xi),\xi\rangle^+$ and $\langle F(\xi),\xi\rangle^-$.

Step 1. Since $0 < q < 1$ and $0 < \gamma < s < 1$, using Holder inequality and by the Sobolev embedding we obtain

$$\int_{\Omega_n^+} a(x)(v_+)^{s+1}(v_+ + 1/\sqrt{n})^{-(\gamma+s)} \le K_1\int_\Omega |v|^{1-\gamma} \le C_2\|v\|^{1-\gamma}_{H_0^1(\Omega)}\,, \tag{3.5}$$

$$\int_{\Omega_n^+}(v_+)^{q+1} \le \int_\Omega |v|^{q+1} = \|v\|^{q+1}_{L^{q+1}(\Omega)} \le C_3\|v\|^{q+1}_{H_0^1(\Omega)}. \tag{3.6}$$

Furthermore,

$$\int_\Omega \phi v \le K_2|\Omega|\|v\|_{L^2(\Omega)} \le C_4\|v\|_{H_0^1(\Omega)}\,. \tag{3.7}$$

By Lemma 2.2(i) and Sobolev embedding, we deduce

$$\int_{\Omega_n^+} f_n(v_+)v_+ \le C\int_\Omega |v_+|^{p+1} \le C\|v\|^{p+1}_{L^{p+1}(\Omega)} \le C_1\|v\|^{p+1}_{H_0^1(\Omega)}\,. \tag{3.8}$$

Thus, it follows from (3.5)–(3.8) that

$$\langle F(\xi),\xi\rangle^+ \ge M(\|v\|^2_{H_0^1(\Omega)})\int_{\Omega_n^+}|\nabla v|^2 - \lambda(C_2\|v\|^{1-\gamma}_{H_0^1(\Omega)} + C_3\|v\|^{q+1}_{H_0^1(\Omega)}) \tag{3.9}$$
$$- C_1\|v\|^{p+1}_{H_0^1(\Omega)} - \frac{C_4}{n}\|v\|_{H_0^1(\Omega)}\,, \tag{3.10}$$

where C_1, C_2, C_3, and C_4 are positive constants that do not depend on n nor m.

Step 2. Since $0 < q < 1$ and $0 < s < 1$, we deduce

$$\int_{\Omega_n^-} a(x)(v_+)^{s+1}(v_+ + 1/\sqrt{n})^{-(\gamma+s)} \le K_1 n^{\frac{\gamma+s}{2}}\int_{\Omega_n^-}|v|^{s+1} \le K_1|\Omega|n^{\frac{\gamma-s}{2}}, \tag{3.11}$$

$$\int_{\Omega_n^-}(v_+)^{q+1} \le |\Omega|\frac{1}{n^{q+1}}. \tag{3.12}$$

By Lemma 2.2(ii) we find that

$$\int_{\Omega_n^-} f_n(v_+)v_+ \le C_2\int_\Omega |v_+|^2 \le C_5|\Omega|\frac{1}{n^2}. \tag{3.13}$$

It follows from (3.11)–(3.13) that

$$\begin{aligned} &\langle F(\xi),\xi\rangle^- \\ &\ge M(\|v\|^2_{H_0^1(\Omega)})\int_{\Omega_n^-}|\nabla v|^2 - \lambda(K_1|\Omega|n^{\frac{\gamma-s}{2}} + |\Omega|\frac{1}{n^{q+1}}) - C_5|\Omega|\frac{1}{n^2}. \end{aligned} \tag{3.14}$$

As a direct consequence of estimates (3.9) and (3.14) we obtain

$$\begin{aligned} &\langle F(\xi),\xi\rangle \\ &\ge M(\|v\|^2_{H_0^1(\Omega)})\int_\Omega |\nabla v|^2 - \lambda(C_2\|v\|^{1-\gamma}_{H_0^1(\Omega)} + C_3\|v\|^{q+1}_{H_0^1(\Omega)}) \\ &\quad - C_1\|v\|^{p+1}_{H_0^1(\Omega)} - \frac{C_4}{n}\|v\|_{H_0^1(\Omega)} - \lambda(K_1|\Omega|n^{\frac{\gamma-s}{2}} + |\Omega|\frac{1}{n^{q+1}}) - C_5|\Omega|\frac{1}{n^2}. \end{aligned} \tag{3.15}$$

Assume that $\|v\|_{H_0^1(\Omega)} = r \le 1$ for some $0 < r \le 1$ to be fixed later. It follows from (1.5) that $M(\|v\|^2_{H_0^1(\Omega)}) = M(r^2) \ge m_0$, and so we deduce

$$\begin{aligned}\langle F(\xi),\xi\rangle &\ge m_0r^2 - \lambda(C_2r^{1-\gamma} + C_3r^{q+1}) - C_1r^{p+1} - \frac{C_4}{n}r \\ &\quad - \lambda(K_1|\Omega|n^{\frac{\gamma-s}{2}} + |\Omega|\frac{1}{n^{q+1}}) - C_2|\Omega|\frac{1}{n^2}.\end{aligned} \tag{3.16}$$

Note that if

$$r \le \Big(\frac{m_0}{2C_1}\Big)^{1/(p-1)},$$

then $m_0r^2 - C_1r^{p+1} \ge m_0r^2/2$. Thus, by considering

$$r = \frac{1}{2}\min\Big\{1, \Big(\frac{m_0}{2C_1}\Big)^{1/(p-1)}\Big\},$$

we obtain

$$\begin{aligned}\langle F(\xi),\xi\rangle &\ge \frac{m_0r^2}{2} - \lambda(C_2r^{1-\gamma} + C_3r^{q+1}) - \frac{C_4}{n}r \\ &\quad - \lambda(K_1|\Omega|n^{\frac{\gamma-s}{2}} + |\Omega|\frac{1}{n^{q+1}}) - C_5|\Omega|\frac{1}{n^2}.\end{aligned}$$

Now, defining $\rho = \frac{m_0r^2}{2} - \lambda(C_2r^{1-\gamma} + C_3r^{q+1})$, we choose $\lambda^* > 0$ such that $\rho > 0$ for every $\lambda < \lambda^*$. Therefore, we may take

$$\lambda^* = \min\Big\{\frac{m_0r^{1+\gamma}}{4C_2}, \frac{m_0r^{1-q}}{4C_3}\Big\} > 0.$$

Moreover, since $\gamma < s$ we may choose $n^* \in \mathbb{N}$ such that

$$\frac{C_4}{n}r + \lambda(K_1|\Omega|n^{\frac{\gamma-s}{2}} + |\Omega|\frac{1}{n^{q+1}}) + C_5|\Omega|\frac{1}{n^2} < \frac{\rho}{2},\ \forall n \ge n^*.$$

Let $\xi \in \mathbb{R}^m$, such that $|\xi| = r$, then for $\lambda < \lambda^*$ and $n \ge n^*$ we have

$$\langle F(\xi),\xi\rangle \ge \frac{\rho}{2} > 0.$$

For every $n \in \mathbb{N}$, f_n is a Lipschitz function, and then by Lemma 2.6, for every $m \in \mathbb{N}$ there exists $y \in \mathbb{R}^m$ with $|y|_m \le r$ such that $F(y) = 0$, that is, there exists $v_m \in W_m$ satisfying

$$\|v_m\|_{H_0^1(\Omega)} \le r \quad \text{for every } m \in \mathbb{N},$$

and such that

$$\begin{aligned}&M\Big(\int_\Omega |\nabla v_m|^2\Big)\int_\Omega \nabla v_m\nabla w \\ &= \lambda\int_\Omega \Big(a(x)(v_{m+})^s(v_{m+} + 1/\sqrt{n})^{-(\gamma+s)} + (v_{m+})^q\Big)w \\ &\quad - \int_\Omega f_n(v_{m+})w - \frac{1}{n}\int_\Omega \phi w,\ \forall w \in W_m.\end{aligned} \tag{3.17}$$

Since $W_m \subset H_0^1(\Omega)$ for every $m \in \mathbb{N}$ and r does not depend on m, it follows that (v_m) is a bounded sequence in $H_0^1(\Omega)$. Thus, for some subsequence, there exist $0 \le t_0 \le 1$ and $v_n \in H_0^1(\Omega)$ (denoting v_n by v) such that

$$\|v_m\|^2_{H_0^1(\Omega)} \to t_0, \tag{3.18}$$

$$v_m \rightharpoonup v \quad \text{weakly in } H_0^1(\Omega). \tag{3.19}$$

From (3.19) and Sobolev compact embedding, we have

$$v_m \to v \quad \text{in } L^2(\Omega) \text{ and a.e. in } \Omega. \tag{3.20}$$

Let $k \in \mathbb{N}$, then for every $k \geq m$ we obtain that $W_k \subset W_m$ and

$$\begin{aligned} &M\Big(\int_\Omega |\nabla v_m|^2\Big) \int_\Omega \nabla v_m \nabla w_k \\ &= \lambda \int_\Omega \Big(a(x)(v_{m+})^s(v_{m+} + 1/\sqrt{n})^{-(\gamma+s)} + (v_{m+})^q\Big) w_k \\ &\quad - \int_\Omega f_n(v_{m+}) w_k - \frac{1}{n}\int_\Omega \phi w_k, \ \forall w_k \in W_k. \end{aligned} \tag{3.21}$$

It follows from (3.19) that

$$\int_\Omega \nabla v_m \nabla w_k \to \int_\Omega \nabla v \nabla w_k \quad \text{as } m \to \infty. \tag{3.22}$$

Also, using (3.18) and continuity of M we deduce that

$$M\Big(\int_\Omega |\nabla v_m|^2\Big) \to M(t_0) \quad \text{as } m \to \infty. \tag{3.23}$$

By (3.20),

$$a(x)(v_{m+})^s(v_{m+} + 1/\sqrt{n})^{-(\gamma+s)} \to a(x)(v_+)^s(v_+ + 1/\sqrt{n})^{-(\gamma+s)} \quad \text{a.e. in } \Omega$$

and $(v_{m+})^s|w_k| \to (v_+)^s|w_k|$ a.e. in Ω. Furthermore,

$$a(x)(v_{m+})^s(v_{m+} + 1/\sqrt{n})^{-(\gamma+s)} \leq K_1 C_n (v_{m+})^s |w_k|, \quad \forall m \in \mathbb{N},$$

and by the Sobolev compact embedding one obtains

$$\int_\Omega (v_{m+})^s |w_k| \to \int_\Omega (v_+)^s |w_k| \quad \text{as } m \to \infty.$$

Therefore, from generalized dominate convergence theorem,

$$\int_\Omega a(x)(v_{m+})^s(v_{m+} + 1/\sqrt{n})^{-(\gamma+s)} \to \int_\Omega a(x)(v_+)^s(v_+ + 1/\sqrt{n})^{-(\gamma+s)}, \tag{3.24}$$

as $m \to \infty$. Also, thanks to (3.20) and Lemma 2.1 (ii) we obtain

$$\int_\Omega f_n(v_{m+}) w_k \to \int_\Omega f_n(v_+) w_k \quad \text{as } m \to \infty. \tag{3.25}$$

Thus, by (3.22), (3.24), and (3.25), letting $m \to \infty$ we deduce that

$$\begin{aligned} &\lambda \int_\Omega \Big(a(x)(v_{m+})^s(v_{m+} + 1/\sqrt{n})^{-(\gamma+s)} + (v_{m+})^q\Big) w_k + \int_\Omega f_n(v_{m+}) w_k \\ &+ \frac{1}{n}\int_\Omega \phi w_k \\ &\to \lambda \int_\Omega \Big(a(x)(v_+)^s(v_+ + 1/\sqrt{n})^{-(\gamma+s)} + (v_+)^q\Big) w_k \\ &\quad + \int_\Omega f_n(v_+) w_k + \frac{1}{n}\int_\Omega \phi w_k. \end{aligned} \tag{3.26}$$

It follows from (3.21), (3.22), (3.23), and (3.26) that

$$\begin{aligned} M(t_0)\int_\Omega \nabla v\nabla w_k &= \lambda\int_\Omega \Big(a(x)(v_+)^s(v_+ + 1/\sqrt{n})^{-(\gamma+s)} + (v_+)^q\Big)\,w_k \\ &\quad + \int_\Omega f_n(v_+)w_k + \frac{1}{n}\int_\Omega \phi w_k,\ \forall w_k\in W_k. \end{aligned} \tag{3.27}$$

Since $[W_k]_{k\in\mathbb{N}}$ is dense in $W_0^{1,N}(\Omega)$, we derive that

$$\begin{aligned} M(t_0)\int_\Omega \nabla v\nabla w &= \lambda\int_\Omega \Big(a(x)(v_+)^s(v_+ + 1/\sqrt{n})^{-(\gamma+s)} + (v_+)^q\Big)\,w \\ &\quad + \int_\Omega f_n(v_+)w + \frac{1}{n}\int_\Omega \phi w,\ \forall w\in H_0^1(\Omega). \end{aligned} \tag{3.28}$$

We claim that $t_0 = \|v_m\|^2_{H_0^1(\Omega)}$. Indeed, taking $w = v_m$ in (3.17) we obtain

$$\begin{aligned} &M\Big(\int_\Omega |\nabla v_m|^2\Big)\int_\Omega |\nabla v_m|^2 \\ &= \lambda\int_\Omega \Big(a(x)(v_{m+})^s(v_{m+} + 1/\sqrt{n})^{-(\gamma+s)} + (v_{m+})^q\Big)\,v_m \\ &\quad + \int_\Omega f_n(v_{m+})v_m + \frac{1}{n}\int_\Omega \phi v_m, \quad \forall w\in W_m. \end{aligned} \tag{3.29}$$

So that, passing to the limit as $m\to\infty$ in (3.29), we find

$$\begin{aligned} M(t_0)t_0 &= \lambda\int_\Omega \Big(a(x)(v_+)^s(v_+ + 1/\sqrt{n})^{-(\gamma+s)} + (v_+)^q\Big)\,v \\ &\quad + \int_\Omega f_n(v_+)v + \frac{1}{n}\int_\Omega \phi v. \end{aligned} \tag{3.30}$$

Therefore, taking $w = v$ in (3.28), it follows from (3.30) that $t_0 = \|v\|^2_{H_0^1(\Omega)}$, since $M(t_0)\ge m_0 > 0$. The claim is proved.

Moreover, $v\ge 0$ a.e. in Ω. In fact since $v_-\in H_0^1(\Omega)$, by (3.28), we obtain

$$\begin{aligned} M\left(\int_\Omega |\nabla v|^2\right)\int_\Omega \nabla v\nabla v_- &= \lambda\int_\Omega \Big(a(x)(v_+)^s(v_+ + 1/\sqrt{n})^{-(\gamma+s)} + (v_+)^q\Big)\,v_- \\ &\quad - \int_\Omega f_n(v_+)v_- - \frac{1}{n}\int_\Omega \phi v_-. \end{aligned}$$

Hence,

$$\begin{aligned} -M\left(\|v\|^2_{H_0^1(\Omega)}\right)\|v_-\|^2_{H_0^1(\Omega)} &= -M\left(\|v\|^2_{H_0^1(\Omega)}\right)\int_\Omega \nabla v\nabla v_- \\ &= \int_\Omega f_n(v_+)v_- + \frac{1}{n}\int_\Omega \phi v_- \ge 0, \end{aligned}$$

then $v_- = 0$ a.e. in Ω, since $M(\|v\|^2_{H_0^1(\Omega)})\ge M(t_0) > 0$. This completes the proof. □

Proof of Lemma 3.2. Define

$$g(x) = \frac{1}{M(t_0)}\Big[\lambda\Big(a(x)v(x)^s(v(x) + 1/\sqrt{n})^{-(\gamma+s)} + v(x)^q\Big) + f_n(v(x)) + \frac{\phi(x)}{n}\Big].$$

Clearly,

$$|g| \leq \frac{1}{M(t_0)}\Big[K_1 n^{\frac{\gamma+s}{2}}|v|^s + |v|^q + |f_n(v)| + \frac{K_2}{n}\Big], \tag{3.31}$$

where $K_1 = \|a\|_{L^\infty(\Omega)}$ and $K_2 = \max\{\phi(x) : x \in \overline{\Omega}\}$. Notice that

$$|v|^s \leq 1 + |v|^{\beta-1}, \quad |v|^q \leq 1 + |v|^{\beta-1}, \tag{3.32}$$

where $2 \leq \beta \leq \frac{2N}{N-2}$. Moreover, since f_n is a Lipschitz function and $f_n(0) = 0$, we obtain $|f_n(v)| \leq C_n|v|$, and so

$$|f_n(v)| \leq C_n(1 + |v|^{\beta-1}). \tag{3.33}$$

It follows from (3.31), (3.32), and (3.33) that

$$|g| \leq C_1 + C_2|v|^{\beta-1}, \tag{3.34}$$

where

$$C_1 = \frac{1}{M(t_0)}\Big[K_1 n^{\frac{\gamma+s}{2}} + \lambda + C_n + \frac{K_2}{n}\Big], \quad C_2 = \frac{1}{M(t_0)}\Big[K_1 n^{\frac{\gamma+s}{2}} + \lambda + C_n\Big].$$

Therefore, by applying bootstrap arguments and using (3.34), similar to those found in [19], we conclude that $v \in C^{2,\alpha}(\overline{\Omega})$ for some $\alpha \in (0,1)$. The proof is complete. $\square$

Remark 3.4. Since $\phi/n \neq 0$, we deduce that $v \neq 0$ in Ω. Therefore, it follows from maximum principle that $v > 0$ in Ω, since $v \geq 0$ in Ω.

Proof of Lemma 3.3. Let $g\colon \mathbb{R} \to \mathbb{R}$ be a function defined by $g(t) = \frac{1}{M(t_0)}(\lambda t^q + f_n(t))$. Notice that g is a locally Lipschitz function because t^q and f_n are Lipschitz functions. According to Lemma 2.3 the solutions $v_n \in C^{2,\alpha}(\overline{\Omega})$ of problem (3.1) satisfy (3.2) and (3.3). $\square$

4. Proof of Theorem 1.1

We will use the existence of a unique solution z to the problem

$$\begin{gathered} -\Delta z = z^q, \quad x \in \Omega, \\ z > 0, \quad x \in \Omega, \\ z = 0, \quad x \in \partial\Omega, \end{gathered} \tag{4.1}$$

where $0 < q < 1$, to show that $v_n \geq az$ in Ω, with a a constant independent of n. This implies that the limit of the sequence v_n of solutions to the approximate problem (3.1) is positive. See for instance [5] for the details of problem (4.1).

Also, we use Lemma 2.5 together with the Hardy-Sobolev inequality, see Lemma 2.4, to show that v_n converges to a positive solution v to (1.1).

Proof of Theorem 1.1. By Lemmas 3.1 and 3.2, equation (3.1) has a solution $v_n \in C^{2,\alpha}(\overline{\Omega})$, for some $0 < \alpha < 1$, for each $n \in \mathbb{N}$. From (3.19) we know that

$$v_m \rightharpoonup v_n \quad \text{weakly in } H_0^1(\Omega) \text{ as } m \to \infty. \tag{4.2}$$

Thus

$$\|v_n\|_{H_0^1(\Omega)} \leq \liminf_{m\to\infty} \|v_m\|_{H_0^1(\Omega)} \leq r \leq 1 \quad \text{for every } n \in \mathbb{N},$$

and r does not depend on n. Therefore, up to a subsequence, there exist $0 \leq \widetilde{t}_0 \leq 1$ and $v \in H_0^1(\Omega)$ such that

$$\|v_n\|^2_{H_0^1(\Omega)} \to \widetilde{t}_0 \quad \text{as } n \to \infty, \tag{4.3}$$

$$v_n \rightharpoonup v \quad \text{weakly in } H_0^1(\Omega), \tag{4.4}$$

and, by the Sobolev embedding for $1 \le \sigma < +\infty$,

$$v_n \to v \quad \text{in } L^\sigma(\Omega) \text{ and a.e. in } \Omega. \tag{4.5}$$

Moreover, since M is continuous we obtain, by (4.3), that

$$M(\|v_n\|^2_{H_0^1(\Omega)}) \to M(\widetilde{t}_0). \tag{4.6}$$

Now, it follows from (1.5) that $0 < m_0 \le M(\|v_n\|^2_{H_0^1(\Omega)}) \le m_\infty$ and since $v_n > 0$, then by taking $\mu = \lambda m_\infty^{-1}$ we find that

$$\begin{gathered} -\Delta v_n \ge \mu v_n^q, \quad x \in \Omega, \\ v_n > 0, \quad x \in \Omega, \\ v_n = 0, \quad x \in \partial\Omega. \end{gathered}$$

Thus, by defining $z_n = \mu^{\frac{1}{1-q}} v_n$ we deduce that

$$-\Delta\Big(\frac{z_n}{\mu^{\frac{1}{1-q}}}\Big) = \mu\Big(\frac{z_n}{\mu^{\frac{1}{1-q}}}\Big)^q,$$

that is, $-\Delta z_n \ge z_n^q$. By [2, Lemma 3.3], it follows that $z_n \ge z$ for every $n \in \mathbb{N}$, implying

$$v_n \ge \mu^{\frac{1}{1-q}} z, \quad \forall n \in \mathbb{N}. \tag{4.7}$$

Letting $n \to \infty$ in (4.7), we have $v \ge \mu^{\frac{1}{1-q}} z$ in Ω. Therefore, $v > 0$ a.e. in Ω.

We claim that v is a solution of (1.1). Since $v_n \to v$ a.e. in Ω we have

$$f_n(v_n(x)) \to f(v(x)) \quad \text{a.e. in } \Omega \tag{4.8}$$

by the uniform convergence of Lemma 2.1(iii).

Observe that

$$\int_\Omega |f_n(v_n) v_n| \le C, \quad \forall n \in \mathbb{N}, \tag{4.9}$$

where $C > 0$ is a constant independent of n. Indeed, let z be a solution to (4.1). In view of the maximum principle we have

$$\frac{\partial z}{\partial \nu} < 0 \quad \text{in } \partial\Omega.$$

So that, by (4.7) and using [23, Lemma 2.6], we find

$$v_n(x) \ge \lambda^{\frac{1}{1-q}} z(x) \ge C d(x) > 0, \tag{4.10}$$

where C is a positive constant. Furthermore, using the Hardy-Sobolev inequality, Lemma 2.4, we deduce that $v_n/d^\gamma \in L^\sigma(\Omega)$ with $\frac{1}{\sigma} = \frac{1}{2} - \frac{1-\gamma}{N}$, and

$$\|\frac{v_n}{d^\gamma}\|_{L^\sigma(\Omega)} \le C\|\nabla v_n\|_{L^2(\Omega)}.$$

Using the estimate $\|v_n\|_{H_0^1(\Omega)} \le r$, we obtain $\|\frac{v_n}{d^\gamma}\|_{L^\sigma(\Omega)} \le Cr$ and so, by (4.10) and Hölder's inequality, we have

$$\begin{aligned} \int_\Omega a(x)(v_n)^s(v_n + 1/\sqrt{n})^{-(\gamma+s)} &\le K_1 \int_\Omega \frac{v_n}{(v_n)^\gamma} \le \int_\Omega \frac{v_n}{C d^\gamma} \\ &\le C_1 \Big(\int_\Omega \Big(\frac{v_n}{C d^\gamma}\Big)^\sigma \Big)^{1/\sigma} \le C, \end{aligned} \tag{4.11}$$

where $C = C_1 r$ is a constant independent of n.

Recall from (3.28) that

$$\begin{aligned} &M(\|v_n\|^2_{H_0^1(\Omega)})\int_\Omega \nabla v_n \nabla w \\ &= \lambda \int_\Omega \Big(a(x)(v_n)^s(v_n+1/\sqrt{n})^{-(\gamma+s)} + (v_n)^q\Big) w \\ &\quad + \int_\Omega f_n(v_n)w + \frac{1}{n}\int_\Omega \phi w, \quad \forall w \in H_0^1(\Omega). \end{aligned} \tag{4.12}$$

Talking $w = v_n$ in (4.12) and since v_n is bounded in $H_0^1(\Omega)$, by the Sobolev compact embedding we obtain (4.9).

By (4.8), (4.9), and the expression of f_n in (2.1), the assumptions of Lemma 2.5 are satisfied, implying $f(v) \in L^1(\Omega)$ and

$$f_n(v_n) \to f(v) \text{ in } L^1(\Omega).$$

Furthermore, since $v_n \to v$ a.e. in Ω, from (4.11) and the dominated converge theorem we have

$$\int_\Omega a(x)(v_n)^s(v_n+1/\sqrt{n})^{-(\gamma+s)}w \to \int_\Omega a(x)v^{-\gamma}w, \quad \forall w \in H_0^1(\Omega).$$

Note that, by (4.8), we have $v(x) \geq Cd(x)$ a.e. in Ω and in addition it follows from the Hardy-Sobolev inequality that $v^{-\gamma}w \in L^1(\Omega)$, since $0 < \gamma < 1$.

Finally, letting $n \to 1$ in (4.12), we have

$$M(\widetilde{t}_0)\int_\Omega \nabla v \nabla w = \lambda \int_\Omega (a(x)v^{-\gamma} + v^q)w + \int_\Omega f_n(v)w, \quad \forall w \in H_0^1(\Omega). \tag{4.13}$$

Observe that, similarly to Lemma 3.1 we can show that $\widetilde{t}_0 = \|v\|^2_{H_0^1(\Omega)}$. Thus, we conclude from (4.13) that $v \in H_0^1(\Omega)$ is a positive solution to problem (1.1). This completes the proof. □

5. Symmetric, monotonicity, and asymptotic behavior of the solutions

In this section we show Propositions 1.2 and 1.4.

Proof of Proposition 3.3. Since $v_n \to v$ a.e. in Ω, where $v \in H_0^1(\Omega)$ is a solution of (1.1), by Lemma 3.3, letting $n \to \infty$, we have $v(x_1, x') \leq v(-x_1, x')$ for every $x = (x_1, x') \in B_\varrho$ such that $x_1 > 0$. Similarly, we may show that $v(-x_1, x') \leq v(x_1, x')$. Furthermore,

$$\frac{\partial u}{\partial x_1} < 0 \quad \text{for every } x \in B_\varrho \text{ with } x_1 > 0.$$

Therefore, v is symmetric with respect to the hyperplane $x_1 = 0$ and decreasing in the direction x_1 with $x_1 > 0$, where $x = (x_1, x') \in B_\varrho$. □

Remark 5.1. If we consider f instead of f_n, we cannot apply Theorem 2.3 because the function $g(t) = \lambda t^q + f(t)$ is not necessarily Lipschitz continuous.

Proof of Proposition 1.4. It follows from definition of solution to problem (1.1), considering $\varphi = u$ as a test function, that

$$\begin{aligned} M(\|u\|_{H_0^1(\Omega)})\int_\Omega |\nabla u|^2 &= \lambda \int_\Omega a(x)(u^{1-\gamma} + u^{q+1}) + \int_\Omega f_n(u)u \\ &\leq \lambda(C_3\|u\|^{1-\gamma}_{H_0^1(\Omega)} + C_4\|u\|^{q+1}_{H_0^1(\Omega)}) + C_1\|u\|^{p+1}_{H_0^1(\Omega)}, \end{aligned} \tag{5.1}$$

where C_1 is give by (3.8).

Since $u \neq 0$, it follows from (5.1) that

$$\|u\|_{H_0^1(\Omega)}^{1+\gamma}\big[m_0 - C_1\|u\|_{H_0^1(\Omega)}^{p-2}\big] \leq \lambda(C_3 + C_4\|u\|_{H_0^1(\Omega)}^{q+\gamma}). \tag{5.2}$$

On the other hand, by the choice of r in Lemma 3.1 we deduce that

$$m_0 - C_1\|u\|_{H_0^1(\Omega)}^{p-2} \geq m_0 - C_1 r^{p-2} \geq 2.$$

Then, from (5.2) we find that $\|u\|_{H_0^1(\Omega)} \leq C\lambda^{\frac{1}{1+\gamma}}$. Therefore $\|u\|_{H_0^1(\Omega)} \to 0$ as $\lambda \to 0^+$. The proof is complete. $\square$

Acknowledgments. The author express their gratitude to the referees for their helpful suggestions.

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