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STABILITY OF BISTABLE TRAVELING WAVEFRONTS FOR A NONLOCAL DISPERSAL EPIDEMIC SYSTEM

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Abstract. This article concerns the stability of traveling wavefronts for a nonlocal dispersal epidemic system. Under a bistable assumption, we first construct a pair of upper-lower solutions and employ the comparison principle to prove that the traveling wavefronts are Lyapunov stable. Then, applying the squeezing technique combining with appropriate upper-lower solutions, we show that the traveling wavefronts are globally exponentially stable. As a corollary, the uniqueness of traveling wavefronts is obtained.

1. Introduction

In this article, we investigate the stability of traveling wavefronts of the nonlocal dispersal epidemic system

$$
\begin{aligned}
\frac{\partial u_1(x,t)}{\partial t} &= d_1\mathcal{D}_1[u_1](x,t) - \alpha u_1(x,t) + h(u_2(x,t)),\\
\frac{\partial u_2(x,t)}{\partial t} &= d_2\mathcal{D}_2[u_2](x,t) - \beta u_2(x,t) + g(u_1(x,t)),
\end{aligned}
\tag{1.1}
$$

where $x \in \mathbb{R}$, $t \in \mathbb{R}$, α and β are all positive constants, $d_i \geq 0$, $\mathcal{D}_i[\cdot]$ model nonlocal dispersal represented by the convolution operators

$$
\mathcal{D}_i[u_i](x,t) := J_i * u_i(x,t) - u_i(x,t) = \int_{\mathbb{R}} J_i(x-y)u_i(y,t)dy - u_i(x,t), \quad i = 1,2.
$$

The variables $u_1(x,t)$ and $u_2(x,t)$ respectively stand for densities of the infectious agents and the infectious human population at location x and at time t; $-\alpha u_1$ is the natural death rate of the bacterial population and $h(u_2)$ is the contribution of the infective humans to the growth rate of the bacteria; $-\beta u_2$ is the natural diminishing rate of the infective population because of the finite mean duration of the infectious population and $g(u_1)$ is the infection rate of the human population under the assumption that the total susceptible human population is constant during the evolution of the epidemic; d_1 and d_2 are diffusion coefficients.

The environmental pollution by an infective human population can lead to the spread of the infectious diseases, which is regarded as one of the main factors of relevant epidemics, such as cholera and malaria [2]. Capasso and Paveri-Fontana

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[4] proposed a model to describe the spread of cholera epidemic which happened in the European Mediterranean regions in 1973,

$$\begin{aligned} \frac{du_1(t)}{dt} &= -\alpha u_1(t) + a u_2(t), \\ \frac{du_2(t)}{dt} &= -\beta u_2(t) + g(u_1(t)), \end{aligned} \tag{1.2}$$

where $a > 0$ is a constant.

By considering the mobility of the bacteria and neglecting the small mobility of the infectious population, Capasso and Maddalena [3] gave the system

$$\begin{aligned} \frac{\partial u_1(x,t)}{\partial t} &= d\frac{\partial^2 u_1(x,t)}{\partial x^2} - \alpha u_1(x,t) + a u_2(x,t), \\ \frac{\partial u_2(x,t)}{\partial t} &= -\beta u_2(x,t) + g(u_1(x,t)). \end{aligned} \tag{1.3}$$

Zhao and Wang [33] established the existence of monotone traveling waves and the minimal wave speed of (1.3) with monostable nonlinearity. Xu and Zhao [26] proved the existence, uniqueness and global exponential stability of traveling waves of (1.3) with bistable nonlinearity.

In 2012, Hsu and Yang [10] studied the epidemic system

$$\begin{aligned} \frac{\partial u_1(x,t)}{\partial t} &= d_1\frac{\partial^2 u_1(x,t)}{\partial x^2} - \alpha u_1(x,t) + h(u_2(x,t)), \\ \frac{\partial u_2(x,t)}{\partial t} &= d_2\frac{\partial^2 u_2(x,t)}{\partial x^2} - \beta u_2(x,t) + g(u_1(x,t)), \end{aligned} \tag{1.4}$$

and proved the existence, uniqueness, monotonicity and asymptotic behaviour of traveling wave solutions of (1.4) under the monostable assumptions. Wu and Hsu [22] investigated the existence of entire solutions for delayed monostable epidemic models (1.4) with and without the quasi-monotone condition. We also refer readers to [28] for existence and stability of traveling waves of (1.4) with discrete diffusion.

Note that the Laplacian operator $\frac{\partial^2 u}{\partial x^2}$, which is used to describe the diffusion of the infectious agents in (1.3) and (1.4) only depicts a local and short range diffusion process. However, in reality, the migration or diffusion of the individuals is not just limited in a local or short range, more details can refer Lee et al. [11] and Murray [16]. So it is not enough or very accurate to formulate the diffusion process of individuals in a long range by Laplacian operator. To consider the spatial migration and describe this model reasonably, the following nonlocal operator

$$(\mathcal{D}u)(x,t) = (J*u)(x,t) - u(x,t) = \int_{\mathbb{R}} J(x-y)[u(y,t) - u(x,t)]dy$$

was introduced in [7, 32, 29, 31]. For example, Zhang and Wang [31] proposed the nonlocal dispersal epidemic system with time delay

$$\begin{aligned} \frac{\partial u_1(x,t)}{\partial t} &= d(J*u_1 - u_1)(x,t) - \alpha u_1(x,t) + a u_2(x,t), \\ \frac{\partial u_2(x,t)}{\partial t} &= -\beta u_2(x,t) + g(u_1(x,t-\tau)). \end{aligned} \tag{1.5}$$

In the quasi-monotone monostable case, Zhang and Wang [31] proved the existence of traveling wavefronts of (1.5) in both isotropic dispersal case and anisotropic dispersal case by constructing appropriate upper and lower solutions. Later on, Zhang, Li and Wu [32] considered system (1.5) without delay, and studied the multi-type entire solutions, when g is monotone increasing monostable and bistable nonlinearity, respectively. More recently, Zhang, Li and Feng [29] proved the stability of traveling waves of (1.5) in the quasi-monotone monostable case and the non-quasi-monotone monostable case. To our best knowledge, the stability of traveling waves of (1.1) and (1.5) is not addressed before in the bistable case.

In this article, we shall focus our attention on the stability of bistable traveling wavefronts of system (1.1). Our main assumptions are as follows:

- (A1) $J_i \in C^1(\mathbb{R},\mathbb{R})$, $J_i(x) = J_i(-x) \geq 0$, $x \in \mathbb{R}$ and $\int_{\mathbb{R}} J_i(x)dx = 1$.
- (A2) For every $\lambda \in \mathbb{R}$, $\int_{\mathbb{R}} J_i(x)e^{-\lambda x}dx < +\infty$.
- (A3) $g, h \in C^1(\mathbb{R}_+,\mathbb{R}_+)$, $g(0) = h(0) = 0$, $v_i^* = g(u_i^*)/\beta$, $h(g(u_i^*)/\beta) = \alpha u_i^*$, $i = 1, 2$, where $u_1^* < u_2^*$ are two positive constants.
- (A4) $g'(0)h'(0) < \alpha\beta$, $g'(u_2^*)h'(v_2^*) < \alpha\beta$ and $g'(u_1^*)h'(v_1^*) > \alpha\beta$.
- (A5) $g'(0) = 0$, $h'(0) = 0$, $g'(u) > 0$ for $u \in (0, u_2^*]$, $h'(v) > 0$ for $v \in (0, v_2^*]$.

A typical example of g and h satisfying the conditions (A3)-(A5) is $g(u) = \frac{u^2}{1+u^2}$ and $h(v) = av^{3/2}$ with $a > 0$. The spatially homogeneous system associated with system (1.1) is written as follows

$$
\begin{aligned}
\frac{du_1(t)}{dt} &= -\alpha u_1(t) + h(u_2(t)),\\
\frac{du_2(t)}{dt} &= -\beta u_2(t) + g(u_1(t)).
\end{aligned} \tag{1.6}
$$

By (A3), this system has three equilibria $E^- = (0, 0)$, $E^0 = (u_1^*, v_1^*)$, and $E^+ = (u_2^*, v_2^*)$. By (A4), E^0 is a saddle point, E^- and E^+ are stable nodes, and hence, the system (1.1) is a bistable system.

A traveling wave solution (in short, traveling wave) of (1.1) has the special form $(u_1(x,t), u_2(x,t)) = (\phi_1(\xi), \phi_2(\xi))$, $\xi = x + ct$, where $c \in \mathbb{R}$ is the wave speed and $(\phi_1(\xi), \phi_2(\xi))$ is the wave profile. Moreover, we say $(\phi_1(\xi), \phi_2(\xi))$ is a traveling wavefront if $(\phi_1(\xi), \phi_2(\xi))$ is monotone in $\xi \in \mathbb{R}$. We want to find the traveling wavefronts of (1.1) connecting E^- and E^+. It is well known that (1.1) has a traveling wave solution $\phi(\xi) = (\phi_1(\xi), \phi_2(\xi))$ which connects E^- and E^+ if and only if $\phi(\xi)$ satisfies the wave profile system

$$
\begin{aligned}
c\phi_1'(\xi) &= d_1\mathcal{D}_1[\phi_1](\xi) - \alpha\phi_1(\xi) + h(\phi_2(\xi)),\\
c\phi_2'(\xi) &= d_2\mathcal{D}_2[\phi_2](\xi) - \beta\phi_2(\xi) + g(\phi_1(\xi)),
\end{aligned} \tag{1.7}
$$

with

$$
\lim_{\xi\to-\infty}(\phi_1(\xi), \phi_2(\xi)) = E^-, \quad \lim_{\xi\to+\infty}(\phi_1(\xi), \phi_2(\xi)) = E^+. \tag{1.8}
$$

By the abstract theory in Fang and Zhao [8], we know that under assumptions (A1)–(A5), there exists a unique constant $c \in \mathbb{R}$ such that (1.1) has a unique increasing traveling wavefront $\phi(\cdot) = (\phi_1(\cdot), \phi_2(\cdot))$ connecting E^- and E^+.

As mentioned before, the main goal of this article is to show the stability of traveling wavefronts of (1.1). Hence, we need to provide further details on the progress of stability of traveling waves in this direction. The stability of traveling wave solutions of systems with Laplace diffusions and nonlocal dispersals has been well studied in the past few years. We refer to [6, 9, 12, 13, 14, 15, 17, 18, 19, 20, 23, 24, 25, 26] for Laplace diffusions and [1, 27, 29, 30] for nonlocal dispersals. As we know, there are three classical methods which have been used to prove the stability of traveling wave solutions. The first one is spectral analysis, see [17, 24] for the non-critical speed case and [9, 25] for the critical speed case. The second one is the method of weighted energy together with the comparison principle, see [12, 14, 15, 27, 30] and the references cited therein. The third one is the squeezing technique developed by Chen [5]. We can refer to [13, 18, 20] for bistable equations. Motivated by [5, 13, 18, 20], we generalize the squeezing technique to nonlocal dispersal system (1.1) for proving the global stability of bistable traveling wavefronts. We should remark that the method used here can be also applied to (1.5) and the global stability of bistable traveling wavefronts can be similarly obtained.

This article is organized as follows. In Section 2, we study the comparison principle of the solutions of the initial value problem corresponding to (1.1), and then show some properties of traveling wavefronts of (1.1). In Section 3, we prove the Lyapunov stability traveling wavefronts of (1.1). In Section 4, we establish the global stability and uniqueness of traveling wavefronts of (1.1).

2. Preliminaries

In this section, we present two lemmas that will be useful later. In view of [21, Lemma 3.1], the solution semigroup of the following nonlocal dispersal equation

$$\begin{gathered}\frac{\partial u(x,t)}{\partial t} = (J*u-u)(x,t), \quad x\in\mathbb{R},\ t>0,\\ u(x,0)=\hat{\varphi}(x), \quad x\in\mathbb{R}.\end{gathered}$$

is given by

$$P(t)[\hat{\varphi}](x) = e^{-t}\sum_{m=0}^{\infty}\frac{t^m}{m!}a_m(\hat{\varphi})(x), \tag{2.1}$$

where $a_0(\hat{\varphi})(x)=\hat{\varphi}(x)$, $a_m(\hat{\varphi})(x)=\int_{\mathbb{R}} J(x-y)a_{m-1}(\hat{\varphi})(y)dy$, for all $m\ge 1$.

Let $\chi = BUC(\mathbb{R},\mathbb{R}^2)$ be a Banach space of bounded and uniformly continuous vector-valued function from $\mathbb{R}$ to $\mathbb{R}^2$ with the general norm $\|\cdot\|$ and

$$\chi_1 = \{\varphi(x)\in\chi : E^- \le \varphi(x)\le E^+,\ \forall x\in\mathbb{R}\},$$

where $\varphi(x) = (\varphi_1(x),\varphi_2(x))$. Define

$$T_i(t)[\cdot](x) = P(d_it)[\cdot](x), \quad x\in\mathbb{R},\ t>0,\ i=1,2,$$

where $P(d_it)$ is defined as in (2.1) with $J=J_i$, $i=1,2$. Let $T(t)=(T_1(t),T_2(t))$. It is easy to see that $T(t):\chi_1\to\chi_1$ is a positive and analytic semigroup.

Now we consider the initial value problem

$$\begin{gathered}\frac{\partial u_1(x,t)}{\partial t} = d_1\mathcal{D}_1[u_1](x,t)-\alpha u_1(x,t)+h(u_2(x,t)),\\ \frac{\partial u_2(x,t)}{\partial t} = d_2\mathcal{D}_2[u_2](x,t)-\beta u_2(x,t)+g(u_1(x,t)),\\ u_1(x,0)=\varphi_1(x), \quad u_2(x,0)=\varphi_2(x),\ x\in\mathbb{R}.\end{gathered} \tag{2.2}$$

Integrating the first two equations of (2.2) with $\varphi(x) := (\varphi_1(x),\varphi_2(x))^T$, it can be derived that the initial value problem (2.2) is equivalent to the integral equation

$$u(x,t) = T(t)\varphi(x) + \int_0^t T(t-s)Q(u(x,s))ds, \tag{2.3}$$

where

$$u(x,t) := \begin{pmatrix} u_1(x,t) \\ u_2(x,t) \end{pmatrix}, \quad T(t) := \begin{pmatrix} T_1 & 0 \\ 0 & T_2 \end{pmatrix},$$
$$Q(u(x,t)) := \begin{pmatrix} Q_1(u(x,t)) \\ Q_2(u(x,t)) \end{pmatrix} = \begin{pmatrix} -\alpha u_1(x,t) + h(u_2(x,t)) \\ -\beta u_2(x,t) + g(u_1(x,t)) \end{pmatrix}.$$

Definition 2.1. A continuous function $u(x,t) = (u_1(x,t), u_2(x,t)) : \mathbb{R} \times [\tau, T] \to \mathbb{R}^2$, $\tau < T$, is called an upper (lower) solution of system (1.1) on $[\tau, T)$ if

$$u(x,t) \geq (\leq) T(t-s)u(x,s) + \int_s^t T(t-r)Q(u(x,r))ds,$$

for any $\tau \leq s < t < T$.

Remark 2.2. If a continuous function $u(x,t) = (u_1(x,t), u_2(x,t))$ is C^1 with $t \geq 0$ and satisfies

$$\begin{aligned} \frac{\partial u_1(x,t)}{\partial t} &\geq (\leq) d_1 \mathcal{D}_1[u_1](x,t) - \alpha u_1(x,t) + h(u_2(x,t)), \\ \frac{\partial u_2(x,t)}{\partial t} &\geq (\leq) d_2 \mathcal{D}_2[u_2](x,t) - \beta u_2(x,t) + g(u_1(x,t)), \end{aligned} \tag{2.4}$$

then $u(x,t)$ is an upper (lower) solution of (1.1).

Now we give the first lemma, i.e., the strong comparison principle.

Lemma 2.3. *Let $u(x,t) = (u_1(x,t), u_2(x,t))$ and $v(x,t) = (v_1(x,t), v_2(x,t))$ be two solutions of (2.2) with $u(x,0) = \varphi(x)$ and $v(x,0) = \tilde{\varphi}(x)$, respectively, where*

$$\varphi(x) = (\varphi_1(x), \varphi_2(x)); \ \tilde{\varphi}(x) = (\tilde{\varphi}_1(x), \tilde{\varphi}_2(x)) \in C(\mathbb{R}, \mathbb{R}^+),$$

with $E^- \leq \tilde{\varphi}(x) \leq \varphi(x) \leq E^+$, $x \in \mathbb{R}$. Then for any $(x,t) \in \mathbb{R} \times (0,\infty)$,

$$E^- \leq v(x,t) \leq u(x,t) \leq E^+$$

and

$$u_i(x,t) - v_i(x,t) \geq \mathcal{N}_i(L, t-t_0) \int_y^{y+1} (u_i(z,t_0) - v_i(z,t_0))dz \geq 0, \tag{2.5}$$

for $L \geq 0$, $x, y \in \mathbb{R}$ satisfying $|x-y| \leq L$ and $t > t_0 \geq 0$, $i = 1,2$, where

$$\mathcal{N}_1(L, t-t_0) = C_1(L)e^{-(\alpha+d_1)(t-t_0)}(t-t_0),$$
$$\mathcal{N}_2(L, t-t_0) = C_2(L)e^{-(\beta+d_2)(t-t_0)}(t-t_0),$$

where $C_i(L) = \min_{x \in [-L-1, L+1]} J_i(x)$, $i = 1,2$.

Proof. The first assertion of the lemma can be proved by the properties of the monotone semiflow, see [7, 27]. So we omit it here. We shall prove that the inequality (2.5) holds.

Let $w_i(x,t) = u_i(x,t) - v_i(x,t)$, $i = 1,2$. Then $w_i(x,t) \geq 0$ for $(x,t) \in \mathbb{R} \times (0, +\infty)$. For any given $0 \leq t_0 < t$ and $x, y \in \mathbb{R}$ satisfying $|x-y| \leq L$, it follows that

$$\begin{aligned} &w_1(x,t) \\ &= T_1(t-t_0)w_1(x,t_0) + \int_{t_0}^t T_1(t-r)[-\alpha w_1(x,r) + h(u_2(x,r)) - h(v_2(x,r))]dr \end{aligned}$$

$$
\begin{aligned}
&= T_1(t-t_0)w_1(x,t_0) + \int_{t_0}^{t} T_1(t-r)[-\alpha w_1(x,r) + h'(\xi)w_2(x,r)]dr \\
&\geq T_1(t-t_0)w_1(x,t_0) - \alpha \int_{t_0}^{t} T_1(t-r)w_1(x,r)dr.
\end{aligned}
$$

Let

$$
z(x,t) = e^{-\alpha(t-t_0)}T_1(t-t_0)w_1(x,t_0), \quad t \geq t_0.
$$

Then $z(x,t)$ is the solution of the nonlocal dispersal equation

$$
\frac{\partial u}{\partial t} = d_1(J_1 * u - u) - \alpha u.
$$

Thus,

$$
z(x,t) = T_1(t-t_0)z(x,t_0) - \alpha \int_{t_0}^{t} T_1(t-r)z(x,r)dr, \ t \geq t_0.
$$

It then follows that $w_1(x,t) \geq z(x,t), \ t \geq t_0$, and hence,

$$
\begin{aligned}
w_1(x,t) &\geq e^{-\alpha(t-t_0)}T_1(t-t_0)w_1(x,t_0) \\
&= e^{-\alpha(t-t_0)}e^{-d_1(t-t_0)} \sum_{m=0}^{\infty} \frac{(t-t_0)^m}{m!} a_m[w_1(x,t_0)] \\
&= e^{-(\alpha+d_1)(t-t_0)} \sum_{m=0}^{\infty} \frac{(t-t_0)^m}{m!} \int_{\mathbb{R}} J_1(x-y)a_{m-1}[w_1(y,t_0)]dy \\
&\geq e^{-(\alpha+d_1)(t-t_0)}(t-t_0) \int_{y}^{y+1} J_1(x-z)w_1(z,t_0)dz \\
&\geq C_1(L)e^{-(\alpha+d_1)(t-t_0)}(t-t_0) \int_{y}^{y+1} w_1(z,t_0)dz,
\end{aligned}
$$

where $C_1(L) = \min_{x\in[-L-1,L+1]} J_1(x)$.

Similarly, one has

$$
\begin{aligned}
&w_2(x,t) \\
&= T_2(t-t_0)w_2(x,t_0) + \int_{t_0}^{t} T_2(t-r)[-\beta w_2(x,r) + g(u_1(x,r)) - g(v_1(x,r))]dr \\
&= T_1(t-t_0)w_2(x,t_0) + \int_{t_0}^{t} T_2(t-r)[-\beta w_2(x,r) + g'(\xi)w_1(x,r)]dr \\
&\geq T_2(t-t_0)w_2(x,t_0) - \beta \int_{t_0}^{t} T_2(t-r)w_2(x,r)dr,
\end{aligned}
$$

and then

$$
\begin{aligned}
w_2(x,t) &\geq e^{-\beta(t-t_0)}T_2(t-t_0)w_2(x,t_0) \\
&\geq e^{-(\beta+d_2)(t-t_0)}(t-t_0) \int_{\mathbb{R}} J_2(x-y)w_2(y,t_0)dy \\
&\geq C_2(L)e^{-(\beta+d_2)(t-t_0)}(t-t_0) \int_{y}^{y+1} w_2(z,t_0)dz,
\end{aligned}
$$

where $C_2(L) = \min_{x\in[-L-1,L+1]} J_2(x)$. The proof is complete. □

The second lemma is about the limit behavior of the derivative of wave profiles at $\pm\infty$. For the convenience, in what follows of this paper, we always denote $E^+ = (u_2^*, v_2^*) := (k_1, k_2)$.

Lemma 2.4. *Let $\phi(\xi) = (\phi_1(\xi), \phi_2(\xi))$ be any traveling wavefront of (1.1) satisfying $0 \le \phi_i(\xi) \le k_i$. Then $\lim_{|\xi|\to+\infty} \phi'(\xi) = 0$.*

3. Lyapunov stability of traveling wavefronts

In this section, we prove the Lyapunov stability of traveling wavefronts of (1.1). By (A3)–(A5), we can find sufficiently small constants $p_i > 0, i = 1, 2$, such that

$$\alpha p_1 > \rho p_2, \quad \beta p_2 > \rho p_1, \tag{3.1}$$

where $\rho = \max\{\rho_1, \rho_2\}$ with

$$\rho_1 = \max\big\{h'(x) \ge 0 : x \in [0, p_2] \cup [k_2 - p_2, k_2]\big\} > 0,$$
$$\rho_2 = \max\big\{g'(x) \ge 0 : x \in [0, p_1] \cup [k_1 - p_1, k_1]\big\} > 0.$$

We construct an upper solution and a lower solution of (1.1).

Lemma 3.1. *Assume that (A1)–(A5) hold. Let $\phi(x+ct) = (\phi_1(x+ct), \phi_2(x+ct))$ be a traveling wavefront of (1.1). Define $w^\pm(x,t) = (w_1^\pm(x,t), w_2^\pm(x,t))$ by*

$$w_i^+(x,t) = \min\big\{\phi_i(\eta^+(x,t)) + \delta p_i e^{-\beta_0 t}, k_i\big\},$$
$$w_i^-(x,t) = \max\big\{\phi_i(\eta^-(x,t)) - \delta p_i e^{-\beta_0 t}, 0\big\},$$

where $\eta^\pm(x,t) = x + ct + \xi_0 \pm \sigma_0\delta(1 - e^{-\beta_0 t})$, $i = 1, 2$. Then there exist $\sigma_0 > 0$, $\beta_0 > 0$, $\delta_0 > 0$ such that for any $\delta \in (0, \delta_0]$ and every ξ_0, $w^+(x,t)$ and $w^-(x,t)$ are an upper solution and a lower solution of (1.1), respectively.

Proof. We only verify that $w^+(x,t)$ is an upper solution of (1.1), since the lower solution $w^-(x,t)$ can be treated similarly. Note that when $w_i^+(x,t) = k_i$ for $i = 1, 2$, it is easy to see that w_i^+ satisfies (2.4). Hence, in what follows, we consider the case $w_i^+(x,t) = \phi_i(\eta^+(x,t)) + \delta p_i e^{-\beta_0 t}$.

For simplicity, we denote $\eta^+(x,t)$ by η. Let

$$\mu := \min\Big\{\frac{\alpha p_1 - \rho p_2}{p_1}, \frac{\beta p_2 - \rho p_1}{p_2}\Big\} > 0.$$

Fix $\beta_0 \in (0, \mu)$ and $\delta^* \in (0, p_0)$ with $p_0 := \max\{p_1, p_2\}$. Then there exists $M = M(\Phi, \beta_0, \delta^*) > 0$ large enough such that

$$\phi_i(\eta) + \delta p_i \ge k_i - \delta^*, \quad \forall \delta \in (0, \delta^*],\ \eta \ge M,$$
$$\phi_i(\eta) - \delta p_i \le \delta^*, \quad \forall \delta \in (0, \delta^*],\ \eta \le -M,\ i = 1, 2.$$

We can directly calculate that

$$\frac{\partial w_i^+(x,t)}{\partial t} = c\phi_i'(\eta) + \beta_0\sigma_0\delta e^{-\beta_0 t}\phi_i'(\eta) - \beta_0\delta p_i e^{-\beta_0 t}.$$

By (2.4), we need to prove that

$$\begin{aligned} \frac{\partial w_1^+(x,t)}{\partial t} &\ge d_1\mathcal{D}_1[w_1^+] - \alpha w_1^+ + h(w_2^+), \\ \frac{\partial w_2^+(x,t)}{\partial t} &\ge d_2\mathcal{D}_2[w_2^+] - \beta w_2^+ + g(w_1^+). \end{aligned} \tag{3.2}$$

The first inequality of (3.2) holds if and only if

$$\begin{aligned}
&c\phi_1'(\eta)+\beta_0\sigma_0\delta e^{-\beta_0 t}\phi_1'(\eta)-\beta_0\delta p_1 e^{-\beta_0 t}\\
&\geq d_1\mathcal{D}_1[\phi_1(\eta)+\delta p_1 e^{-\beta_0 t}]-\alpha[\phi_1(\eta)+\delta p_1 e^{-\beta_0 t}]+h(w_2^+)\\
&= d_1\big\{J_1*(\phi_1(\eta)+\delta p_1 e^{-\beta_0 t})-(\phi_1(\eta)+\delta p_1 e^{-\beta_0 t})\big\}\\
&\quad -\alpha[\phi_1(\eta)+\delta p_1 e^{-\beta_0 t}]+h(w_2^+)\\
&= d_1[J_1*\phi_1(\eta)-\phi_1(\eta)]-\alpha\phi_1(\eta)-\alpha\delta p_1 e^{-\beta_0 t}+h(w_2^+)\\
&= d_1\mathcal{D}_1[\phi_1](\eta)-\alpha\phi_1(\eta)-\alpha\delta p_1 e^{-\beta_0 t}+h(w_2^+)+h(\phi_2(\eta))-h(\phi_2(\eta)).
\end{aligned}$$

In view of (1.7), we only need to verify

$$\begin{aligned}
&\delta e^{-\beta_0 t}[\beta_0\sigma_0\phi_1'(\eta)-\beta_0 p_1]\geq \delta e^{-\beta_0 t}(-\alpha p_1)+[h(w_2^+)-h(\phi_2(\eta))]\\
&\Leftrightarrow\ \beta_0\sigma_0\phi_1'(\eta)-\beta_0 p_1\geq -\alpha p_1+\delta^{-1}e^{\beta_0 t}[h(w_2^+)-h(\phi_2(\eta))]\\
&\Leftrightarrow\ \beta_0\sigma_0\phi_1'(\eta)-\beta_0 p_1\geq -\alpha p_1+\delta^{-1}e^{\beta_0 t}h'(\theta)\delta p_2 e^{-\beta_0 t}=-\alpha p_1+h'(\theta)p_2,
\end{aligned}$$

where $\theta\in[\phi_2(\eta),w_2^+(x,t)]$. For $|\eta|\geq M$, by the choice of M, it suffices to show that

$$\sigma_0\beta_0\phi_1'(\eta)-\beta_0 p_1\geq -\alpha p_1+\rho p_2.$$

Obviously, the above inequality holds by $\phi_1'(\eta)>0$ and the choice of β_0. Since $\phi_i'(\eta)>0$, $i=1,2$, for $|\eta|\leq M$, we let

$$m_0:=\min_{i=1,2}\min\{\phi_i'(\eta)||\eta|\leq M\}>0.$$

In addition, we take

$$\begin{gathered}
s_1:=\max\{h'(\xi)\big|\xi\in[0,k_2]\},\quad s_2:=\max\{g'(\xi)\big|\xi\in[0,k_1]\},\\
s_0:=\max\{s_1,s_2\},\quad \sigma_0:=\frac{p_0(\beta_0+s_0)}{m_0\beta_0}>0,\quad \delta_0:=\min\big\{\delta^*,\frac{1}{\sigma_0}\big\}.
\end{gathered}$$

Then for $|\eta|\leq M$, we obtain

$$\sigma_0\beta_0\phi_1'(\eta)-\beta_0 p_1+\alpha p_1-h'(\theta)p_2\geq \sigma_0\beta_0 m_0-p_0(\beta_0+s_0)=0.$$

The second inequality of (3.2) holds if and only if

$$\begin{aligned}
&c\phi_2'(\eta)+\beta_0\sigma_0\delta e^{-\beta_0 t}\phi_2'(\eta)-\beta_0\delta p_2 e^{-\beta_0 t}\\
&\geq d_2\mathcal{D}_2[\phi_2(\eta)+\delta p_2 e^{-\beta_0 t}]-\beta[\phi_2(\eta)+\delta p_2 e^{-\beta_0 t}]+g(w_1^+)\\
&= d_2\big[J_2*(\phi_2(\eta)+\delta p_2 e^{-\beta_0 t})-(\phi_2(\eta)+\delta p_2 e^{-\beta_0 t})\big]\\
&\quad -\beta[\phi_2(\eta)+\delta p_2 e^{-\beta_0 t}]+g(w_1^+)\\
&= d_2\big\{[J_2*\phi_2(\eta)-\phi_2(\eta)]+\delta e^{-\beta_0 t}[J_2*p_2-p_2]\big\}-\beta\phi_2(\eta)-\beta\delta p_2 e^{-\beta_0 t}+g(w_1^+)\\
&= d_2\mathcal{D}_2[\phi_2](\eta)-\beta\phi_2(\eta)-\beta\delta p_2 e^{-\beta_0 t}+g(w_1^+)+g(\phi_1(\eta))-g(\phi_1(\eta)).
\end{aligned}$$

By (1.7), we only need to verify that

$$\begin{aligned}
&\delta e^{-\beta_0 t}[\beta_0\sigma_0\phi_2'(\eta)-\beta_0 p_2]\geq \delta e^{-\beta_0 t}(-\beta p_2)+[g(w_1^+)-g(\phi_1(\eta))]\\
&\Leftrightarrow\ \beta_0\sigma_0\phi_2'(\eta)-\beta_0 p_2\geq -\beta p_2+\delta^{-1}e^{\beta_0 t}[g(w_1^+)-g(\phi_1(\eta))]\\
&\Leftrightarrow\ \beta_0\sigma_0\phi_2'(\eta)-\beta_0 p_2\geq -\beta p_2+\delta^{-1}e^{\beta_0 t}g'(\theta)\delta p_1 e^{-\beta_0 t}=-\beta p_2+g'(\theta)p_1,
\end{aligned}$$

where $\theta \in [\phi_1(\eta), w_1^+(x,t)]$. For $|\eta| \geq M$, by the choice of M, it is equivalent to show that

$$\sigma_0\beta_0\phi_2'(\eta) - \beta_0 p_2 \geq -\beta p_2 + \rho p_1.$$

It is easy to see that the above inequality holds due to $\phi_2'(\eta) \geq 0$ and the choice of β_0. For $|\eta| \leq M$, since $\phi_i'(\eta) > 0, i = 1,2$, taking σ_0 as above, we obtain

$$\sigma_0\beta_0\phi_2'(\eta) - \beta_0 p_2 + \beta p_2 - g'(\theta)p_1 \geq m_0\sigma_0\beta_0 - p_0[\beta_0 + s_0] = 0.$$

The proof is complete. □

With the help of the upper and lower solution in Lemma 3.1 and the comparison principle, we can observe the following Lyaponov stability theorem.

Theorem 3.2 (Lyapunov Stability)**.** *Assume that* (A1)–(A5) *hold. Let* $u(x,t;\varphi)$ *be the solution of* (1.1) *with the initial value* φ. *Then the traveling wavefront* (ϕ, c) *of* (1.1) *is Lyapunov stable in the sense that for any* $\varepsilon > 0$, *there exists* $\delta > 0$ *such that* $\|u(\cdot,t;\varphi) - \phi(\cdot + ct)\| < \varepsilon$ *provided that* $\|\varphi - \phi\| \leq \delta$.

Proof. It is clear that $\phi_i'(\xi)$ is continuous in $\xi \in \mathbb{R}$, $i = 1,2$. Then by Lemma 2.4, we obtain that $\phi_i(\cdot)$ is uniformly continuous on $\mathbb{R}$. Hence, for any $\varepsilon > 0$, there exists $\delta_1 = \delta_1(\varepsilon) > 0$ such that for any $|y| \leq \delta_1$, it holds

$$|\phi_i(\cdot + y) - \phi_i(\cdot)| < \frac{\varepsilon}{4}. \tag{3.3}$$

Choose $\delta = \delta(\varepsilon) \in (0, \min\{\frac{\varepsilon}{4(1+\max\{p_1,p_2\})}, \frac{\delta_1}{\sigma_0}, \delta_0\})$, where σ_0 and δ_0 are given in Lemma 3.1. Then for every $\varphi := (\varphi_1, \varphi_2)$ with $\|\varphi - \phi\| < \delta$, we have

$$\max\{\phi_i(x) - \delta p_i, 0\} \leq \varphi_i(x) \leq \min\{\phi_i(x) + \delta p_i, k_i\}, \quad \forall x \in \mathbb{R},\ i = 1,2.$$

By Lemma 3.1 and the comparison principle, one has

$$\begin{aligned} &\max\{\phi_i(\eta^-(x,t)) - \delta p_i e^{-\beta_0 t}, 0\} \\ &\leq u_i(x,t;\varphi_i) \\ &\leq \min\{\phi_i(\eta^+(x,t)) + \delta p_i e^{-\beta_0 t}, k_i\}, \quad i = 1,2, \end{aligned} \tag{3.4}$$

where $x \in \mathbb{R}$, $t \geq 0$, and $\eta^\pm(x,t) := x + ct \pm \sigma_0\delta(1 - e^{-\beta_0 t})$. It is easy to see that

$$|\pm\sigma_0\delta(1 - e^{-\beta_0 t})| \leq \sigma_0\delta < \delta_1, \quad \forall t \geq 0.$$

Hence, combining with (3.3) and (3.4), we obtain

$$\phi_i(x + ct) - \frac{\varepsilon}{2} \leq u_i(x,t;\varphi_i) \leq \phi_i(x + ct) + \frac{\varepsilon}{2}, \quad \forall x \in \mathbb{R},\ t \geq 0,\ i = 1,2,$$

which implies that

$$\|u(\cdot,t;\varphi) - \phi(\cdot + ct)\| < \varepsilon, \quad \forall t \geq 0.$$

The proof is complete. □

4. Global stability of traveling wavefronts

In this section, we establish the global stability of traveling wavefronts of (1.1) by applying the squeezing technique, and then show the uniqueness of traveling wavefronts. To obtain the global stability of traveling wavefronts, we construct another pair of upper and lower solutions which are different from that in Lemma 3.1. Let $\zeta(\cdot) \in C^\infty(\mathbb{R})$ be a fixed function with the following properties:

$$\begin{gathered}\zeta(s) = 0 \quad \forall s \in (-\infty, 0]; \quad \zeta(s) = 1, \quad \forall s \in [4, \infty);\\ 0 < \zeta'(s) < 1, \quad |\zeta''(s)| \le 1, \quad \forall s \in (0, 4).\end{gathered}$$

We define $v^\pm(x,t) = (v_1^\pm(x,t), v_2^\pm(x,t))$ by

$$\begin{aligned}v_i^+(x,t) &= \min\big\{k_i + \delta p_i - [k_i - (1-2\delta)p_i e^{-\varepsilon t}]\zeta(\varsigma^+_{\varepsilon,C}(x,t)), k_i\big\},\\ v_i^-(x,t) &= \max\big\{-\delta p_i + [k_i - (1-2\delta)p_i e^{-\varepsilon t}]\zeta(\varsigma^-_{\varepsilon,C}(x,t)), 0\big\},\end{aligned}$$

where $\zeta(\varsigma^\pm_{\varepsilon,C}(x,t) = \mp\varepsilon(x - \xi \pm Ct)$, $i = 1, 2$.

Lemma 4.1. *Assume that* (A1)–(A5) *hold. Then, for any* $\delta \in (0, 1/2]$*, there exist two positive numbers* $\varepsilon = \varepsilon(\delta)$ *and* $C = C(\delta)$ *such that, for any* $\xi \in \mathbb{R}$*, the functions* $v^+(x,t)$ *and* $v^-(x,t)$ *are an upper solution and a lower solution of* (1.1)*, respectively.*

Proof. We only prove that $v^+(x,t)$ is an upper solution of (1.1), since the lower solution $v^-(x,t)$ can be showed in a similar way. We define

$$\begin{aligned}\mathcal{L}_1[v_1^+] &:= \frac{\partial v_1^+}{\partial t} - d_1\mathcal{D}_1 v_1^+ + \alpha v_1^+ - h(v_2^+),\\ \mathcal{L}_2[v_2^+] &:= \frac{\partial v_2^+}{\partial t} - d_2\mathcal{D}_2 v_2^+ + \beta v_2^+ - g(v_1^+).\end{aligned}$$

For simplicity, set $\nu = x - \xi + Ct$. It is easy to calculate that

$$\begin{aligned}\frac{\partial v_i^+}{\partial t} &= \varepsilon C[k_i - (1-2\delta)p_i e^{-\varepsilon t}]\zeta'(-\varepsilon\nu) - \varepsilon(1-2\delta)p_i e^{-\varepsilon t}\zeta(-\varepsilon\nu)\\ &\ge \varepsilon C(k_i - p_i)\zeta'(-\varepsilon\nu) - k_i\varepsilon, \ i = 1, 2.\end{aligned}$$

In view of (3.1), we take $\varepsilon = \varepsilon(\delta) > 0$ sufficiently small such that

$$\begin{aligned}&-k_1\varepsilon - d_1k_1\varepsilon\int_{\mathbb{R}} J_1(y)|y|dy + \delta(\alpha p_1 - \rho p_2) > 0,\\ &-k_2\varepsilon - d_2k_2\varepsilon\int_{\mathbb{R}} J_2(y)|y|dy + \delta(\beta p_2 - \rho p_1) > 0.\end{aligned} \tag{4.1}$$

Since p_1, p_2 are small enough such that $\frac{\delta p_1}{2k_1} + \frac{\delta p_2}{2k_2} < 1$ and $\zeta'(s) > 0$ for $\zeta(s) \in (0,1)$, we can choose $C = C(\delta)$ large enough such that

$$\begin{aligned}\min\Big\{&\varepsilon C(k_1 - p_1)\zeta'(-\varepsilon\nu) - k_1\varepsilon - d_1k_1\varepsilon\int_{\mathbb{R}} J_1(y)|y|dy + \alpha v_1^+ - h(v_2^+) :\\ &\frac{\delta p_1}{2k_1} \le \zeta(-\varepsilon\nu) \le 1 - \frac{\delta p_2}{2k_2},\ v_1^+ \in [\delta p_1, k_1],\ v_2^+ \in [\delta p_2, k_2]\Big\} > 0\end{aligned} \tag{4.2}$$

and

$$\begin{aligned}\min\Big\{&\varepsilon C(k_2-p_2)\zeta'(-\varepsilon\nu)-k_2\varepsilon-d_2k_2\varepsilon\int_{\mathbb{R}}J_1(y)|y|dy+\beta v_2^+-g(v_1^+):\\&\frac{\delta p_2}{2k_2}\le\zeta(-\varepsilon\nu)\le 1-\frac{\delta p_1}{2k_1},\ v_1^+\in[\delta p_1,k_1],\ v_2^+\in[\delta p_2,k_2]\Big\}>0.\end{aligned}\tag{4.3}$$

Let

$$\tilde\rho_1:=\max\{h'(x)>0|x\in(0,p_2]\},\quad \tilde\rho_2:=\max\{g'(x>0|x\in(0,p_1]\}.$$

Then by (A3) and (A5), we have

$$\alpha k_1-\tilde\rho_1k_2>0,\quad \beta k_2-\tilde\rho_2k_1>0.\tag{4.4}$$

We prove that $\mathcal{L}_1[v_1^+]\ge 0$ by distinguishing three cases:

Case(i): $\zeta(-\varepsilon\nu)\le\frac{\delta p_1}{2k_1}$. It is easy to see that $v_1^+(x,t)=k_1$, and $\mathcal{L}_1[v_1^+]\ge 0$ by (A3) and (A5).

Case(ii): $\zeta(-\varepsilon\nu)>1-\frac{\delta p_2}{2k_2}$. In this case, we have $\delta p_2<v_2^+<p_2-\frac{\delta p_2}{2}$. In view of (3.1), (4.1) and (4.4), we obtain

$$\begin{aligned}&\mathcal{L}_1[v_1^+]\\&\ge\varepsilon C(k_1-p_1)\zeta'(-\varepsilon\nu)-k_1\varepsilon-d_1k_1\big\{J_1*\zeta(-\varepsilon\nu)-\zeta(-\varepsilon\nu)\big\}\\&\quad+d_1(1-2\delta)p_1e^{-\varepsilon t}\big\{J_1*\zeta(-\varepsilon\nu)-\zeta(-\varepsilon\nu)\big\}\\&\quad+\alpha\big\{k_1+\delta p_1-[k_1-(1-2\delta)p_1e^{-\varepsilon t}]\zeta(-\varepsilon\nu)\big\}-h(v_2^+)\\&\ge\varepsilon C(k_1-p_1)\zeta'(-\varepsilon\nu)-k_1\varepsilon-d_1k_1\int_{\mathbb{R}}J_1(y)|\zeta(-\varepsilon(\nu-y))-\zeta(-\varepsilon\nu)|dy\\&\quad+\alpha\big\{k_1+\delta p_1-[k_1-(1-2\delta)p_1e^{-\varepsilon t}]\zeta(-\varepsilon\nu)\big\}-h'(\theta)v_2^+\\&\ge\varepsilon C(k_1-p_1)\zeta'(-\varepsilon\nu)-k_1\varepsilon-d_1k_1\varepsilon\int_{\mathbb{R}}J_1(y)|y|dy\\&\quad+\alpha\big\{k_1+\delta p_1-[k_1-(1-2\delta)p_1e^{-\varepsilon t}]\zeta(-\varepsilon\nu)\big\}\\&\quad-\tilde\rho_1\Big\{k_2+\delta p_2-[k_2-(1-2\delta)p_2e^{-\varepsilon t}]\zeta(-\varepsilon\nu)\big\}\\&=\varepsilon C(k_1-p_1)\zeta'(-\varepsilon\nu)-k_1\varepsilon-d_1k_1\varepsilon\int_{\mathbb{R}}J_1(y)|y|dy+(\alpha k_1-\tilde\rho_1k_2)[1-\zeta(-\varepsilon\nu)]\\&\quad+\delta(\alpha p_1-\tilde\rho_1p_2)+(1-2\delta)(\alpha p_1-\tilde\rho_1p_2)e^{-\varepsilon t}\zeta(-\varepsilon\nu)\\&\ge -k_1\varepsilon-d_1k_1\varepsilon\int_{\mathbb{R}}J_1(y)|y|dy+\delta(\alpha p_1-\rho p_2)>0,\end{aligned}$$

where $\theta\in[0,v_2^+(x,t)]$.

Case(iii): $\frac{\delta p_1}{2k_1}\le\zeta(-\varepsilon\nu)\le 1-\frac{\delta p_2}{2k_2}$. By (4.2), one has

$$\begin{aligned}&\mathcal{L}_1[v_1^+]\\&\ge\varepsilon C(k_1-p_1)\zeta'(-\varepsilon\nu)-k_1\varepsilon-d_1k_1\varepsilon\int_{\mathbb{R}}J_1(y)|y|dy+\alpha v_1^+-h(v_2^+)\\&\ge\min\Big\{\varepsilon C(k_1-p_1)\zeta'(-\varepsilon\nu)-k_1\varepsilon-d_1k_1\varepsilon\int_{\mathbb{R}}J_1(y)|y|dy+\alpha v_1^+-h(v_2^+)\Big\}>0.\end{aligned}$$

Next, we verify that $\mathcal{L}_2[v_2^+]\ge 0$ by considering the following three cases.

Case(i): $\zeta(-\varepsilon\nu) \le \frac{\delta p_2}{2k_2}$. It is clear that $v_2^+(x,t) = k_2$, and hence, $\mathcal{L}_2[v_2^+] \ge 0$.

Case(ii): $\zeta(-\varepsilon\nu) > 1 - \frac{\delta p_1}{2k_1}$. In this case, we have $\delta p_1 < v_1^+ < p_1 - \frac{\delta p_1}{2}$. In view of (3.1), (4.1) and (4.4), we obtain

$$\begin{aligned}
&\mathcal{L}_2[v_2^+]\\
&\ge \varepsilon C(k_2-p_2)\zeta'(-\varepsilon\nu) - k_2\varepsilon - d_2k_2\big\{J_2 * \zeta(-\varepsilon\nu) - \zeta(-\varepsilon\nu)\big\}\\
&\quad + d_2(1-2\delta)p_2e^{-\varepsilon t}\big\{J_2 * \zeta(-\varepsilon\nu) - \zeta(-\varepsilon\nu)\big\}\\
&\quad + \beta\big\{k_2 + \delta p_2 - [k_2 - (1-2\delta)p_2e^{-\varepsilon t}]\zeta(-\varepsilon\nu)\big\} - g(v_1^+)\\
&\ge \varepsilon C(k_2-p_2)\zeta'(-\varepsilon\nu) - k_2\varepsilon - d_2k_2\int_{\mathbb{R}} J_2(y)|\zeta(-\varepsilon(\nu-y)) - \zeta(-\varepsilon\nu)|dy\\
&\quad + \beta\big\{k_2 + \delta p_2 - [k_2 - (1-2\delta)p_2e^{-\varepsilon t}]\zeta(-\varepsilon\nu)\big\} - g'(\theta)v_1^+\\
&\ge \varepsilon C(k_2-p_2)\zeta'(-\varepsilon\nu) - k_2\varepsilon - d_2k_2\varepsilon\int_{\mathbb{R}} J_2(y)|y|dy\\
&\quad + \beta\big\{k_2 + \delta p_2 - [k_2 - (1-2\delta)p_2e^{-\varepsilon t}]\zeta(-\varepsilon\nu)\big\}\\
&\quad - \tilde{\rho}_2\big\{k_1 + \delta p_1 - [k_1 - (1-2\delta)p_1e^{-\varepsilon t}]\zeta(-\varepsilon\nu)\big\}\\
&= \varepsilon C(k_2-p_2)\zeta'(-\varepsilon\nu) - k_2\varepsilon - d_2k_2\varepsilon\int_{\mathbb{R}} J_2(y)|y|dy + (\beta k_2 - \tilde{\rho}_2k_1)[1-\zeta(-\varepsilon\nu)]\\
&\quad + \delta(\beta p_2 - \tilde{\rho}_2p_1) + (1-2\delta)(\beta p_2 - \tilde{\rho}_2p_1)e^{-\varepsilon t}\zeta(-\varepsilon\nu)\\
&\ge -k_2\varepsilon - d_2k_2\varepsilon\int_{\mathbb{R}} J_2(y)|y|dy + \delta(\beta p_2 - \rho p_1) > 0,
\end{aligned}$$

where $\theta \in [0, v_1^+(x,t)]$.

Case(iii): $\frac{\delta p_2}{2k_2} \le \zeta(-\varepsilon\nu) \le 1 - \frac{\delta p_1}{2k_1}$. By (4.3), we derive

$$\begin{aligned}
&\mathcal{L}_2[v_2^+]\\
&\ge \varepsilon C(k_2-p_2)\zeta'(-\varepsilon\nu) - k_2\varepsilon - d_2k_2\varepsilon\int_{\mathbb{R}} J_2(y)|y|dy + \beta v_2^+ - g(v_1^+)\\
&\ge \min\big\{\varepsilon C(k_2-p_2)\zeta'(-\varepsilon\nu) - k_2\varepsilon - d_2k_2\varepsilon\int_{\mathbb{R}} J_1(y)|y|dy + \beta v_2^+ - g(v_1^+)\big\} > 0.
\end{aligned}$$

The proof is complete. □

Remark 4.2. Clearly, the functions $v_i^{\pm}(x,t)$, $i = 1,2$ in Lemma 4.1 have the following properties:

$$\begin{gathered}
v_i^+(x,0) = k_i, \quad \forall x \in [\xi, \infty),\\
v_i^+(x,0) \ge (1-\delta)p_i, \quad \forall x \in (-\infty, \infty),\\
v_i^+(x,t) \le \delta p_i + (1-2\delta)p_ie^{-\varepsilon t}, \quad \forall (x,t) \in (-\infty, \xi - Ct - 4\varepsilon^{-1}] \times \mathbb{R}^+,\\
v_i^-(x,0) = 0, \quad \forall x \in (-\infty, \xi],\\
v_i^-(x,0) \le k_i - (1-\delta)p_i, \quad \forall x \in (-\infty, \infty),\\
v_i^-(x,t) \ge k_i - \delta p_i - (1-2\delta)p_ie^{-\varepsilon t}, \quad \forall (x,t) \in [\xi + Ct + 4\varepsilon^{-1}, \infty) \times \mathbb{R}^+.
\end{gathered}$$

By Lemma 3.1, we define $w^{\pm}(x,t,\xi_0,\delta) = (w_1^{\pm}(x,t,\xi_0,\delta), w_2^{\pm}(x,t,\xi_0,\delta))$ where

$$\begin{aligned}
w_i^+(x,t,\xi_0,\delta) &:= \min\big\{\phi_i(x+ct+\xi_0+\sigma_0\delta(1-e^{-\beta_0t})) + \delta p_ie^{-\beta_0t}, k_i\big\},\\
w_i^-(x,t,\xi_0,\delta) &:= \max\big\{\phi_i(x+ct+\xi_0-\sigma_0\delta(1-e^{-\beta_0t})) - \delta p_ie^{-\beta_0t}, 0\big\}.
\end{aligned}$$

Lemma 4.3. *Assume that* (A1)–(A5) *hold. Let* $\phi(x+ct) = (\phi_1(x+ct), \phi_2(x+ct))$ *be a traveling wavefront of* (1.1). *Then there exists* $\varepsilon^* > 0$ *such that, if* $u(x,t) = (u_1(x,t), u_2(x,t))$ *is a solution of* (1.1) *on* $[0,\infty)$ *with the initial data* $u(x,0)$ *satisfying* $0 \le u_i(x,0) \le k_i$ *for all* $x \in \mathbb{R}$, $i = 1,2$, *and the following is true:*

$$w^-(x,0,cT+\xi,\delta) \le u(x,T) \le w^+(x,0,cT+\xi+h,\delta)$$

on $\mathbb{R}$ *provided that for some* $\xi \in \mathbb{R}$, $T \ge 0$, $h > 0$ *and* $\delta \in (0, \min\{\frac{\delta_0}{2}, \frac{1}{\sigma_0}\})$, *then for every* $t \ge T+1$, *there exist* $\hat{\xi}(t)$, $\hat{\delta}(t)$ *and* $\hat{h}(t)$ *satisfying*

$$w^-(x,0,ct+\hat{\xi}(t),\hat{\delta}(t)) \le u(x,t) \le w^+(x,0,ct+\hat{\xi}(t)+\hat{h}(t),\hat{\delta}(t)), \tag{4.5}$$

where $\hat{\xi}(t)$, $\hat{\delta}(t)$ *and* $\hat{h}(t)$ *are defined as follows*

$$\begin{gathered}\hat{\xi}(t) \in [\xi - \sigma_0\delta, \xi + h + \sigma_0\delta],\\ \hat{h}(t) \in [0, h - \sigma_0\varepsilon^* \min\{h,1\} + 2\sigma_0\delta],\\ \hat{\delta}(t)) \in (\delta e^{-\beta_0} + \varepsilon^* \min\{h,1\}) e^{-\beta_0(t-(T+1))}.\end{gathered}$$

Proof. By Lemma 3.1, we can see that $w^+(x,t,cT+\xi+h,\delta)$ and $w^-(x,t,cT+\xi,\delta)$, respectively, are upper and lower solutions of (1.1). It is clear that $\tilde{u}(x,t) = u(x,T+t)$, $t \ge 0$ is a solution of (1.1) with the initial value $\tilde{u}(x,0) = u(x,T)$ for $x \in \mathbb{R}$. Then by the comparison principle, we have

$$w^-(x,t,cT+\xi,\delta) \le u(x,T+t) \le w^+(x,t,cT+\xi+h,\delta), \quad \forall (x,t) \in \mathbb{R} \times [0,+\infty),$$

i.e.,

$$\begin{aligned}&\max\{\phi_i(\eta^-(x,t,T)) - \delta p_i e^{-\beta_0 t}, 0\}\\ &\le u_i(x,T+t)\\ &\le \min\{\phi_i(\eta^+(x,t,T)+h) + \delta p_i e^{-\beta_0 t}, k_i\}, \quad \forall (x,t) \in \mathbb{R} \times [0,+\infty),\end{aligned} \tag{4.6}$$

where $i = 1,2$, $\eta^\pm(x,t) = x + c(T+t) + \xi \pm \sigma_0\delta(1 - e^{-\beta_0 t})$. Take $y = -cT - \xi$. Then by the comparison principle, we have that for every nonnegative constant L, any $x \in \mathbb{R}$ satisfying $|x-y| \le L$ and every $t > 0$, $i = 1,2$,

$$\begin{aligned}&u_i(x,T+t) - w_i^-(x,t,cT+\xi,\delta)\\ &\ge \mathcal{N}_i(L,t) \int_y^{y+1} (u_i(z,T) - w_i^-(z,0,cT+\xi,\delta))dz.\end{aligned} \tag{4.7}$$

Note that $\lim_{|x|\to+\infty} \phi_i'(x) = 0$ in Lemma 2.4, $i = 1,2$. We can choose $M > 0$ large enough such that $\phi_i'(x) \le \frac{\min\{p_1,p_2\}}{2\sigma_0}$ for all $|x| \ge M$. Let

$$L = M + |c| + 1, \quad \bar{h} = \min\{h,1\}, \quad \varepsilon_1 = \frac{1}{2}\min\{\phi_1'(x), \phi_2'(x) \big| |x| \le 2\} > 0.$$

Since

$$w_i^-(z,0,-y,\delta) < \phi_i(z-y), \quad w_i^+(z,0,-y+\bar{h},\delta) > \phi_i(z-y+\bar{h}), \quad i = 1,2,$$

we obtain

$$\begin{aligned}&\int_y^{y+1} [w_i^+(z,0,cT+\xi+\bar{h},\delta) - w_i^-(z,0,cT+\xi,\delta)]dz\\ &\ge \int_y^{y+1} [\phi_i(z+cT+\xi+\bar{h}) - \phi_i(z+cT+\xi)]dz\end{aligned}$$

$$= \int_y^{y+1} [\phi_i(z+\bar{h}) - \phi_i(z)]dz$$
$$= \int_y^{y+1} \phi_i'(\nu)\bar{h}dz \geq 2\varepsilon_1\bar{h}.$$

Then at least one of the following two assertions is true:

(i) $\int_y^{y+1}[u_i(z,T) - w_i^-(z,0,cT+\xi,\delta)]dz \geq \varepsilon_1\bar{h}$;

(ii) $\int_y^{y+1}[w_i^+(z,0,cT+\xi+\bar{h},\delta) - u_i(z,T)]dz \geq \varepsilon_1\bar{h}$;

Subsequently, we consider only the case (i). The case (ii) is similar and thus omitted. For any $|x-y| \leq L$, letting $t=1$ in (4.7), it holds

$$\begin{aligned} u_i(x,T+1) &\geq w_i^-(x,1,cT+\xi,\delta) + \mathcal{N}_i(L,1)\varepsilon_1\bar{h} \\ &\geq \phi_i(x-y+c-\sigma_0\delta(1-e^{-\beta_0})) - \delta p_i e^{-\beta_0} + \mathcal{N}_0(L)\varepsilon_1\bar{h}, \quad i=1,2, \end{aligned}$$

where $\mathcal{N}_0(L) = \min\{\mathcal{N}_1(L,1), \mathcal{N}_2(L,1)\}$. Let

$$L_1 = L + |c| + 2, \quad \varepsilon^* = \min_{1\leq i\leq 2}\Big\{\min_{|x|\leq L_1} \frac{\mathcal{N}_0(L)\varepsilon_1}{2\sigma_0\phi_i'(x)}, \frac{1}{2\sigma_0}, \frac{\delta_0}{2}\Big\}.$$

Then for $|x-y| \leq L$, by the mean value theorem, we have

$$\begin{aligned} &\phi_i(x-y+c+2\sigma_0\varepsilon^*\bar{h} - \sigma_0\delta(1-e^{-\beta_0})) - \phi_i(x-y+c-\sigma_0\delta(1-e^{-\beta_0})) \\ &= \phi_i'(x-y+c+2\theta_i\sigma_0\varepsilon^*\bar{h} - \sigma_0\delta(1-e^{-\beta_0}))2\sigma_0\varepsilon^*\bar{h} \\ &\leq \mathcal{N}_0(L)\varepsilon_1\bar{h}, \quad i=1,2,\ \theta_i \in (0,1). \end{aligned}$$

Thus,

$$u_i(x,T+1) \geq \phi_i(\eta^-(x,1,T) + 2\sigma_0\varepsilon^*\bar{h}) - \delta p_i e^{-\beta_0}, \quad i=1,2. \tag{4.8}$$

From the mean value theorem and the definitions of M and L, we obtain that for $|x-y| \geq L$,

$$\begin{aligned} &\phi_i(\eta^-(x,1,T)) - \phi_i(\eta^-(x,1,T) + 2\sigma_0\varepsilon^*\bar{h}) \\ &= \phi_i'(\eta^-(x,1,T) - 2\theta_i\sigma_0\varepsilon^*\bar{h})(-2\sigma_0\varepsilon^*\bar{h}) \\ &\geq -\varepsilon^*\bar{h}p_i,\ i=1,2,\ \theta_i \in (0,1). \end{aligned}$$

That is, for all $|x-y| \geq L$,

$$\phi_i(\eta^-(x,1,T)) \geq \phi_i(\eta^-(x,1,T) + 2\sigma_0\varepsilon^*\bar{h})) - \varepsilon^*\bar{h}p_i, \quad i=1,2,$$

and therefore, by (4.6) with $t=1$, it holds

$$u_i(x,T+1) \geq \max\big\{\phi_i(\eta^-(x,1,T) + 2\sigma_0\varepsilon^*\bar{h}) - (\delta e^{-\beta_0} + \varepsilon^*\bar{h})p_i, 0\big\} \tag{4.9}$$

for all $|x-y| \geq L$, $i=1,2$. By (4.8) and (4.9), we obtain that for all $x \in \mathbb{R}$, $i=1,2$,

$$\begin{aligned} u_i(x,T+1) &\geq \max\big\{\phi_i(\eta^-(x,1,T) + 2\sigma_0\varepsilon^*\bar{h}) - (\delta e^{-\beta_0} + \varepsilon^*\bar{h})p_i, 0\big\} \\ &= \max\big\{\phi_i(x+\iota) - (\delta e^{-\beta_0} + \varepsilon^*\bar{h})p_i, 0\big\}, \end{aligned}$$

where

$$\iota = c(T+1) + 2\sigma_0\varepsilon^*\bar{h} + \xi + \bar{\xi}, \quad \bar{\xi} = \sigma_0\delta(e^{-\beta_0} - 1). \tag{4.10}$$

Then

$$u(x,T+1) \geq w^-(x,0,\iota,\bar{\mu}), \quad \forall x \in \mathbb{R},$$

where $\bar{\mu} = \delta e^{-\beta_0} + \varepsilon^*\bar{h} \leq \delta_0$.

In view of the comparison principle and by the choice of ε^*, one has

$$w^-(x,\tilde t,\iota,\bar\mu) \le u(x,T+1+\tilde t),\ \forall \tilde t \ge 0. \tag{4.11}$$

Then for all $t \ge T+1$, setting $\tilde t = t-(T+1)$ in (4.11), we have

$$\begin{aligned} u_i(x,t) &\ge w_i^-(x,t-(T+1),\iota,\bar\mu)\\ &= \phi_i[x+ct-c(T+1)+\iota-\sigma_0\bar\mu(1-e^{-\beta_0(t-(T+1))})] - \bar\mu p_i e^{-\beta_0(t-(T+1))}\\ &\ge \phi_i[x+ct-c(T+1)+\iota-\sigma_0\bar\mu] - \hat\delta(t)p_i,\ i=1,2, \end{aligned}$$

where $\hat\delta(t) = \bar\mu e^{-\beta_0(t-(T+1))}$. Since $\phi_i(\cdot)$ is monotone increasing, together with the choice of η and (4.10), it holds

$$u_i(x,t) \ge w_i^-(x,0,ct+\hat\xi,\hat\delta(t)),\ \forall x\in\mathbb{R},\ i=1,2, \tag{4.12}$$

where

$$\hat\xi = 2\sigma_0\varepsilon^*\bar h + \xi - \sigma_0\delta(1-e^{-\beta_0}) - \sigma_0\bar\mu = \sigma_0\varepsilon^*\bar h + \xi - \sigma_0\delta.$$

By simple computations, we obtain the following estimate of $\hat\xi$,

$$\xi - \sigma_0\delta \le \hat\xi(t) \le \xi + \sigma_0\varepsilon^*\bar h \le \xi + h + \sigma_0\delta.$$

For every $t \ge T$, in view of (4.6), it follows that

$$\begin{aligned} u_i(x,t) &\le \min\big\{\phi_i(\eta^+(x,t-T,T)+h) + \delta p_i e^{-\beta_0(t-T)}, k_i\big\}\\ &\le \min\big\{\phi_i(x+ct+\xi+h+\sigma_0\delta) + \hat\delta(t)p_i, k_i\big\}, \quad \forall x\in\mathbb{R},\ i=1,2. \end{aligned}$$

Hence, for any $t \ge T+1$, one has

$$u_i(x,t) \le w_i^+(x,0,ct+\hat\xi(t)+\hat h(t),\hat\delta(t)), \quad \forall x\in\mathbb{R},\ i=1,2,$$

that is, for $x \in \mathbb{R}$,

$$u(x,t) \le w^+(x,0,ct+\hat\xi(t)+\hat h(t),\hat\delta(t)), \tag{4.13}$$

where

$$\hat h(t) = \xi + h + \sigma_0\delta - \hat\xi(t) = h - \sigma_0\varepsilon^*\bar h + 2\sigma_0\delta.$$

From the definition of ε^*, it holds $h - \sigma_0\varepsilon^*\bar h \ge h - \sigma_0\varepsilon^* h > 0$, and thus,

$$\hat h(t) \in (0, h - \sigma_0\varepsilon^*\bar h + 2\sigma_0\delta].$$

Therefore, (4.12) and (4.13) imply that (4.5) holds. The proof is complete. □

Lemma 4.4. *Assume that (A1)–(A5) hold. Let $\phi(x+ct) = (\phi_1(x+ct),\phi_2(x+ct))$ be a traveling wavefront of* (1.1) *and $\varphi(x) = (\varphi_1(x),\varphi_2(x))$ with $\varphi_i \in [0,k_i]$ be such that*

$$\lim_{x\to+\infty}\varphi_i(x) > k_i - p_i, \quad \lim_{x\to-\infty}\varphi_i(x) < p_i, \quad i=1,2.$$

Then, for every $\delta > 0$, there exist $T = T(\varphi,\delta) > 0$, $\xi = \xi(\varphi,\delta) \in \mathbb{R}$ and $h = h(\varphi,\delta) > 0$ such that

$$w^-(x,0,cT+\xi,\delta) \le u(x,T;\varphi) \le w^+(x,0,cT+\xi+h,\delta), \quad \forall x\in\mathbb{R}.$$

Proof. By the comparison principle, $u(x,t;\varphi) = (u_1(x,t;\varphi_1),u_2(x,t;\varphi_2))$ exists on $\mathbb{R}^+$ and $0 \le u_i(x,t;\varphi_i) \le k_i$ for any $(x,t)\in\mathbb{R}\times\mathbb{R}^+$, $i=1,2$. For every $\delta>0$, one can take $\delta_1 = \delta_1(\delta,\varphi) \in (0,\min\{\delta,\delta_0\})$ satisfying

$$\lim_{x\to+\infty}\varphi_i(x) > k_i - (1-\delta_1)p_i, \quad \lim_{x\to-\infty}\varphi_i(x) < (1-\delta_1)p_i, \quad i=1,2.$$

Thus, there exists a constant $M = M(\varphi, \delta_1) > 0$ such that, for $i = 1, 2$,

$$\varphi_i(x) \le (1-\delta_1)p_i, \quad \forall x \le -M; \quad \varphi_i(x) \ge k_i - (1-\delta_1)p_i, \quad \forall x \ge M. \tag{4.14}$$

Let $\varepsilon = \varepsilon(\delta_1)$, $C = C(\delta_1)$ and $v^{\pm}(x,t)$ be described by Lemma 4.1 with δ replaced by δ_1 and $\xi = \xi^{\pm}$, where $\xi^{\pm} = \mp M$. Together with (4.14) and Remark 4.2, we obtain that for $i = 1, 2$,

$$\begin{gathered} \varphi_i(x) \le (1-\delta_1)p_i \le v_i^+(x,0), \quad \forall x \le -M, \\ \varphi_i(x) \le k_i = v_i^+(x,0), \quad \forall x \ge -M, \end{gathered}$$

and

$$\begin{gathered} \varphi_i(x) \ge k_i - (1-\delta_1)p_i \ge v_i^-(x,0), \quad \forall x \ge M, \\ \varphi_i(x) \ge 0 = v_i^-(x,0), \quad \forall x \le M. \end{gathered}$$

Thus,

$$v^-(x,0) \le \varphi(x) \le v^+(x,0), \quad \forall x \in \mathbb{R}.$$

By Lemma 4.1 and the comparison principle, one has

$$v^-(x,t) \le u(x,t;\varphi) \le v^+(x,t), \quad \forall (x,t) \in \mathbb{R} \times \mathbb{R}^+.$$

Since $\delta_1 < \delta$, we can take $T > 0$ sufficiently large such that, for any $t \ge T$,

$$\delta_1 p_i + (1-2\delta_1)p_i e^{-\varepsilon t} < \delta p_i, \quad k_i - \delta_1 p_i + (1-2\delta_1)p_i e^{-\varepsilon t} > k_i - \delta p_i, \quad i = 1, 2.$$

and hence, by Remark 4.2 again, for $i = 1, 2$,

$$\begin{gathered} u_i(x,t;\varphi_i) \le v_i^+(x,t) < \delta p_i, \ \forall x \le x^-(t), \\ u_i(x,t;\varphi_i) \ge v_i^-(x,t) > k_i - \delta p_i, \ \forall x \ge x^+(t), \end{gathered} \tag{4.15}$$

where $x^{\pm}(t) = \xi^{\mp} \pm Ct \pm 4\varepsilon^-$. By (4.15), we have

$$\begin{gathered} u_i(x,T;\varphi_i) < \delta p_i, \quad \forall x \le x^-(T), \\ u_i(x,T;\varphi_i) > k_i - \delta p_i, \quad \forall x \ge x^+(T), \quad i = 1, 2. \end{gathered} \tag{4.16}$$

Since $\lim_{x\to-\infty} \phi_i(x) = 0$ and $\lim_{x\to+\infty} \phi_i(x) = k_i$, $i = 1, 2$, we can choose $H > 0$ large enough such that $\frac{H}{2} > x^+(T)$, $-\frac{H}{2} < x^-(T)$, and

$$\begin{gathered} \phi_i(x) + \delta p_i > k_i, \quad \forall x \ge \frac{H}{2}, \\ \phi_i(x) - \delta p_i < 0, \quad \forall x \le -\frac{H}{2}. \end{gathered} \tag{4.17}$$

Since $0 \le \phi_i(x) \le k_i$ and $0 \le u_i(x,t;\varphi_i) \le k_i$ for every $(x,t) \in \mathbb{R} \times \mathbb{R}^+$, and together with (4.16) and (4.17), we have that for $i = 1, 2$,

$$\max\{\phi_i(-H+x) - \delta p_i, 0\} \le u_i(x,T;\varphi_i) \le \min\{\phi_i(H+x) + \delta p_i, k_i\}, \tag{4.18}$$

for all $x \in \mathbb{R}$.

Let $-H = \xi_0 + cT$, $h_0 = 2H > 0$. It is clear that (4.18) implies that for $i = 1, 2$,

$$\begin{aligned} &\max\left\{\phi_i(x + cT + \xi_0) - \delta p_i, 0\right\} \\ &\le u_i(x,T;\varphi_i) \\ &\le \min\{\phi_i(x + cT + \xi_0 + h_0) + \delta p_i, k_i\}, \quad \forall x \in \mathbb{R}. \end{aligned} \tag{4.19}$$

Let $\xi = \xi_0$ and $h = h_0 > 0$. It then follows from (4.19) that

$$u_i(x,T;\varphi_i) \ge w_i^-(x,0,cT+\xi_0,\delta) = w_i^-(x,0,cT+\xi,\delta),$$

$$u_i(x,T;\varphi_i) \le w_i^+(x,0,cT+\xi_0+h_0,\delta) = w_i^+(x,0,cT+\xi+h,\delta), \quad i=1,2.$$

The proof is complete. □

Theorem 4.5 (Global stability). *Assume that* (A1)–(A5) *hold. Let* (ϕ,c) *be a traveling wavefront of* (1.1) *with wave speed* $c\in\mathbb{R}$. *Then* $\phi(x+ct)$ *is globally exponentially stable with shift in the sense that there exists a positive constant* μ *such that for every* $\varphi_i\in[0,k_i]$ *satisfying*

$$\lim_{x\to+\infty}\varphi_i(x) > k_i - p_i, \quad \lim_{x\to-\infty}\varphi_i(x) < p_i, \quad i=1,2,$$

the solution $u(x,t;\varphi)=(u_1(x,t;\varphi_1),u_2(x,t;\varphi_2))$ *of* (1.1) *satisfies*

$$\|u(\cdot,t;\varphi)-\phi(\cdot+ct+\xi_0)\| \le Ke^{-\mu t}, \quad t\ge 0$$

for some $K=K(\varphi)>0$ *and* $\xi_0=\xi_0(\varphi)\in\mathbb{R}$, $\varphi(x)=(\varphi_1(x),\varphi_2(x))$, *where* $\|\cdot\|$ *is the general super norm in* $\mathbb{R}^2$.

Proof. Let β_0, σ_0, δ_0 be as in Lemma 3.1 and let ε^* be as in Lemma 4.3 with ε^* chosen so that $\sigma_0\varepsilon^*<1$. We choose $0<\delta^*<\min\{\frac{\delta_0}{2},\frac{1}{2\sigma_0}\}$ such that

$$0<k^*:=\sigma_0\varepsilon^*-2\sigma_0\delta^*<1,$$

and then fix a $t^*\ge 1$ satisfying $e^{-\beta_0(t^*-1)}(1+\frac{\varepsilon^*}{\delta^*})<1-k^*$. The proof needs the following two claims.

Claim (i). There exist $T^*=T^*(\varphi)>0$ and $\xi^*=\xi^*(\varphi)\in\mathbb{R}$ such that

$$w^-(x,0,cT^*+\xi^*,\delta^*) \le u(x,T^*;\varphi) \le w^+(x,0,cT^*+\xi^*+1,\delta^*), \quad \forall x\in\mathbb{R}. \tag{4.20}$$

In fact, by Lemma 4.4, there exist $T=T(\varphi)>0$, $\xi=\xi(\varphi)>0$ and $h=h(\varphi)>0$ such that

$$w^-(x,0,cT+\xi,\delta^*) \le u(x,T;\varphi) \le w^+(x,0,cT+\xi+h,\delta^*), \quad \forall x\in\mathbb{R}. \tag{4.21}$$

If $h\le 1$, then by the monotonicity of $\phi_i(\cdot)$, $i=1,2$, Claim (i) holds. If $h>1$, we can choose an integer $N:=\max\{m|m\in\mathbb{Z}^+, mk^*<h\}$. Since $k^*\in(0,1)$ and $h>1$, then $N\ge 1$, $h\in(Nk^*,(N+1)k^*]$, furthermore, $0<h-Nk^*\le k^*<1$. Note that $\bar h:=\min\{1,h\}=1$. By (4.21), Lemma 4.3 and the choice of k^* and t^*, we obtain

$$\begin{aligned} &w^-(x,0,c(T+t^*)+\hat\xi(T+t^*),\hat\delta(T+t^*)) \\ &\le u(x,T+t^*;\varphi) \\ &\le w^+(x,0,c(T+t^*)+\hat\xi(T+t^*)+\hat h(T+t^*),\hat\delta(T+t^*)), \quad \forall x\in\mathbb{R}, \end{aligned} \tag{4.22}$$

where

$$\begin{gathered} \hat\xi(T+t^*)\in[\xi-\sigma_0\delta^*,\xi+h+\sigma_0\delta^*], \\ 0\le\hat h(T+t^*)\le h-\sigma_0\varepsilon^*+2\sigma_0\delta^*, \\ \hat\delta(T+t^*)=(\delta^*e^{-\beta_0}+\varepsilon^*)e^{-\beta_0(t^*-1)}\le(1-k^*)\delta^*<\delta^*. \end{gathered}$$

Applying the similar argument N times, we conclude that (4.22), with $T+t^*$ replaced by $T+Nt^*$, holds for some $\xi^*=\hat\xi\in\mathbb{R}$, $\hat\delta\in(0,\delta^*]$ and $0\le\hat h\le h-Nk^*<1$. Then by the monotonicity of $\phi(\cdot)$, (4.20) holds.

Claim (ii). Define $p = 2\sigma_0\delta^* + 1$, $T_n = T^* + nt^*$, $\delta_n^* = (1-k^*)^n\delta^*$ and $h_n = (1-k^*)^n < 1$, $n \ge 0$. Then there exist a sequence $\{\xi_n\}_{n=0}^{\infty} \subset \mathbb{R}$ with $\xi_0 = \xi^*$ such that

$$|\xi_{n+1} - \xi_n| \le ph_n, \quad \forall n \ge 0, \tag{4.23}$$

$$w^-(x, 0, cT_n + \xi_n, \delta_n^*) \le u(x, T_n; \varphi) \le w^+(x, 0, cT_n + \xi_n + h_n, \delta_n^*), \tag{4.24}$$

for all $n \ge 0$ and $x \in \mathbb{R}$. Indeed, we use a mathematical induction to show that (4.24) holds for every $n \ge 0$. Clearly, Claim (i) implies that (4.24) holds when $n = 0$. Suppose that (4.24) holds for some $n = m > 0$. By Lemma 4.3 with $T = T_m$, $\xi = \xi_m$, $h = h_m$, $\delta = \delta_m^*$, and $t = T_m + t^* = T_{m+1}$ (since $t \ge 1$), it follows that

$$w^-(x, 0, cT_{m+1} + \hat{\xi}, \hat{\delta}) \le u(x, T_{m+1}; \varphi) \le w^+(x, 0, cT_{m+1} + \hat{\xi} + \hat{h}, \hat{\delta}), \quad \forall x \in \mathbb{R},$$

where $\hat{\xi} \in [\xi_m - \sigma_0\delta_m^*, \xi_m + h_m + \sigma_0\delta_m^*]$,

$$\begin{aligned}
\hat{\delta} &= (\delta_m^* e^{-\beta_0} + \varepsilon^* h_m) e^{-\beta_0(T_{m+1} - T_m - 1)} \\
&\le (1-k^*)^m \delta^* \big[(1 + \frac{\varepsilon^*}{\delta^*}) e^{-\beta_0(t^*-1)}\big] \\
&\le (1-k^*)^m \delta^* (1-k^*) \\
&= (1-k^*)^{m+1}\delta^* = \delta_{m+1}^*,
\end{aligned}$$

and

$$\hat{h} \le h_m - \sigma_0\varepsilon^* h_m + 2\sigma_0\delta_m^* = (1-k^*)^m[1 - \sigma_0\varepsilon^* + 2\sigma_0\delta^*] = h_{m+1}.$$

Taking $\xi_{m+1} = \hat{\xi}$, we have

$$\begin{aligned}
|\xi_{m+1} - \xi_m| &\le |\xi_m + h_m + \sigma_0\delta_m^* - (\xi_m - \sigma_0\delta_m^*)| \\
&= |2\sigma_0\delta_m^* + h_m| \\
&= |2\sigma_0(1-k^*)^m\delta^* + (1-k^*)^m| \\
&= |(1-k^*)^m(2\sigma_0\delta^* + 1)| = ph_m.
\end{aligned}$$

It follows that (4.23) holds for $n = m$, and (4.24) holds for $n = m+1$. By induction, we obtain that (4.23) and (4.24) hold for all $n \ge 0$.

By (4.24) and the comparison principle, for every $n \ge 0$, all $t \ge T_n$ and $x \in \mathbb{R}$,

$$\begin{aligned}
&\max\big\{\phi_i(\eta_n^-(x,t)) - \delta_n^* p_i e^{-\beta_0(t-T_n)}, 0\big\} \\
&\le u(x, t, \psi_i) \\
&\le \min\big\{\phi_i(\eta_n^+(x,t) + h_n) + \delta_n^* p_i e^{-\beta_0(t-T_n)}, k_i\big\}, \quad i = 1, 2,
\end{aligned} \tag{4.25}$$

where $\eta_n^{\pm}(x,t) = x + ct + \xi_n \pm \sigma_0\delta_n^*(1 - e^{-\beta_0(t-T_n)})$. For $t \ge T^*$, let $n = [\frac{t-T^*}{t^*}]$ be the largest integer not greater than $\frac{t-T^*}{t^*}$, and define $\delta(t) = \delta_n^*$, $\xi(t) = \xi_n - \sigma_0\delta_n^*$, and $h(t) = h_n + 2\sigma_0\delta_n^*$. Then, $T_n = T + nt^* \le t \le T^* + (n+1)t^* = T_{n+1}$, in other words, $t \in [T_n, T_{n+1})$. In view of (4.25), we obtain that for $t \ge T^*$ and $x \in \mathbb{R}$, $i = 1, 2$,

$$\phi_i(x + ct + \xi(t)) - p_i\delta(t) \le u_i(x, t; \varphi_i) \le \phi_i(x + ct + \xi(t) + h(t)) + p_i\delta(t). \tag{4.26}$$

Set $\mu := -\frac{1}{t^*}\ln(1-k^*) > 0$ and $q(t) := e^{(\frac{t-T^*}{t^*} - 1)\ln(1-k^*)}$. Since $0 \le n \le \frac{t-T^*}{t^*} < n+1$, we have

$$(1-k^*)^n < (1-k^*)^{\frac{t-T^*}{t^*} - 1} = q(t).$$

Hence,

$$\delta(t) = \delta_n^* \le \delta^* q(t), \quad \forall t \ge T^*, \tag{4.27}$$

$$h(t) = (2\sigma_0\delta^* + 1)(1-k^*)^n \le (2\sigma_0\delta^* + 1)q(t), \quad \forall t \ge T^*. \tag{4.28}$$

By (4.23) and (4.27), it follows that for $\tau \ge t \ge T^*$,

$$\begin{aligned}
|\xi(\tau) - \xi(t)| &= |\xi_m - \sigma_0\delta_m^* - (\xi_n - \sigma_0\delta_n^*)| \\
&\le \sum_{s=n}^{m-1} |\xi_{s+1} - \xi_s| + 2\sigma_0\delta_n^* \\
&\le \sum_{s=n}^{m-1} ph_s + 2\sigma_0\delta_n^* \\
&\le \frac{ph_n}{1-(1-k^*)} + 2\sigma_0\delta_n^* = \frac{ph_n}{k^*} + 2\sigma_0\delta_n^* \\
&\le \Big(\frac{p}{k^*\delta^*} + 2\sigma_0\Big)\delta(t),
\end{aligned} \tag{4.29}$$

where $m = [\frac{\tau - T^*}{t^*}] \ge n = [\frac{t-T^*}{t^*}]$. From (4.29), the limit $\xi_0 := \lim_{t\to+\infty} \xi(t)$ exists, and

$$|\xi_0 - \xi(t)| \le \Big(\frac{p}{k^*\delta^*} + 2\sigma_0\Big)\delta(t), \quad \forall t \ge T^*. \tag{4.30}$$

Therefore, by combining (4.26)-(4.28) and (4.30), we obtain the assertion of the theorem. The proof is complete. □

As a corollary of Theorem 4.5, we can obtain the uniqueness of traveling wavefront of (1.1).

Corollary 4.6. *Assume that* (A1)–(A5) *hold. Let* $\tilde\phi(x+\tilde c t) = (\tilde\phi_1(x+\tilde c t), \tilde\phi_2(x+\tilde c t))$ *be a traveling wavefront with* $0 \le \tilde\phi_i(x+\tilde c t) \le k_i$. *Then there exists* $\tilde s \in \mathbb{R}$ *such that* $\tilde\phi(\cdot) = \phi(\cdot + \tilde s)$ *and* $\tilde c = c$, *where* $\phi(x+ct) = (\phi_1(x+ct), \phi_2(x+ct))$ *be traveling wavefront of* (1.1).

Proof. It is clear that

$$\lim_{\xi\to+\infty} \tilde\phi_i(\xi) > k_i - p_i, \quad \lim_{\xi\to-\infty} \tilde\phi_i(\xi) < p_i, \quad i = 1, 2.$$

By Theorem 4.5, there exist $K > 0$ and $\tilde s \in \mathbb{R}$ such that

$$\|\tilde\phi(\cdot + \tilde c t) - \phi(\cdot + ct + \tilde s)\| \le Ke^{-\mu t}, \quad \forall t \ge 0. \tag{4.31}$$

We choose $\xi_0 \in \mathbb{R}$ such that $0 < \tilde\phi_i(\xi_0) < k_i$, $i = 1, 2$, and define a set

$$\mathcal{M}(\xi_0) := \{(x,t) \in \mathbb{R} \times [0, +\infty) | x + ct = \xi_0\}.$$

It then follows from (4.31) that for any $(x,t) \in \mathcal{M}(\xi_0)$,

$$\phi_i(\eta) - Ke^{-\mu t} \le \tilde\phi_i(\xi_0) \le \phi_i(\eta) + Ke^{-\mu t}, \quad i = 1, 2, \tag{4.32}$$

where $\eta := \tilde s + \xi_0 + (c - \tilde c)t$. Note that $\lim_{\xi\to+\infty} \phi_i(\xi) = k_i$ and $\lim_{\xi\to-\infty} \phi_i(\xi) = 0$, $i = 1, 2$. Let $t \to +\infty$ in (4.32), we obtain that $\tilde c \ge c$ and $\tilde c \le c$ by the left-hand and right-hand side inequalities, respectively. Then, $\tilde c = c$. Substituting it into (4.31), we obtain that for any $(x,t) \in \mathcal{M}(\xi_0)$,

$$\|\tilde\phi(\cdot) - \phi(\cdot + \tilde s)\| \le Ke^{-\mu t}. \tag{4.33}$$

Hence, it follows from (4.33) that $\tilde\phi(\cdot) = \phi(\cdot + \tilde s)$ as $t \to +\infty$. The proof is complete. □

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