Electronic Journal of Differential Equations, Vol. 2022 (2022), No. 50, pp. 1–23.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu

ASYMPTOTIC BEHAVIOR OF COOPERATIVE SYSTEMS INVOLVING p-LAPLACIAN OPERATORS

PABLO ÁLVAREZ-CAUDEVILLA

Abstract. This work is devoted to the analysis of the asymptotic behavior of a parameter dependent quasilinear cooperative eigenvalue system when a parameter in front of some non-negative potentials approaches infinity. In particular we consider operators of p-Laplacian type. We prove that the eigenfunctions concentrate on the subdomains where those potentials vanish at the limit, while the eigenvalue approaches an upper bound that will depend on those subdomains. We also show several properties for the unusual limiting problems.

1. Introduction

1.1. **Models and preliminaries.** In this article we study the asymptotic behavior of the elliptic cooperative eigenvalue problem with potential terms

$$
\begin{aligned}
(-\Delta_p + \lambda a|u|^{p-2})u - b|u|^{\alpha-1}u|v|^{\beta}v &= \tau|u|^{p-2}u,\\
(-\Delta_q + \lambda d|v|^{q-2})v - c|u|^{\alpha}u|v|^{\beta-1}v &= \tau|v|^{q-2}v,
\end{aligned} \tag{1.1}
$$

for $(u,v)\in W_0^{1,p}(\Omega)\times W_0^{1,q}(\Omega)$ and when the parameter $\lambda\in\mathbb{R}$ approaches infinity. Moreover, Ω is a smooth bounded domain of $\mathbb{R}^N$, $N\geq 1$, with smooth boundary $\partial\Omega$, for example of class C^2 or Lipschitz. Also, we consider $\alpha,\beta\geq 0$ and $p,q>1$ satisfying the relation

$$
\frac{\alpha+1}{p}+\frac{\beta+1}{q}=1. \tag{1.2}
$$

Moreover, the operator $-\Delta_p$ stands for the so called p-Laplacian operator so that

$$
-\Delta_p u = -\operatorname{div}(|\nabla u|^{p-2}\nabla u),
$$

and similarly for $-\Delta_q$ as a q-Laplacian operator. To simplify the notation we might write the eigenvalue problem (1.1) in matrix form

$$
\mathcal{S}(V_1,V_2)\begin{pmatrix}u\\ v\end{pmatrix} := \begin{pmatrix}(-\Delta_p+V_1)u & -b|u|^{\alpha-1}u|v|^{\beta}v\\ -c|u|^{\alpha}u|v|^{\beta-1}v & (-\Delta_q+V_2)v\end{pmatrix} = \tau\begin{pmatrix}|u|^{p-2}u\\ |v|^{q-2}v\end{pmatrix}. \tag{1.3}
$$

In particular, for $V_1=\lambda a|u|^{p-2}$ and $V_2=\lambda d|v|^{q-2}$ we denote $\mathcal{S}(V_1,V_2)$ as

$$
\mathcal{S}_\lambda := \mathcal{S}(\lambda a|u|^{p-2},\lambda d|v|^{q-2}) \quad\text{and}\quad \mathcal{S}=\mathcal{S}(0,0). \tag{1.4}
$$

The exponent of the p-Laplacian (respectively the q-Laplacian) and the rest of the terms in each equation must be consistent so that relation (1.2) is satisfied.

2020 *Mathematics Subject Classification.* 35J47, 35P20, 49R05.
Key words and phrases. Cooperative systems; p-Laplacian; eigenvalue problems.

Submitted December 22, 2021. Published July 15, 2022.

Otherwise the system does not have the proper structure of an eigenvalue problem (further details in [17]).

Furthermore, we denote by

$$\tau_1[\mathcal{S}(V_1, V_2), \Omega] \tag{1.5}$$

the principal eigenvalue of the operator (1.3) under homogeneous Dirichlet boundary conditions and $(\varphi_1, \psi_1)^T$ the principal eigenfunction associated with the principal eigenvalue denoted by (1.5). We will show that such a principal eigenvalue corresponds to the one with an associated positive eigenfunction $(\varphi_1, \psi_1)^T$, in the sense that $\varphi_1 \geq 0$, $\psi_1 \geq 0$ and $\varphi_1 \neq 0$, $\psi_1 \neq 0$. Indeed, the positive eigenfunction will belong to the positive cone $\mathcal{K}$ with positive functions in the interior of the domain Ω and strictly negative outward normal derivatives on the boundary, i.e.

$$\mathcal{K} := \big\{\varphi \in W^{1,p}(\Omega) : \varphi(x) > 0 \text{ for } x \in \Omega,\ \frac{\partial \varphi(x)}{\partial \mathbf{n}} < 0 \text{ for } x \text{ on } \partial\Omega\big\},$$

where $\mathbf{n}$ stands for the unitary outward normal vector to the boundary $\partial\Omega$. In particular, for the eigenvalue problem (1.1) we have that

$$\tau_1(\lambda) = \tau_1[\mathcal{S}(\lambda a|u|^{p-2}, \lambda d|v|^{q-2}), \Omega],$$

as the principal eigenvalue under homogeneous Dirichlet boundary conditions, associated with the principal eigenfunction $(\varphi_\lambda, \psi_\lambda)^T$.

Thus, we are interested in understanding the limit $\lim_{\lambda\to\infty} \tau_1(\lambda)$, under certain assumptions established for the problem (1.1). In particular, we will assume that a, d are potentials as defined in [1], i.e. as Borel functions

$$a, d : \overline{\Omega} \to \mathbb{R}^+ \quad \text{such that} \quad a, d \in \mathcal{Q}(\Omega),$$

with $\mathcal{Q}(\Omega)$ representing the sets of potentials so that the following two properties hold:

(1) $\sup_{x\in\overline{\Omega}} a(x) < +\infty$, $\sup_{x\in\overline{\Omega}} d(x) < +\infty$;
(2) $W_0^{1,p}(\Omega_0^g) = \{u \in W_0^{1,p}(\Omega);\ u = 0 \text{ a.e. on } \Omega_+^g\}$, with $g = a, d$,

Moreover, those potentials a and d are going to be non-negative so that, $a \geq 0$, $d \geq 0$ and we denote the open sets/subdomains of Ω where the potentials a and d vanish, as

$$\Omega_0^a := \{x \in \Omega : a(x) = 0\}, \quad \Omega_0^d := \{x \in \Omega : d(x) = 0\}. \tag{1.6}$$

Also, we denote

$$\Omega_0 := \Omega_0^{a+d} = \Omega_0^a \cap \Omega_0^d = \{x \in \Omega : a(x) = d(x) = 0\},$$

so that the following subdomains come into play

$$\Omega_+^a := \{x \in \Omega : a(x) > 0\} = \overline{\Omega}\backslash\Omega_0^a, \quad \Omega_+^d := \{x \in \Omega : d(x) > 0\} = \overline{\Omega}\backslash\Omega_0^d,$$

and, there exists a subdomain of Ω

$$\Omega_+ = \{x \in \Omega : a(x) + d(x) > 0\},$$

which is also an open set, with $\overline{\Omega}_+ \subset \Omega$. Then, we consider a compact set

$$K_0 = (a + d)^{-1}(0) = \overline{\Omega} \setminus \Omega_+.$$

Note that, the compact set K_0 consists of two compact subsets in $\mathbb{R}^N$ so that

$$\Omega_0^a = \operatorname{Int}(K_0^a) \neq \emptyset, \quad \Omega_0^d = \operatorname{Int}(K_0^d) \neq \emptyset, \tag{1.7}$$

and the subdomains Ω_0^a and Ω_0^d may not be connected, but with a finite number of components. However, we suppose that they are Lipschitz domains on the boundary. Maybe these regularity assumptions on the boundary might be relaxed with extra care on the analysis but for our purposes Lipschitz boundary conditions are sufficient.

We also suppose that $b, c \in C(\overline{\Omega})$ and (1.1) is *strongly cooperative* in the sense that

$$b(x) > 0 \text{ and } c(x) > 0, \quad \text{for all } x \in \overline{\Omega}.$$

1.2. **Motivation.** Most of previous results rely on the variational structure of the problem, for example [16, 20].

Furthermore, focusing on cooperative systems, there are previous works analyzing similar quasilinear systems to (1.1) (see for example [17] and references therein) where the proofs were based on variational techniques and assuming the particular case when the cooperative terms are equal, i.e. $b = c$ in (1.1) multiplied by the proper coefficients depending on the exponents of the coupling terms. Indeed, assuming that those off-diagonal couple terms are the same having the proper variational structure, there are several additional works. We would like to stress the work done by Bozhkov & Mitidieri [7] for a variational system of the form

$$\begin{gathered}
-\Delta_p u = (\alpha+1)c(x)|u|^{\alpha-1}u|v|^{\beta}v + \lambda a(x)|u|^{p-2}u, \quad \text{in } \Omega, \\
-\Delta_q v = (\beta+1)c(x)|u|^{\alpha}u|v|^{\beta-1}v + \mu b(x)|v|^{q-2}v, \quad \text{in } \Omega, \\
(u,v) = (0,0), \quad \text{on } \partial\Omega,
\end{gathered} \tag{1.8}$$

where some results of existence and multiplicity of solutions were obtained via the fibering method. This methodology was introduced by Pohozaev in the 1970s [21, 22], as a convenient generalization of previous versions by Clark [9] and Rabinowitz [23] of variational approaches, and further developed by Drábek and Pohozaev [14] and others in the 1980's to ascertain the existence and multiplicity of solutions for equations with a variational form (in particular and relevant for our work, the p-Laplacian) associated with such equation, i.e. potential operator equations, alternatively with other methods such as bifurcation theory, critical point theory and so on.

However, for non-variational systems such as (1.1) there are only a few results involving pq-Laplacian operators. In this case we would like to emphasize the work done by Clément, Fleckinger, Mitidieri and Thélin [10] where they proved the existence of radial positive solutions for a quasilinear elliptic system

$$\begin{gathered}
-\Delta_p u = |u|^{\alpha}|v|^{\beta}, \quad \text{in } \Omega, \\
-\Delta_q v = |u|^{\gamma}|v|^{\delta}, \quad \text{in } \Omega, \\
(u,v) = (0,0), \quad \text{on } \partial\Omega,
\end{gathered}$$

where the exponents $\alpha, \beta, \delta, \gamma$ are non-negative and imposing the condition $\beta\gamma > 0$, which maintains the system coupled.

As one of the main motivations to study these kinds of problems, system (1.1) is an extension of a similar system assuming the Laplacian operators, which represents the steady states of a parabolic cooperative problem of the form

$$\frac{\partial u}{\partial t} - \Delta u = \lambda u + b(x)v - a(x)f(x,u)u, \quad \text{in } \Omega \times (0,\infty),$$

$$
\begin{gathered}
\frac{\partial v}{\partial t}-\Delta v=c(x)u+\lambda v-d(x)g(x,v)v, \quad \text{in } \Omega\times(0,\infty),\\
(u,v)^T=(0,0)^T \quad \text{on } \partial\Omega\times(0,\infty),\\
(u(\cdot,0),v(\cdot,0))^T=(u_0,v_0)^T>(0,0)^T \quad \text{in } \Omega,
\end{gathered}
$$

where $\gamma\in\mathcal{C}^{\mu}(\overline{\Omega})$, for all $\gamma\in\{a,b,c,d\}$ and some $\mu\in(0,1]$, such that $a,d\geq 0$ and $b,c>0$ in Ω, λ is a real parameter and the nonlinear functions $f,g\in\mathcal{C}^{\mu,1+\mu}(\bar{\Omega}\times[0,\infty))$ satisfy that for any $h\in\{f,g\}$, $h(x,0)=0$ and $\partial_u h(x,u)>0$ for all $x\in\overline{\Omega}$ and $u>0$. This model arises in population dynamics for the analysis of cooperative species where the environments are heterogeneous (see [4] for further details). In those parabolic models the principal eigenvalue of the linearised associated elliptic problem seems to be crucial in ascertaining the long time behaviour of the population, i.e. the linear eigenvalue problem

$$
\begin{gathered}
-\Delta u+\lambda au-bv=\tau u,\\
-\Delta v+\lambda dv-cu=\tau v,
\end{gathered}
\tag{1.9}
$$

for $(u,v)^T\in H_0^1(\Omega)\times H_0^1(\Omega)$, Indeed, there are several works analyzing that asymptotic behavior, when λ goes to infinity, assuming spatial heterogeneities for the potentials in front of the parameter λ, for one single equation such as [3], and for cooperative systems as well such as [2, 12], or for very general spatial heterogeneities conditions [1].

Thus, the problem under consideration here might represent the steady state solutions of the parabolic problem

$$
\begin{gathered}
\frac{\partial u}{\partial t}+(-\Delta_p+\lambda a|u|^{p-2})u-b|u|^{\alpha-1}u|v|^{\beta}v=\tau|u|^{p-2}u, \quad \text{in } \Omega\times(0,\infty),\\
\frac{\partial v}{\partial t}+(-\Delta_q+\lambda d|v|^{q-2})v-c|u|^{\alpha}u|v|^{\beta-1}v=\tau|v|^{q-2}v, \quad \text{in } \Omega\times(0,\infty),\\
(u,v)^T=(0,0)^T \quad \text{on } \partial\Omega\times(0,\infty),\\
(u(\cdot,0),v(\cdot,0))^T=(u_0,v_0)^T>(0,0)^T \quad \text{in } \Omega,
\end{gathered}
\tag{1.10}
$$

where, again, the principal eigenvalue of the elliptic problem (1.1) will play an important role in the dynamical behavior of the model. In relation to the convergence of similar problems, however assuming fractional operators, in [19] the authors obtained some multiplicity results and the convergence of those solutions when a parameter is passed to the limit to a limiting problem with infinitely many solutions.

1.3. **Main results.** Here, as mentioned above, we assume a non-variational cooperative system (1.1) assuming that the off-diagonal terms b and c of the pq-Laplacian system (1.1) are different, so considering a non-self-adjoint operator of the form (1.3) with

$$
b(x)\neq c(x), \quad \text{for all } x\in\overline{\Omega},
$$

and, then, obtaining the convergence of the principal eigenvalue under the very general heterogenous assumptions (1) and (2) for the non-negative potentials a and d set up above, and under a less restrictive system as the one assumed in [17]. In [17] the authors obtained such a convergence for a system with the same cooperative coefficients, applying variational arguments and under several constraints. Consequently, the analysis shown here is completely different from the previous works.

Although, our methods will depend on certain convergences, other important elements are based on operator theory that will provide us with several important properties of the principal eigenvalue of our problem.

Indeed, these properties will play a crucial role in obtaining the asymptotic behavior of problem (1.1), as well as having their own interest since they have never been proved before. Especially due to the non-variational nature of problem (1.1). Indeed, despite the fact that our problem lacks such a variational structure we are able to prove the following crucial issue: the principal eigenvalue is the smallest, simple, positive eigenvalue and, also, isolated. To do so we use different approaches for linear problems that we adapt for the nonlinear system (1.1) as well as some arguments shown in [15]. Note that those properties for the eigenvalue problem (1.1) might be established as well for the unusual limiting problem obtained here; see details below.

Consequently, under the assumptions established above we state the main result of this paper.

Theorem 1.1. *Let $\Omega \subset \mathbb{R}^N$ be an open set and assume that a and d are two potentials for which conditions (1) and (2) are fulfilled. Then*

$$\lim_{\lambda\to+\infty} \tau_1(\lambda) = \tau_1, \tag{1.11}$$

where $\tau_1(\lambda)$ is the principal eigenvalue of the pq-Laplacian system (1.1)*, and the limiting principal eigenvalue τ_1 is the one corresponding to the limiting pq-Laplacian system*

$$\begin{aligned} -\Delta_p u - b|u|^{\alpha-1}u|\mathcal{P}v|^{\beta}\mathcal{P}v &= \tau_1|u|^{p-2}u,\\ -\Delta_q v - c|\mathcal{P}u|^{\alpha}\mathcal{P}u|v|^{\beta-1}v &= \tau_1|v|^{q-2}v, \end{aligned} \tag{1.12}$$

for $(u,v)^T \in W_0^{1,p}(\Omega_0^a)\times W_0^{1,q}(\Omega_0^d)$ and where $\mathcal{P}$ stands for the projection on the subdomain where both potentials vanish at the same time, i.e.

$$\mathcal{P}w = \chi_{\Omega_0^a\cap\Omega_0^d} w, \quad \textit{with } w = u, v.$$

In fact, if the intersection of the subdomains where the potentials a and d vanish were empty

$$\Omega_0^a \cap \Omega_0^d = \emptyset,$$

$\mathcal{P}$ is defined to be zero and the limiting principal eigenvalue τ_1 will be the infimum among the principal eigenvalues corresponding to the uncoupled system

$$\begin{aligned} -\Delta_p u &= \tau_1|u|^{p-2}u \quad \textit{in } \Omega_0^a,\\ -\Delta_q v &= \tau_1|v|^{q-2}v \quad \textit{in } \Omega_0^d, \end{aligned} \tag{1.13}$$

under homogeneous Dirichlet boundary conditions and such that

$$\tau_1 = \inf\{\tau_1[-\Delta_p,\Omega_0^a], \tau_1[-\Delta_q,\Omega_0^d]\}.$$

In addition, any sequence of normalized eigenfunctions $\{(\varphi_\lambda,\psi_\lambda)^T\}$ associated with $\tau_1(\lambda)$ admits a subsequence that converges strongly in $W_0^{1,p}(\Omega)\times W_0^{1,q}(\Omega)$ to the normalized eigenfunction $(\varphi_*,\psi_*)^T$ in $L^p(\Omega)\times L^q(\Omega)$ associated with τ_1, in the sense that

$$\int_\Omega |\varphi_*|^p + \int_\Omega |\psi_*|^q = 1.$$

Remark 1.2. The principal eigenvalue $\tau_1(\ell)$ is simple, isolated and positive (see details below) so the whole sequence $\{(\varphi_\lambda, \psi_\lambda)^T\}$ converges strongly in $W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega)$ to the corresponding normalized eigenfunction $(\varphi_*, \psi_*)^T \in W_0^{1,p}(\Omega_0^a) \times W_0^{1,q}(\Omega_0^d)$ (e.g. [2]). We will prove this fact as part of Theorem 1.1.

Moreover, we cannot forget that for this quasilinear system (1.1) we have the possibility of existence of semi-trivial solutions.

We define the spaces $W_0^{1,p}(E)$ for a measurable set $E \subseteq \mathbb{R}^N$ in terms of the capacity. In other words,

$$W_0^{1,p}(E) := \{u \in W^{1,p}(\mathbb{R}^N) \text{ so that } u = 0 \text{ q.e. on } \mathbb{R}^N \backslash E\}, \tag{1.14}$$

where q.e. means "quasi everywhere" (with respect to a set of capacity zero), $E \subseteq \mathbb{R}^N$ is any measurable subset (not necessarily open). Observe, that particularly for any open set $\Omega \subset \mathbb{R}^N$ there is a nice characterization of $W_0^{1,p}(\Omega)$ using capacity (see for instance [8, Theorem 4.1.2] or [24]), namely we have

$$u \in W_0^{1,p}(\Omega) \Leftrightarrow (u \in W^{1,p}(\mathbb{R}^N) \text{ and } u = 0 \text{ q.e. on } \mathbb{R}^N \backslash \Omega).$$

Note that $W_0^{1,p}(E)$ is a closed subspace of $W^{1,p}(\mathbb{R}^N)$ and inherits its structure. Consequently, the imbedding $W_0^{1,p}(E)$ into $L^p(E)$ remains compact. Notice also that according to our definition $W_0^{1,p}(E)$ is never empty, because it always contains the function identically equal to 0. That could be also deduced from the condition (1.7). It is also clear from the definition that $W_0^{1,p}(E) = \{0\}$ when the capacity of E is zero.

However, when the potentials are continuous functions we must recall that the definition for the space $W_0^{1,p}(\Omega)$ denoted above by (1.14) is equivalent to

$$W_0^{1,p}(\Omega) := \{u \in W^{1,p}(\mathbb{R}^N) \text{ so that } u = 0 \text{ a.e. on } \mathbb{R}^N \backslash \Omega\},$$

since the set Ω will be open and the zero set has capacity zero. This implies that part of the zero set of the potentials a and d with measure zero could be inside Ω_+^a and Ω_+^d respectively. Thus, under hypothesis (1.6) we know that

$$\begin{aligned} W_0^{1,p}(K_a) &:= W^{1,p}(\mathbb{R}^N) \cap \{u = 0 \text{ q.e. in } \mathbb{R}^N \setminus K_a\} \\ &= W^{1,p}(\mathbb{R}^N) \cap \{u = 0 \text{ a.e. in } \mathbb{R}^N \setminus K_a\}. \end{aligned}$$

Indeed, the only kind of regularity assumption that is contained in condition (2) which can be understood as a stability-type property for $W_0^{1,p}$ (resp. $W_0^{1,q}$).

2. p-Laplacian eigenvalue problem

We assume that the Banach space $W_0^{1,p}(\Omega)$ with $1 < p < \infty$, for a bounded domain Ω in $\mathbb{R}^N$, is equipped with the norm

$$\|u\|_{W_0^{1,p}(\Omega)} := \Big(\int_\Omega |\nabla u|^p \Big)^{1/p}.$$

Thanks to Poincaré's inequality this norm is equivalent to the standard one for the Sobolev spaces $W_0^{1,p}(\Omega)$ in bounded domains. The pair $(u,v)^T \in W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega)$

is called a weak solution of the problem (1.1) if the following integral identities hold

$$\begin{aligned} &\int_\Omega |\nabla u|^{p-2}\nabla u \nabla \nu_1 + \lambda \int_\Omega a|u|^{p-2}u\,\nu_1 - \int_\Omega b|u|^{\alpha}|v|^{\beta+1}\nu_1 \\ &= \tau(\lambda)\int_\Omega |u|^{p-2}u\,\nu_1, \\ &\int_\Omega |\nabla v|^{q-2}\nabla v \nabla \nu_2 + \lambda \int_\Omega d|v|^{q-2}v\,\nu_2 - \int_\Omega c|u|^{\alpha+1}|v|^{\beta}\nu_2 \\ &= \tau(\lambda)\int_\Omega |v|^{q-2}v\,\nu_2, \end{aligned} \tag{2.1}$$

for any $(\nu_1,\nu_2)^T \in W_0^{1,p}(\Omega)\times W_0^{1,q}(\Omega)$. In addition, a real number τ is called an eigenvalue under homogeneous Dirichlet boundary conditions and $(u,v)^T$ is its associated eigenfunction of the system

$$\begin{gathered} -\Delta_p u - b|u|^{\alpha-1}u|v|^{\beta}v = \tau|u|^{p-2}u, \quad \text{in } \Omega, \\ -\Delta_q v - c|u|^{\alpha}u|v|^{\beta-1}v = \tau|v|^{q-2}v, \quad \text{in } \Omega, \\ (u,v) = (0,0), \quad \text{on } \partial\Omega, \end{gathered} \tag{2.2}$$

if

$$\begin{aligned} &\int_\Omega |\nabla u|^{p-2}\nabla u\nabla \nu_1 + \int_\Omega |\nabla v|^{q-2}\nabla v \nabla \nu_2 \\ &= \tau\Big(\int_\Omega |u|^{p-2}u\,\nu_1 + \int_\Omega |v|^{q-2}v\,\nu_2\Big) + \int_\Omega b|u|^{\alpha}|v|^{\beta+1}\nu_1 + \int_\Omega c|u|^{\alpha+1}|v|^{\beta}\nu_2, \end{aligned} \tag{2.3}$$

for every $(\nu_1,\nu_2)^T \in W_0^{1,p}(\Omega)\times W_0^{1,q}(\Omega)$.

2.1. **Operator properties.** We first obtain the semicontinuity and monotonicity of the operators for the single equation (2.8). Indeed, assuming the problem

$$\begin{gathered} (-\Delta_p + \lambda a|u|^{p-2})u = f, \quad \text{in } \Omega, \\ u = 0, \quad \text{on } \partial\Omega, \end{gathered} \tag{2.4}$$

for $p>1$ and f in the dual space of $Y = W_0^{1,p}(\Omega)$, denoted by $Y' = W^{-1,p'}(\Omega)$ with $1 = \frac{1}{p}+\frac{1}{p'}$ we obtain the following result.

Lemma 2.1. *Let $f \in W^{-1,p'}(\Omega)$ be for problem (2.4). Then, the operator*

$$\mathcal{B}(\cdot) = -\Delta_p(\cdot) + \lambda a|\cdot|^{p-2}(\cdot), \tag{2.5}$$

is continuous and monotone.

Proof. Take a bounded sequence $\{u_n\}$ in $W_0^{1,p}(\Omega)$. Then, because of the compactness we can find a subsequence, again labeled $\{u_n\}$, which converges weakly in $W_0^{1,p}(\Omega)$ and strongly in $L^p(\Omega)$ to a certain u. Moreover,

$$\begin{aligned} \|\mathcal{B}(u_n)-\mathcal{B}(u)\|_{\mathcal{M}(Y)} &= \sup_{\|h\|_Y\le 1}\|\langle \mathcal{B}(u_n)-\mathcal{B}(u),h\rangle\|_Y \\ &= \sup_{\|h\|_Y\le 1}\Big\|\int_\Omega [|\nabla u_n|^{p-2}\nabla u_n - |\nabla u|^{p-2}\nabla u]\cdot\nabla h \\ &\quad + \lambda\int_\Omega a[|u_n|^{p-2}u_n - |u|^{p-2}u]h\Big\|_Y, \end{aligned}$$

with $\mathcal{M}(Y)$ denoting the space of bounded operators from Y to Y. Applying Hölder's inequality yields

$$
\begin{aligned}
&\|\mathcal{B}(u_n)-\mathcal{B}(u)\|_{\mathcal{M}(Y)}\\
&\le \Big(\int_\Omega \big||\nabla u_n|^{p-2}\nabla u_n - |\nabla u|^{p-2}\nabla u\big|^{p'}\Big)^{1/p'}\Big(\int_\Omega |\nabla h|^p\Big)^{1/p}\\
&\quad + \lambda\Big(\int_\Omega \big|a[|u_n|^{p-2}u_n - |u|^{p-2}u]\big|^{p'}\Big)^{1/p'}\Big(\int_\Omega |h|^p\Big)^{1/p}\\
&\le K\Big[\Big(\int_\Omega \big||\nabla u_n|^{p-2}\nabla u_n - |\nabla u|^{p-2}\nabla u\big|^{p'}\Big)^{1/p'}\\
&\quad + \ell\Big(\int_\Omega \big|a[|u_n|^{p-2}u_n - |u|^{p-2}u]\big|^{p'}\Big)^{1/p'}\Big].
\end{aligned}
\tag{2.6}
$$

Note that

$$
\frac{1}{p}+\frac{1}{p'}=1, \quad \text{so that} \quad p'=\frac{p}{p-1}.
$$

Thus, we find that (2.6) certainly converges to zero because of the convergence of the sequence $\{u_n\}$ in $L^p(\Omega)$ and, also, by the continuity of the Nemytski operator

$$
M(s)=|s|^{p-2}s, \quad \text{from } L^p(\Omega) \text{ to } L^{p'}(\Omega).
$$

Now, we see that the operator $\mathcal{B}$ is also a monotone operator. Indeed, for $u\neq v$

$$
\begin{aligned}
&\langle \mathcal{B}(u)-\mathcal{B}(v), u-v\rangle\\
&=\int_\Omega [|\nabla u|^{p-2}\nabla u - |\nabla v|^{p-2}\nabla v]\cdot(\nabla u-\nabla v)\\
&\quad + \lambda\int_\Omega a[|u|^{p-2}u-|v|^{p-2}v](u-v)\\
&\ge \int_\Omega |\nabla u|^p - \int_\Omega |\nabla v|^{p-2}\nabla v\cdot\nabla u + \int_\Omega |\nabla v|^p - \int_\Omega |\nabla u|^{p-2}\nabla u\cdot\nabla v\\
&\ge \int_\Omega |\nabla u|^p - \Big(\int_\Omega |\nabla v|^{p'}\Big)^{1/p'}\Big(\int_\Omega |\nabla u|^p\Big)^{1/p}\\
&\quad + \int_\Omega |\nabla v|^p - \Big(\int_\Omega |\nabla u|^{p'}\Big)^{1/p'}\Big(\int_\Omega |\nabla v|^p\Big)^{1/p}\\
&= [\|u\|^{p-1}-\|v\|^{p-1}][\|u\|-\|v\|]\ge 0,
\end{aligned}
$$

since $s\mapsto |s|^{p-1}$ is an increasing function on $(0,+\infty)$. Therefore, the operator $\mathcal{B}$ is monotone and continuous. □

Remark 2.2. Consequently, by Lemma 2.1 we have the existence of solutions for a problem of the form (2.4) from Browder's Theorem [13, Th. 5.3.22], since the operator is also semicontinuous, bounded and coercive.

The conclusions established above for the operators $\mathcal{B}$ (2.5) corresponding to the single equation (2.4) can be extended to a pq-Laplacian cooperative system

$$
\begin{gathered}
(-\Delta_p+\lambda a|u|^{p-2})u - b|u|^{\alpha-1}u|v|^{\beta}v = f, \quad \text{in } \Omega,\\
(-\Delta_q+\lambda d|v|^{q-2})v - c|u|^{\alpha}u|v|^{\beta-1}v = g, \quad \text{in } \Omega,\\
u=v=0, \quad \text{on } \partial\Omega,
\end{gathered}
\tag{2.7}
$$

for $p > 1$, $q > 1$. Thus, we prove, following similar arguments and under certain extra assumptions, that the operator corresponding to the cooperative system is continuous. However, in general, we must stress that for large cooperative terms b and c the monotonicity is not true. Hence, only the continuity can be obtained.

Lemma 2.3. *Assume a cooperative system of the form* (2.7) *such that f belongs to the dual space of $Y_1 = W_0^{1,p}(\Omega)$ denoted by*

$$Y_1' = W^{-1,p'}(\Omega) \quad \text{with } 1 = \frac{1}{p} + \frac{1}{p'},$$

g belongs to the dual space of $Y_2 = W_0^{1,q}(\Omega)$ denoted by

$$Y_2' = W^{-1,q'}(\Omega) \quad \text{with} \quad 1 = \frac{1}{q} + \frac{1}{q'},$$

and condition (1.2) *is also satisfied. Then, the operator $\mathcal{S}_\lambda$ acting in* (1.3) *is continuous.*

Proof. Taking a bounded sequence

$$\{(u_n, v_n)^T\} \in W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega),$$

by compactness we can find a subsequence, again labeled $\{(u_n, v_n)^T\}$, which converges weakly in $W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega)$ and strongly in $L^p(\Omega) \times L^q(\Omega)$ to a certain $(u, v)^T$. Moreover,

$$\begin{aligned}
&\Big\| \mathcal{S}_\lambda \begin{pmatrix} u_n \\ v_n \end{pmatrix} - \mathcal{S}_\lambda \begin{pmatrix} u \\ v \end{pmatrix} \Big\|_{\mathcal{M}(W)} \\
&= \sup_{\substack{\|h_i\|_{Y_i} \le 1 \\ i=1,2}} \Big\| \Big\langle \mathcal{S}_\lambda \begin{pmatrix} u_n \\ v_n \end{pmatrix} - \mathcal{S}_\lambda \begin{pmatrix} u \\ v \end{pmatrix}, \begin{pmatrix} h_1 \\ H_2 \end{pmatrix} \Big\rangle \Big\|_W \\
&= \sup_{\substack{\|h_i\|_{Y_i} \le 1 \\ i=1,2}} \Big\| \int_\Omega [|\nabla u_n|^{p-2} \nabla u_n - |\nabla u|^{p-2} \nabla u] \cdot \nabla h_1 \\
&\quad + \int_\Omega [|\nabla v_n|^{q-2} \nabla v_n - |\nabla v|^{q-2} \nabla v] \cdot \nabla h_2 \\
&\quad + \lambda \Big(\int_\Omega a[|u_n|^{p-2} u_n - |u|^{p-2} u]\, h_1 + \int_\Omega d[|v_n|^{q-2} v_n - |v|^{q-2} v]\, h_2 \Big) \\
&\quad - \int_\Omega b|u|^{\alpha-1} u|v|^\beta v\, h_1 - \int_\Omega c|u|^\alpha u|v|^{\beta-1} v\, h_2 \Big\|_W,
\end{aligned}$$

where $W := Y_1 \times Y_2 = W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega)$. It is now clear that it converges to zero, once we apply the Hölder's inequality and the cooperative character of the system. Indeed,

$$\begin{aligned}
&\Big\| \mathcal{S}_\lambda \begin{pmatrix} u_n \\ v_n \end{pmatrix} - \mathcal{S}_\lambda \begin{pmatrix} u \\ v \end{pmatrix} \Big\|_{\mathcal{M}(W)} \\
&\le \Big(\int_\Omega \big| |\nabla u_n|^{p-2} \nabla u_n - |\nabla u|^{p-2} \nabla u \big|^{p'} \Big)^{1/p'} \Big(\int_\Omega |\nabla h_1|^p \Big)^{1/p} \\
&\quad + \Big(\int_\Omega \big| |\nabla v_n|^{q-2} \nabla v_n - |\nabla v|^{q-2} \nabla v \big|^{q'} \Big)^{1/q'} \Big(\int_\Omega |\nabla h_2|^q \Big)^{1/q}
\end{aligned}$$

$$
\begin{aligned}
&+\lambda\Big(\int_\Omega \big|a[|u_n|^{p-2}u_n-|u|^{p-2}u]\big|^{p'}\Big)^{1/p'}\Big(\int_\Omega |h_1|^p\Big)^{1/p}\\
&+\lambda\Big(\int_\Omega \big|d[|v_n|^{q-2}v_n-|v|^{q-2}v]\big|^{q'}\Big)^{1/q'}\Big(\int_\Omega |h_2|^q\Big)^{1/q}\\
\le\ & K\Big(\Big(\int_\Omega \big||\nabla u_n|^{p-2}\nabla u_n-|\nabla u|^{p-2}\nabla u\big|^{p'}\Big)^{1/p'}\\
&+\lambda\Big(\int_\Omega \big|a[|u_n|^{p-2}u_n-|u|^{p-2}u]\big|^{p'}\Big)^{1/p'}\Big(\int_\Omega \big||\nabla v_n|^{q-2}\nabla v_n-|\nabla v|^{q-2}\nabla v\big|^{q'}\Big)^{1/q'}\\
&+\ell\Big(\int_\Omega \big|d[|v_n|^{q-2}v_n-|v|^{q-2}v]\big|^{q'}\Big)^{1/q'}\Big).
\end{aligned}
$$

Therefore, the operator $\mathcal{S}_\lambda$ is continuous. $\square$

2.2. **Principal eigenvalue properties.** In this section we establish a result which is a counterpart of the already existent one corresponding to the single equation (see e.g. [5, 6, 20]) of the form

$$
-\Delta_p u=\tau|u|^{p-2}u, \tag{2.8}
$$

under homogeneous Dirichlet boundary conditions. For which it is known that it admits, a unique positive first eigenvalue τ_1 with a non-negative eigenfunction. Moreover, the principal eigenvalue of problem (2.8) is also isolated and simple as a consequence of its variational characterization.

For cooperative systems we can consider the variational cooperative eigenvalue problem

$$
\begin{gathered}
-\Delta_p u-(\alpha+1)B|u|^{\alpha-1}u|v|^{\beta}v=\tau_0|u|^{p-2}u,\quad \text{in } \Omega,\\
-\Delta_q v-(\beta+1)B|u|^{\alpha}u|v|^{\beta-1}v=\tau_0|v|^{q-2}v,\quad \text{in } \Omega,\\
(u,v)=(0,0)\quad \text{on } \partial\Omega,
\end{gathered}\tag{2.9}
$$

where $\tau_0=\tau_0[\mathcal{S}_0,\Omega]$ stands for the principal eigenvalue for the symmetric operator

$$
\mathcal{S}_0\begin{pmatrix}u\\ v\end{pmatrix}:=\begin{pmatrix}-\Delta_p u & -(\alpha+1)B|u|^{\alpha-1}u|v|^{\beta}v\\ -(\beta+1)B|u|^{\alpha}u|v|^{\beta-1}v & -\Delta_q v\end{pmatrix}.
$$

Furthermore, since problem (2.9) has a variational structure, in this case we have an expression for the first eigenvalue based on the Rayleigh quotient, i.e.

$$
\tau_0=\inf_{\substack{u\in W_0^{1,p}(\Omega),\\ v\in W_0^{1,q}(\Omega)}}\frac{\int_\Omega|\nabla u|^p+\int_\Omega|\nabla v|^q-(\alpha+\beta+2)\int_\Omega B|u|^{\alpha+1}|v|^{\beta+1}}{\int_\Omega u^p+\int_\Omega v^q}. \tag{2.10}
$$

Hence, applying the results obtained by Kawohl & Lindqvist [16] and Lindqvist [20] we can find that the first eigenvalue of the problem (2.9) is unique, positive and isolated for any arbitrary domain Ω in $\mathbb{R}^N$. Moreover, its associated eigenfunction is unique and positive in Ω and with maximal regularity $C^{1,\alpha}(\Omega)$, for some $0<\alpha<1$ by elliptic regularity. Those facts can be proved following the arguments nicely shown in [16] for problem (2.9); see also [17, Lemma 2.3].

Thus, we state a similar result for the cooperative system of the form (2.2) proving that there is no positive eigenvalue below τ_1 (denoting the smallest eigenvalue) as well as to being an isolated eigenvalue from above, associated with a positive eigenfunction $(\varphi_1,\psi_1)^T\in W_0^{1,p}(\Omega)\times W_0^{1,q}(\Omega)$, so that $\varphi_1>0$ and $\psi_1>0$, with negative outward normal derivatives on the boundary. However, we must note that our problem is not variational so a different approach must be followed. The results actually rely on the strong maximum principle for p-Laplacian problems.

To do so we introduce the following notation

$$-\Delta_p u - b|u|^{\alpha-1}u|v|^{\beta}v = \mathcal{A}_1 w, \quad -\Delta_q v - c|u|^{\alpha}u|v|^{\beta-1}v = \mathcal{A}_2 w,$$

where $w = (u, v)$ and $\mathcal{A}_i$, for $i = 1, 2$ the operator involved in each equation so that (2.2) becomes

$$\begin{aligned} \mathcal{A}_1 w &= \tau(\mathcal{D}w)_1, \\ \mathcal{A}_2 w &= \tau(\mathcal{D}w)_2. \end{aligned} \tag{2.11}$$

with

$$\tau|u|^{p-2}u = \tau(\mathcal{D}w)_1, \quad \tau|v|^{q-2}v = \tau(\mathcal{D}w)_2.$$

According to Lemma 2.3 the operator $\mathcal{S}$ is continuous in the positive cone $\mathcal{K}$ corresponding to the Banach space $W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega)$. From condition (1.2) on the exponents we find that

$$\begin{aligned} \text{either} \quad & p - 1 < \alpha + \beta + 1 < q - 1, \\ \text{or} \quad & q - 1 < \alpha + \beta + 1 < p - 1, \end{aligned} \tag{2.12}$$

depending on the assumption of $p \le q$ or $q \le p$. Then assuming (2.12) we introduce a "pseudo-homogenous" condition for the operator $\mathcal{S}$ saying that

$$\mathcal{S}(t\mathbf{w}) \ge t^{r-1}\mathcal{S}(\mathbf{w}), \tag{2.13}$$

where $\mathbf{w} = (u, v)^T$, with $r = p$ or $r = q$, depending on the different possible situations, and if $t \ge 0$. In particular, it follows that

$$\mathcal{A}_1(t\mathbf{w}) \ge t^{r-1}\mathcal{A}_1\mathbf{w}, \quad \mathcal{A}_2(t\mathbf{w}) \ge t^{r-1}\mathcal{A}_2\mathbf{w}.$$

The last property is similar to the equivalent one for linear operators

$$\mathcal{L}(\mu\mathbf{w}) = \mu\mathcal{L}(\mathbf{w}),$$

which means that the operator is homogeneous of degree one or simply homogeneous.

Furthermore, thanks to [15, Lemma 5.5] and (2.12) we actually have that weak solutions are uniformly bounded in $L^\infty(\Omega) \times L^\infty(\Omega)$. Thus, due to [18, Theorem 1, page 1203] the solutions of system (2.2) belong to $C^{1,\alpha}(\Omega) \times C^{1,\alpha}(\Omega)$, for some $0 < \alpha < 1$.

It is important to point out that the next result is also valid for the limiting problem (1.12) under the heterogenous assumptions established at the beginning of this work.

Lemma 2.4. *Let τ_1 be the smallest eigenvalue of the problem* (2.2)*, under homogeneous Dirichlet boundary conditions and denoted by*

$$\tau_1 := \tau_1[\mathcal{S}, \Omega].$$

Moreover, τ_1 is algebraically simple, isolated and it possesses a unique positive eigenfunction (up to a multiplicative constant), denoted by $(\varphi_1, \psi_1)^T$. Furthermore, $(\varphi_1, \psi_1)^T$ is strictly positive and there is not any other eigenvalue τ of (2.2) *satisfying $\tau < \tau_1$.*

Proof. **Positivity of the first eigenvalue**. To prove this we claim that if it exists $(u,v)^T \in \mathcal{K}\setminus\{0\}$ so that

$$\mathcal{S}\begin{pmatrix}u\\ v\end{pmatrix} \geq \mu \begin{pmatrix}|u|^{p-2}u\\ |v|^{q-2}v\end{pmatrix},$$

with $\mu > 0$, then, there is $\tau_1 \geq \mu$ and $(\varphi_1,\psi_1)^T \in \mathcal{K}\setminus\{0\}$ such that

$$\mathcal{S}\begin{pmatrix}\varphi_1\\ \psi_1\end{pmatrix} = \tau_1 \begin{pmatrix}|\varphi_1|^{p-2}\varphi_1\\ |\psi_1|^{q-2}\psi_1\end{pmatrix}.$$

In other words, the topological degree on the positive cone $\mathcal{K}$ for the operator

$$\Big(\operatorname{diag}(|u|^{p-2}u, |v|^{q-2}v) - \frac{1}{\mu}\mathcal{S}\begin{pmatrix}u\\ v\end{pmatrix}\Big),$$

in the unit ball changes. Hence, the problem

$$\begin{pmatrix}|u|^{p-2}u\\ |v|^{q-2}v\end{pmatrix} = \frac{1}{\mu}\mathcal{S}\begin{pmatrix}u\\ v\end{pmatrix} + t\begin{pmatrix}|\varphi_1|^{p-2}\varphi_1\\ |\psi|^{q-2}\psi_1\end{pmatrix}, \tag{2.14}$$

has no solution in $\mathcal{K}$ if $t > 0$. To prove this, assume that if $(u,v)^T$ is a solution of (2.14) in $\mathcal{K}\setminus\{0\}$ we have that

$$\begin{pmatrix}|u|^{p-2}u\\ |v|^{q-2}v\end{pmatrix} \geq t\begin{pmatrix}|\varphi_1|^{p-2}\varphi_1\\ |\psi|^{q-2}\psi_1\end{pmatrix} \quad \text{and} \quad \begin{pmatrix}|u|^{p-2}u\\ |v|^{q-2}v\end{pmatrix} \geq \tilde{t}\begin{pmatrix}|\varphi_1|^{p-2}\varphi_1\\ |\psi|^{q-2}\psi_1\end{pmatrix},$$

where $\tilde{t} = \tilde{s}^{r-1}$ is the maximum among all the t's. Then

$$\begin{pmatrix}|u|^{p-2}u\\ |v|^{q-2}v\end{pmatrix} \geq \frac{1}{\mu}\mathcal{S}\begin{pmatrix}\tilde{t}\varphi_1\\ \tilde{t}\psi_1\end{pmatrix} + t\begin{pmatrix}|\varphi_1|^{p-2}\varphi_1\\ |\psi|^{q-2}\psi_1\end{pmatrix},$$

and, hence, since by (2.13) and assuming (2.12),

$$\mathcal{S}\begin{pmatrix}\tilde{t}\varphi_1\\ \tilde{t}\psi_1\end{pmatrix} \geq \tilde{s}^{r-1}\mathcal{S}\begin{pmatrix}\varphi_1\\ \psi_1\end{pmatrix},$$

and, by definition of $(\varphi_1,\psi_1)^T$, we find that

$$\begin{aligned}\begin{pmatrix}|u|^{p-2}u\\ |v|^{q-2}v\end{pmatrix} &\geq \frac{\tilde{s}^{r-1}}{\mu}\mu\begin{pmatrix}|\varphi_1|^{p-2}\varphi_1\\ |\psi|^{q-2}\psi_1\end{pmatrix} + t\begin{pmatrix}|\varphi_1|^{p-2}\varphi_1\\ |\psi|^{q-2}\psi_1\end{pmatrix}\\ &= (\tilde{t}+t)\begin{pmatrix}|\varphi_1|^{p-2}\varphi_1\\ |\psi|^{q-2}\psi_1\end{pmatrix},\end{aligned}$$

which is a contradiction with the maximality of $\tilde{t}$. Therefore, there is no solution to equation (2.14) if $t > 0$ or the topological degree on the positive cone $\mathcal{K}$ of the operator

$$\Big(\operatorname{diag}(|u|^{p-2}u, |v|^{q-2}v) - \frac{1}{\mu}\mathcal{S}\begin{pmatrix}u\\ v\end{pmatrix}\Big),$$

in the unit ball is zero if $t > 0$. Consequently, $t = 0$ and, hence,

either, there exists a positive solution, or the topological degree on the positive cone $\mathcal{K}$ changes.

In the first situation, we have that the eigenvalue is positive. On the other hand, for the second situation we have that the degree is 1 and we find a positive eigenvalue. Also, the degree is 1 if $\tau_1 = 0$. However, if that is the case we will find that if there exists $\lambda \in [0, \frac{1}{\mu}]$ such that

$$\begin{pmatrix}|u|^{p-2}u\\ |v|^{q-2}v\end{pmatrix} = \lambda\mathcal{S}(\mathbf{w}),$$

where $\mathbf{w} = (u,v)^T$, with normalized $\mathbf{w} \in \mathcal{K}$, $\mathbf{w} = (u,v)^T$, then we actually have that $\lambda = \frac{1}{\mu}$. To prove it, let us take a sequence $\lambda_n \to \frac{1}{\mu}$, as $n \to \infty$ and $\mathbf{w}_n$ positive solutions so that

$$\mathbf{w}_n = \lambda_n \mathcal{S}(\mathbf{w}_n), \quad \text{with} \quad \mathbf{w}_n \to \mathbf{w}_0, \quad \text{as} \quad n \to \infty,$$

where $\mathbf{w}_0$ is a positive solution. Thus,

$$\mathbf{w}_n = \lambda_n \mathcal{S}(\mathbf{w}_n) = \frac{1}{\mu}\mathcal{S}(\mathbf{w}_n) + \Big(\lambda_n - \frac{1}{\mu}\Big)\mathcal{S}(\mathbf{w}_n).$$

Therefore, passing to the limit we obtain only positive solutions if $\lambda = 1/\mu$, proving that the first eigenvalue is strictly positive.

Non-existence of positive eigenvalue smaller (or bigger) than τ_1. Next we prove that there is no other positive eigenvalue smaller than τ_1, assuming that for the first eigenvalue its associated eigenfunction $(\varphi_1,\psi_1)^T$ has both components non-negative, i.e.

$$\begin{pmatrix} |\varphi_1|^{p-2}\varphi_1 \\ |\psi_1|^{q-2}\psi_1 \end{pmatrix} = \frac{1}{\tau_1}\mathcal{S}\begin{pmatrix} \varphi_1 \\ \psi_1 \end{pmatrix}, \quad \text{where} \quad (\varphi_1,\psi_1)^T \in \mathcal{K}.$$

Subsequently, to prove that τ_1 is the smallest positive eigenvalue we argue by contradiction. Thus, let us assume that there exists a positive eigenfunction $(u,v)^T$ so that $u > 0$ and $v > 0$ and satisfying the eigenvalue problem (2.2), i.e.

$$\begin{pmatrix} |u|^{p-2}u \\ |v|^{q-2}v \end{pmatrix} = \frac{1}{\tau}\mathcal{S}\begin{pmatrix} u \\ v \end{pmatrix}, \quad \text{where } (u,v)^T \in \mathcal{K}, \tag{2.15}$$

and with $0 < \tau < \tau_1$. In other words, there is another positive eigenvalue smaller than τ_1 with an associated positive eigenfunction. Then, by definition we have that

$$\begin{aligned} \begin{pmatrix} |u|^{p-2}u \\ |v|^{q-2}v \end{pmatrix} &= \frac{1}{\tau}\mathcal{S}\begin{pmatrix} u \\ v \end{pmatrix} = \frac{1}{\tau_1}\mathcal{S}\begin{pmatrix} u \\ v \end{pmatrix} + \Big(\frac{1}{\tau} - \frac{1}{\tau_1}\Big)\mathcal{S}\begin{pmatrix} u \\ v \end{pmatrix} \\ &= \frac{1}{\tau_1}\mathcal{S}\begin{pmatrix} u \\ v \end{pmatrix} + \Big(\frac{1}{\tau} - \frac{1}{\tau_1}\Big)\tau\begin{pmatrix} |u|^{p-2}u \\ |v|^{q-2}v \end{pmatrix} \\ &\geq \frac{1}{\tau_1}\mathcal{S}\begin{pmatrix} u \\ v \end{pmatrix} + \Big(1 - \frac{\tau}{\tau_1}\Big)^{r-1}\begin{pmatrix} |u|^{p-2}u \\ |v|^{q-2}v \end{pmatrix}. \end{aligned}$$

Note that

$$0 < 1 - \frac{\tau}{\tau_1} < 1, \quad \text{since} \quad \tau < \tau_1.$$

Next, we show that for a certain k (to be determined below) it follows that

$$\mathcal{S}\begin{pmatrix} u \\ v \end{pmatrix} \geq \mathcal{S}\begin{pmatrix} k\varphi_1 \\ k\psi_1 \end{pmatrix}. \tag{2.16}$$

To prove so we use the weak formulation of the eigenvalue problem (2.3) i.e.

$$\begin{aligned} &\int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla \nu_1 - k^{p-1}\int_\Omega |\nabla \varphi_1|^{p-2}\nabla\varphi_1 \cdot \nabla\nu_1 - \int_\Omega b|u|^\alpha|v|^{\beta+1}\nu_1 \\ &+ k^{\alpha+\beta+1}\int_\Omega b|\varphi_1|^\alpha|\psi_1|^{\beta+1}\nu_1 \\ &= \tau\int_\Omega |u|^{p-2}u\,\nu_1 - k^{p-1}\int_\Omega |\varphi_1|^{p-2}\varphi_1\,\nu_1, \end{aligned}$$

$$
\begin{aligned}
&\int_\Omega |\nabla v|^{q-2}\nabla v\cdot\nabla\nu_2 - k^{q-1}\int_\Omega |\nabla\psi_1|^{q-2}\nabla\psi_1\cdot\nabla\nu_2 - \int_\Omega c|u|^{\alpha+1}|v|^{\beta}\nu_2\\
&+ k^{\alpha+\beta+1}\int_\Omega c|\varphi_1|^{\alpha+1}|\psi_1|^{\beta}\nu_2\\
&= \tau\int_\Omega |v|^{q-2}v\,\nu_2 - k^{q-1}\int_\Omega |\psi_1|^{q-2}\psi_1\,\nu_2,
\end{aligned}
$$

for a test function $(\nu_1,\nu_2)^T$. Since the eigenfunctions are positive and the system is cooperative ($b(x)>0$ and $c(x)>0$) taking

$$
B = \max_{x\in\overline{\Omega}}\{b(x), c(x)\},
$$

we have that (we also add $\alpha+1$ in the first equation and $b+1$ in the second, for the coupling terms)

$$
\begin{aligned}
&\int_\Omega |\nabla u|^{p-2}\nabla u\cdot\nabla\nu_1 - k^{p-1}\int_\Omega |\nabla\varphi_1|^{p-2}\nabla\varphi_1\cdot\nabla\nu_1 - \int_\Omega b|u|^{\alpha}|v|^{\beta+1}\nu_1\\
&+ k^{\alpha+\beta+1}\int_\Omega b|\varphi_1|^{\alpha}|\psi_1|^{\beta+1}\nu_1\\
&\geq \int_\Omega |\nabla u|^{p-2}\nabla u\cdot\nabla\nu_1\\
&\quad - (\alpha+1)\int_\Omega B|u|^{\alpha}|v|^{\beta+1}\nu_1 - k^{p-1}\int_\Omega |\nabla\varphi_1|^{p-2}\nabla\varphi_1\cdot\nabla\nu_1,\\
&\int_\Omega |\nabla v|^{q-2}\nabla v\cdot\nabla\nu_2 - k^{q-1}\int_\Omega |\nabla\psi_1|^{q-2}\nabla\psi_1\cdot\nabla\nu_2 - \int_\Omega c|u|^{\alpha+1}|v|^{\beta}\nu_2\\
&+ k^{\alpha+\beta+1}\int_\Omega c|\varphi_1|^{\alpha+1}|\psi_1|^{\beta}\nu_2\\
&\geq \int_\Omega |\nabla v|^{q-2}\nabla v\cdot\nabla\nu_2\\
&\quad - (\beta+1)\int_\Omega B|u|^{\alpha+1}|v|^{\beta}\nu_2 - k^{q-1}\int_\Omega |\nabla\psi_1|^{q-2}\nabla\psi_1\cdot\nabla\nu_2.
\end{aligned}
$$

Thanks to the variational cooperative eigenvalue problem (2.9) and the Rayleigh quotient of the first eigenvalue τ_0 (2.10) it follows that

$$
\begin{aligned}
&\int_\Omega |\nabla u|^{p-2}\nabla u\cdot\nabla\nu_1 - k^{p-1}\int_\Omega |\nabla\varphi_1|^{p-2}\nabla\varphi_1\cdot\nabla\nu_1 - \int_\Omega b|u|^{\alpha}|v|^{\beta+1}\nu_1\\
&+ k^{\alpha+\beta+1}\int_\Omega b|\varphi_1|^{\alpha}|\psi_1|^{\beta+1}\nu_1\\
&\geq \tau_0\int_\Omega |u|^{p-2}u\,\nu_1 - k^{p-1}\int_\Omega |\nabla\varphi_1|^{p-2}\nabla\varphi_1\cdot\nabla\nu_1,\\
&\int_\Omega |\nabla v|^{q-2}\nabla v\cdot\nabla\nu_2 - k^{q-1}\int_\Omega |\nabla\psi_1|^{q-2}\nabla\psi_1\cdot\nabla\nu_2 - \int_\Omega c|u|^{\alpha+1}|v|^{\beta}\nu_2\\
&+ k^{\alpha+\beta+1}\int_\Omega c|\varphi_1|^{\alpha+1}|\psi_1|^{\beta}\nu_2\\
&\geq \tau_0\int_\Omega |v|^{q-2}v\,\nu_2 - k^{q-1}\int_\Omega |\nabla\psi_1|^{q-2}\nabla\psi_1\cdot\nabla\nu_2.
\end{aligned}
$$

Hence, to arrive at the inequality (2.16) we need that

$$k \le \min\Big\{\Big(\frac{\tau_0\int_\Omega |u|^{p-2}u\,\nu_1}{\int_\Omega |\nabla\varphi_1|^{p-2}\nabla\varphi_1\cdot\nabla\nu_1}\Big)^{\frac{1}{p-1}}, \Big(\frac{\tau_0\int_\Omega |v|^{q-2}v\,\nu_2}{\int_\Omega |\nabla\psi_1|^{q-2}\nabla\psi_1\cdot\nabla\nu_2}\Big)^{\frac{1}{q-1}}\Big\}, \tag{2.17}$$

with $u,\varphi_1,\nu_1 \in W_0^{1,p}(\Omega)$ and $v,\psi_1,\nu_2 \in W_0^{1,q}(\Omega)$. The maximal value for k that satisfies (2.17) is denoted by

$$\kappa = k_{\max}.$$

In particular, expression (2.17) provides us with the condition

$$(|u|^{p-2}u, |v|^{q-2}v)^T \ge \kappa^{r-1}(|\varphi_1|^{p-2}\varphi_1, |\psi_1|^{q-2}\psi_1)^T, \tag{2.18}$$

with the appropriate $r = p$ or $r = q$ depending on (2.12). Consequently, thanks to (2.15) and (2.18) we find that

$$\begin{aligned}
\begin{pmatrix} |u|^{p-2}u \\ |v|^{q-2}v \end{pmatrix} &\ge \frac{1}{\tau_1}\mathcal{S}\begin{pmatrix} u \\ v \end{pmatrix} + \Big(1-\frac{\tau}{\tau_1}\Big)^{r-1}\begin{pmatrix} |u|^{p-2}u \\ |v|^{q-2}v \end{pmatrix} \\
&\ge \frac{1}{\tau_1}\mathcal{S}\begin{pmatrix} \kappa\varphi_1 \\ \kappa\psi_1 \end{pmatrix} + \Big(1-\frac{\tau}{\tau_1}\Big)^{r-1}\kappa^{r-1}\begin{pmatrix} |\varphi_1|^{p-2}\varphi_1 \\ |\psi_1|^{q-2}\psi_1 \end{pmatrix} \\
&\ge \kappa^{r-1}\begin{pmatrix} |\varphi_1|^{p-2}\varphi_1 \\ |\psi_1|^{q-2}\psi_1 \end{pmatrix} + \Big(1-\frac{\tau}{\tau_1}\Big)^{r-1}\kappa^{r-1}\begin{pmatrix} |\varphi_1|^{p-2}\varphi_1 \\ |\psi_1|^{q-2}\psi_1 \end{pmatrix} \\
&= \Big(\kappa^{r-1} + \Big(1-\frac{\tau}{\tau_1}\Big)^{r-1}\kappa^{r-1}\Big)\begin{pmatrix} |\varphi_1|^{p-2}\varphi_1 \\ |\psi_1|^{q-2}\psi_1 \end{pmatrix},
\end{aligned}$$

which contradicts the maximality of κ^{r-1} and proving that there is no positive eigenvalue below τ_1.

Similar arguments show that there is no eigenvalue bigger that τ_1 which has a positive associated eigenfunction.

Simplicity. To show that the principal eigenvalue τ_1 is a simple eigenvalue we assume that, apart from the eigenfunction $(\varphi_1,\psi_1)^T \in \mathcal{K}$, there exists another eigenfunction $(u,v)^T \in \mathcal{K}$. Arguing as above we arrive at $(u,v)^T \ge K_1(\varphi_1,\psi_1)^T$ and, also, the opposite inequality $(u,v)^T \le K_2(\varphi_1,\psi_1)^T$, so that $K_1K_1 = 1$, showing that both eigenfunctions are proportional. Therefore, the principal eigenvalue is simple.

Isolated principal eigenvalue τ_1. To prove it, we assume a sequence of eigenvalues $\{\tau_n\}$ of problem (2.2) such that $\tau_n \to \tau_1$, as $n \to \infty$. Moreover, the sequence of eigenfunctions $\{(u_n,v_n)^T\}$ belongs to the positive cone $\mathcal{K}$, normalized $\|(u_n,v_n)^T\|_{L^\infty(\Omega\times L^\infty(\Omega)} = 1$. Thanks to regularity theory and maximal principles we actually have that the sequence is bounded in $C^{1,\alpha}$, with $\alpha \in (0,1)$. Hence, we have the convergence of such a sequence in $C^{1,\alpha}$, up to a subsequence, i.e.

$$(u_n,v_n)^T \to (\varphi_1,\psi_1)^T, \quad \text{as } n \to \infty,$$

where $(\varphi_1,\psi_1)^T$ is the associated eigenfunction with the principal eigenvalue τ_1. Therefore, since the elements of the sequence will belong to the positive cone, and τ_1 is the only eigenvalue with a positive eigenfunction we arrive at a contradiction, proving that such an eigenvalue is isolated. $\square$

Remark 2.5. Lemma 2.4 is also true for the system with

$$\begin{aligned}
(-\Delta_p + \lambda a|u|^{p-2})u - b|u|^{\alpha-1}u|v|^{\beta}v &= \mathcal{A}_1(\lambda)\mathbf{w}, \\
(-\Delta_q + \lambda d|v|^{q-2})v - c|u|^{\alpha}u|v|^{\beta-1}v &= \mathcal{A}_2(\lambda)\mathbf{w}.
\end{aligned}$$

Also, thanks to the monotonicity of the principal eigenvalue with respect to the potential and with respect to the domain we know that the function $\tau_1(\lambda)$ is continuous and increasing with respect to the operator $\mathcal{S}(V_1, V_2)$ denoted by (1.3) (cf. [17]) so that the limit $\lim_{\lambda\to\infty}\tau_1(\lambda)$ exists. Indeed, under the spatial heterogenous conditions for the potentials a and d this limit is bounded above. Otherwise the limit could be possibly $\pm\infty$.

Furthermore, for the limiting system we consider the problem

$$\begin{aligned} -\Delta_p u - b|u|^{\alpha-1}u|v|^{\beta}v &= \tau_1|u|^{p-2}u,\\ -\Delta_q v - c|u|^{\alpha}u|v|^{\beta-1}v &= \tau_1|v|^{q-2}v, \end{aligned}$$

for $(u,v)^T \in W_0^{1,p}(\Omega_0^a)\times W_0^{1,q}(\Omega_0^d)$. We say that $(u,v)^T$ is a solution of this system when each equation is satisfied in the sense/framework

$$(-\Delta_p u + \lambda a|u|^{p-2})u = f, \quad u \in W_0^{1,p}(A),$$

so that A satisfies the spatial heterogeneous conditions under consideration in this paper and f in the dual space $W^{-1,p'}(A)$

3. Proof of the main results

To prove Theorem 1.1, we do the follow in 3 steps. First we prove the convergence of the eigenfunctions in $W_0^{1,p}(\Omega)\times W_0^{1,q}(\Omega)$.

Step 1. Convergence of the eigenfunctions $\{(\varphi_\lambda,\psi_\lambda)^T\}$ in $X := W_0^{1,p}(\Omega)\times W_0^{1,q}(\Omega)$. Let $\{\ell_n\}_{n\ge 1}$ be any increasing unbounded sequence, i.e. $0<\lambda_n<\lambda_m$ if $n<m$, and $\lim_{n\to\infty}\ell_n=\infty$. Then, for every $n\ge 1$ we consider a sequence $\{(\varphi_{\lambda_n},\psi_{\lambda_n})^T\}$ of normalized solutions in $Y := L^p(\Omega)\times L^q(\Omega)$ for system (1.1) in the sense that

$$\int_\Omega |\varphi_{\lambda_n}|^p + \int_\Omega |\psi_{\lambda_n}|^q = 1,$$

associated with $\tau_1(\lambda_n)$ for problem (1.1). Then, multiplying (1.1) by $(\varphi_{\lambda_n},\psi_{\lambda_n})^T$ and integrating by parts yields

$$\begin{aligned} &\int_\Omega |\nabla\varphi_{\lambda_n}|^p + \int_\Omega |\nabla\psi_{\lambda_n}|^q + \lambda_n\int_\Omega (a|\varphi_{\lambda_n}|^p + d|\psi_{\lambda_n}|^q)\\ &= \tau_1(\lambda_n) + \int_\Omega (b+c)|\varphi_{\lambda_n}|^{\alpha+1}|\psi_{\lambda_n}|^{\beta+1}. \end{aligned}$$

It is now clear that by Hölder's inequality, condition (1.2), and the cooperative assumptions on the coefficients b and c, we find that

$$\int_\Omega (b+c)|\varphi_{\lambda_n}|^{\alpha+1}|\psi_{\lambda_n}|^{\beta+1} \le C\Big(\int_\Omega |\varphi_{\lambda_n}|^p\Big)^{\frac{\alpha+1}{p}}\Big(\int_\Omega |\psi_{\lambda_n}|^q\Big)^{\frac{\beta+1}{q}},$$

for a positive constant $C>0$, so that by construction we have

$$\int_\Omega |\nabla\varphi_{\lambda_n}|^p \le K, \quad \int_\Omega |\nabla\psi_{\lambda_n}|^q \le K, \quad \lambda_n\int_\Omega (a\varphi_{\lambda_n}^p + d\psi_{\lambda_n}^q) \le K, \tag{3.1}$$

for a positive constant K. We point out that the principal eigenvalue $\tau_1(\lambda)$ is bounded above, thanks to the monotonicity of the principal eigenvalue with respect to the domain, by the principal eigenvalue for the operator $\mathcal{S}$ under Dirichlet homogeneous boundary conditions in the subdomain Ω_0 and denoted by (1.4), i.e.

$$\tau_1(\lambda) \le \tau_1[\mathcal{S};\Omega_0].$$

Hence, the sequence $\{(\varphi_{\lambda_n},\psi_{\lambda_n})^T\}$ is bounded in $X := W_0^{1,p}(\Omega)\times W_0^{1,q}(\Omega)$. As the imbedding $W_0^{1,r}(\Omega)\hookrightarrow L^r(\Omega)$ with $r=p$, $r=q$, is compact, we can extract a subsequence, again labeled $\{(\varphi_{\lambda_n},\psi_{\lambda_n})^T\}$, weakly convergent in X and strongly in Y to some function $(\varphi_*,\psi_*)\in Y$, i.e.

$$\lim_{\lambda\to\infty}\|(\varphi_{\lambda_n},\psi_{\lambda_n})^T-(\varphi_*,\psi_*)^T\|_Y=0,\quad Y=L^p(\Omega)\times L^q(\Omega).$$

In fact, we will prove in the sequel that the sequence $\{(\varphi_{\lambda_n},\psi_{\lambda_n})^T\}$ is actually a Cauchy sequence in X. In other words, we have the strong convergence of that subsequence in U, this implies that

$$\lim_{\lambda\to\infty}\|(\varphi_{\lambda_n},\psi_{\lambda_n})^T-(\varphi_*,\psi_*)^T\|_X=0,\quad X=W_0^{1,p}(\Omega)\times W_0^{1,q}(\Omega).$$

with $(\varphi_*,\psi_*)^T\in X$ and such that

$$\int_\Omega|\varphi_*|^p+\int_\Omega|\psi_*|^q=1.$$

Subsequently, fix $n<m$ so that $0<\lambda_n<\lambda_m$ and set

$$D_{n,m}:=\int_\Omega\big|\nabla(\varphi_{\lambda_n}-\varphi_{\lambda_m})\big|^p+\int_\Omega\big|\nabla(\psi_{\lambda_n}-\psi_{\lambda_m})\big|^q. \tag{3.2}$$

To obtain the convergence we consider the so-called Clarkson's inequality i.e.

$$\frac{|u_1-u_2|^p}{2^{p-1}}+\frac{|u_1+u_2|^p}{2^{p-1}}\le|u_1|^p+|u_2|^p. \tag{3.3}$$

Also, thanks to the strict convexity of the mapping $u\mapsto|u|^p$ it follows that

$$|u_2|^p>|u_1|^p+p|u_1|^{p-2}u_1(u_2-u_1),\quad\text{for points in }\mathbb{R}^N\text{ and }u_1\neq u_2,\ p>1. \tag{3.4}$$

Hence, according to (3.4) for u_1+u_2,

$$\frac{|u_1+u_2|^p}{2^p}\ge|u_1|^p+\frac{1}{2}p|u_1|^{p-2}u_1(u_2-u_1). \tag{3.5}$$

Consequently, combining both inequalities (3.3), (3.5) we find that

$$\frac{|u_1-u_2|^p}{2^{p-1}}\le|u_2|^p-|u_1|^p-p|u_1|^{p-2}u_1(u_2-u_1),$$

and, hence, if $\nabla(\varphi_{\lambda_n}-\varphi_{\lambda_m})=u_1-u_2$ it yields

$$\frac{|\nabla\varphi_{\lambda_n}-\nabla\varphi_{\lambda_m}|^p}{2^{p-1}}\le|\nabla\varphi_{\lambda_m}|^p-|\nabla\varphi_{\lambda_n}|^p-p|\nabla\varphi_{\lambda_n}|^{p-2}\nabla\varphi_{\lambda_n}\cdot(\nabla\varphi_{\lambda_m}-\nabla\varphi_{\lambda_n}).$$

Indeed, integrating ver Ω,

$$\begin{aligned}&\frac{1}{2^{p-1}}\int_\Omega|\nabla\varphi_{\lambda_n}-\nabla\varphi_{\lambda_m}|^p\\&\le\int_\Omega|\nabla\varphi_{\lambda_m}|^p-\int_\Omega|\nabla\varphi_{\lambda_n}|^p-\int_\Omega p|\nabla\varphi_{\lambda_n}|^{p-2}\nabla\varphi_{\lambda_n}\cdot(\nabla\varphi_{\lambda_m}-\nabla\varphi_{\lambda_n}).\end{aligned}$$

Then

$$\begin{aligned}D_{n,m}\le&\int_\Omega|\nabla\varphi_{\lambda_m}|^p-\int_\Omega|\nabla\varphi_{\lambda_n}|^p-\int_\Omega p|\nabla\varphi_{\lambda_n}|^{p-2}\nabla\varphi_{\lambda_n}\cdot(\nabla\varphi_{\lambda_m}-\nabla\varphi_{\lambda_n})\\&+\int_\Omega|\nabla\psi_{\lambda_m}|^q-\int_\Omega|\nabla\psi_{\lambda_n}|^q-\int_\Omega q|\nabla\psi_{\lambda_n}|^{q-2}\nabla\psi_{\lambda_n}\cdot(\nabla\psi_{\lambda_m}-\nabla\psi_{\lambda_n}).\end{aligned}$$

Now, applying the weak formulation (2.1) for the eigenvalue problem (1.1) with the principal eigenfunctions $\{(\varphi_{\lambda_n},\psi_{\lambda_n})^T\}$ (respectively $\{(\varphi_{\lambda_m},\psi_{\lambda_m})^T\}$) and rearranging terms we find that

$$
\begin{aligned}
&D_{n,m}\\
&\le \tau_1(\lambda_m)\int_\Omega|\varphi_{\lambda_m}|^p-\lambda_m\int_\Omega a|\varphi_{\lambda_m}|^p+\int_\Omega b|\varphi_{\lambda_m}|^{\alpha+1}|\psi_{\lambda_m}|^{\beta+1}\\
&\quad+\tau_1(\lambda_m)\int_\Omega|\psi_{\lambda_m}|^q-\lambda_m\int_\Omega d|\psi_{\lambda_m}|^q+\int_\Omega c|\varphi_{\lambda_m}|^{\alpha+1}|\psi_{\lambda_m}|^{\beta+1}\\
&\quad+(p-1)\Big[\tau_1(\lambda_n)\int_\Omega|\varphi_{\lambda_n}|^p-\lambda_n\int_\Omega a|\varphi_{\lambda_n}|^p+\int_\Omega b|\varphi_{\lambda_n}|^{\alpha+1}|\psi_{\lambda_n}|^{\beta+1}\Big]\\
&\quad+(q-1)\Big[\tau_1(\lambda_n)\int_\Omega|\psi_{\lambda_n}|^q-\lambda_n\int_\Omega d|\psi_{\lambda_n}|^q+\int_\Omega c|\varphi_{\lambda_n}|^{\alpha+1}|\psi_{\lambda_n}|^{\beta+1}\Big]\\
&\quad-p\Big[\tau_1(\lambda_n)\int_\Omega|\varphi_{\lambda_n}|^{p-1}\varphi_{\lambda_m}-\lambda_n\int_\Omega a|\varphi_{\lambda_n}|^{p-1}\varphi_{\lambda_m}+\int_\Omega b|\varphi_{\lambda_n}|^{\alpha}|\psi_{\lambda_n}|^{\beta+1}\varphi_{\lambda_m}\Big]\\
&\quad-q\Big[\tau_1(\lambda_n)\int_\Omega|\psi_{\lambda_n}|^{q-1}\psi_{\lambda_m}-\lambda_n\int_\Omega a|\psi_{\lambda_n}|^{q-1}\psi_{\lambda_m}+\int_\Omega c|\varphi_{\lambda_n}|^{\alpha+1}|\psi_{\lambda_n}|^{\beta}\psi_{\lambda_m}\Big].
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
D_{n,m}&\le \tau_1(\lambda_m)\Big(\int_\Omega|\varphi_{\lambda_m}|^p+\int_\Omega|\psi_{\lambda_m}|^q\Big)-\tau_1(\lambda_n)\Big(\int_\Omega|\varphi_{\lambda_n}|^p+\int_\Omega|\psi_{\lambda_n}|^q\Big)\\
&\quad+p\tau_1(\lambda_n)\int_\Omega|\varphi_{\lambda_n}|^{p-1}(\varphi_{\lambda_n}-\varphi_{\lambda_m})+q\tau_1(\lambda_n)\int_\Omega|\psi_{\lambda_n}|^{q-1}(\psi_{\lambda_n}-\psi_{\lambda_m})\\
&\quad+p\lambda_n\int_\Omega a|\varphi_{\lambda_n}|^{p-1}(\varphi_{\lambda_m}-\varphi_{\lambda_n})+q\lambda_n\int_\Omega d|\psi_{\lambda_n}|^{q-1}(\psi_{\lambda_m}-\psi_{\lambda_n})\\
&\quad+\lambda_n\int_\Omega a|\varphi_{\lambda_n}|^p-\lambda_m\int_\Omega a|\varphi_{\lambda_m}|^p+\lambda_n\int_\Omega d|\psi_{\lambda_n}|^q-\lambda_m\int_\Omega d|\psi_{\lambda_m}|^q\\
&\quad+p\int_\Omega b|\varphi_{\lambda_n}|^{\alpha}|\psi_{\lambda_n}|^{\beta+1}(\varphi_{\lambda_n}-\varphi_{\lambda_m})+q\int_\Omega c|\varphi_{\lambda_n}|^{\alpha+1}|\psi_{\lambda_n}|^{\beta}(\psi_{\lambda_n}-\psi_{\lambda_m}).
\end{aligned}
$$

For the terms involving the non-negative potentials a and d and thanks to the convexity property (3.4), it follows that

$$
\begin{aligned}
&\lambda_n\int_\Omega a|\varphi_{\lambda_n}|^p+p\lambda_n\int_\Omega a|\varphi_{\lambda_n}|^{p-1}(\varphi_{\lambda_m}-\varphi_{\lambda_n})-\lambda_m\int_\Omega a|\varphi_{\lambda_m}|^p\\
&<(\lambda_n-\lambda_m)\int_\Omega a|\varphi_{\lambda_m}|^p\le 0,
\end{aligned}
$$

and

$$
\begin{aligned}
&\lambda_n\int_\Omega d|\psi_{\lambda_n}|^q+q\lambda_n\int_\Omega d|\psi_{\lambda_n}|^{q-1}(\psi_{\lambda_m}-\psi_{\lambda_n})-\lambda_m\int_\Omega d|\psi_{\lambda_m}|^q\\
&<(\lambda_n-\lambda_m)\int_\Omega d|\psi_{\lambda_m}|^q\le 0,
\end{aligned}
$$

since, by construction $n<m$ and, then $\lambda_n<\lambda_m$. Hence, after adding and subtracting some appropriate terms it yields

$$
D_{n,m}\le(\tau_1(\lambda_m)-\tau_1(\lambda_n))\Big(\int_\Omega|\varphi_{\lambda_m}|^p+\int_\Omega|\psi_{\lambda_m}|^q\Big)
$$

$$
\begin{aligned}
&+\tau_1(\lambda_n)\Big(\int_\Omega(|\varphi_{\lambda_m}|^p-|\varphi_{\lambda_n}|^p+\int_\Omega(|\psi_{\lambda_m}|^q-|\psi_{\lambda_n}|^q)\Big)\\
&+p\tau_1(\lambda_n)\int_\Omega|\varphi_{\lambda_n}|^{p-1}(\varphi_{\lambda_n}-\varphi_{\lambda_m})+q\tau_1(\lambda_n)\int_\Omega|\psi_{\lambda_n}|^{q-1}(\psi_{\lambda_n}-\psi_{\lambda_m})\\
&+p\int_\Omega b|\varphi_{\lambda_n}|^\alpha|\psi_{\lambda_n}|^{\beta+1}(\varphi_{\lambda_n}-\varphi_{\lambda_m})+q\int_\Omega c|\varphi_{\lambda_n}|^{\alpha+1}|\psi_{\lambda_n}|^\beta(\psi_{\lambda_n}-\psi_{\lambda_m}).
\end{aligned}
$$

For the second term we use again the convexity property (3.4), thus

$$
\begin{aligned}
\tau_1(\lambda_n)\Big(\int_\Omega(|\varphi_{\lambda_m}|^p-|\varphi_{\lambda_n}|^p)\Big)&\le\tau_1(\lambda_n)p\int_\Omega|\varphi_{\lambda_m}|^{p-2}\varphi_{\lambda_m}(\varphi_{\lambda_m}-\varphi_{\lambda_n}),\\
\tau_1(\lambda_n)\Big(\int_\Omega(|\psi_{\lambda_m}|^q-|\psi_{\lambda_n}|^q)\Big)&\le\tau_1(\lambda_n)q\int_\Omega|\psi_{\lambda_m}|^{q-2}\psi_{\lambda_m}(\psi_{\lambda_m}-\psi_{\lambda_n}).
\end{aligned}
$$

Consequently, applying Hölder's inequality and the fact that the eigenfunctions are bounded in $Y=L^p(\Omega)\times L^q(\Omega)$ to the different terms we have that for a positive constant C,

$$
\begin{aligned}
(\tau_1(\lambda_m)-\tau_1(\lambda_n))\Big(\int_\Omega|\varphi_{\lambda_m}|^p+\int_\Omega|\psi_{\lambda_m}|^q\Big)&\le C(\tau_1(\lambda_m)-\tau_1(\lambda_n)),\\
\tau_1(\lambda_n)\int_\Omega(|\varphi_{\lambda_m}|^p-|\varphi_{\lambda_n}|^p)&\le\tau_1(\lambda_n)p\int_\Omega|\varphi_{\lambda_m}|^{p-2}\varphi_{\lambda_m}(\varphi_{\lambda_m}-\varphi_{\lambda_n})\\
&\le C\|\varphi_{\lambda_m}-\varphi_{\lambda_n}\|_{L^p(\Omega)},\\
\tau_1(\lambda_n)\int_\Omega(|\psi_{\lambda_m}|^q-|\psi_{\lambda_n}|^q)&\le\tau_1(\lambda_n)q\int_\Omega|\psi_{\lambda_m}|^{q-2}\psi_{\lambda_m}(\psi_{\lambda_m}-\psi_{\lambda_n})\\
&\le C\|\psi_{\lambda_m}-\psi_{\lambda_n}\|_{L^q(\Omega)},\\
p\tau_1(\lambda_n)\int_\Omega|\varphi_{\lambda_n}|^{p-1}(\varphi_{\lambda_n}-\varphi_{\lambda_m})&\le C\Big(\int_\Omega|\varphi_{\lambda_n}|^{p'}\Big)^{1/p'}\Big(\int_\Omega|\varphi_{\lambda_n}-\varphi_{\lambda_m}|^p\Big)^{1/p}\\
&\le C\|\varphi_{\lambda_n}-\varphi_{\lambda_m}\|_{L^p(\Omega)},\\
q\tau_1(\lambda_n)\int_\Omega|\psi_{\lambda_n}|^{q-1}(\psi_{\lambda_n}-\psi_{\lambda_m})&\le C\Big(\int_\Omega|\psi_{\lambda_n}|^{q'}\Big)^{1/q'}\Big(\int_\Omega|\psi_{\lambda_n}-\psi_{\lambda_m}|^q\Big)^{1/q}\\
&\le C\|\psi_{\lambda_n}-\psi_{\lambda_m}\|_{L^q(\Omega)}.
\end{aligned}
$$

Also, for the terms with the cooperative coefficients b and c, we have

$$
p\int_\Omega b|\varphi_{\lambda_n}|^\alpha|\psi_{\lambda_n}|^{\beta+1}(\varphi_{\lambda_n}-\varphi_{\lambda_m})\quad\text{and}\quad q\int_\Omega c|\varphi_{\lambda_n}|^{\alpha+1}|\psi_{\lambda_n}|^\beta(\psi_{\lambda_n}-\psi_{\lambda_m}).
$$

we will apply Young's and Hölder's inequalities. Indeed, by Young's inequality and assuming the cooperative term $b(x)$ is bounded in Ω, we find that

$$
\begin{aligned}
&p\int_\Omega b|\varphi_{\lambda_n}|^\alpha|\psi_{\lambda_n}|^{\beta+1}(\varphi_{\lambda_n}-\varphi_{\lambda_m})\\
&\le C\int_\Omega\left(\frac{\alpha+1}{p}|\varphi_{\lambda_n}|^{\frac{\alpha p}{\alpha+1}}+\frac{\beta+1}{q}|\psi_{\lambda_n}|^q\right)(\varphi_{\lambda_n}-\varphi_{\lambda_m}).
\end{aligned}
$$

Moreover, thanks to Hölder's inequality,

$$
\begin{aligned}
&\int_\Omega\Big(\frac{\alpha+1}{p}|\varphi_{\lambda_n}|^{\frac{\alpha p}{\alpha+1}}+\frac{\beta+1}{q}|\psi_{\lambda_n}|^q\Big)(\varphi_{\lambda_n}-\varphi_{\lambda_m})\\
&\le C\Big(\int_\Omega|\varphi_{\lambda_n}|^{\frac{\alpha(p-1)}{\alpha+1}}\Big)^{1/p'}\Big(\int_\Omega|\varphi_{\lambda_n}-\varphi_{\lambda_m}|^p\Big)^{1/p}
\end{aligned}
$$

$$+ C\Big(\int_\Omega |\psi_{\lambda_n}|^{qp'}\Big)^{1/p'}\Big(\int_\Omega |\varphi_{\lambda_n}-\varphi_{\lambda_m}|^p\Big)^{1/p}.$$

Thus, by [11, Lemma 2] it follows that

$$\|\psi_{\lambda_n}\|_{qp'}^q \le \epsilon\int_\Omega |\nabla\psi_{\lambda_n}|^q + M_\epsilon\int_\Omega \omega|\psi_{\lambda_n}|^q,$$

for $\epsilon>0$, a positive constant M_ϵ depending on ϵ and a bounded positive weight ω. Then, we finally have that

$$\begin{aligned}
&p\int_\Omega b|\varphi_{\lambda_n}|^\alpha|\psi_{\lambda_n}|^{\beta+1}(\varphi_{\lambda_n}-\varphi_{\lambda_m})\\
&\le C\Big(\int_\Omega |\varphi_{\lambda_n}|^{\frac{\alpha(p-1)}{\alpha+1}}\Big)^{1/p'}\|\varphi_{\lambda_n}-\varphi_{\lambda_m}\|_{L^p(\Omega)}\\
&\quad + C\Big(\epsilon\int_\Omega |\nabla\psi_{\lambda_n}|^q + M_\epsilon\int_\Omega \omega|\psi_{\lambda_n}|^q\Big)\|\varphi_{\lambda_n}-\varphi_{\lambda_m}\|_{L^p(\Omega)}.
\end{aligned}$$

Similarly, for the term with the cooperative coefficient c we find that

$$\begin{aligned}
&q\int_\Omega c|\varphi_{\lambda_n}|^{\alpha+1}|\psi_{\lambda_n}|^{\beta}(\psi_{\lambda_n}-\psi_{\lambda_m})\\
&\le C\Big(\int_\Omega |\psi_{\lambda_n}|^{\frac{\beta(q-1)}{\beta+1}}\Big)^{1/q'}\Big(\int_\Omega |\psi_{\lambda_n}-\psi_{\lambda_m}|^q\Big)^{1/q}\\
&\quad + C\Big(\epsilon\int_\Omega |\nabla\varphi_{\lambda_n}|^p + M_\epsilon\int_\Omega \omega|\varphi_{\lambda_n}|^p\Big)\Big(\int_\Omega |\psi_{\lambda_n}-\psi_{\lambda_m}|^q\Big)^{1/q},
\end{aligned}$$

supposing that

$$\|\varphi_{\lambda_n}\|_{pq'}^p \le \epsilon\int_\Omega |\nabla\varphi_{\lambda_n}|^p + M_\epsilon\int_\Omega \omega|\varphi_{\lambda_n}|^p,$$

Hence, since the eigenfunctions are bounded in $Y=L^p(\Omega)\times L^q(\Omega)$,

$$\frac{\alpha(p-1)}{\alpha+1}<p \quad\text{and}\quad \frac{\beta(q-1)}{\beta+1}<q,$$

and thanks to (3.1) we finally obtain that

$$\begin{gathered}
p\int_\Omega b|\varphi_{\lambda_n}|^\alpha|\psi_{\lambda_n}|^{\beta+1}(\varphi_{\lambda_n}-\varphi_{\lambda_m}) \le C\|\varphi_{\lambda_n}-\varphi_{\lambda_m}\|_{L^p(\Omega)},\\
q\int_\Omega c|\varphi_{\lambda_n}|^{\alpha+1}|\psi_{\lambda_n}|^{\beta}(\psi_{\lambda_n}-\psi_{\lambda_m}) \le C\|\psi_{\lambda_n}-\psi_{\lambda_m}\|_{L^q(\Omega)}.
\end{gathered}$$

Therefore, by the previous inequalities we find that there exists a positive constant C such that

$$\begin{aligned}
&\int_\Omega \big|\nabla(\varphi_{\lambda_n}-\varphi_{\lambda_m})\big|^p + \int_\Omega \big|\nabla(\psi_{\lambda_n}-\psi_{\lambda_m})\big|^q\\
&\le C(\tau_1(\lambda_m)-\tau_1(\lambda_n)) + C\|\varphi_{\lambda_n}-\varphi_{\lambda_m}\|_{L^p(\Omega)} + C\|\psi_{\lambda_n}-\psi_{\lambda_m}\|_{L^q(\Omega)}.
\end{aligned}$$

Thus, thanks to the convergence of the sequence $\{\varphi_{\lambda_n}\}$ in L^p, the sequence $\{\psi_{\lambda_n}\}$ in L^q and the fact that the function $\tau_1(\lambda)$ is convergent, since it is an increasing function in λ and bounded above, by the monotonicity of the principal eigenvalue with respect to the potential and the domain, we actually have that the sequence $\{(\varphi_{\lambda_n},\psi_{\lambda_n})^T\}$ is a Cauchy sequence in X so that the limit $(\varphi_*,\psi_*)^T$ satisfies that

$$(\varphi_*,\psi_*)\ge(0,0), \quad\text{and}\quad \int_\Omega |\varphi_*|^p + \int_\Omega |\psi_*|^q = 1.$$

Step 2. $\{(\varphi_*, \psi_*)^T\}$ belongs to the functional space $W_0^{1,p}(\Omega_0^a) \times W_0^{1,q}(\Omega_0^d)$. First thanks to (3.1) we have that the sequence $\{(\varphi_{\lambda_n}, \psi_{\lambda_n})^T\}$ is bounded in X. Then, as performed above we can extract a subsequence, again labeled $\{(\varphi_{\lambda_n}, \psi_{\lambda_n})^T\}$, weakly convergent in X and strongly in $L^p(\Omega) \times L^q(\Omega)$ to some function $(\varphi_*, \psi_*)^T \in L^p(\Omega) \times L^q(\Omega)$. Actually, we have proved that the sequence converges strongly in $W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega)$.

Furthermore, according to (3.1) it follows that

$$\lambda_n \int_\Omega a\varphi_{\lambda_n}^p \leq K \quad \text{and} \quad \lambda_n \int_\Omega d\psi_{\lambda_n}^q \leq K.$$

Then, since $\{(\varphi_{\lambda_n}, \psi_{\lambda_n})^T\}$ converges strongly in $L^p(\Omega) \times L^q(\Omega)$ to $(\varphi_*, \psi_*)^T \in L^p(\Omega) \times L^q(\Omega)$. Indeed,

$$\lim_{n\to\infty} \int_\Omega a\varphi_{\lambda_n}^p = 0 \quad \text{and} \quad \lim_{n\to\infty} \int_\Omega d\psi_{\lambda_n}^q = 0,$$

from which we easily deduce that $\varphi_* = 0$ a.e. in Ω_+^a and, $\psi_* = 0$ a.e. in Ω_+^d. Consequently, we can conclude that

$$(\varphi_*, \psi_*)^T \in W_0^{1,p}(\Omega_0^a) \times W_0^{1,q}(\Omega_0^d).$$

Step 3. $(\varphi_*, \psi_*)^T$ eigenfunction of the limiting problem in $W_0^{1,p}(\Omega_0^a) \times W_0^{1,q}(\Omega_0^d)$. As a consequence of Step 1 it follows that

$$\begin{gathered}\tau_1(\lambda_n)|\varphi_{\lambda_n}|^{p-2}\varphi_{\lambda_n} + b|\varphi_{\lambda_n}|^{\alpha-1}\varphi_{\lambda_n}|\psi_{\lambda_n}|^\beta\psi_{\lambda_n}, \\ \tau_1(\lambda_n)|\psi_{\lambda_n}|^{q-2}\psi_{\lambda_n} + c|\varphi_{\lambda_n}|^\alpha\varphi_{\lambda_n}|\psi_{\lambda_n}|^{\beta-1}\psi_{\lambda_n},\end{gathered}$$

converge strongly in $L^p(\Omega) \times L^q(\Omega)$ to

$$\tau_1|\varphi_*|^{p-2}\varphi_* + b|\varphi_*|^{\alpha-1}\varphi_*|\psi_*|^\beta\psi_* \quad \text{and} \quad \tau_1|\psi_*|^{q-2}\psi_* + c|\varphi_*|^\alpha\varphi_*|\psi|^{\beta-1}\psi_*,$$

respectively. Therefore, from the assumptions on the potentials a and d and Step 2 we find that up to a subsequence, $\{(\varphi_{\lambda_n}, \psi_{\lambda_n})^T\}$ converges strongly in $W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega)$ and the limit

$$(\varphi_*, \psi_*)^T \in W_0^{1,p}(\Omega_0^a) \times W_0^{1,q}(\Omega_0^d),$$

is a solution of the eigenvalue problem

$$\begin{gathered}-\Delta_p\varphi_* - b|\varphi|^{\alpha-1}\varphi_*|\mathcal{P}\psi_*|^\beta\mathcal{P}\psi_* = \tau_1\varphi_* \\ -\Delta_q\psi_* - c|\mathcal{P}\varphi_*|^\alpha\mathcal{P}\varphi_*|\psi_*|^{\beta-1}\psi_* = \tau_1\psi_*\end{gathered} \tag{3.6}$$

for $(\varphi_*, \psi_*)^T \in W_0^{1,p}(\Omega_0^a) \times W_0^{1,q}(\Omega_0^d)$ and where $\mathcal{P}$ stands for the projection on the subdomain where both potentials vanish at the same time, i.e.

$$\mathcal{P}\phi = \chi_{\Omega_0^a\cap\Omega_0^d}\phi, \quad \text{with} \quad \phi = \varphi_*, \psi_*.$$

Thus, $\mathcal{P}$ is defined to be zero if the intersection is null, $\Omega_0^a \cap \Omega_0^d = \emptyset$. Furthermore, by the uniqueness of the principal eigenvalue of a singular p-Laplacian equation the sequence converges to the eigenfunction associated with the principal eigenvalue τ_1 for the uncoupled system (1.13), i.e. the principal eigenfunction whose components are the corresponding eigenfunctions for each equation in (1.13). Actually, we observe that if that was the case the limiting eigenvalue would be the infimum between the corresponding eigenvalues of the following uncoupled system

$$\begin{aligned} -\Delta_p \varphi_* &= \tau_1 |\varphi_*|^{p-2}\varphi_* \quad \text{in } \Omega_0^a, \\ -\Delta_q \psi_* &= \tau_1 |\psi_*|^{q-2}\psi_* \quad \text{in } \Omega_0^d, \end{aligned}$$

under homogeneous boundary conditions and such that

$$\tau_1 = \inf\{\tau_1[-\Delta_p, \Omega_0^a], \tau_1[-\Delta_q, \Omega_0^d]\}.$$

Acknowledgements. This research was partially supported by the Ministry of Economy and Competitiveness of Spain, under research project PID2019-106122GB-I00.

References

[1] P. Álvarez-Caudevilla, A. Lemenant; *Asymptotic analysis for some linear eigenvalue problems via Gamma–convergence*, Adv. Diff. Equations, **15**, (2010), 649–688.
[2] P. Álvarez-Caudevilla, J. López-Gómez; *Asymptotic behaviour of principal eigenvalues for a class of cooperative systems, "Corrigendum" in [J. Differential Equations* ***245*** *(2008), no. 2, 566–567]*, J. Differential Equations, **244**, (2008), 1093–1113.
[3] P. Álvarez-Caudevilla, J. López-Gómez; *Semiclassical analysis for highly degenerate potentials*, Proc. Amer. Math. Soc., **136**, (2008), 665–675.
[4] P. Álvarez-Caudevilla, J. López-Gómez; *Metasolutions in cooperative systems*, Nonlinear Anal.: Real World Appl., **9** (2008), 1119–1157.
[5] A. Anane, *Simplicité et isolation de la première valeur propre du p-laplacien avec poids*, C. R. Acad. Sci. Paris Sér. I Math., **305**, (1987), 725–728.
[6] G. Barles; *Remarks on uniqueness results of the first eigenvalue of the p-Laplacian*, Ann. Fac. Sci. Toulouse Math. (5), **9**, (1988), 65–75.
[7] Y. Bozhkov, E. Mitidieri; *Existence of multiple solutions for quasilinear systems via fibering method*, Journal of Differential Equations, **190**, (2003), 239–267.
[8] D Bucur, G. Buttazzo; *Variational methods in shape optimization problems*, Progress in Nonlinear Differential Equations and their Applications, 65, Birkhäuser Boston Inc., Boston, MA, 2005.
[9] D. C. Clark; *A variant of Lusternik–Schnirelman theory*, Indiana Univ. Math. J., **22** (1972), 65–74.
[10] P. Clément, J. Fleckinger, E. Mitidieri, F. de Thélin; *Existence of positive solutions for a non–variational quasilinear system*, Journal of Differential Equations, **166**, (2000), 455–477.
[11] M. Cuesta, H. Ramos Quoirin; *A weighted eigenvalue problem for the p-Laplacian plus a potential*, NoDEA - Nonlinear Differential Equations and Applications, **16**, (2009), no. 4, 469–491.
[12] E. N. Dancer; *On the least point spectrum of certain cooperative linear systems with a large parameter*, J. Differential Equations, **250** (2011), no. 1, 33–38.
[13] P. Drábek, J. Milota; *Methods of Nonlinear Analysis. Application to Differential Equations*, Birkhaäuser Advanced Texts (2007).
[14] P. Drábek, S.I. Pohozaev; *Positive solutions for the p-Laplacian: application of the fibering method*, Proc. Roy. Soc. Edinburgh, **127A** (1997), 703–726.
[15] J. García-Melián, J. Sabina de Lis, P. Takáč; *Dirichlet problems for the p-Laplacian with a convection term*, Revista Matemática Complutense, **30** (2017), no. 2, 313–334.
[16] B. Kawohl, P. Lindqvist; *Positive eigenfunctions for the p-Laplace operator revisited*, Analysis. International Mathematical Journal of Analysis and its Applications, **26** (2006), no. 4, 545–550.
[17] L. Leadi, H. Ramos Quoirin; *Principal eigenvalue for quasilinear cooperative elliptic systems*, Differential Integral Equations, **24**, (2011), 1107–1124.
[18] G. Lieberman; *Boundary regularity for solutions of degenerate elliptic equations*, Nonlinear Analysis. Theory, Methods and Applications, **12** (1988), no. 11, 1203–1219.

[19] X. Lin, S. Zheng; *Multiplicity and asymptotic behavior of solutions to fractional* (p, q)-*Kirchhoff type problems with critical Sobolev-Hardy exponent*, Electron. J. Differential Equations, Vol. 2021 (2021), No. 66, pp. 1–20.
[20] P. Lindqvist; *On the equation* $\operatorname{div}(|\nabla u|^{p-2}\nabla u) + \lambda|u|^{p-2}u = 0$, Proc. Amer. Math. Soc., **109** (1990), no. 1, 157–164.
[21] S. I. Pohozaev; *On an approach to nonlinear equations*, Soviet Math. Dokl., **20** (1979), 912–916.
[22] S. I. Pohozaev; *The fibering method in nonlinear variational problems*, Pitman Research Notes in Math., Vol. **365**, Pitman, 1997, pp. 35–88.
[23] P. Rabinowitz; *Variational methods for nonlinear eigenvalue problems*, In: Eigenvalue of Nonlinear Problems, Edizioni Cremonese, Rome, 1974, pp. 141–195.
[24] W. P. Ziemer; *Weakly differentiable functions*, Graduate Texts in Mathematics, **120**, Sobolev spaces and functions of bounded variation, Springer-Verlag, New York, 1989.

Pablo Álvarez-Caudevilla
Departamento de Matemáticas, Universidad Carlos III de Madrid, Av. Universidad 30, 28911, Leganés, Madrid, Spain
Email address: pcaudev@math.uc3m.es