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EXISTENCE OF GLOBAL SOLUTIONS AND BLOW-UP FOR p-LAPLACIAN PARABOLIC EQUATIONS WITH LOGARITHMIC NONLINEARITY ON METRIC GRAPHS

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Abstract. In this article, we study the initial-boundary value problem for a p-Laplacian parabolic equation with logarithmic nonlinearity on compact metric graphs. Firstly, we apply the Galerkin approximation technique to obtain the existence of a unique local solution. Secondly, by using the potential well theory with the Nehari manifold, we establish the existence of global solutions that decay to zero at infinity for all $p > 1$, and solutions that blow up at finite time when $p > 2$ and at infinity when $1 < p \leq 2$. Furthermore, we obtain decay estimates of the global solutions and lower bound on the blow-up rate.

1. Introduction

This article is devoted to studying the existence global solutions and blow-up for the p-Laplacian parabolic equations with logarithmic nonlinearity on compact metric graphs. A metric graph $\mathcal{G} = (E, V)$ is a connected metric space and consists of a set of edges E and a set of vertices V, where some of the endpoints of the edges are glued together at some vertices. Any edge $\mathrm{e} \in E$ is identified either with a closed bounded interval $I_{\mathrm{e}} = [0, \ell_{\mathrm{e}}]$ of length $\ell_{\mathrm{e}} > 0$, or with a closed half-line $I_{\mathrm{e}} = [0, \infty)$. A metric graph is compact if and only if it has a finite number of edges and vertices, and there are no half-lines.

In recent decades, motivated from various applications in physics, chemistry, biology, and engineering, the study of partial differential equations on metric graphs has attracted much attention. In particular, the results for parabolic equations on metric graphs (also called networks or one-dimensional ramified spaces in this case) first appeared in 1980s, see for example [36, 39, 43, 46]. Since then, there have been many studies on this subject, see [7, 11, 13, 22, 23, 26, 38] and the references therein. Around the same period, there were a lot of research conducted for nonlinear Schrödinger equations on metric graphs (also called quantum graphs), the interested reader is referred to [2, 3, 6, 8, 11, 25, 27, 28, 40]. For other evolution equations on metric graphs, we refer the reader to [4, 5, 9, 16, 31].

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In this article, we are concerned with the p-Laplacian parabolic problem on a compact metric graph $\mathcal{G} = (E, V)$,

$$
\begin{gathered}
(u_{\rm e})_t - (|u_{\rm e}'|^{p-2}u_{\rm e}')' + |u_{\rm e}|^{p-2}u_{\rm e} = |u_{\rm e}|^{p-2}u_{\rm e}\log|u_{\rm e}| \quad \text{for each } {\rm e} \in E,\ t > 0,\\
\sum_{{\rm e}\succ {\rm v}} |\frac{du_{\rm e}}{dx_{\rm e}}({\rm v}, t)|^{p-2}\frac{du_{\rm e}}{dx_{\rm e}}({\rm v}, t) = 0 \quad \text{at each } {\rm v} \in V,\ t > 0,\\
u_{{\rm e}_i}({\rm v}, t) = u_{{\rm e}_j}({\rm v}, t) \quad \text{if } {\rm e}_i \succ {\rm v}, {\rm e}_j \succ {\rm v},\ t > 0,\\
u_{\rm e}(x, 0) = (u_0)_{\rm e}(x) \quad \text{for each } {\rm e} \in E,
\end{gathered}
\tag{1.1}
$$

where $p > 1$, $u_{\rm e} := u_{\rm e}(x, t)$ is the restriction of $u : \mathcal{G} \times [0, +\infty) \to \mathbb{R}$ on the edge e for each $t \geq 0$ (see Sect. 2.1 for details), ${\rm e} \succ {\rm v}$ means that the sum above is extended to all edges e incident at v, and $\frac{du_{\rm e}}{dx_{\rm e}}({\rm v}, t)$ stands for $u_{\rm e}'(0, t)$ if $x_{\rm e} = 0$ at v and $-u_{\rm e}'(\ell_{\rm e}, t)$ if $x_{\rm e} = \ell_{\rm e}$ at v. The initial value $(u_0)_{\rm e}$ is the restriction of $u_0 \in W^{1,p}(\mathcal{G})$ on e, where $W^{1,p}(\mathcal{G})$ is a Banach space that will be introduced in Sect. 2.1. Here the second equation is known as the Kirchhoff boundary condition (see [17, 18, 21]), and the third equation is the so called continuity condition at each vertex v. Our goal in the present paper is to establish a sufficient criterion for the global existence and blow-up of solutions to (1.1).

The existence of global solutions, and the finite time blow-up of parabolic problems in Euclidean space have been studied extensively by many scholars. Among them there are a great many studies devoted to the problems with power nonlinearity (see [1, 24, 32, 35, 45, 48, 50] and the references therein), based on the potential well method introduced by Payne and Sattinger in [41] (see also [44]). In recent years, the logarithmic parabolic equations received considerable interest, see [10, 12, 14, 15, 29, 30]. In fact, the logarithmic nonlinearity has wide applications in physics and other applied science, and it describes different blow-up mechanism from the case of power nonlinearity. In [29], the following p-Laplacian parabolic equation with logarithmic nonlinearity was investigated:

$$
\begin{gathered}
u_t - \Delta_p u = |u|^{p-2}u\log|u| \quad \text{in } \Omega \times (0, T),\\
u(x, t) = 0 \quad \text{on } \partial\Omega \times (0, T),\\
u(x, 0) = u_0(x) \quad \text{on } \Omega,
\end{gathered}
$$

where $p > 2$, $\Delta_p u := \operatorname{div}(|\nabla u|^{p-2}\nabla u)$, $\Omega \subset \mathbb{R}^N (N \geq 2)$ is a bounded domain with smooth boundary, while the case $p = 2$ was studied in [14], and the case $1 < p < 2$ was studied in [12]. In these studies, by using the logarithmic Sobolev inequality (see [19, 20, 33]) and the potential well method, it was shown that the blow-up of solutions occurs at finite time when $p > 2$ and at infinity when $1 < p \leq 2$.

Motivated by above-mentioned results, it is natural to ask whether similar results hold for parabolic problems on metric graphs. Recently, in [13], Cazazu et al. studied the existence and asymptotic behavior of solutions for large times of a parabolic equation with linear diffusion and a power nonlinear term on a star-shaped graph. As far as we know, there are no results yet on Cauchy problems of nonlinear parabolic equations derived by the quasi-linear operator $(|u'|^{p-2}u')'$ on metric graphs.

To deal with the logarithmic nonlinear term in (1.1), we note that, because of the presence of Kirchhoff boundary condition, the logarithmic Sobolev inequality involving L^p estimates on real line (see [42]) is not applicable, and hence we can not follow directly the strategy as in [12, 14, 29], where the logarithmic Sobolev inequality is very useful.

In this article, by combining the potential well method and the Nehari manifold, together with some modified estimates without the use of logarithmic Sobolev inequality, we obtain the existence of bounded global solutions that decay to zero at infinity for all $p > 1$. Under suitable conditions, we shall show there exist finite time blow-up solutions when $p > 2$, and global solutions that blow up at infinity when $1 < p \leq 2$. In addition, we also obtain the decay estimates of the bounded global solutions and lower bound for the blow-up rate.

To study the dynamic properties of the solutions to problem (1.1), we define the energy functional $\Phi : W^{1,p}(\mathcal{G}) \to \mathbb{R}$ by

$$\Phi(u) := \frac{1}{p}\int_{\mathcal{G}} |u'|^p dx + \frac{1}{p}\int_{\mathcal{G}} |u|^p dx + \frac{1}{p^2}\int_{\mathcal{G}} |u|^p dx - \frac{1}{p}\int_{\mathcal{G}} |u|^p \log|u| dx.$$

We define

$$f(t) := \begin{cases} |t|^{p-2}t\log|t|, & \text{if } t \neq 0, \\ 0, & \text{if } t = 0. \end{cases}$$

Since $p > 1$, $f : \mathbb{R} \to \mathbb{R}$ is continuous. By elementary computations, for any $q \in (p, \infty)$, there exists $C_q > 0$ such that

$$|f(t)| \leq C_q(1 + |t|^{q-1}).$$

Identifying the logarithmic nonlinear term by $f(u)$, using Lemma 2.1 and standard arguments as in [47, Lemma 2.16], we have $\Phi \in C^1(W^{1,p}(\mathcal{G}), \mathbb{R})$.

The Nehari functional $I : W^{1,p}(\mathcal{G}) \to \mathbb{R}$ is defined by

$$I(u) := \langle \Phi'(u), u\rangle = \int_{\mathcal{G}} |u'|^p dx + \int_{\mathcal{G}} |u|^p dx - \int_{\mathcal{G}} |u|^p \log|u|\, dx,$$

where $\langle\cdot,\cdot\rangle$ denotes the dual pairing between $W^{1,p}(\mathcal{G})$ and its dual space $W^{-1,p'}(\mathcal{G})$. Clearly,

$$\Phi(u) = \frac{1}{p}I(u) + \frac{1}{p^2}\|u\|_p^p. \tag{1.2}$$

We define the potential well $\mathcal{W}$ and its outside, respectively, by

$$\mathcal{W} := \{u \in X_0 : \Phi(u) < d,\, I(u) > 0\},$$
$$\mathcal{Z} := \{u \in X_0 :\ \Phi(u) < d,\ I(u) < 0\},$$

where d is the depth of the potential well $\mathcal{W}$ and $X_0 := W^{1,p}(\mathcal{G}) \setminus \{0\}$.

We define the Nehari manifold by

$$\mathcal{N} := \{u \in X_0 :\ I(u) = 0\}.$$

Set

$$d = \inf_{u\in\mathcal{N}} \Phi(u). \tag{1.3}$$

To state our main results, we introduce the following definition.

Definition 1.1. A function $u := u(x,t)$ is called a weak solution of problem (1.1) on $\mathcal{G} \times (0, T_*)$, if $u \in L^\infty(0, T_*; W^{1,p}(\mathcal{G}))$ with $u_t \in L^2(0, T_*; L^2(\mathcal{G}))$ satisfies (1.1) in the distribution sense, i.e.,

$$\int_{\mathcal{G}} u_t v dx + \int_{\mathcal{G}} |u'|^{p-2}u'v' dx + \int_{\mathcal{G}} |u|^{p-2}uv dx = \int_{\mathcal{G}} |u|^{p-2}uv\log|u| dx,$$

for all $v \in W^{1,p}(\mathcal{G})$, a.e. $t \in (0, T_*)$, where $u(x,0) = u_0(x) \in X_0$.

Our first result is about the existence of a local solution.

Theorem 1.2. *Let $\mathcal{G}$ be a compact metric graph and $u_0 \in X_0$. Then there exists a positive constant T_* such that problem (1.1) has a unique weak solution $u := u(x,t)$ on $\mathcal{G} \times (0, T_*)$ in the sense of Definition 1.1. Furthermore, u satisfies the energy inequality*

$$\int_0^t \|u_s(\cdot,s)\|_2^2 ds + \Phi(u(x,t)) \le \Phi(u_0), \quad a.e.\ t \in [0, T_*]. \tag{1.4}$$

The existence of a global solution for problem (1.1) can be stated as follows.

Theorem 1.3. *Let $\mathcal{G}$ be a compact metric graph and $u_0 \in \mathcal{W}$. Then problem (1.1) admits a global weak solution $u := u(x,t)$ satisfying the energy estimate*

$$\int_0^t \|u_s(\cdot,s)\|_2^2 ds + \Phi(u(x,t)) \le \Phi(u_0), \quad a.e.\ t \ge 0. \tag{1.5}$$

Moreover, there exists a constant $R \le d$ such that

(i) *if $p > 2$ and $\Phi(u_0) < R$, then*

$$\|u(\cdot,t)\|_2 \le \|u_0\|_2 \Big(\frac{p}{2(1+\zeta(p-2)\|u_0\|_2^{p-2} t)}\Big)^{\frac{1}{p-2}}, \quad \forall t \ge 0,$$

where $\zeta := |\mathcal{G}|^{\frac{2-p}{2}} \left(\frac{1}{2} - C(\frac{1}{2})(p^2\Phi(u_0))^{\gamma-1}\right) > 0$ with $\gamma > 1$ given in Sect. 2.2;

(ii) *if $1 < p \le 2$ and $\Phi(u_0) < R$, then*

$$\|u(\cdot,t)\|_2 \le \|u_0\|_2 e^{\frac{1}{2} - \frac{1}{C\alpha} t}, \ \forall t \ge 0,$$

where $\alpha := \frac{1}{2} - C(\frac{1}{2})(p^2\Phi(u_0))^{\gamma-1} > 0$.

The next theorem is about the blow-up results of problem (1.1).

Theorem 1.4. *Let $\mathcal{G}$ be a compact metric graph and $u_0 \in \mathcal{Z}$. Assume that $u := u(x,t)$ is the local weak solution of problem (1.1) corresponding to this initial function and satisfies the energy inequality*

$$\int_0^t \|u_s(\cdot,s)\|_2^2 ds + \Phi(u(x,t)) \le \Phi(u_0), \quad \forall t \in [0, T_{\max}). \tag{1.6}$$

Then the following statements hold:

(i) *if $p > 2$ and $\Phi(u_0) \le 0$, then u blows up at finite time, i.e.,*

$$\lim_{t \to T_{\max}^-} \|u(\cdot,t)\|_2^2 = +\infty,$$

where $T_{\max} := \frac{2p}{p-2} |\mathcal{G}|^{\frac{p-2}{2}} \|u_0\|_2^{2-p}$;

(ii) *if $1 < p \le 2$ and $\Phi(u_0) \le 0$, then u blows up at infinity, i.e.,*

$$\lim_{t \to +\infty} \|u(\cdot,t)\|_2^2 = +\infty.$$

Furthermore, if $\|u_0\|_2^2 \le -p^2\Phi(u_0)$, the lower bound for the blow-up rate can be estimated by

$$\|u(\cdot,t)\|_2^2 \ge \|u_0\|_2^2.$$

In the sequel, for the sake of simplicity, we denote $u(x,t)$ by $u(t)$. This paper is organized as follows. In Sect. 2, we first present some notation, definitions and lemmas that will be used throughout the paper, and then obtain the existence of ground state solutions for a stationary problem associated with (1.1). In Sect. 3, we obtain the existence and uniqueness of local weak solutions of problem (1.1). In Sect. 4, we establish the existence of global solutions of problem (1.1). Moreover, we give the decay estimates. In Sect. 5, under some appropriate conditions, we prove that, problem (1.1) admits a finite time blow-up weak solution when $p > 2$, and a global weak solution that blows up at infinity when $1 < p \le 2$, respectively. The lower bound for the blow-up rate is also given.

2. Preliminaries

2.1. **Sobolev Space.** A function u on a metric graph $\mathcal{G}$ is a map $u : \mathcal{G} \to \mathbb{R}$, and it can be interpreted as a family of functions $\{u_\mathrm{e}\}_{\mathrm{e}\in E}$, where $u_\mathrm{e} : I_\mathrm{e} \to \mathbb{R}$ is the restriction of u to the interval I_e such that $u|_\mathrm{e} = u_\mathrm{e}$. Similarly, a function $u : \mathcal{G} \times [0,+\infty)$ can be seen as the family $\{u_\mathrm{e}(x,t)\}_{\mathrm{e}\in E}$ with $u_\mathrm{e} : I_\mathrm{e} \times [0,+\infty) \to \mathbb{R}$ is the restriction of u to the interval I_e for each $t \ge 0$.

If u is defined on $\mathcal{G}$ ($\mathcal{G} \times [0,+\infty)$ respectively), we denote by u' ($u'(x,t)$ respectively) the functions with restriction to every I_e given by u'_e ($u'_\mathrm{e}(x,t)$ respectively), which is the derivative with respect to x. Throughout this paper, we will also denote $u'(x,t)$ by $u'(t)$ for simplicity if there is no confusion.

Let $1 \le p \le \infty$. Endowing each edge with Lebesgue measure, one can define the space $L^p(\mathcal{G})$ in a natural way, i.e., we say that $u \in L^p(\mathcal{G})$ if $u_\mathrm{e} \in L^p(I_\mathrm{e})$ for all $\mathrm{e} \in E$ and

$$\|u\|_p^p := \|u\|_{L^p(\mathcal{G})}^p := \sum_{\mathrm{e}\in E} \|u_\mathrm{e}\|_{L^p(I_\mathrm{e})}^p < \infty.$$

The Sobolev space $W^{1,p}(\mathcal{G})$ is defined as the space of continuous functions u on $\mathcal{G}$ such that $u_\mathrm{e} \in W^{1,p}(I_\mathrm{e})$ for all $\mathrm{e} \in E$ and

$$\|u\|_{1,p}^p := \|u\|_{W^{1,p}(\mathcal{G})}^p := \sum_{\mathrm{e}\in E} \left(\|u_\mathrm{e}\|_{L^p(I_\mathrm{e})}^p + \|u'_\mathrm{e}\|_{L^p(I_\mathrm{e})}^p\right) < \infty.$$

By [18], we know that $W^{1,p}(\mathcal{G})$ is a Banach space for $1 \le p \le \infty$. It is reflexive for $1 < p < \infty$ and separable for $1 \le p < \infty$. Throughout the paper, we will denote by C a positive constant that may vary from place to place. The following two lemmas will be needed later.

Lemma 2.1 ([18])**.** *Let $\mathcal{G}$ be a compact metric graph and $1 < p < \infty$. Then the injection $W^{1,p}(\mathcal{G}) \subset L^q(\mathcal{G})$ is compact for all $1 \le q \le \infty$.*

Lemma 2.2 ([29])**.** *Let η be a positive number. Then*

$$\log s \le \frac{e^{-1}}{\eta} s^\eta, \quad \forall s \in [1,+\infty).$$

By Lemma 2.1 and the interpolation inequality, we obtain the following Gagliardo-Nirenberg type inequality involving L^p norm on compact metric graphs.

Lemma 2.3. *Let $1 < p < +\infty$ and $r > 0$. Then, for any $u \in W^{1,p}(\mathcal{G})$, there exists $C > 0$ such that*

$$\|u\|_{p+r} \le C\|u\|_{1,p}^\theta \|u\|_p^{(1-\theta)},$$

where $\theta := \frac{p+kr}{k(p+r)} \in (0,1)$ *with* $k \geq 2$ *a positive integer, and* $C > 0$ *depends only on* $\mathcal{G}$, p, r, *and* k.

2.2. **Potential well.** In this subsection, we consider the minimization problem (1.3). Let $u \in X_0$. We consider the fibering map $j : \mathbb{R}^+ \to \mathbb{R}$ given by

$$j(\lambda) = \Phi(\lambda u), \quad \forall \lambda > 0.$$

Clearly,

$$j(\lambda) = \frac{\lambda^p}{p}\|u\|_{1,p}^p + \frac{\lambda^p}{p^2}\|u\|_p^p - \frac{\lambda^p \log \lambda}{p}\|u\|_p^p - \frac{\lambda^p}{p}\int_{\mathcal{G}} |u|^p \log |u| dx.$$

Moreover, we have the following properties.

Lemma 2.4. *Let* $u \in X_0$. *Then we have*

1. $\lim_{\lambda \to 0^+} j(\lambda) = 0$ *and* $\lim_{\lambda \to +\infty} j(\lambda) = -\infty$*;*
2. *there is a unique* $\lambda^* := \lambda^*(u) > 0$ *such that* $j'(\lambda^*) = 0$*;*
3. $j(\lambda)$ *is increasing on* $(0, \lambda^*)$, *decreasing on* $(\lambda^*, +\infty)$ *and attains the maximum at* λ^**;*
4. $I(\lambda u) > 0$ *for* $0 < \lambda < \lambda^*$, $I(\lambda u) < 0$ *for* $\lambda^* < \lambda < +\infty$ *and* $I(\lambda^* u) = 0$.

Proof. Fix $u \in X_0$. (1) is obvious. Differentiating $j(\lambda)$, we obtain

$$\begin{aligned} \frac{d}{d\lambda} j(\lambda) &= \lambda^{p-1}\Big(\|u\|_{1,p}^p - \log \lambda \|u\|_p^p - \int_{\mathcal{G}} |u|^p \log |u| dx\Big) \\ &= \lambda^{p-1}\big(I(u) - \log \lambda \|u\|_p^p\big). \end{aligned}$$

Therefore, taking $\lambda^* := \lambda^*(u) := \exp\big(\frac{I(u)}{\|u\|_p^p}\big)$, we obtain that (2) and (3) hold. To show (4), it suffices to check that

$$I(\lambda u) = \lambda^p\Big(\|u\|_{1,p}^p - \log \lambda \|u\|_p^p - \int_{\mathcal{G}} |u|^p \log |u| dx\Big) = \lambda j'(\lambda). \qquad \square$$

For the Nehari functional I, we have the following result.

Lemma 2.5. *Let* $u \in X_0$. *Then there exists a constant* $l := l(p) > 0$ *such that*

1. *if* $0 < \|u\|_p < l$, *then* $I(u) > 0$*;*
2. *if* $I(u) < 0$, *then* $\|u\|_p > l$*;*
3. *if* $I(u) = 0$, *then* $\|u\|_p \geq l$.

Proof. By Lemma 2.2, we obtain

$$\begin{aligned} \int_{\mathcal{G}} |u|^p \log |u| dx &= \int_{\mathcal{G}_1} |u|^p \log |u| dx + \int_{\mathcal{G}_2} |u|^p \log |u| dx \\ &\leq \frac{1}{\eta e}\int_{\mathcal{G}_2} |u|^{p+\eta} dx \\ &\leq \frac{1}{\eta e}\|u\|_{p+\eta}^{p+\eta}, \end{aligned}$$

where $\eta \in (0, \frac{p}{2})$, $\mathcal{G}_1 := \{x \in \mathcal{G} : |u(x)| \leq 1\}$, and $\mathcal{G}_2 := \{x \in \mathcal{G} : |u(x)| > 1\}$. Then, by Lemma 2.3 it follows that, for some $C > 0$,

$$\int_{\mathcal{G}} |u|^p \log |u| dx \leq \frac{C}{\eta e}\|u\|_{1,p}^{\theta(p+\eta)}\|u\|_p^{(1-\theta)(p+\eta)},$$

where $\theta \in (0,1)$ is given in Lemma 2.3 with $k=2$. Note that $\theta(p+\eta) < p$, applying the Young inequality with ϵ, for any $\epsilon > 0$, there exists $C(\epsilon) > 0$ such that

$$\int_{\mathcal{G}} |u|^p \log |u| dx \le \epsilon \|u\|_{1,p}^p + C(\epsilon)\|u\|_p^{p\gamma}, \tag{2.1}$$

where

$$\gamma := \frac{(1-\theta)(p+\eta)}{p-\theta(p+\eta)} > 1.$$

Thus, by (2.1) and the definition of I, it follows that

$$\begin{aligned} I(u) &\ge (1-\epsilon)\|u\|_{1,p}^p - C(\epsilon)(\|u\|_p^p)^\gamma \\ &= (1-\epsilon)\|u'\|_p^p + \|u\|_p^p[(1-\epsilon) - C(\epsilon)\|u\|_p^{p(\gamma-1)}]. \end{aligned}$$

Taking $\epsilon = 1/2$ yields

$$I(u) \ge \|u\|_p^p[\frac{1}{2} - C(\frac{1}{2})\|u\|_p^{p(\gamma-1)}]. \tag{2.2}$$

Set

$$l := \Big(\frac{1}{2C(\frac{1}{2})}\Big)^{\frac{1}{p(\gamma-1)}}.$$

Then, in view of (2.2), it is easily seen that (1)–(3) hold. □

From Lemma 2.4, it is obvious that $\mathcal{N}$ is not empty. Moreover, since

$$\Phi(u) = \frac{1}{p^2}\|u\|_p^p, \quad \forall u \in \mathcal{N},$$

by (2.1), we have the following lemma.

Lemma 2.6. Φ *is coercive on* $\mathcal{N}$.

Next, we prove that the infimum in (1.3) can be attained at some $u_0 \in \mathcal{N}$. Then, by standard arguments as in [47], we know that u_0 is a critical point of Φ and thus it is a ground state solution of the stationary problem associated with (1.1).

Lemma 2.7. *The following statements hold:*

1. $d = \inf_{u\in X_0}\sup_{\lambda>0}\Phi(\lambda u) \ge R := \frac{l^p}{p^2}$;
2. *there exists* $u_0 \in \mathcal{N}$ *such that* $\Phi(u_0) = d$.

Proof. For (1), by Lemma 2.4, for any $u \in X_0$, there exists a unique $\lambda^* > 0$ such that $\lambda^* u \in \mathcal{N}$. Specially, $\lambda^* = 1$ if $u \in \mathcal{N}$. Furthermore,

$$d \le \Phi(\lambda^* u) = \inf_{u\in X_0}\sup_{\lambda>0}\Phi(\lambda u) \le \inf_{u\in\mathcal{N}}\sup_{\lambda>0}\Phi(\lambda u) = \inf_{u\in\mathcal{N}}\Phi(u) = d. \tag{2.3}$$

By Lemma 2.4, we have $I(\lambda^* u) = 0$, which implies that $\|\lambda^* u\|_p \ge l$. Then, by (2.3), we obtain

$$d = \Phi(\lambda^* u) = \frac{1}{p^2}\|\lambda^* u\|_p^p \ge R.$$

To show (2), let $\{u_n\}_{n=1}^\infty \subset \mathcal{N}$ be a minimizing sequence for Φ such that

$$\lim_{n\to\infty}\Phi(u_n) = d.$$

By Lemma 2.4, $\{u_n\}_{n=1}^{\infty}$ is bounded in $W^{1,p}(\mathcal{G})$. Using Lemma 2.1, up to a subsequence if necessary, there exists $u_0 \in W^{1,p}(\mathcal{G})$ such that

$$\begin{gathered} u_n \rightharpoonup u_0 \quad \text{in } W^{1,p}(\mathcal{G}), \\ u_n \to u_0 \quad \text{in } L^q(\mathcal{G})\ (1 \le q \le \infty), \\ u_n \to u_0 \quad \text{a.e. in } \mathcal{G}. \end{gathered} \tag{2.4}$$

Then, by Lemma 2.2, (2.4) and similar arguments as in [47, Lemma A.1], we can apply the Lebesgue dominated convergence theorem to get

$$\lim_{n\to\infty} \int_{\mathcal{G}} |u_n|^p \log |u_n| dx = \int_{\mathcal{G}} |u_0|^p \log |u_0| dx.$$

Hence, we deduce that

$$\begin{aligned} \Phi(u_0) &= \frac{1}{p}\|u_0\|_{1,p}^p - \frac{1}{p}\int_{\mathcal{G}} |u_0|^p \log |u_0| dx + \frac{1}{p^2}\|u_0\|_p^p \\ &\le \liminf_{n\to\infty} \Big(\frac{1}{p}\|u_n\|_{1,p}^p - \frac{1}{p}\int_{\mathcal{G}} |u_n|^p \log |u_n| dx + \frac{1}{p^2}\|u_n\|_p^p\Big) \\ &= \liminf_{n\to\infty} \Phi(u_n) = d. \end{aligned}$$

It suffices to show that $I(u_0) = 0$. By similar arguments as above, we have

$$\begin{aligned} I(u_0) &= \|u_0\|_{1,p}^p - \int_{\mathcal{G}} |u_0|^p \log |u_0| dx \\ &\le \liminf_{n\to\infty} \Big(\|u_n\|_{1,p}^p - \int_{\mathcal{G}} |u_n| \log |u_n| dx\Big) \\ &= \liminf_{n\to\infty} I(u_n) = 0. \end{aligned}$$

We assume by contradiction that $I(u_0) < 0$. By Lemma 2.4, there exists a unique $\lambda^* := \exp(\frac{I(u_0)}{\|u_0\|_p^p}) < 1$ such that $I(\lambda^* u_0) = 0$. Then, using (2.3),

$$d \le \frac{1}{p^2}\|\lambda^* u\|_p^p \le \frac{(\lambda^*)^p}{p^2} \liminf_{n\to\infty} \|u_n\|_p^p = (\lambda^*)^p \liminf_{n\to\infty} \Phi(u_n) = (\lambda^*)^p d < d.$$

A contradiction. This completes the proof. □

3. Existence and uniqueness of local weak solutions

In this section, we prove Theorem 1.2 by using the Galerkin approximation. The proof is divided into four steps.

Step 1. Approximate problem. Let $\{X_m\}_{m\in\mathbb{N}}$ be a Galerkin scheme of the separable Banach space $W^{1,p}(\mathcal{G})$, i.e.,

$$X_m := \operatorname{span}\{\varphi_1, \varphi_2, \dots, \varphi_m\},$$

where $\{\varphi_j\}_{j=1}^{\infty}$ is an orthonormal basis in $L^2(\mathcal{G})$. Take $u_0 \in X_0$. Then there exists $u_{0m} \in X_m$ such that

$$u_m(0) := u_{0m} \to u_0 \quad \text{strong in } W^{1,p}(\mathcal{G}) \text{ as } m \to \infty. \tag{3.1}$$

We find the approximate solution $u_m(t) := u_m(x,t)$ of problem (1.1) in the form

$$u_m(x,t) := \sum_{j=1}^{m} g_{jm}(t)\varphi_j(x),$$

where the coefficients $g_{jm}(t)$ satisfy

$$\begin{aligned} &\int_{\mathcal{G}} u_{mt}(t)\varphi_i dx + \int_{\mathcal{G}} |u_m'(t)|^{p-2}u_m'(t)\varphi_i' dx + \int_{\mathcal{G}} |u_m(t)|^{p-2}u_m(t)\varphi_i dx \\ &= \int_{\mathcal{G}} |u_m(t)|^{p-2}u_m(t)\varphi_i \log|u_m(t)| dx \end{aligned} \tag{3.2}$$

with the initial conditions

$$u_m(0) = \sum_{j=1}^{m} g_{jm}(0)\varphi_j(x) = u_{0m}. \tag{3.3}$$

Equivalently, it suffices to consider the initial value problem

$$\begin{gathered} g_{im}'(t) = F_i(g(t)), \quad i = 1,2,\dots,m,\ t \in [0,t_0], \\ g_{im}(0) = a_{im}, \quad i = 1,2,\dots,m, \end{gathered} \tag{3.4}$$

where

$$\begin{aligned} F_i(g(t)) := &-\int_{\mathcal{G}} |u_m'(t)|^{p-2}u_m'(t)\varphi_i' dx - \int_{\mathcal{G}} |u_m(t)|^{p-2}u_m(t)\varphi_i dx \\ &+ \int_{\mathcal{G}} |u_m(t)|^{p-2}u_m(t)\varphi_i \log|u_m(t)| dx. \end{aligned}$$

By the Peano's theorem in standard ODEs theory, there is $t_{0,m} > 0$ depending on $|a_{jm}|$ such that problem (3.4) admits a local solution $g_{jm} \in C^1([0,t_{0,m}])$.

Step 2. A priori estimates. Multiplying the i^{th} equation in (3.2) by $g_{im}(t)$ and summing over i, we obtain

$$\frac{1}{2}\frac{d}{dt}\|u_m(\cdot,t)\|_2^2 + \|u_m(\cdot,t)\|_{1,p}^p = \int_{\mathcal{G}} |u_m(t)|^p \log|u_m(t)| dx. \tag{3.5}$$

On the other hand, by Lemma 2.3 and similar arguments as in Sect. 2, for some $\hat{\gamma} > 1$, we have

$$\int_{\mathcal{G}} |u_m(t)|^p \log|u_m(t)| dx \le \epsilon\|u_m(\cdot,t)\|_{1,p}^p + C_\epsilon\|u_m(\cdot,t)\|_2^{2\hat{\gamma}}, \quad \forall t \in [0,t_{0,m}]. \tag{3.6}$$

Combining (3.5) with (3.6), we obtain

$$\frac{1}{2}\frac{d}{dt}\|u_m(\cdot,t)\|_2^2 + (1-\epsilon)\|u_m(\cdot,t)\|_{1,p}^p \le C_\epsilon\|u_m(\cdot,t)\|_2^{2\hat{\gamma}}.$$

Taking $\epsilon = 1/2$, there exists $\tilde{C}_1 > 0$ such that

$$\frac{d}{dt}\|u_m(\cdot,t)\|_2^2 + \|u_m(\cdot,t)\|_{1,p}^p \le 2\tilde{C}_1\|u_m(\cdot,t)\|_2^{2\hat{\gamma}},$$

which implies that

$$\|u_m(\cdot,t)\|_2^2 \le \Big(\frac{1}{\tilde{C}_2^{1-\hat{\gamma}} - 2\tilde{C}_1(\hat{\gamma}-1)t}\Big)^{\frac{1}{\hat{\gamma}-1}}$$

for all $0 < t < T_0 := \frac{\tilde{C}_2^{1-\hat{\gamma}}}{2\tilde{C}_1(\hat{\gamma}-1)}$, where $\tilde{C}_2 := \sup_{m\in\mathbb{N}^*}\|u_{0m}\|_2^2$. Hence,

$$\|u_m(\cdot,t)\|_2^2 \le 2^{\frac{1}{\hat{\gamma}-1}}\tilde{C}_2, \quad \forall t \le \min\{t_{0,m}, \frac{T_0}{2}\}.$$

Then, for some $C > 0$,

$$\|u_m(\cdot,t_{0,m})\|_2^2 \le 2^{\frac{1}{\hat{\gamma}-1}}(\tilde{C}_2 + C).$$

Therefore, by extending the solution to the interval $[0, T_*]$ with $T_* := \frac{T_0}{2}$, we obtain

$$\|u_m(\cdot,t)\|_2^2 \le 2^{\frac{1}{\hat{\gamma}-1}} \tilde{C}_2, \quad \forall t \in [0, T_*],$$

which together with (3.6) implies that, for any $\epsilon \in (0,1)$,

$$\int_{\mathcal{G}} |u_m(t)|^p \log |u_m(t)| dx \le \epsilon \|u_m(\cdot,t)\|_{1,p}^p + 2^{\frac{\gamma}{\gamma-1}} C_\epsilon \tilde{C}_2^{\hat{\gamma}}, \quad \forall t \in [0, T_*]. \tag{3.7}$$

Multiplying (3.2) by $g'_{im}(t)$, summing over i, and integrating over $(0,t)$ we obtain

$$\int_0^t \|u_{ms}(\cdot,s)\|_2^2 ds + \Phi(u_m(t)) = \Phi(u_m(0)). \tag{3.8}$$

Using (3.1) and the continuity of Φ, there exists $C > 0$ such that

$$\Phi(u_m(0)) \le C, \quad \forall m.$$

From (3.7), we have

$$\Phi(u_m(t)) \ge \frac{1-\epsilon}{p} \|u_m(\cdot,t)\|_{1,p}^p + \frac{1}{p^2} \|u_m(\cdot,t)\|_p^p - \frac{1}{p} 2^{\frac{\gamma}{\gamma-1}} C_\epsilon \tilde{C}_2^{\hat{\gamma}}.$$

Then, using (3.8), it follows that, for some $C > 0$,

$$\|u_m\|_{L^\infty(0,T_*;W^{1,p}(\mathcal{G}))} \le C, \tag{3.9}$$

$$\|u_{mt}\|_{L^2(0,T_*;L^2(\mathcal{G}))} \le C. \tag{3.10}$$

Step 3. Passage to the limit. By the prior estimates (3.9) and (3.10), there exists $u \in L^\infty(0,T_*;W^{1,p}(\mathcal{G}))$ such that, up to a subsequence,

$$u_m \to u \quad \text{weakly}^* \text{ in } L^\infty(0,T_*;W^{1,p}(\mathcal{G})), \tag{3.11}$$

$$u_{mt} \to u_t \quad \text{weakly}^* \text{ in } L^2(0,T_*;L^2(\mathcal{G})), \tag{3.12}$$

$$|u'_m|^{p-2} u'_m \to \chi \quad \text{weakly}^* \text{ in } L^\infty(0,T_*;W^{-1,p'}(\mathcal{G})). \tag{3.13}$$

Since $\{u_m\}_{m=1}^\infty \subset L^\infty(0,T_*;W^{1,p}(\mathcal{G}))$ and $\{u_{mt}\}_{m=1}^\infty \subset L^2(0,T_*;L^2(\mathcal{G}))$, by the Aubin-Lions compactness theorem (see [34]) we obtain, up to a subsequence,

$$u_m \to u \quad \text{in } C([0,T_*];L^r(\mathcal{G})),\ \forall r \in [2,\infty).$$

By (3.1) it follows that, for a.e. $(x,t) \in \mathcal{G} \times (0,T_*)$,

$$|u_m(t)|^{p-2} u_m(t) \log |u_m(t)| \to |u(t)|^{p-2} u(t) \log |u(t)|. \tag{3.14}$$

By Lemma 2.2,

$$\begin{aligned}
&\int_{\mathcal{G}} \big||u_m(t)|^{p-2} u_m(t) \log |u_m(t)|\big|^{p'} dx \\
&= \int_{\mathcal{G}_1} \big||u_m(t)|^{p-2} u_m(t) \log |u_m(t)|\big|^{p'} dx \\
&\quad + \int_{\mathcal{G}_2} \big||u_m(t)|^{p-2} u_m(t) \log |u_m(t)|\big|^{p'} dx \\
&\le C[|\mathcal{G}| + \int_{\mathcal{G}_2} |u_m(t)|^q dx] \\
&\le C[|\mathcal{G}| + \|u_m(\cdot,t)\|_{1,p}^q] \le C_{T_*},
\end{aligned} \tag{3.15}$$

where $q = (p-1+\eta)p' > p$, $p' = \frac{p}{p-1}$, and $\mathcal{G}_1 := \{x \in \mathcal{G} : |u_m(t)| \le 1\}$, $\mathcal{G}_2 := \{x \in \mathcal{G} : |u_m(t)| > 1\}$. Then, by (3.14)-(3.15) and [34, Lemma 1.3], we obtain

$$|u_m|^{p-2}u_m \log|u_m| \to |u|^{p-2}u \log|u| \quad \text{weakly}^* \text{ in } L^\infty(0,T_*;L^{p'}(\mathcal{G})). \tag{3.16}$$

Similarly,

$$|u_m|^{p-2}u_m \to |u|^{p-2}u \quad \text{weakly}^* \text{ in } L^\infty(0,T_*;L^{p'}(\mathcal{G})). \tag{3.17}$$

Using (3.11)-(3.12) and [49, Lemma 3.1.7], we have

$$u_m(0) \to u(0) \quad \text{weakly}^* \text{ in } L^2(\mathcal{G}).$$

However, by (3.1) it follows that $u_m(0) \to u_0$ in $L^2(\mathcal{G})$. Hence, $u(0) = u_0$, which implies that u satisfies the initial condition.

In what follows, we take $m \to \infty$ in (3.2). For $\phi \in L^2(0,T_*)$ and $\varphi_i \in X_m$, we have

$$\begin{aligned}
&\int_0^{T_*}\int_{\mathcal{G}} u_{mt}(t)\varphi_i dx\phi(t)dt + \int_0^{T_*}\int_{\mathcal{G}} |u_m'(t)|^{p-2}u_m'(t)\varphi_i' dx\phi(t)dt \\
&+ \int_0^{T_*}\int_{\mathcal{G}} |u_m(t)|^{p-2}u_m(t)\varphi_i dx\phi(t)dt \\
&= \int_0^{T_*}\int_{\mathcal{G}} |u_m(t)|^{p-2}u_m(t)\varphi_i \log|u_m(t)| dx\phi(t)dt.
\end{aligned} \tag{3.18}$$

Letting $m \to \infty$ in (3.18) and using (3.12), (3.13), (3.16) and (3.17), we deduce that

$$\begin{aligned}
&\int_0^{T_*}\int_{\mathcal{G}} u_t(t)\varphi_i dx\phi(t)dt + \int_0^{T_*}\int_{\mathcal{G}} \chi(t)\varphi_i' dx\phi(t)dt \\
&+ \int_0^{T_*}\int_{\mathcal{G}} |u(t)|^{p-2}u(t)\varphi_i dx\phi(t)dt \\
&= \int_0^{T_*}\int_{\mathcal{G}} |u(t)|^{p-2}u(t)\varphi_i \log|u(t)| dx\phi(t)dt,
\end{aligned}$$

which implies that

$$\begin{aligned}
&\int_{\mathcal{G}} u_t(t)\varphi_i dx + \int_{\mathcal{G}} \chi(t)\varphi_i' dx + \int_{\mathcal{G}} |u(t)|^{p-2}u(t)\varphi_i dx \\
&= \int_{\mathcal{G}} |u(t)|^{p-2}u(t)\varphi_i \log|u(t)| dx, \text{ a.e. in } (0,T_*).
\end{aligned}$$

By the density of X_m in $W^{1,p}(\mathcal{G})$, it follows that

$$\begin{aligned}
&\int_{\mathcal{G}} u_t(t)\omega dx + \int_{\mathcal{G}} \chi(t)\omega' dx + \int_{\mathcal{G}} |u(t)|^{p-2}u(t)\omega dx \\
&= \int_{\mathcal{G}} |u(t)|^{p-2}u(t)\omega \log|u(t)| dx, \quad \text{a.e. in } (0,T_*),\ \forall \omega \in W^{1,p}(\mathcal{G}).
\end{aligned} \tag{3.19}$$

It suffices to show that $\chi = |u'|^{p-2}u'$ in the weak sense, which can be proven to be true by using standard arguments from the theory of monotone operators. Hence, u is the desired solution of problem (1.1).

Step 4. Uniqueness. For $p \ge 2$, the uniqueness is derived from the locally Lipschitz continuity of the nonlinearity $f(u) = |u|^{p-2}u \log|u|$. For $1 < p < 2$, f is just Hölder continuous but not locally Lipschitz continuous, and we need different arguments. Assume by contradiction that (1.1) admits two weak solution $u_1(t)$ and $u_2(t)$. Set $w(t) = u_1(t) - u_2(t)$. Then $w(t)$ satisfies the equation

$$\begin{aligned}&(w_{\rm e})_t(t) - ((p-1)|\widetilde{w}_{\rm e}'(t)|^{p-2}w_{\rm e}'(t))' + (p-1)|\widetilde{w}_{\rm e}(t)|^{p-2}w_{\rm e}(t)\\&= ((p-1)\log|\widetilde{w}_{\rm e}(t)|+1)|\widetilde{w}_{\rm e}(t)|^{p-2}w_{\rm e}(t), \quad \forall t>0,\end{aligned} \tag{3.20}$$

on each edge ${\rm e}\in E$, and the initial-boundary value conditions

$$\begin{gathered}\sum_{{\rm e}\succ{\rm v}}|\frac{d\widetilde{w}_{\rm e}}{dx_{\rm e}}({\rm v},t)|^{p-2}\frac{dw_{\rm e}}{dx_{\rm e}}({\rm v},t)=0 \quad \text{at each } {\rm v}\in V,\ t>0,\\ w_{{\rm e}_i}({\rm v},t)=w_{{\rm e}_j}({\rm v},t) \quad \text{if } {\rm e}_i\succ{\rm v},{\rm e}_j\succ{\rm v},\ t>0,\\ w_{\rm e}(x,0)=0 \quad \text{for each } {\rm e}\in E,\end{gathered} \tag{3.21}$$

where $\widetilde{w}(t) = \tau u_1(t) + (1-\tau)u_2(t)$ with $\tau\in[0,1]$.

Multiplying (3.20) by $w_{\rm e}(t)$, integrating on $I_{\rm e}$ and summing over all ${\rm e}\in E$, by (3.21) we have

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int_{\mathcal G}|w(t)|^2dx + \int_{\mathcal G}(p-1)|\widetilde{w}'(t)|^{p-2}|w'(t)|^2dx + \int_{\mathcal G}(p-1)|\widetilde{w}(t)|^{p-2}|w(t)|^2dx\\&= \int_{\mathcal G}((p-1)\log|\widetilde{w}(t)|+1)|\widetilde{w}(t)|^{p-2}|w(t)|^2dx.\end{aligned}$$

Integrating on $(0,t)$ with $t\in(0,T)$, by $1<p<2$ it follows that

$$\begin{aligned}\frac{1}{2}\int_{\mathcal G}|w(t)|^2dx &\le \int_0^t\int_{\mathcal G}((p-1)\log|\widetilde{w}(s)|+1)|\widetilde{w}(s)|^{p-2}|w(s)|^2\,dx\,ds\\&\le C\int_0^t\int_{\mathcal G}|w(s)|^2\,dx\,ds\end{aligned}$$

for some constant $C>0$ independent of $u_1(t)$ and $u_2(t)$. By the Gronwall's inequality we obtain $\int_{\mathcal G}|w(s)|^2dx = 0$ for a.e. $s\in(0,t)$. Hence $w(t)=0$ a.e. in $\mathcal G\times(0,T)$.

Finally, we prove (1.4). Let $\psi\in C([0,T_*])$ such that $\psi\ge 0$. By (3.8) we obtain

$$\begin{aligned}&\int_0^{T_*}\psi(t)\int_0^t\|u_{ms}(\cdot,s)\|_2^2\,ds\,dt + \int_0^{T_*}\Phi(u_m(t))\psi(t)dt\\&= \int_0^{T_*}\Phi(u_m(0))\psi(t)dt.\end{aligned} \tag{3.22}$$

Using Lemma 2.1 and (3.1), we obtain

$$\int_0^{T_*}\Phi(u_m(0))\psi(t)dt \to \int_0^{T_*}\Phi(u_0)\psi(t)dt \quad \text{as } m\to\infty.$$

From this and (3.22), it follows that

$$\int_0^{T_*}\psi(t)\int_0^t\|u_s(\cdot,s)\|_2^2\,ds\,dt + \int_0^{T_*}\Phi(u(t))\psi(t)dt \le \int_0^{T_*}\Phi(u_0)\psi(t)dt.$$

Since ψ is arbitrary, we obtain

$$\int_0^t\|u_s(\cdot,s)\|_2^2ds + \Phi(u(t)) \le \Phi(u_0), \quad \text{a.e. } t\in[0,T_*].$$

4. Existence of global weak solutions

In this section, we prove Theorem 1.3. When $u_0 \in \mathcal{W}$, we prove that problem (1.1) admits a global weak solution for all $p > 1$. Furthermore, we show that the norm $\|u(\cdot,t)\|_2$ decays polynomially when $p > 2$, and decays exponentially when $1 < p \le 2$. We first recall the following result.

Lemma 4.1 ([37]). *Let $h : \mathbb{R}^+ \to \mathbb{R}^+$ be a non-increasing function, σ, ρ are constants such that $\rho > 0$, $\sigma \ge 0$ and*

$$\int_t^{+\infty} h^{1+\sigma}(s)ds \le \frac{1}{\varrho} f^\sigma(0)h(t), \quad \forall t \ge 0.$$

Then

1. *if $\sigma = 0$, then $h(t) \le h(0)e^{1-\varrho t}$, for all $t \ge 0$;*
2. *if $\sigma > 0$, then $h(t) \le h(0)\big(\frac{1+\sigma}{1+\varrho\sigma t}\big)^{1/\sigma}$, for all $t \ge 0$.*

Proof of Theorem 1.3. Assume that $u_0 \in \mathcal{W}$. Clearly, $\Phi(u_0) > 0$. Denote by T_m the maximal time of existence of $u_m(t)$. Then, by (3.8), we have

$$\int_0^t \|u_{ms}(\cdot,s)\|_2^2 ds + \Phi(u_m(t)) = \Phi(u_{0m}), \quad 0 \le t < T_m. \tag{4.1}$$

Since $\Phi(u_0) < d$, by (3.1) and the continuity of Φ it follows that

$$\int_0^t \|u_{ms}(\cdot,s)\|_2^2 ds + \Phi(u_m(t)) < d, \quad 0 \le t < T_m \tag{4.2}$$

for sufficiently large m.

In the following, we shall prove that $T_m = +\infty$ and, for m large enough,

$$u_m(t) \in \mathcal{W}, \quad \forall t \ge 0. \tag{4.3}$$

In fact, if not, there exists $t_* \in [0, T_m)$ such that $u_m(t_*) \in \partial\mathcal{W}$, which implies that

$$\Phi(u_m(t_*)) = d \quad \text{or} \quad I(u_m(t_*)) = 0. \tag{4.4}$$

By (4.2) we obtain $I(u_m(t_*)) = 0$. Then by the definition of d, it follows that $\Phi(u_m(t_*)) \ge d$, which is contrary to (4.2). Hence (4.3) holds. Thus

$$\|u_m(\cdot,t)\|_p^p < dp^2 \quad \text{and} \quad \int_0^t \|u_{ms}(\cdot,s)\|_2^2 ds < d \tag{4.5}$$

hold for sufficiently large m and $t \in [0, T_m)$. Moreover, by (2.1), for any $\epsilon \in (0,1)$, we have

$$\begin{aligned}
\|u_m(\cdot,t)\|_{1,p}^p &= p\Phi(u_m(t)) + \int_{\mathcal{G}} |u_m(t)|^p \log|u_m(t)| dx - \frac{1}{p}\|u_m(\cdot,t)\|_p^p \\
&\le pd + \epsilon\|u_m(\cdot,t)\|_{1,p}^p + C(\epsilon)(dp^2)^\gamma,
\end{aligned}$$

which implies that for some $C_d > 0$,

$$\|u_m(\cdot,t)\|_{1,p}^p \le C_d, \quad \forall t \in [0, T_m). \tag{4.6}$$

Thus $T_m = +\infty$ for all m. Now, using (4.5), (4.6), and (3.1), by similar arguments as in previous section, it follows that problem (1.1) admits a global solution $u := u(t)$.

In what follows, we consider the decay of $\|u(\cdot,t)\|_2$. Let R be as in Lemma 2.7 and assume $\Phi(u_0) < R$. By $u_m(t) \in \mathcal{W}$ and (4.1) we obtain (4.5) holds. Then, using (2.1) and similar arguments as above, we obtain

$$I(u_m(t)) \ge (\frac{1}{2} - C(\frac{1}{2})(p^2\Phi(u_{0m}))^{\gamma-1})\|u_m(\cdot,t)\|_p^p. \tag{4.7}$$

On the other hand, for each $T > 0$, multiplying both sides of (3.2) by $g_{im}(t)$, summing over $i \in \{1, 2, \dots, m\}$ and integrating over (t, T) yields

$$\int_t^T I(u_m(s))ds = -\int_t^T \int_{\mathcal{G}} u_{sm}(s)u_m(s)\,dx\,ds \le \frac{1}{2}\|u_m(\cdot,t)\|_2^2, \quad \forall t \in [0,T]. \tag{4.8}$$

Case 1: $p > 2$**.** By (4.7) and the Hölder's inequality, we have

$$I(u_m(t)) \ge |\mathcal{G}|^{\frac{2-p}{2}}(\frac{1}{2} - C(\frac{1}{2})(p^2\Phi(u_{0m}))^{\gamma-1})\|u_m(\cdot,t)\|_2^p = \zeta_m\|u_m(\cdot,t)\|_2^p, \tag{4.9}$$

where $\zeta_m := |\mathcal{G}|^{\frac{2-p}{2}}(\frac{1}{2} - C(\frac{1}{2})(p^2\Phi(u_{0m}))^{\gamma-1}) > 0$. By (4.8) and (4.9) it follows that

$$\int_t^T \|u_m(\cdot,s)\|_2^p ds \le \frac{1}{2\zeta_m}\|u_m(\cdot,t)\|_2^2, \quad \forall t \in [0,T]. \tag{4.10}$$

On the other hand, by (4.5) and (4.6), we have $\{u_m\}_{m=1}^\infty \subset L^\infty(0,T;W^{1,p}(\mathcal{G}))$ and $\{u_{mt}\}_{m=1}^\infty \subset L^2(0,T;L^2(\mathcal{G}))$. By (3.1) and the continuity of Φ, we may assume that $\zeta_m \to \zeta$ as $m \to \infty$. Letting $m \to \infty$ in (4.10), by the Aubin-Lions compactness theorem it follows that

$$\int_t^T \|u(\cdot,s)\|_2^p ds \le \frac{1}{2\zeta}\|u(\cdot,t)\|_2^2, \quad \forall t \in [0,T].$$

Letting $T \to \infty$, by Lemma 4.1(2), we obtain

$$\|u(\cdot,t)\|_2 \le \|u_0\|_2\Big(\frac{p}{2(1+\zeta(p-2)\|u_0\|_2^{p-2}t)}\Big)^{\frac{1}{p-2}}, \quad \forall t \ge 0.$$

Case 2: $1 < p \le 2$**.** By Lemma 2.1 and (4.6), we have

$$\int_{\mathcal{G}} |u_m(t)|^2 dx \le \sup_{\mathcal{G}} |u_m(t)|^{2-p} \int_{\mathcal{G}} |u_m(t)|^p dx \le C\|u_m(\cdot,t)\|_p^p.$$

Using (4.7), we obtain

$$I(u_m(t)) \ge C(\frac{1}{2} - C(\frac{1}{2})(p^2\Phi(u_{0m}))^{\gamma-1})\|u_m(\cdot,t)\|_2^2 = C\alpha_m\|u_m(\cdot,t)\|_2^2, \tag{4.11}$$

where $\alpha_m := \frac{1}{2} - C(\frac{1}{2})(p^2\Phi(u_{0m}))^{\gamma-1} > 0$. Combining (4.8) with (4.11) yields

$$\int_t^T \|u_m(\cdot,s)\|_2^2 ds \le \frac{1}{C\alpha_m}\|u_m(\cdot,t)\|_2^2, \quad \forall t \in [0,T]. \tag{4.12}$$

Assume $\alpha_m \to \alpha$ as $m \to \infty$. Letting $m \to \infty$ in (4.12) we obtain

$$\int_t^T \|u(\cdot,s)\|_2^2 ds \le \frac{1}{C\alpha}\|u(\cdot,t)\|_2^2, \quad \forall t \in [0,T].$$

Letting $T \to \infty$, by Lemma 4.1 (1) it follows that

$$\|u(\cdot,t)\|_2 \le \|u_0\|_2 e^{\frac{1}{2} - \frac{1}{C\alpha}t}, \quad \forall t \ge 0.$$

This completes the proof. $\square$

5. Blow-up

In this section, we prove Theorem 1.4. When $u_0 \in \mathcal{Z}$, we prove that the local solution of problem (1.1) blows up at finite time when $p > 2$, and the global solution of problem (1.1) blows up at infinity when $1 < p \leq 2$.

Proof Theorem 1.4. Assume that $u(t)$ is the unique local solution obtained in Theorem 1.2, which admits a maximal existence interval $[0, T_{\max})$ for some $T_{\max} > 0$. Clearly, (1.6) holds on $[0, T_{\max})$.

We claim that

$$\text{if } u_0 \in \mathcal{Z}, \text{ then } u(t) \in \mathcal{Z} \text{ for all } t \in [0, T_{\max}). \tag{5.1}$$

Note that, by similar arguments as in Theorem 1.3, we can prove that

$$u_m(t) \in \mathcal{Z}, \quad \forall t \in [0, T_{\max})$$

for sufficiently large m. By (3.11), (3.12) and similar arguments as in previous section, we obtain

$$u(t) \in \mathcal{Z}, \quad \forall t \in [0, T_{\max}).$$

Hence the claim holds.

We define $M(t) := \|u(\cdot, t)\|_2^2$. Then from (5.1), we obtain

$$\begin{aligned} M'(t) &= 2 \int_{\mathcal{G}} u_t(t) u(t) dx \\ &= 2\Big(- \|u(\cdot, t)\|_{1,p}^p + \int_{\mathcal{G}} |u(t)|^p \log |u(t)| dx \Big) \\ &= -2I(u(t)) > 0, \quad \forall t \in [0, T_{\max}). \end{aligned} \tag{5.2}$$

Furthermore, using (1.6) and (5.2), we have

$$M'(t) \geq 2p \int_0^t \|u_s(\cdot, s)\|_2^2 ds + \frac{2}{p} \|u(\cdot, t)\|_p^p - 2p\Phi(u_0). \tag{5.3}$$

Case 1: $p > 2$. By $\Phi(u_0) \leq 0$ and (5.3) we obtain

$$M'(t) \geq \frac{2}{p} \|u(\cdot, t)\|_p^p \geq \frac{2}{p} |\mathcal{G}|^{\frac{2-p}{2}} \|u(\cdot, t)\|_2^p = \frac{2}{p} |\mathcal{G}|^{\frac{2-p}{2}} M(t)^{\frac{p}{2}},$$

which implies that

$$\|u(\cdot, t)\|_2^2 \geq \Big(\frac{1}{\|u_0\|_2^{2-p} - \kappa t} \Big)^{\frac{2}{p-2}},$$

where $\kappa := \frac{p-2}{2p} |\mathcal{G}|^{\frac{2-p}{2}}$. Hence, letting $T_{\max} := \|u_0\|_2^{2-p} / \kappa$, we obtain

$$\lim_{t \to T_{\max}^-} \|u(\cdot, t)\|_2^2 = +\infty.$$

Case 2: $1 < p \leq 2$. By (1.6), (5.2) and $\Phi(u_0) \leq 0$, it follows that

$$M'(t) \geq p \int_0^t \|u_s(\cdot, s)\|_2^2 ds. \tag{5.4}$$

We claim that, for any $t_0 > 0$,

$$\int_0^{t_0} \|u_s(\cdot, s)\|_2^2 ds > 0.$$

If not, we may assume that there exists $t_0 > 0$ such that $\int_0^{t_0} \|u_s(\cdot, s)\|_2^2 ds = 0$, which implies that $u_t(t) = 0$ for a.e. $(x,t) \in \mathcal{G} \times (0, t_0)$. Then $I(u(t)) = 0$ for a.e. $t \in (0, t_0)$, and thus

$$\Phi(u(t)) = \frac{1}{p^2} \|u(\cdot, t)\|_p^p.$$

Since $\Phi(u(t)) \le \Phi(u_0) \le 0$, we obtain $\|u(\cdot, t)\|_p = 0$ for all $t \in [0, t_0]$, which produces a contradiction. Hence, the claim holds.

Fix $t_0 > 0$. Clearly, $\mu := \int_0^{t_0} \|u_s(\cdot, s)\|_2^2 ds > 0$. Integrating (5.4) over (t_0, t), we obtain

$$\begin{aligned} M(t) &\ge M(t_0) + p \int_{t_0}^{t} \int_0^{\tau} \|u_s(\cdot, s)\|_2^2 ds d\tau \\ &\ge \|u(\cdot, t_0)\|_2^2 + p\mu(t - t_0) \\ &\ge p\mu(t - t_0). \end{aligned}$$

Hence $\lim_{t\to\infty} M(t) = \infty$. Furthermore, by (5.2), we have

$$M'(t) = -2I(u(t)) \ge -2p\Phi(u_0) - \frac{2}{p}\|u(\cdot, t)\|_2^2 = -2p\Phi(u_0) - \frac{2}{p} M(t),$$

which implies that

$$\big(e^{\frac{2}{p}t} M(t)\big)' \ge -2pe^{2t/p}\Phi(u_0).$$

Integrating over $(0, t)$, by $\Phi(u_0) \le 0$ and $\|u_0\|_2^2 \le -p^2\Phi(u_0)$, we obtain

$$M(t) \ge \|u_0\|_2^2.$$

The proof is complete. □

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