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# DOUBLE PHASE EQUATIONS WITH AN INDEFINITE CONCAVE TERM

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Abstract. We consider a Dirichlet problem having a double phase differential operator with unbalanced growth and reaction involving the combined effects of a concave (sublinear) and of a convex (superlinear) terms. We allow the coefficient $\mathcal{E} \in L^\infty(\Omega)$ of the concave term to be sign changing. We show that when $\|\mathcal{E}\|_\infty$ is small the problem has at least two bounded positive solutions.

## 1. Introduction

Let $\Omega \subseteq \mathbb{R}^N$ be a bounded domain with a Lipschitz boundary $\partial\Omega$. In this paper we study the double phase problem

$$\begin{gathered} -\Delta_p^a u(z) - \Delta_q u(z) = \mathcal{E}(z)u(z)^{\tau-1} + f(z,u(z)) \quad \text{in } \Omega, \\ u|_{\partial\Omega} = 0, \quad 1 < \tau < q < p < N, ]; u \geq 0. \end{gathered} \tag{1.1}$$

By $\Delta_p^a$ we denote the weighted $p$-Laplace differential operator with weight $a(\cdot)$, defined by

$$\Delta_p^a u = \operatorname{div}(a(z)|Du|^{p-2}Du).$$

By $\Delta_q$ we denote the standard $q$-Laplace differential operator

$$\Delta_q u = \operatorname{div}(|Du|^{q-2}Du).$$

Problem (1.1) has the sum of these two operators. So, the differential operator in (1.1), is not homogeneous. In the reaction tern (right hand side) of (1.1), we have the combined effects of two nonlinearities of different nature. One is the function $x \to \mathcal{E}(z)x^{\tau-1}, x \geq 0$ with $\mathcal{E} \in L^\infty(\Omega)$. Since $\tau < q < p$, this is a "concave" ($(q-1)$-sublinear) term, while the perturbation $f(z,x)$ is a Caratheodory function (that is, for all $x \in \mathbb{R}, z \to f(z,x)$ is measurable and for a.a. $z \in \Omega, x \to f(z,x)$ is continuous) which is assumed to be $(p-1)$-superlinear as $x \to \infty$ but without satisfying the Ambrosetti-Rabinowitz condition (the AR-condition for short). So, we have a "concave-convex" reaction, with the distinguishing feature that the coefficient $\mathcal{E} \in L^\infty(\Omega)$ of the concave term, is in general sign changing. In the standard concave-convex problem $\mathcal{E}(\cdot) \equiv \lambda > 0$ and we can prove existence and multiplicity of positive solutions and the result is global with respect to the parameter $\lambda > 0$ (a bifurcation-type result). We refer to the works of Ambrosetti-Brezis-Cerami

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[1] and Anello [2] (equations driven by the Laplacian), Garcia Azoreio-Manfredi-Peral Alonso [4] (equations driven by the $p$-Laplacian) and Liu-Papageorgiou [9] (anisotropic $(p,q)$-equations).

We do not assume that the weight function $a \in L^\infty(\Omega)$ is bounded away from zero, that is, we do not require that $0 < \operatorname{ess\,inf}_\Omega a(\cdot)$. Therefore the integrand in the energy functional corresponding to the differential operator

$$\eta(z,t) = a(z)t^p + t^q \quad \text{for all } z \in \Omega, \text{ all } t \geq 0$$

is a Caratheodory function which exhibits unbalanced growth with respect to $t \geq 0$, namely we have

$$t^q \leq \eta(z,t) \leq c_0[1+t^p] \quad \text{for a.a. } z \in \Omega, \text{ all } t \geq 0, \text{ some } c_0 > 0.$$

Such functionals were first investigated in the context of problems of the calculus of variations and nonlinear elasticity theory by Marcellini [10] and Zhikov [16]. For unbalanced elliptic problems there is no global regularity theory (up to the boundary), analogous to the one existing for balanced problems (see, for example, [5]). There are only local regularity results, see Baroni-Colombo-Mingione [3], Marcellini [11] and the nice survey paper of Mingione-Rădulescu [12]. The unbalanced growth of $\eta(z,\cdot)$ leads to a functional framework for double phase problems which is based on generalized Orlicz spaces. In the next section we discuss this family of spaces. Details can be found in the book of Harjulehto-Hästo [7].

Using variational tools in the framework of such spaces, we show that when $\|\mathcal{E}\|_\infty$ is small problem (1.1) has at least two bounded positive solutions. The lack of a global regularity theory for double phase unbalanced growth problems and the fact that the coefficient function of the concave term is nodal (sign-changing), are features that make problem 1.1 more difficult to handle and also more interesting.

## 2. Mathematical background

As we already mentioned in the Introduction, the functional framework for the analysis of problem (1.1) is provided by generalized Orlicz spaces. Recall that by $C^{0,1}(\overline{\Omega})$ we denote the space of Lipschitz continuous functions. Our hypotheses on the weight $a(\cdot)$ and the exponents $\tau, q, p$ are the following:

(H0) $a \in C^{0,1}(\overline{\Omega}), a \not\equiv 0, a(z) \geq 0$ for all $z \in \overline{\Omega}$, $1 < \tau < q < p < N, \frac{p}{q} < 1 + \frac{1}{N}$.

**Remark 2.1.** The above inequality restricting the exponents $q$ $p$ is common in Dirichlet double phase problems and it implies that $p < q^* = \frac{Nq}{N-q}$. This leads to some useful compact embeddings of spaces. Moreover, since $a \in C^{0,1}(\overline{\Omega})$, we know that the Poincare inequality is valid on the relevant Orlicz-Sobolev space (see [7, p.138]).

Let $M(\Omega)$ be the linear space of all measurable functions $u : \Omega \to \mathbb{R}$. As usual we identify two such functions which differ only on a Lebesgue-null set. Recall that

$$\eta(z,t) = a(z)t^p + t^q.$$

Then the Orlicz-Lebesghe space is

$$L^\eta(\Omega) = \{u \in M(\Omega) : \rho_\eta(u) < \infty\},$$

where $\rho_\eta(\cdot)$ is the modular function

$$\rho_\eta(u) = \int_\Omega \eta(z,|u|)dz.$$

We equip $L^\eta(\Omega)$ with the so called "Luxemburg norm"

$$\|u\|_\eta = \inf\big[\lambda > 0 : \rho_\eta(\frac{u}{\lambda}) \le 1\big].$$

Then $L^\eta(\Omega)$ becomes a Banach space which is separable and reflexive. In fact the uniform convexity of $\eta(z,\cdot)$ implies the uniform convexity of $L^\eta(\Omega)$. Moreover, $L^\eta(\Omega)^* = L^{\eta^*}(\Omega)$ with $\eta^*(z,y)$ being the convex conjugate of $\eta(z,\cdot)$ and we have the following version of the Hölder's inequality

$$\int_\Omega |hg|dz \le 2\|h\|_\eta\|g\|_{\eta^*} \quad \text{for all } h \in L^\eta(\Omega), \ \ \text{all } g \in L^{\eta^*}(\Omega).$$

Using $L^\eta(\Omega)$ we can define the corresponding Orlicz-Sobolev space

$$W^{1,\eta}(\Omega) = \{u \in L^\eta(\Omega) : |Du| \in L^\eta(\Omega)\}$$

with $Du$ denoting the weak gradient of $u$. We equip this space with the norm

$$\|u\|_{1,\eta} = \|u\|_\eta + \|Du\|_\eta$$

where $\|Du\|_\eta = \||Du|\|_\eta$. Let $W_0^{1,\eta}(\Omega) = \overline{C_0^\infty(\Omega)}^{\|\cdot\|_{1,\eta}}$. For this space the Poincaré inequality is true and so on $W_0^{1,\eta}(\Omega)$ we can consider the equivalent norm

$$\|u\| = \|Du\|_\eta \quad \text{for all } u \in W_0^{1,\eta}(\Omega).$$

We have the following useful embeddings.

**Proposition 2.2.** *If* (H0) *holds, then*

(a) $L^\eta(\Omega) \hookrightarrow L^r(\Omega)$ *and* $W_0^{1,\eta}(\Omega) \hookrightarrow W_0^{1,r}(\Omega)$ *continuously for all* $r \in [1,q]$;
(b) $W_0^{1,\eta}(\Omega) \hookrightarrow L^r(\Omega)$ *continuously (resp. compactly) for all* $r \in [1,q^*]$ *(resp.* $r \in [1,q^*)$*);*
(c) $L^p(\Omega) \hookrightarrow L^\eta(\Omega)$ *continuously.*

The norm $\|\cdot\|$ and the modular function $\rho_\eta(\cdot)$ are closely related.

**Proposition 2.3.** *If hypotheses (H0) holds, then*

(a) *for* $u \in L^\eta(\Omega) \setminus \{0\}$ *we have* $\|u\|_\eta = \lambda \Leftrightarrow \rho_\eta(\frac{u}{\lambda}) = 1$;
(b) $\|u\|_\eta < 1(resp. = 1, > 1) \Leftrightarrow \rho_\eta(u) < 1(resp. = 1, > 1)$;
(c) $\|u\|_\eta < 1 \Rightarrow \|u\|_\eta^p \le \rho_\eta(u) \le \|u\|_\eta^q$;
(d) $\|u\|_\eta > 1 \Rightarrow \|u\|_\eta^q \le \rho_\eta(u) \le \|u\|_\eta^p$;
(e) $\|u\|_\eta \to 0(resp. \ \to +\infty) \Leftrightarrow \rho_\eta(u) \to 0(resp. \ \to +\infty)$.

We consider the nonlinear operator $V : W_0^{1,\eta}(\Omega) \to W_0^{1,\eta}(\Omega)^*$ defined by

$$\langle V(u), h\rangle = \int_\Omega [a(z)|Du|^{p-2} + |Du|^{q-2}](Du, Dh)_{\mathbb{R}^N} dz, \ \text{for all } u, h \in W_0^{1,\eta}(\Omega).$$

This operator has the following properties (see Liu-Dai [8]).

**Proposition 2.4.** *If hypotheses* (H0) *holds, then* $V(\cdot)$ *is bounded (that is, maps bounded sets to bounded sets), continuous, strictly monotone (thus maximal monotone too) and of type* $(S)_+$, *that is, "*$u_n \xrightarrow{w} u$ *in* $W_0^{1,\eta}(\Omega)$ *and* $\limsup_{n\to\infty}\langle V(u_n), u_n - u\rangle \le 0$ *imply that* $u_n \to u$ *in* $W_0^{1,\eta}(\Omega)$."

If $u \in M(\Omega)$, then $u^{\pm} = \max\{\pm u(z), 0\}$, for all $z \in \Omega$ and we have $u = u^+ - u^-$, $|u| = u^+ + u^-$ and if $u \in W_0^{1,\eta}(\Omega)$, then $u^{\pm} \in W_0^{1,\eta}(\Omega)$.

If $X$ is a Banach space and $\varphi \in C^1(X)$, then $K_\varphi = \{u \in X : \varphi'(u) = 0\}$ (the critical set of $\varphi$). Also, we say that $\varphi(\cdot)$ satisfies the C-condition, if the following property hold:

> every sequence $\{u_n\}_{n\in\mathbb{N}} \subseteq X$ such that $\{\varphi(u_n)\}_{n\in\mathbb{N}} \subseteq \mathbb{R}$ is bounded and $(1 + \|u_n\|_X)\varphi'(u_n) \to 0$ in $X^*$ as $n \to +\infty$ admits a strongly convergent subsequence.

This is a compactness-type condition on the functional $\varphi(\cdot)$ which compensates for the fact the ambient space need not be locally compact (being in general infinite dimensional). It leads to a deformation lemma from which follow the minimax theorems for the critical values of $\varphi(\cdot)$ (see Papageorgiou-Rădulescu-Repovš [4, Chaper 5]).

By $\widehat{\lambda}_1(q)$ we denote the principal eigenvalue of $(-\Delta_q, W_0^{1,q}(\Omega))$. So, we consider the following nonlinear eigenvalue problem

$$-\Delta_q u(z) = \widehat{\lambda}|u(z)|^{q-2}u(z) \quad \text{in } \Omega, u|_{\partial\Omega} = 0. \tag{2.1}$$

We say $\widehat{\lambda}$ is an "eigenvalue" of $(-\Delta_q, W_0^{1,q}(\Omega))$ if problem (2.1) has a nontrivial solution known as an "eigenfunction" corresponding to the eigenvalue $\widehat{\lambda}$. We know that there is a smallest eigenvalue $\widehat{\lambda}_1(q) > 0$ given by

$$\widehat{\lambda}_1(q) = \inf\big[\frac{\|Du\|_q^q}{\|u\|_q^q} : u \in W_0^{1,q}(\Omega), u \neq 0\big]. \tag{2.2}$$

The eigenvalue is isolated and simple (that is, if $u, v$ are eigenfunctions corresponding to $\widehat{\lambda}_1(q) > 0$, then $u = \theta v$ for some $\theta \in \mathbb{R} \setminus \{0\}$). The infimum in (2.2) is realized on the corresponding one-dimensional eigenspace the elements of which have constant sign and belong in $C^1(\Omega)$. So, if $\widehat{u}$ is an eigenfunction corresponding to $\widehat{\lambda}_1(q) > 0$, then $\widehat{u}(z) > 0$ or $\widehat{u}(z) < 0$ for all $z \in \Omega$ (by the nonlinear maximum principle). For details we refer to Gasiński-Papageorgiou [5, Chapter 6]. Using the aforementioned properties of $\widehat{\lambda}_1(q) > 0$ and of its corresponding eigenfunctions, we show the following proposition (see Mugnai-Papageorgiou [13, Lemma 4.11]).

**Proposition 2.5.** *If $\theta \in L^\infty(\Omega)$, $\theta(z) \le \widehat{\lambda}_1(q)$ for a.a. $z \in \Omega$ and $\theta \not\equiv \widehat{\lambda}_1(q)$, then there exists $c_0 > 0$ such that*

$$c_0\|Du\|_q^q \le \|Du\|_q^q - \int_\Omega \theta(z)|u|^q dz \quad \text{for all } u \in W_0^{1,q}(\Omega).$$

Next we introduce our hypotheses on the rest of the data of (1.1).

(H1) $\mathcal{E} \in L^\infty(\Omega), \mathcal{E}^+ \not\equiv 0$ and there exists $\widetilde{u} \in W_0^{1,\eta}(\Omega)$ with $\widetilde{u}(z) > 0$, for a.a. $z \in \Omega$ and $\int_\Omega \mathcal{E}(z)\widetilde{u}^\tau dz > 0$.

**Remark 2.6.** If $\mathcal{E}(\cdot) \equiv \lambda > 0$ (that is, we have the standard concave-convex problem), then hypotheses (H1) is satisfied with any $u \in W_0^{1,\eta}(\Omega)$, $u \not\equiv 0, u \ge 0$.

(H2) $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function such that $f(z, 0) = 0$ for a.a. $z \in \Omega$ and
  - (i) $|f(z, x)| \le \widehat{a}(z)[1 + x^{r-1}]$ for a.a. $z \in \Omega$, all $x \ge 0$, with $\widehat{a} \in L^\infty(\Omega)$, $p < r < q^*$;

(ii) If $F(z,x)=\int_0^x f(z,s)ds$, then $\lim_{x\to+\infty}\frac{F(z,x)}{x^p}=+\infty$ uniformly for a.a. $z\in\Omega$ and there exists $\mu\in\big((r-q)\frac{N}{q},q^*\big)$ such that $\tau<\mu$ and

$$0<\beta_0\le\liminf_{x\to+\infty}\frac{f(z,x)x-pF(z,x)}{x^\mu}$$

uniformly for a.a. $z\in\Omega$;

(iii) there exist $\delta>0$ and $\theta\in L^\infty(\Omega)$ such that

$$\theta(z)\le\widehat{\lambda}_1(q)\quad\text{for a.a. } z\in\Omega,\ \theta\not\equiv\widehat{\lambda}_1(q),$$
$$f(z.x)\ge 0\quad\text{for a.a. } z\in\Omega,\ \text{all } 0\le x\le\delta,$$
$$\limsup_{x\to0^+}\frac{qF(z,x)}{x^q}\le\theta(z)\quad\text{uniformly for a.a. } z\in\Omega.$$

Since we look for positive solutions and the above hypotheses concern the positive semiaxis $\mathbb{R}_+=[0,+\infty)$, without any loss of generality, we may assume that $f(z,x)=0$ for a.a. $z\in\Omega$ and all $x\le 0$.

## 3. Positive solutions

In the section we show that when $\|\mathcal{E}\|_\infty$ is small, then problem (1.1) has at least two bounded positive solutions.

Let $\varphi:W_0^{1,\eta}(\Omega)\to\mathbb{R}$ be the energy functional for problem (1.1) defined by

$$\varphi(u)=\frac{1}{p}\rho_a(Du)+\frac{1}{q}\|Du\|_q^q-\frac{1}{\tau}\int_\Omega\mathcal{E}(z)(u^+)^\tau dz-\int_\Omega f(z,u^+)dz,$$

for all $u\in W_0^{1,\eta}(\Omega)$, with $\rho_a(Du)=\int_\Omega a(z)|Du|^p dz$. We know that $\varphi\in C^1(W_0^{1,\eta}(\Omega))$.

**Proposition 3.1.** *If hypotheses* (H0)–(H2) *hold, then* $\varphi(\cdot)$ *satisfies the C-condition.*

*Proof.* We consider a sequence $\{u_n\}_{n\in\mathbb{N}}\subseteq W_0^{1,\eta}(\Omega)$ such that

$$|\varphi(u_n)|\le c_1\quad\text{for some } c_1>0,\ \text{and all } n\in\mathbb{N},\tag{3.1}$$
$$(1+\|u_n\|)\varphi'(u_n)\to 0\quad\text{in } W_0^{1,\eta}(\Omega)^*\text{ as } n\to+\infty.\tag{3.2}$$

From (3.1) we have

$$\rho_a(Du_n)+\frac{p}{q}\|Du_n\|_q^q-\frac{p}{\tau}\int_\Omega\mathcal{E}(z)(u_n^+)^\tau dz-\int_\Omega pF(z,u_n^+)dz\le pc_1\tag{3.3}$$

for all $n\in\mathbb{N}$. Also from (3.2) we have

$$|\langle V(u_n),h\rangle-\int_\Omega\mathcal{E}(z)(u_n^+)^{\tau-1}hdz-\int_\Omega f(z,u_n^+)hdz|\le\frac{\varepsilon_n\|h\|}{1+\|u_n\|}\tag{3.4}$$

for all $h\in W_0^{1,\eta}(\Omega)$ and all $\varepsilon_n\to0^+$.

In (3.4) we choose $h=u_n\in W_0^{1,\eta}(\Omega)$ and obtain

$$-\rho_a(Du_n)-\|Du_n\|_q^q+\int_\Omega\mathcal{E}(z)(u_n^+)^\tau dz+\int_\Omega f(z,u_n^+)u_n^+dz\le\varepsilon_n\tag{3.5}$$

for all $n\in\mathbb{N}$. When we add (3.3) and (3.5), and recall that $\tau<q<p$, we obtain

$$\int_\Omega[f(z,u_n^+)u_n^+-pF(z,u_n^+)]dz\le[\frac{p}{\tau}-1]\|\mathcal{E}\|_\infty\|u_n^+\|_\tau^\tau+c_2\tag{3.6}$$

for some $c_2>0$ and all $n\in\mathbb{N}$.

Hypotheses (H2)(i),(ii) imply that we can find $\widehat{\beta}_0 \in (0, \beta_0)$ and $c_3 > 0$ such that

$$\widehat{\beta}_0 x^\mu - c_3 \le f(z,x)x - pF(z,x) \quad \text{for a.a. } z \in \Omega, \text{ all } x \ge 0. \tag{3.7}$$

We return to (3.6) and use (3.7) to obtain

$$\begin{aligned} &\|u_n^+\|_\mu^\mu \le c_4\Big[1 + \|u_n^+\|_\mu^\tau\Big] \quad \text{for some } c_4 > 0 \text{ all } n \in \mathbb{N} \text{ (recall that } \tau < \mu),\\ &\Rightarrow \{u_n^+\}_{n\in\mathbb{N}} \subseteq L^\mu(\Omega) \quad \text{is bounded.} \end{aligned} \tag{3.8}$$

From hypothesis (H2)(ii) we see that we can always assume that $\mu < r < q^*$. Hence we can find $t \in (0,1)$ such that

$$\frac{1}{r} = \frac{1-t}{\mu} + \frac{t}{q^*} \tag{3.9}$$

Invoking the interpolation inequality (Papageorgiou-Winkert [15, p. 116]), we have

$$\begin{aligned} &\|u_n^+\|_r \le \|u_n^+\|_\mu^{1-t} \|u_n^+\|_{q^*}^t \text{ for all } n \in \mathbb{N},\\ &\Rightarrow \|u_n^+\|_r^r \le c_5 \|u_n^+\|^{tr} \text{ for some } c_5 > 0, \text{ all } n \in \mathbb{N} \end{aligned} \tag{3.10}$$

(recall that $W_0^{1,\eta}(\Omega) \hookrightarrow L^{q^*}(\Omega)$ continuously, see Proposition 2.2). In (3.4) we use the test function $h = u_n^+ \in W_0^{1,\eta}(\Omega)$. Then

$$\rho_a(Du_n^+) \le \varepsilon_n + \int_\Omega f(z, u_n^+) u_n^+ dz + \|\mathcal{E}\|_\infty \|u_n^+\|_\tau^\tau \tag{3.11}$$

for all $n \in \mathbb{N}$, see hypotheses (H1).

Since our aim is to show that $\{u_n^+\}_{n\in\mathbb{N}} \subseteq W_0^{1,\eta}(\Omega)$ is bounded, we may assume without any loss of generality that $\|u_n^+\| \ge 1$ for all $n \in \mathbb{N}$. Then from (3.11) and using Proposition 2.3, we have

$$\|u_n^+\|^q \le c_6[1 + \|u_n\|^{tr}] \quad \text{for some } c_6 > 0, \quad \text{all } n \in \mathbb{N} \tag{3.12}$$

(see hypothesis (H2)(i),(ii), (3.10) and recall that $\tau < \mu$).) Since $q^* = \frac{Nq}{N-q}$, from (3.9) we have

$$tr = \frac{q^*(r-\mu)}{q^* - \mu} = \frac{Nq(r-\mu)}{Nq - N\mu + q\mu} < q \quad \text{(see hypothesis (H2)(ii)).} \tag{3.13}$$

From (3.13) and (3.12) it follows that

$$\{u_n^+\}_{n\in\mathbb{N}} \subseteq W_0^{1,\eta}(\Omega) \quad \text{is bounded.} \tag{3.14}$$

Next in (3.4) we choose $h = u_n^- \in W_0^{1,\eta}(\Omega)$. Then

$$\begin{aligned} &\rho_a(Du_n^-) + \|Du_n^-\|_q^q \le \varepsilon_n \quad \text{for all } n \in \mathbb{N}\\ &\Rightarrow u_n^- \to 0 \quad \text{in } W_0^{1,\eta}(\Omega) \text{ as } n \to +\infty. \end{aligned} \tag{3.15}$$

(see Proposition 2.3 and use Poincaré inequality). From (3.14) and (3.15), we infer that

$$\{u_n\}_{n\in\mathbb{N}} \subseteq W_0^{1,\eta}(\Omega) \quad \text{is bounded.}$$

So, by passing to a suitable subsequence if necessary, we assume that

$$u_n \xrightarrow{w} u \text{ in } W_0^{1,\eta}(\Omega) \text{ and } u_n \to u \text{ in } L^r(\Omega). \tag{3.16}$$

In (3.4) we choose $h = u_n - u \in W_0^{1,\eta}(\Omega)$, pass to the limit as $n \to +\infty$ and use (3.16). We obtain

$$\lim_{n\to+\infty} \langle V(u_n), u_n - u\rangle = 0;$$

$$\Rightarrow u_n \to u \text{ in } W_0^{1,\eta}(\Omega) \text{ (see Proposition 2.4).}$$

This proves that $\varphi(\cdot)$ satisfies the C-condition. □

Next we show that when $\|\mathcal{E}\|_\infty$ is small, then the function $\varphi(\cdot)$ satisfied the mountain pass geometry.

**Proposition 3.2.** *If hypotheses* (H0)—(H2) *hold, then there exists* $\lambda^* > 0$ *such that if* $\|\mathcal{E}\|_\infty < \lambda^*$*, we can find* $\widehat{\rho} > 0$ *such that* $\varphi(u) \geq \widehat{m} > 0$ *for all* $\|u\| = \widehat{\rho}$*.*

*Proof.* On account of hypotheses (H2)(i),(iii), given $\varepsilon > 0$, we can find $c_7 = c_7(\varepsilon)$ such that

$$F(z,x) \leq \frac{1}{q}[\theta(z) + \varepsilon]x^p + c_7 x^r \quad \text{for a.a. } z \in \Omega, \text{ all } x \geq 0. \tag{3.17}$$

Then for $u \in W_0^{1,\eta}(\Omega)$ we have

$$\begin{aligned}\varphi(u) \geq \frac{1}{p}\rho_a(Du) + \frac{1}{q}\Big[\|Du\|_q^q - \int_\Omega \theta(z)|u|^q dz - \varepsilon c_8 \|Du\|_q^q\Big] \\ - c_9[\|\mathcal{E}\|_\infty \|u\|^\tau + \|u\|^r]\end{aligned} \tag{3.18}$$

for some $c_8, c_9 > 0$ (see (3.17), (2.2)). From Proposition 2.5, we know that

$$\|Du\|_q^q - \int_\Omega \theta(z)|u|^q dz \geq c_0 \|Du\|_q^q \quad \text{for all } u \in W_0^{1,\eta}(\Omega).$$

So, choosing $\varepsilon \in (0, c_0/c_8)$ we see that

$$\|Du\|_q^q - \int_\Omega \theta(z)|u|^q dz - \varepsilon c_8 \|Du\|_q^q \geq 0. \tag{3.19}$$

Assume that $\|u\| \leq 1$. Then using Proposition 2.3 and (3.19), from (3.18) we have

$$\begin{aligned}\varphi(u) &\geq \frac{1}{p}\|u\|^p - c_9\|\mathcal{E}\|_\infty \|u\|^\tau - c_9\|u\|^r \\ &\geq [\frac{1}{p} - c_9(\|\mathcal{E}\|_\infty \|u\|^{\tau-p} + \|u\|^{r-p})]\|u\|^p.\end{aligned} \tag{3.20}$$

Let $\gamma(t) = \|\mathcal{E}\|_\infty t^{\tau-p} + t^{r-p}$ for all $t > 0$. Evidently $\gamma \in C^1(0,\infty)$ and since $\tau < p < r$, we have $\gamma(t) \to +\infty$ as $t \to 0^+$ and as $t \to +\infty$. Therefore we can find $t_0 > 0$ such that

$$\begin{aligned}&\gamma(t_0) = \min_{t>0} \gamma(t), \\ &\Rightarrow \gamma'(t_0) = 0, \\ &\Rightarrow (p-\tau)\|\mathcal{E}\|_\infty = (r-p)t_0^{r-\tau}, \\ &\Rightarrow t_0 = [\frac{(p-\tau)\|\mathcal{E}\|_\infty}{r-p}]^{\frac{1}{r-\tau}}.\end{aligned}$$

So, we have

$$\gamma(t_0) = \|\mathcal{E}\|_\infty [\frac{r-p}{(p-\tau)\|\mathcal{E}\|_\infty}]^{\frac{p-\tau}{r-\tau}} + [\frac{(p-\tau)\|\mathcal{E}\|_\infty}{r-p}]^{\frac{r-p}{r-\tau}}.$$

Since $\frac{p-\tau}{r-\tau} < 1$, we see that $\|\mathcal{E}\|_\infty \to 0^+ \Rightarrow \gamma(t_0) \to 0^+$. Therefore we can find $\lambda^* > 0$ such that

$$\|\mathcal{E}\|_\infty < \lambda^* \Rightarrow \gamma(t_0) < \frac{1}{c_9 p}.$$

Using this in (3.20) we see that

$$\varphi(u) \geq \widehat{m} > 0 \text{ for all } \|u\| = \widehat{\rho} = t_0(\|\mathcal{E}\|_\infty).$$

If $\|u\| > 1$, then the same argument works with $p$ replaced by $q$ (since now $\rho_a(Du) \geq \|u\|^q$, see Proposition 2.3). $\square$

**Proposition 3.3.** *If hypotheses* (H0)–(H2) *hold, then for $t > 0$ small we have $\varphi(t\widetilde{u}) < 0$ with $\widetilde{u} \in W_0^{1,\eta}(\Omega)$ as postulated by hypotheses* (H1).

*Proof.* On account of hypotheses (H2)(i), (iii), we can find $c_{10} > 0$ such that

$$F(z,x) \geq -c_{10}x^r \quad \text{for a.a. } z \in \Omega, \text{ all } x \geq 0. \tag{3.21}$$

Then for $t > 0$, we have

$$\varphi(t\widetilde{u}) \leq \frac{t^p}{p}\rho_a(D\widetilde{u}) + \frac{t^q}{q}\|D\widetilde{u}\|_q^q + c_{11}t^r\|\widetilde{u}\|^r - \frac{t^\tau}{\tau}\int_\Omega \mathcal{E}(z)\widetilde{u}^\tau dz$$

for some $c_{11} > 0$, see (3.1). For $t \in (0,1)$ we have

$$\varphi(t\widetilde{u}) \leq c_{12}t^q - \frac{t^\tau}{\tau}\int_\Omega \mathcal{E}(z)\widetilde{u}^\tau dz \quad \text{for some } c_{12} > 0.$$

Using hypotheses (H1) and since $\tau < q$, for $t \in (0,1)$ small, we have $\varphi(t\widetilde{u}) < 0$. $\square$

Now we are ready to prove the multiplicity theorem when $\|\mathcal{E}\|_\infty$ is small. One solution will be produced using the mountain pass theorem and the other will be a local minimizer of $\varphi(\cdot)$.

**Theorem 3.4.** *If hypotheses* (H0)–(H2) *hold, then for $\|\mathcal{E}\|_\infty$ small problem* (1.1) *has at least two nontrivial solutions $u_0, \widehat{u} \in W_0^{1,\eta}(\Omega) \cap L^\infty(\Omega)$ such that*

$$u_0(z), \widehat{u}(z) \geq 0 \quad \textit{for a.a. } z \in \Omega.$$

*Proof.* From Proposition 3.2 we know that if $\|\mathcal{E}\|_\infty$ is small, then we can find $\widehat{\rho} > 0$ such that

$$0 < \widehat{m} \leq \varphi(u) \quad \text{for all } u \in W_0^{1,\eta}(\Omega),\ \|u\| = \widehat{\rho}.$$

Consider the closed ball $\overline{B}_{\widehat{\rho}} = \{u \in W_0^{1,\eta}(\Omega) : \|u\| \leq \widehat{\rho}\}$. The reflexivity of $W_0^{1,\eta}(\Omega)$ implies that $\overline{B}_{\widehat{\rho}}$ is weakly compact and then by the Eberlein-Smulian theorem, we have that $\overline{B}_{\widehat{\rho}}$ is sequentially weakly compact. Using Proposition 2.2, we see that $\varphi(\cdot)$ is sequentially weakly lower semicontinuous. So, by the Weierstrass-Tonelli theorem we know that there exists $u_0 \in W_0^{1,\eta}(\Omega)$ such that

$$\varphi(u_0) = \inf[\varphi(u) : u \in \overline{B}_{\widehat{\rho}}]. \tag{3.22}$$

From Propositions 3.2 and 3.3, we have that

$$\begin{aligned}
&0 < \|u_0\| < \widehat{\rho}, \\
&\Rightarrow \varphi'(u_0) = 0 \text{ (see(25))} \\
&\Rightarrow |\langle V(u_0), h\rangle = \int_\Omega \mathcal{E}(z)(u_0^+)^{\tau-1}hdz + \int_\Omega f(z, u_0^+)hdz
\end{aligned} \tag{3.23}$$

for all $h \in W_0^{1,\eta}(\Omega)$.

In (3.18) we use the test function $h = -u_0^- \in W_0^{1,\eta}(\Omega)$ and obtain that $u_0 \geq 0, u_0 \neq 0$. Hence $u_0$ is a nontrivial positive solution of problem (1.1). Invoking [6, Theorem 3.1] we have that $u_0 \in W_0^{1,\eta}(\Omega) \cap L^\infty(\Omega)$.

Consider $u \in W_0^{1,\eta}(\Omega)$ with $u(z) > 0$ for a.a. $z \in \Omega$. Hypothesis (H2)(ii) implies that

$$\varphi(tu) \to -\infty \quad \text{as } t \to +\infty.$$

This fact and Propositions 3.1 and 3.2, permit the use the mountain pass theorem. So, we can find $\widehat{u} \in W_0^{1,\eta}(\Omega)$ such that

$$\widehat{u} \in K_\varphi \quad \text{and} \quad \varphi(u_0) < 0 = \varphi(0) < \widehat{m} \le \varphi(\widehat{u}).$$

Then $\widehat{u} \ge 0, \widehat{u} \ne 0$, is a solution of (1.1) and as before $\widehat{u} \in L^\infty(\Omega)$. □

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