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GROUND STATE SOLUTIONS FOR FRACTIONAL p-KIRCHHOFF EQUATION

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Abstract. We study the fractional p-Kirchhoff equation

$$\Big(a+b\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}\,dx\,dy\Big)(-\Delta)_p^s u-\mu|u|^{p-2}u=|u|^{q-2}u,\quad x\in\mathbb{R}^N,$$

where $(-\Delta)_p^s$ is the fractional p-Laplacian operator, a and b are strictly positive real numbers, $s\in(0,1)$, $1<p<\frac{N}{s}$, and $p<q<p_s^*-2$ with $p_s^*=\frac{Np}{N-ps}$. By using the variational method, we prove the existence and uniqueness of global minimum or mountain pass type critical points on the L^p-normalized manifold $S(c):=\big\{u\in W^{s,p}(\mathbb{R}^N):\int_{\mathbb{R}^N}|u|^p dx=c^p\big\}$.

1. Introduction and statement of main results

In this article, we consider the fractional p-Kirchhoff equation

$$\Big(a+b\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}\,dx\,dy\Big)(-\Delta)_p^s u-\mu|u|^{p-2}u=|u|^{q-2}u, \tag{1.1}$$

for $x\in\mathbb{R}^N$, where $(-\Delta)_p^s u$ is the fractional p-Laplacian operator, a and b are strictly positive real numbers, $s\in(0,1)$, $1<p<\frac{N}{s}$, and $p<q<p_s^*-2$ with $p_s^*=\frac{Np}{N-ps}$.

Equation (1.1) is related to stationary solutions of

$$u_{tt}+\Big(a+b\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}\,dx\,dy\Big)(-\Delta)_p^s u=f(x,u), \tag{1.2}$$

where $f(x,u)$ is a general nonlinearity. Kirchhoff's equation was suggested as a model for the transverse oscillations of a stretched string of the form [13]

$$\rho h\,u_{tt}-\Big(p_0+\frac{\mathcal{E}h}{2L}\int_0^L|\nabla u|^2\,dx\Big)\Delta u+\delta u_t+f(x,u)=0 \tag{1.3}$$

for $0<x<L$ and $t\geq 0$, where $u=u(x,t)$ is the lateral displacement at position x and at time t, L is the length of the string, h is the cross section area, ρ is the mass density, p_0 is the initial stress tension, $\mathcal{E}$ is the Young modulus, δ is the resistance modulus and f is the external force. Comparing with the semilinear equations , it

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is way more challenge and fascinating to research equations (1.1) and (1.2) visible of the existence of the nonlocal term

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}\,dx\,dy(-\Delta)_p^s u.$$

In recent years, many authors have dealt with Kirchhoff-type problems in the context of classical Laplace operators and proved results concerning the existence, multiplicity and properties of the solutions by variational methods. The existence, nonexistence and multiplicity of nontrivial solutions of fractional Kirchhoff-type equation with Hardy-Littlewood-Sobolev critical exponent were presented in [20]. By using a fibering-type approach, Che and Wu [3] obtained several quantitative results for the problem

$$-\Big(a+b\int_{\mathbb{R}^N}|\nabla u|^2\,dx\Big)\Delta u+u=k(x)|u|^{p-2}u+m(x)|u|^{q-2}u\quad\text{in } \mathbb{R}^N, \tag{1.4}$$

where $N\geq 3$, $a,b>0$, $1<q<2<p<\min\{4,2^*\}$. The three positive solutions are obtained mainly by using the Ekeland variational principle and the innovative constraint method of Nehari manifolds. For the p-Laplace operators, the uniqueness of the positive solution of the p-Laplace equation with Hardy potential and the asymptotic behavior was established [8].

For instance, replacing the term $|u|^p u$ with a general nonlinearity $f(x,u)$, there are many results on the existence of solutions for such equations, one can refer to [1, 5, 10, 15, 16] and the references therein. For the fractional Laplace equation, Feng and Su [7] establishes a generalized version of the lion-type theorem for the fractional Laplace that obtains the ground state solution. The existence of ground state solutions of fractional equations can also be found in Su and Feng [21] recent article. However, there is little literature concerned about the normalized solutions for the fractional p-Kirchhoff equation. With regard to the point, we attempt to study this kind of problem in this paper.

By treating μ as an unknown Lagrange multiplier, Equation (1.1) can be viewed as an eigenvalue problem. From this perspective, we can solve it by studying some constraint variational problems (1.1) and obtain a normalized solution. Inspired by [2, 12, 23], we first consider the following minimization problem

$$I(c):=\inf_{u\in S(c)}E_p(u) \tag{1.5}$$

where

$$S(c):=\big\{u\in W^{s,p}(\mathbb{R}^N):\int_{\mathbb{R}^N}|u|^p dx=c^p\big\}.$$

and

$$\begin{aligned} E_p(u)=\frac{a}{p}&\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}\,dx\,dy\\ &+\frac{b}{2p}\Big(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}\,dx\,dy\Big)^2-\frac{1}{q}\int_{\mathbb{R}^N}|u|^q dx \end{aligned} \tag{1.6}$$

A normalized solution to problem (1.5) exists if $u\in S(c)$ is a minimizer of problem (1.5) such that there exists $\mu\in\mathbb{R}$ such that $E_p'(u)=\mu|u|^{p-2}u$, i.e., $u\in S(c)$ is a solution of (1.1) for some $\mu\in\mathbb{R}$.

For the case of $p=2$ and $s=1$, scholars have made in-depth research. For example, Ye [23] according to the principle of concentrated compactness, it is proved that there exists $c_s^*>0$ such that if $c>c_s^*$, problem (1.5) is reachable, where the constant c is related to the ground state solution of equation (1.7) below. Zeng and Zhang [24] reproved some of the results in [22] by applying some simple energy estimates. They also showed that the minimum element of problem (1.5) (if it exists) is unique and is a telescopic translation of the ground state solution of equation (1.7).

This article intends to prove the existence and uniqueness of the minimal element of problem (1.5) and extend the results of paper [11, 24] to the case of $p \in (1, \frac{N}{s})$. To do this, we first study the equation

$$(-\Delta)_p^s u + \big[\frac{pqs}{N(q-p)} - 1\big]|u|^{p-2}u = |u|^{q-2}u, \quad 0 < q < p_s^*. \tag{1.7}$$

Note that if $p \neq 2$ or $s \neq 1$, the operator $(-\Delta)_p^s$ is no longer linear, which leads to some different properties from the case of $p = 2$ and $s = 1$. For example, when $p = 2$, equation (1.7) has a unique positive radial symmetric solution; it is not clear whether the positive radial solution of (1.7) with general $p \in (1, \frac{N}{s})$ is unique. This brings some new difficulties to the study of problems (1.5) and (1.7). First, we introduce some known results about equation (1.7). The energy functional of (1.7) can be defined as

$$G(u) = \frac{1}{p}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}\,dx\,dy + \frac{1}{p}\big[\frac{pqs}{N(q-p)} - 1\big]\int_{\mathbb{R}^N}|u|^p dx - \frac{1}{q}\int_{\mathbb{R}^N}|u|^q dx.$$

Moreover, all nontrivial solutions of (1.7) can be expressed as

$$\mathcal{W} := \big\{u \in W^{s,p}(\mathbb{R}^N)\setminus\{0\} : \langle G'(u), \varphi\rangle = 0, \forall \varphi \in W^{s,p}(\mathbb{R}^N)\big\}.$$

We say that $Q(x) \in W^{s,p}(\mathbb{R}^N)$ is a ground state solution of (1.7), if u satisfies

$$Q(x) \in \mathcal{N} := \big\{u \in \mathcal{W} : G(u) = \inf_{v\in\mathcal{W}} G(v)\big\} = \big\{u \in \mathcal{W} : G(u) = \inf_{v\in\mathcal{W}} \frac{s}{N}\int_{\mathbb{R}^N}|v|^p dx\big\}.$$

Combining with the Pohozaev and Nehari identity, $u(x)$ satisfies

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|Q(x)-Q(y)|^p}{|x-y|^{N+ps}}\,dx\,dy = \int_{\mathbb{R}^N}|Q(x)|^p dx = \frac{N(q-p)}{pqs}\int_{\mathbb{R}^N}|Q(x)|^q dx. \tag{1.8}$$

Before stating our main results, we introduce the fractional Gagliar do-Nirenberg inequality [9]

$$\begin{aligned}\int_{\mathbb{R}^N}|u|^q dx \le \frac{pqs}{N(q-p)\|Q\|_{L^p}^{q-p}}\Big(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}\,dx\,dy\Big)^{\frac{N(q-p)}{p^2 s}} \\ \times\Big(\int_{\mathbb{R}^N}|u|^p dx\Big)^{\frac{q}{p}-\frac{N(q-p)}{p^2 s}}, \quad \forall u \in W^{s,p}(\mathbb{R}^N).\end{aligned} \tag{1.9}$$

Furthermore, Q is an optimizer of the fractional Gagliardo-Nirenberg inequality. We note that, in a similar way to the literature [6], we can prove that all optimizers of (1.9) are in fact the scaling and translations of $Q(x)$, i.e., belong to the set

$$\{\lambda Q(\alpha x + y) : \ \alpha, \lambda \in \mathbb{R}^+, y \in \mathbb{R}, Q \in \mathcal{W}\}. \tag{1.10}$$

Remark 1.1. If $p \in (1, \frac{N}{s})$, it is known from the conclusion in [14, 6, 18] that the radially symmetric ground state solution of equation (1.7) is unique (up to translations). Accordingly, $Q(|x|)$ is the unique (up to translations) radially symmetric positive solution of the following equation (1.7) in $W^{s,p}(\mathbb{R}^N)$.

The following theorem discusses the existence and uniqueness of the reachable elements of equation (1.5).

Theorem 1.2. (i) *Suppose that* $0 < q < p + \frac{p^2 s}{N}$*, problem* (1.5) *has a unique minimizer* u_c *(up to translations). Moreover, the function* u_c *satisfies*

$$u_c = \frac{c\lambda_p^{N/p}}{\|Q\|_{L^p}} Q(\lambda_p x),$$

where $\lambda = (\frac{t_p}{c^p})^{\frac{1}{ps}}$ *with* t_p *being the unique minimum point of the function*

$$f_p(t) = \frac{a}{p}t + \frac{b}{2p}t^2 - \frac{psc^{\left(q - \frac{N(q-p)}{ps}\right)}}{N(q-p)\|Q\|_{L^p}^{(q-p)}} t^{\frac{N(q-p)}{p^2 s}}, \quad t \in (0, +\infty). \tag{1.11}$$

(ii) *Suppose that* $q = p + \frac{p^2 s}{N}$*, if* $c > \left(\frac{aN(q-p)}{p^2 s}\right)^{\frac{1}{q-p}} \|Q\|_{L^p}$*, then problem* (1.5) *has a unique minimizer* u_c *(up to translations). Moreover,*

$$u_c = \frac{c\lambda_p^{N/p}}{\|Q\|_{L^p}} Q(\lambda_p x), \tag{1.12}$$

where

$$\lambda_p = \Big[\frac{p^2 s c^{(q-p)} - aN(q-p)\|Q\|_{L^p}^{q-p}}{bN(q-p)c^p\|Q\|_{L^p}^{q-p}}\Big]^{\frac{1}{ps}}.$$

On the contrary, problem (1.5) *has no minimizer if* $c \le [\frac{aN(q-p)}{p^2 s}]^{\frac{1}{q-p}} \|Q\|_{L^p}$.

(iii) *Suppose that* $p + \frac{p^2 s}{N} < q < min\{p + \frac{2p^2 s}{N}, p_s^*\}$*, if* $c \ge c^*$*, then* (1.5) *has a unique minimizer* u_c *(up to translations). Moreover,*

$$u_c = \frac{c\lambda_p^{N/p}}{\|Q\|_{L^p}} Q(\lambda_p x), \tag{1.13}$$

with

$$\lambda_p = \Big\{\frac{2\big[N(q-p) - p^2 s\big]a}{\big[2p^2 s - N(q-p)\big]c^p b}\Big\}^{\frac{1}{ps}},$$

$$I(c) = \frac{(c^*)^{q - \frac{N(q-p)}{ps}} - c^{q - \frac{N(q-p)}{ps}}}{\frac{N(q-p)\|Q\|_{L^p}^{q-p}}{ps}} \Big\{\frac{2[N(q-p) - p^2 s]a}{[2p^2 s - N(q-p)]b}\Big\}^{\frac{N(q-p)}{p^2 s}}$$

for all $c \ge c^*$*. Conversely, problem* (1.5) *has no minimizer if* $c < c^*$*, where* c^* *is given by* (3.6).

(iv) *Suppose that* $p + \frac{2p^2 s}{N} \le q < p_s^*$*, problem* (1.5) *has no minimizer for all* $c > 0$.

By the above theorem, we first obtain a complete classification with respect to the exponent q with the L^p-normalized solutions of problem (1.5). Moreover, all these solutions are unique up to translations, our proof relies only on some simple energy estimates and avoids the use of the concentration-compactness principle.

Theorem 1.2 shows us that the minimizer of (1.5) must be a scaling of $Q(x)$, which extends [11, Theorem 1.1], also the existence of the minimizer of (1.5) is discussed therein. Moreover, we see that problem (1.5) has no minimizer if $q \ge p + \frac{2p^2 s}{N}$. Thus, to obtain the normalized solutions for (1.1), one may search for saddle point for functional (1.6). Inspired by other studies [11, 24], we examine the mountain pass type critical point for $E_p(\cdot)$ on $S(c)$. Before stating our second result, we introduce the following definition.

Definition 1.3. Functional $E_p(\cdot)$ is said to have the mountain pass geometry on $S(c)$ given $c>0$, if there exists $K(c)>0$ such that

$$\gamma(c):=\inf_{h\in\Gamma(c)}\max_{t\in[0,1]}E_p\big(h(t)\big)>\max\{E_p(h(0)),E_p(h(1))\} \tag{1.14}$$

holds in the set

$$\Gamma(c)=\Big\{h\in C\big([0,1];S(c)\big):h(0)\in A_{K(c)}\text{ and }E\big(h(1)\big)<0\Big\},$$

where

$$A_{K(c)}=\big\{u\in S(c):\Big(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}\,dx\,dy\Big)^{1/p}\le K(c)\big\}.$$

Since only $q\ge p+\frac{2p^2s}{N}$ is considered, we assume that q satisfies one of the conditions below:

(A1) $q>p+\frac{2p^2s}{N}$;

(A2) $q=p+\frac{2p^2s}{N}$ and $c>c_*:=[\frac{bN(q-p)\|Q\|_{L^p}^{q-p}}{2p^2s}]^{\frac{ps}{pqs-N(q-p)}}$.

As well as by studying some analytical properties of $\gamma(c)$ and involving rigorous arguments, we will prove separately that $E_p(\cdot)$ possesses the mountain pass geometry on $S(c)$, see lemma 2.3 below for details. In addition, there exists $u_c\in S(c)$ such that $E_p(u_c)=\gamma(c)$, and u_c is a solution of (1.1) with some $\lambda\in\mathbb{R}^-$. Inspired by this fact and the proof of our first theorem, we try to study some characteristics of $\gamma(c)$ by bringing in some new estimates of the observations and energies. Furthermore, as a side effect, we show that a critical point on the level $\gamma(c)$ is known to be unique if $u_c\in S(c)$ is a critical point of $E_p(\cdot)$ by indeed a scaling of $Q(x)$. So, we have the following theorem.

Theorem 1.4. *Suppose conditions* (A1) *or* (A2) *hold, and that $\bar t_p$ be the unique maximum point of $f_p(t)$ at $(0,+\infty)$. Then*

$$\gamma(c)=f_p(\bar t_p),$$

which can be achieved by $\bar u_c=\frac{c\bar\lambda_p^{\frac{N}{p}}}{\|Q\|_{L^p}}Q(\bar\lambda_p x)$, where $\bar\lambda_p=\Big(\frac{\bar t_p}{c^p}\Big)^{\frac{1}{ps}}$. Moreover, $\bar u_c$ is also a solution of (1.1) *for some $\lambda\in\mathbb{R}^-$.*

Remark 1.5. Still let $f_p(\cdot)$ be given by (1.11) and note that it has a unique maximum point in $(0,+\infty)$ once (A1) or (A2) is assumed. In Theorem 1.4 , $\bar u_c$ is the unique solution of (1.14). The significance is as follows: if

$$E_p'(\bar u)|_{S(c)}=0\quad\text{and}\quad E_p(\bar u)=\gamma(c). \tag{1.15}$$

i.e., $\bar u\in S(c)$ is a critical point of $E_p(\cdot)$ on $S(c)$ and its energy equal to $\gamma(c)$. Then, up to translations, $\bar u=\bar u_c$.

2. Preliminaries

We first give some useful notation and basic results for fractional Sobolev spaces. Let $0<s<1<p<\infty$ be real numbers. The fractional Sobolev space $W^{s,p}(\mathbb{R}^N)$ is defined by

$$W^{s,p}(\mathbb{R}^N)=\big\{u\in L^p(\mathbb{R}^N):\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}\,dx\,dy\big\},$$

equipped with the norm

$$\|u\|_{W^{s,p}(\mathbb{R}^N)} = \Big(\|u\|^p_{L^p(\mathbb{R}^N)} + \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}\,dx\,dy\Big)^{1/p}.$$

Now, we introduce the fractional Gagliardo-Nirenberg-Sobolev inequality; for more details, see [4, 17, 19].

Lemma 2.1 ([4]). *If $u \in W^{s,p}(\mathbb{R}^N)$ and $p < q < p_s^* - 2$, then*

$$\begin{aligned}&\int_{\mathbb{R}^N} |u|^q dx\\ &\le \frac{pqs}{N(q-p)\|Q\|^{q-p}_{L^p}}\Big(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}\,dx\,dy\Big)^{\frac{N(q-p)}{p^2 s}}\Big(\int_{\mathbb{R}^N}|u|^p dx\Big)^{\frac{q}{p}-\frac{N(q-p)}{p^2 s}}.\end{aligned}$$

Moreover, Q is an optimizer of the fractional Gagliardo-Nirenberg inequality.

The Pohozaev identity plays an important role in our discussion. We give it in the following lemma.

Lemma 2.2 ([17, Lemma 2]). *Let $u \in W^{s,p}(\mathbb{R}^N)$, $N \ge 2$, satisfy the equation*

$$(-\Delta)^s_p u = f(u).$$

then

$$\frac{(N-ps)}{p}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}\,dx\,dy = N\int_{\mathbb{R}^N} F(u)dx,$$

where $F(s) = \int_0^s f(t)dt$.

To prove the theorem 1.4, we first introduce the following lemma, which indicates that if hypothesis (A1) or (A2) hold, $E_p(\cdot)$ has mountain path geometry.

Lemma 2.3. *Assume that* (A1) *or* (A2) holds. *Then there exists $K(c) \in (0,1)$ such that*

$$\gamma(c) := \inf_{h\in\Gamma(c)}\max_{t\in[0,1]} E_p\big(h(t)\big) > \max\{E_p(h(0)), E_p(h(1))\}.$$

Proof. On the one hand, for any $u \in S(c)$ and $\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}\,dx\,dy \le \frac{2a}{b}$, we have

$$\begin{aligned}&E_p(u)\\ &\le \frac{a}{p}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}\,dx\,dy + \frac{b}{2p}\Big(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}\,dx\,dy\Big)^2\\ &\le \frac{2a}{p}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}\,dx\,dy.\end{aligned} \tag{2.1}$$

On the other hand, if $\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}\,dx\,dy$ is small enough, from (A2), we have

$$\begin{aligned}E_p(u) &\ge \frac{a}{p}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}\,dx\,dy + \frac{b}{2p}\Big(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}\,dx\,dy\Big)^2\\ &\quad - \frac{psc^{\left(q-\frac{N(q-p)}{ps}\right)}}{N(q-p)\|Q\|^{(q-p)}_{L^p}}\Big(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}\,dx\,dy\Big)^{\frac{N(q-p)}{p^2 s}}\\ &\ge \frac{a}{2p}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}\,dx\,dy.\end{aligned}$$

This and (2.1) imply that

$$E_p(u) \to 0 \quad \text{as} \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}\,dx\,dy \to 0$$

and for $K(c)$ small enough; moreover, if $K(c) \le \frac{2a}{b}$, we have

$$\begin{aligned} \sup_{u\in A_{K(c)}} E_p(u) &\le \frac{2a}{p}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}\,dx\,dy \\ &\le \frac{2a}{p}K^p(c) = \frac{a}{2p}4K^p(c) \\ &\le \inf_{u\in\partial A_{4K(c)}} E_p(u). \end{aligned} \tag{2.2}$$

where

$$\partial A_{4K(c)} = \Big\{u\in S(c) : \Big(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}\,dx\,dy\Big)^{1/p} = 4K(c)\Big\}.$$

Furthermore, for all $u\in A_{4K(c)}$, (2.1) indicates that $E_p(u)\ge 0$.

Next, we prove $\Gamma(c)\neq\emptyset$. Set

$$u_\lambda(x) = \frac{c\lambda^{N/p}}{\|Q\|_{L^p}}Q(\lambda x), \tag{2.3}$$

where $\lambda>0$ will be determined later. Then $u_\lambda\in S(c)$ and it follows from (1.8) that

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u_\lambda(x)-u_\lambda(y)|^p}{|x-y|^{N+ps}}\,dx\,dy = c^p\lambda^{ps}, \quad \int_{\mathbb{R}^N}|u_\lambda|^q dx = \frac{(pqs)c^q\lambda^{\frac{N(q-p)}{p}}}{N(q-p)\|Q\|_{L^p}^{q-p}}.$$

Consequently,

$$E_p(u_\lambda) = \frac{a}{p}(c^p\lambda^{ps}) + \frac{b}{2p}(c^p\lambda^{ps})^2 - \frac{psc^{\left[q-\frac{N(q-p)}{ps}\right]}}{N(q-p)\|Q\|_{L^p}^{q-p}}(c^p\lambda^{ps})^{\frac{N(q-p)}{p^2s}}. \tag{2.4}$$

Thus,

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u_{\lambda_1}(x)-u_{\lambda_1}(y)|^p}{|x-y|^{N+ps}}\,dx\,dy < K(c), \quad \text{if } \lambda_1 < \Big(\frac{K(c)}{c^p}\Big)^{\frac{1}{ps}},$$

which implies that $u_{\lambda_1}\in A_{K(c)}$. As a consequence of (A1) or (A2) and (2.4), it is easy to check that $E_p(u)\to-\infty$ as $\lambda\to\infty$. Hence, we choose $\lambda_2>0$ large enough, such that

$$E_p(u_{\lambda_2}) < 0.$$

Taking $g(t) = u_{((1-t)\lambda_1+t\lambda_2)}$, we have

$$g(0) = u_{\lambda_1}\in A_{K(c)}, \quad g(0) = u_{\lambda_2}, \quad E_p(u_{\lambda_2}) < 0.$$

These means that $g(t)\in\Gamma(c)\neq\emptyset$.

For any $g(t)\in\Gamma(c)$, we know that

$$g(0)\in A_{K(c)} \quad \text{and} \quad E_p(g(1)) < 0.$$

Since $g(t)$ is continuous, then there exists a $\bar t\in(0,1)$, such that

$$g(\bar t)\in\partial A_{4K(c)}$$

According to (2.2), we have

$$\max_{t\in[0,1]} E_p\big(g(t)\big) \geq E_p\big(g(\bar{t})\big) > \max\{E_p(g(0)), E_p(g(1))\}.$$

Moreover,

$$\gamma(c) := \inf_{h\in\Gamma(c)} \max_{t\in[0,1]} E_p\big(h(t)\big) > \max\{E_p(h(0)), E_p(h(1))\}.$$

The proof is complete. □

3. Proof of main results

In this section, we prove Theorems 1.2 and 1.4 by using some energy estimates and the Gagliardo-Nirenberg inequality (1.9). We first note that by simply rescaling, we can easily prove that

$$I(c) \leq 0 \quad \text{for all } c > 0 \quad \text{and} \quad 0 < q < p_s^*. \tag{3.1}$$

Furthermore, using (1.9), we observe that for any $u \in S(c)$,

$$\begin{aligned} E_p(u) &\geq \frac{a}{p}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}\,dx\,dy + \frac{b}{2p}\Big(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}\,dx\,dy\Big)^2 \\ &\quad - \frac{psc^{\left[q-\frac{N(q-p)}{ps}\right]}}{N(q-p)\|Q\|_{L^p}^{(q-p)}}\Big(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}\,dx\,dy\Big)^{\frac{N(q-p)}{p^2 s}} \\ &= f_p(t), \end{aligned} \tag{3.2}$$

where $f_p(\cdot)$ is given by (1.11) and let $t = \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}\,dx\,dy$.

Proof of Theorem 1.2. (i) Because $0 < q < p + \frac{p^2 s}{N}$, we can readily check that $f_p(t)$ ($t \in (0,\infty)$) is minimized at a unique point, denoted by t_p. Thus, from (3.2) we obtain

$$I(c) = \inf_{u\in S(c)} E_p(u) \geq f_p(t_p). \tag{3.3}$$

On the other hand, choosing $\lambda = (\frac{t_p}{c^p})^{\frac{1}{ps}}$, i.e., $c^p\lambda^{ps} = t_p$, it can be seen from (2.4) that

$$I(c) \leq E_p(u_\lambda) = f_p(t_p).$$

From this and (3.3), we infer that

$$I(c) = f_p(t_p) = \inf_{t\in\mathbb{R}^+} f_p(t), \tag{3.4}$$

and u_λ with $\lambda = (\frac{t_p}{c^p})^{\frac{1}{ps}}$, i.e.,

$$u_\lambda = u_c = \frac{c}{\|Q\|_{L^p}}\big(\frac{t_p}{c^p}\big)^{\frac{N}{p^2 s}} Q\Big(\big(\frac{t_p}{c^p}\big)^{\frac{1}{ps}} x\Big)$$

is a minimizer of (1.5).

All that remains is to prove that u_c (up to translations) is the unique minimizer of (1.5). In fact, if $u_0 \in S(c)$ is a minimizer, then it can be shown from (3.2) that

$$I(c) = E_p(u_0) \geq f_p(t_0), \quad \text{with} \quad t_0 := \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u_0(x)-u_0(y)|^p}{|x-y|^{N+ps}}\,dx\,dy,$$

where the "=" in the second inequality holds if and only if u_0 is an optimizer of (1.9). Which together with (3.4) further means that $t_0 = t_p$ and $f_p(t_0) =$

$E_p(u_0)$. Therefore, u_0 is an optimizer of (1.9) and it follows from (1.10) that up to translations, u_0 must be the form of $u_0(x) = \alpha Q(\beta x)$. Using

$$\int_{\mathbb{R}^N} |u_0|^p dx = c^p, \quad \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u_0(x)-u_0(y)|^p}{|x-y|^{N+ps}}\,dx\,dy = t_p,$$

and combining with this (1.8), we have $\alpha = \frac{c}{\|Q\|_{L^p}}\big(\frac{t_p}{c^p}\big)^{\frac{N}{p^2 s}}$ and $\beta = \big(\frac{t_p}{c^p}\big)^{\frac{1}{ps}}$, as a result, $u_0 = u_c$.

(ii) Since $q < p + \frac{p^2 s}{N}$, i.e., $\frac{N(q-p)}{p^2 s} = 1$, from (2.4), we can conclude that

$$f_p(t) = \Big[\frac{a}{p} - \frac{psc^{(q-p)}}{N(q-p)\|Q\|_{L^p}^{q-p}}\Big]t + \frac{b}{2p}t^2. \tag{3.5}$$

If $c \le [\frac{aN(q-p)}{p^2 s}]^{\frac{1}{q-p}}\|Q\|_{L^p}$, we can easily derive from (3.2) that

$$E_p(u) \ge f_p\Big(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}\,dx\,dy\Big) > 0 \quad \text{for all } u \in S(c).$$

In consideration of (3.1), this shows that (1.5) has no minimizer.

In the next step, we move to the case of $c > \Big[\frac{aN(q-p)}{p^2 s}\Big]^{\frac{1}{q-p}}\|Q\|_{L^p}$. We know from (3.5) that $f_p(t)(t \in (0,+\infty))$ attains its minimum at the unique point

$$t_p = \frac{p^2 sc^{(q-p)} - aN(q-p)\|Q\|_{L^p}^{q-p}}{bN(q-p)\|Q\|_{L^p}^{q-p}}.$$

Following a similar argument as in part (i), we can demonstrate that, up to translations that

$$u_c = \frac{c\lambda_p^{N/p}}{\|Q\|_{L^p}}Q(\lambda_p x),$$

where

$$\lambda_p = \Big[\frac{p^2 sc^{(q-p)} - aN(q-p)\|Q\|_{L^p}^{q-p}}{bN(q-p)c^p\|Q\|_{L^p}^{q-p}}\Big]^{\frac{1}{ps}}.$$

Therefore, u_c is the unique minimizer of (1.5).

(iii) For the case $p + \frac{p^2 s}{N} < q < min\{p + \frac{2p^2 s}{N}, p_s^*\}$, i.e., $1 < \frac{N(q-p)}{p^2 s} < 2$, let

$$\alpha = \frac{2p^2 s - N(q-p)}{p^2 s}, \quad \beta = 1-\alpha = \frac{N(q-p)-p^2 s}{p^2 s}.$$

It is obvious from Young's inequality that for any $t > 0$, one has

$$\begin{aligned}
\frac{a}{p}t + \frac{b}{2p}t^2 &= \alpha\Big(\frac{a}{p\alpha}t\Big) + \beta\Big(\frac{b}{2p\beta}t^2\Big)\\
&\ge \Big(\frac{a}{p\alpha}\Big)^{\alpha}\Big(\frac{b}{2p\beta}\Big)^{\beta}t^{\alpha+2\beta}\\
&= \Big[\frac{aps}{2p^2 s - N(q-p)}\Big]^{\frac{2p^2 s - N(q-p)}{p^2 s}}\Big[\frac{bps}{2N(q-p)-2p^2 s}\Big]^{\frac{N(q-p)-p^2 s}{p^2 s}} t^{\frac{N(q-p)}{p^2 s}}.
\end{aligned}$$

where the "=" in the second inequality holds if and only if

$$\frac{a}{p\alpha}t = \frac{b}{2p\beta}t^2, \quad \text{i.e., } t = t_0 := \frac{2\beta a}{\alpha b} = \frac{2[N(q-p)-p^2 s]a}{[2p^2 s - N(q-p)]b}.$$

We set

$$\begin{aligned} c^* = &\Big\{ \frac{N(q-p)\|Q\|_{L^p}^{q-p}}{ps} \Big[\frac{aps}{2p^2s - N(q-p)} \Big]^{\frac{2p^2s - N(q-p)}{p^2s}} \Big\}^{\frac{ps}{pqs - N(q-p)}} \\ &\times \Big\{ \Big[\frac{bps}{2N(q-p) - 2p^2s} \Big]^{\frac{N(q-p)-p^2s}{p^2s}} \Big\}^{\frac{ps}{pqs - N(q-p)}}. \end{aligned} \tag{3.6}$$

In view of (3.2) and (3.6), we consequently have

$$E_p(u) \geq \frac{(c^*)^{q - \frac{N(q-p)}{ps}} - c^{q - \frac{N(q-p)}{ps}}}{\frac{N(q-p)\|Q\|_{L^p}^{q-p}}{ps}} \big(t_0\big)^{\frac{N(q-p)}{p^2s}} = f_p(t_0) \quad \text{for all } u \in S(c). \tag{3.7}$$

If $c \geq c^*$, on the one hand, we deduce from (3.7) that $I(c) \geq f_p(t_0)$. On the other hand, let $u_\lambda(x)$ be as in (2.3) and set $\lambda = (\frac{t_0}{c^p})^{\frac{1}{ps}}$, then

$$I(c) \leq E_p(u_\lambda) = f_p(t_0).$$

Which shows that u_λ is a minimizer of (1.5) and that for any $c \geq c^*$,

$$I(c) = f_p(t_0) = \frac{(c^*)^{q - \frac{N(q-p)}{ps}} - c^{q - \frac{N(q-p)}{ps}}}{\frac{N(q-p)\|Q\|_{L^p}^{q-p}}{ps}} \Big\{ \frac{2\big[N(q-p) - p^2s\big]a}{\big[2p^2s - N(q-p)\big]b} \Big\}^{\frac{N(q-p)}{p^2s}}.$$

Uniqueness of the minimizers can be proved by the same proofs in part (i).

If $c < c^*$, we then deduce from (3.7) that $E_p(u) > 0$ for all $u \in S(c)$. Thus, problem (1.5) cannot be achieved for (3.1).

(iv) On the one hand if

$$q = p + \frac{2p^2s}{N} \quad \text{and} \quad c > \Big[\frac{bN(q-p)\|Q\|_{L^p}^{q-p}}{2p^2s} \Big]^{\frac{ps}{pqs - N(q-p)}},$$

from (2.3) and (2.4), it follows that

$$I(c) \leq \lim_{\lambda \to +\infty} E_p(u_\lambda) = -\infty,$$

therefore, problem (1.5) cannot be achieved.

On the other hand, if

$$q = p + \frac{2p^2s}{N} \quad \text{and} \quad c \leq \Big[\frac{bN(q-p)\|Q\|_{L^p}^{q-p}}{2p^2s} \Big]^{\frac{ps}{pqs - N(q-p)}},$$

from (3.2) we have $E_p(u) > 0$ for all $u \in S(c)$. This and (3.1) obviously indicate that problem (1.5) cannot be attained.

Finally, if $q > p + \frac{2p^2s}{N}$, it follows (2.3) and (2.4) that

$$I(c) \leq \lim_{\lambda \to +\infty} E_p(u_\lambda) = -\infty,$$

and thus problem (1.5) cannot be attained. The proof is complete. □

Proof of Theorem 1.4. First, for any $q \geq p + \frac{2p^2s}{N}$, we can prove the existence of $K(c) > 0$ by Lemma 2.3 and can choose $K(c)$ small enough so that $E_p(\cdot)$ satisfies mountain pass geometry on $S(c)$ if (A1) or (A2) is assumed. Therefore, in the following, we always hypothesize that $K(c) < \bar{t}_p$, where $\bar{t}_p$ stands for the unique maximum point of $f_p(t)$ in $(0, +\infty)$.

For any $r \in [0,1]$ and $h(r) \in \Gamma(c)$, we can derive from (3.2) that

$$E_p(h(r)) \geq f_p\Big(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|\big(h(r)\big)(x) - \big(h(r)\big)(y)|^p}{|x-y|^{N+ps}}\,dx\,dy\Big), \tag{3.8}$$

where "=" holds if and only if $h(r) \in S(c)$ is an optimizer of (1.9), i.e., up to translations,

$$(h(r))(x) = \frac{c\beta^{N/p}}{\|Q\|_{L^p}} Q(\beta x) \quad \text{for some } \beta > 0. \tag{3.9}$$

Since $h(0) \in A_{K(c)}$ with $K(c) < \bar{t}_p$, and note that $f_p(t) > 0$ $\forall t \in (0, \bar{t}_p]$, thus we have

$$\begin{aligned} &\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|\big(h(0)\big)(x) - \big(h(0)\big)(y)|^p}{|x-y|^{N+ps}}\,dx\,dy \\ &< \bar{t}_p < \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|\big(h(1)\big)(x) - \big(h(1)\big)(y)|^p}{|x-y|^{N+ps}}\,dx\,dy. \end{aligned} \tag{3.10}$$

As a result of (3.8) and (3.10), it holds that

$$\max_{r\in[0,1]} E_p\big(h(r)\big) \geq f_p(\bar{t}_p) = \max_{t\in\mathbb{R}^+} f_p(t). \tag{3.11}$$

Thus,

$$\gamma(c) \geq f_p(\bar{t}_p). \tag{3.12}$$

Instead, let $u_\lambda(x)$ be the test function given by (2.3), characterized by

$$\lambda = \bar{\lambda}_p = \Big(\frac{\bar{t}_p}{c^p}\Big)^{\frac{1}{ps}}.$$

Set

$$g(r) := r^{\frac{N}{p^2 s}} u_\lambda(r^{\frac{1}{ps}} x),$$

One can then check that $E_p(g(r)) = f_p(\bar{t}_p r)$. Choosing $0 < \tilde{t}_p < \bar{t}_p$ small enough such that

$$g\Big(\frac{\tilde{t}_p}{\bar{t}_p}\Big) \in A_{K(c)},$$

and select $\hat{t}_p > \bar{t}_p$ such that $f_p(\hat{t}_p) < 0$. Let

$$h(r) = g\Big((1-r)\frac{\tilde{t}_p}{\bar{t}_p} + r\frac{\hat{t}_p}{\bar{t}_p}\Big), \quad \forall r \in (0,1).$$

Then

$$h(0) = g\Big(\frac{\tilde{t}_p}{\bar{t}_p}\Big) \in A_{K(c)} \quad \text{and} \quad E_p\big(h(1)\big) = E_p\Big(g\big(\frac{\hat{t}_p}{\bar{t}_p}\big)\Big) = f_p(\hat{t}_p) < 0.$$

This shows that $h \in \Gamma(c)$, and

$$\gamma(c) \leq \max_{t\in[0,1]} E\big(h(t)\big) = E_p(u_{\bar{\lambda}_p}) = f_p(\bar{t}_p).$$

Combing this with (3.12), we deduce that $\gamma(c) = f_p(\bar{t}_p)$ and

$$u_{\bar{\lambda}_p} = \bar{u}_c(x) = \frac{c}{\|Q\|_{L^p}}\Big(\frac{\bar{t}_p}{c^p}\Big)^{\frac{N}{p^2 s}} Q\Big(\big(\frac{\bar{t}_p}{c^p}\big)^{\frac{1}{ps}} x\Big)$$

is a solution of problem (1.14).

Next we demonstrate that $\bar{u}_c$ satisfies equation (1.1) for some $\mu \in \mathbb{R}^-$. In fact, in light of $f_p'(\bar{t}_p) = 0$ and $\bar{\lambda}_p = \left(\frac{\bar{t}_p}{c^p}\right)^{\frac{1}{ps}}$, we obtain

$$\begin{aligned}\frac{c^{[q-\frac{N(q-p)}{ps}]}}{p\|Q\|_{L^p}^{q-p}}(\bar{t}_p)^{\frac{N(q-p)-p^2s}{p^2s}} &= \frac{a}{p} + \frac{b}{p}\bar{t}_p \\ &= \frac{a}{p} + \frac{b}{p}\Big(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|\bar{u}_c(x)-\bar{u}_c(y)|^p}{|x-y|^{N+ps}}\,dx\,dy\Big).\end{aligned} \tag{3.13}$$

Moreover, since $Q(x)$ is a solution of (1.7) and $\bar{\lambda}_p = \left(\frac{\bar{t}_p}{c^p}\right)^{\frac{1}{ps}}$, it follows that $\bar{u}_c$ satisfies

$$\begin{aligned}&\frac{c^{\left[q-\frac{N(q-p)}{ps}\right]}}{\|Q\|_{L^p}^{q-p}}(\bar{t}_p)^{\frac{N(q-p)-p^2s}{p^2s}}(-\Delta)_p^s\bar{u}_c - |\bar{u}_c|^{q-2}\bar{u}_c \\ &= -\frac{[pqs-N(q-p)](c\bar{\lambda}_p^{\frac{N}{p}})^{q-p}}{N(q-p)\|Q\|_{L^p}^{q-p}}|\bar{u}_c|^{p-2}\bar{u}_c.\end{aligned}$$

From this and (3.13) we conclud that $\bar{u}_c$is a solution of (1.1) with

$$\mu = -\frac{\left[pqs-N(q-p)\right]\left(c\bar{\lambda}_p^{\frac{N}{p}}\right)^{q-p}}{N(q-p)\|Q\|_{L^p}^{q-p}}.$$

We finally prove that $\bar{u}_c$ is a unique solution of $\gamma(c)$ before translation. Assume that $\bar{u}$ is a solution of $\gamma(c)$ and that it satisfies (1.15), then there exists $\mu \in \mathbb{R}$ such that

$$E'(u) = \mu|\bar{u}|^{p-2}\bar{u},$$

so we have

$$\begin{aligned}&a\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|\bar{u}(x)-\bar{u}(y)|^p}{|x-y|^{N+ps}}\,dx\,dy + b\Big(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|\bar{u}(x)-\bar{u}(y)|^p}{|x-y|^{N+ps}}\,dx\,dy\Big)^2 \\ &-\int_{\mathbb{R}^N}|\bar{u}|^q dx \\ &= \mu\int_{\mathbb{R}^N}|\bar{u}|^p dx.\end{aligned} \tag{3.14}$$

It then follows from the Pohozaev identity that

$$\begin{aligned}&\frac{a(N-ps)}{p}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|\bar{u}(x)-\bar{u}(y)|^p}{|x-y|^{N+ps}}\,dx\,dy + \frac{b(N-ps)}{p} \\ &\times\Big(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|\bar{u}(x)-\bar{u}(y)|^p}{|x-y|^{N+ps}}\,dx\,dy\Big)^2 - \frac{N}{q}\int_{\mathbb{R}^N}|\bar{u}|^q dx \\ &= \frac{\mu N}{p}\int_{\mathbb{R}^N}|\bar{u}|^p dx.\end{aligned} \tag{3.15}$$

From (3.14) and (3.15), we deduce that

$$\begin{aligned}&a\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|\bar{u}(x)-\bar{u}(y)|^p}{|x-y|^{N+ps}}\,dx\,dy + b\Big(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|\bar{u}(x)-\bar{u}(y)|^p}{|x-y|^{N+ps}}\,dx\,dy\Big)^2 \\ &-\frac{N(q-p)}{pqs}\int_{\mathbb{R}^N}|\bar{u}|^q dx = 0.\end{aligned} \tag{3.16}$$

Letting $\hat{h}(r) := r^{\frac{N}{p^2 s}}\bar{u}(r^{\frac{1}{ps}}x)$, we obtain

$$\begin{aligned} g(r) &:= E_p(\hat{h}(r)) \\ &= \frac{ar}{p}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|\bar{u}(x)-\bar{u}(y)|^p}{|x-y|^{N+ps}}\,dx\,dy + \frac{br^2}{2p}\Big(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|\bar{u}(x)-\bar{u}(y)|^p}{|x-y|^{N+ps}}\,dx\,dy\Big)^2 \\ &\quad - \frac{r^{\frac{N(q-p)}{p^2 s}}}{q}\int_{\mathbb{R}^N}|\bar{u}|^q dx. \end{aligned}$$

Equality (3.16) shows that $g(r)$ $(r \in (0,+\infty))$ reaches its maximum at the unique point $r = 1$, and

$$\lim_{r\to+\infty} g(r) = -\infty.$$

Choosing $0 < \tilde{s} < 1 < \hat{s}$ such that $\hat{h}(\tilde{s}) \in A_{K(c)}$ and $g(\hat{s}) < 0$, we have

$$h_0(r) := \hat{h}((1-r)\tilde{s} + r\hat{s}) \in \Gamma(c), \quad \max_{r\in[0,1]} E_p(h_0(r)) = E_p(\bar{u}).$$

Through arguments such as (3.8) and (3.11), one sees that

$$f_p(\bar{t}_p) = \gamma(c) = E_p(\bar{u}) = \max_{r\in[0,1]} E_p(h_0(r)) \geq \max_{t\in\mathbb{R}^+} f_p(t) = f_p(\bar{t}_p).$$

From (3.9), this implies that $\bar{u}$ must be the form $\frac{c\beta^{N/p}}{\|Q\|_{L^p}}Q(\beta x)$ for some $\beta > 0$. Translating this into the equality $f_p(\bar{t}_p) = E_p(\bar{u})$, then we can obtain that $\bar{u} = \bar{u}_c$ and $\beta = \bar{\lambda}_p$. The proof is complete. □

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