OPTIMAL DECAY RATES FOR HIGHER-ORDER DERIVATIVES OF SOLUTIONS TO 3D COMPRESSIBLE NAVIER-STOKES-POISSON EQUATIONS WITH EXTERNAL FORCE

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ABSTRACT. We investigate optimal decay rates for higher-order spatial derivatives of solutions to the 3D compressible Navier-Stokes-Poisson equations with external force. The main novelty of this article is twofold: First, we prove the first and second order spatial derivatives of the solutions converge to zero at the L^2-rate $(1 + t)^{-5/4}$, which is faster than the L^2-rate $(1 + t)^{-3/4}$ in Li-Zhang [15]. Second, for well-chosen initial data, we show the lower optimal decay rates of the first order spatial derivative of the solutions. Therefore, our decay rates are optimal in this sense.

1. INTRODUCTION

This study concerns the initial value problem of the isentropic Navier-Stokes-Poisson equations

$$\partial_t \rho + \nabla \cdot (\rho u) = 0,$$
$$\rho[\partial_t u + (u \cdot \nabla)u] + \nabla P(\rho) = \rho \nabla \phi + \mu \Delta u + (\mu + \nu)\nabla(\nabla \cdot u) + \rho F,$$
$$\Delta \phi = \rho - \bar{\rho}, \quad \lim_{|x| \to \infty} \phi(x, t) = 0, \tag{1.1}$$
$$(\rho, u)(x, 0) = (\rho_0, u_0)(x).$$

Here the time variable is $t \geq 0$, and the spatial coordinate is x is in $\mathbb{R}^3$. The unknown functions are the density $\rho > 0$, the velocity u, and the electrostatic potential ϕ. $\bar{\rho} > 0$ stands for the constant background doping profile. The constants μ and ν are the viscosity coefficients satisfying $\mu > 0$ and $2\mu + 3\nu \geq 0$. $F(x) = (F_1(x), F_2(x), F_3(x))$ is a given external force, here, for simplicity, we assume that $F = -\nabla \psi(x)$. $P = P(\rho)$ is the pressure. In this paper, we always assume that $P = P(\rho)$ is a C^2-function in the neighborhood of $\bar{\rho}$ and satisfies $P'(\rho) > 0$ for $\rho > 0$. The typical example is $P(\rho) = A\rho^\gamma$ corresponding to polytropic $(\gamma > 1)$ and isothermal fluid $(\gamma = 1)$. The main purpose of this article is to show optimal decay rates for higher-order spatial derivatives of solutions for (1.1) for the initial

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data around stationary solutions. And the stationary problem takes the form

$$\nabla \cdot (\tilde{\rho}\tilde{u}) = 0,$$
$$\tilde{\rho}(\tilde{u} \cdot \nabla)\tilde{u} + \nabla P(\tilde{\rho}) = \tilde{\rho}\nabla\tilde{\phi} - \tilde{\rho}\nabla\psi + \mu\Delta\tilde{u} + (\mu + \nu)\nabla(\nabla \cdot \tilde{u}),$$
$$\Delta\tilde{\phi} = \tilde{\rho} - \bar{\rho},$$
$$\tilde{\rho} \to \bar{\rho}, \quad \tilde{u} \to 0 \quad \text{as } |x| \to \infty. \tag{1.2}$$

1.1. **History of the problem.** Before stating our main results, let's briefly review some former results which are closely related. Many mathematicians are interested in studying the large time behavior of solutions and the global well-posedness for the compressible Navier-Stokes-Poisson system, see, for example [2, 3, 4, 7, 9, 11, 13, 12, 10, 14, 15, 17, 19, 20, 23, 21, 26, 31, 30] and references therein. In the following, we only review some results about the decay rates for the compressible Navier-Stokes-Poisson system with and without the external potential force.

When there is no external force, Li-Matsumura-Zhang [11] proved that the density of the NSP system converges to its equilibrium state at the L^2-rate $(1+t)^{-3/4}$, but the momentum of the NSP system decays at the L^2-rate $(1+t)^{-\frac{1}{4}}$. Later, they also showed the similar results for the non-isentropic case [13]. Recently, Wang [21] obtained the optimal decay rates of the higher-order spatial derivatives of the solution via the pure energy method. Furthermore, Wang and Wang [26] established global existence and decay estimates of classical solutions to the compressible NSP system in three and higher dimensions, which is faster than ones of [11, 25]. For the the bipolar Navier-Stokes-Poisson (BNSP) system in 3D, since it has non-conservative structure and the interaction of two fluids through the electric field, up to now, there are very few results. By employing a detailed analysis on Green's function of the linearized system and some elaborate energy estimates, Wang-Xu[27] obtained the global existence and the H^s decay rates of classical solutions for the BNSP system. Besides, Wu-Zhang-Zhang [28] established the optimal decay rates of solutions by a regularity interpolation trick and delicate energy methods. It should be noted that Chen-Wu-Zhang in [6] showed the explicit influences of the electric field on the qualitative behaviors of solutions.

When there is an external force $F = -\nabla\psi(x)$, the problem becomes much more complicated due to the appearance of the non-trivial stationary solution. For the BNSP system with external force, Zhao-Li [31] gave the optimal L^p-convergence rates of the solutions towards the stationary solution. Recently, Li and Zhang [15] proved the existence of the solution to the stationary problem (1.2) and long time behavior of the Cauchy problem (1.1). Their main results can be outlined as follows: There exists $\epsilon_1 > 0$ such that if

$$\|\Delta\psi\|_{H^2} + \sum_{k=0}^{1} \|(1 + |x|)\nabla^k \Delta\psi\| \le \epsilon_1, \tag{1.3}$$

then the stationary problem (1.2) has a unique solution $(\tilde{\rho}, \tilde{u}, \tilde{\phi})(x)$ satisfying $\tilde{u} = 0$ and

$$\|\tilde{\rho} - \bar{\rho}\|_{H^4} + \|\nabla\tilde{\phi}\|_{H^3} + \|(1 + |x|)(\tilde{\rho} - \bar{\rho})\|_{H^3} + \|(1 + |x|)\nabla\tilde{\phi}\|_{H^2} \le C\epsilon_1. \tag{1.4}$$

Moreover, if $\|(\rho_0 - \bar{\rho}, u_0)\|_{H^2 \cap L^1}$ is sufficiently small, then the Cauchy problem (1.1) admits a unique global solution (ρ, u, ϕ) satisfying

$$\|\nabla(\rho - \tilde{\rho}, u, \nabla\phi - \nabla\tilde{\phi})(t)\|_{H^1} \le C(1 + t)^{-3/4}. \tag{1.5}$$

However, it is clear that the L^2 decay rate of the second order spatial derivative of the solution $(\rho, u, \nabla\phi)$ is $(1+t)^{-3/4}$ in (1.5), which is the same as that of its first order spatial derivative and is slower than that of the heat equation. Therefore, the decay rate of the second order spatial derivative of the solution $(\rho, u, \nabla\phi)$ is not optimal in this sense. The main motivation of this paper is to provide a general framework that can be used to extract the optimal decay rates of the solution to the Cauchy problem (1.1). More precisely, we will modify the methods developed in [6, 29] to derive the lower optimal decay rates of the first order spatial derivative of the solutions.

1.2. **Main results.** In this article, we use $H^k(\mathbb{R}^3)$ to denote the usual Sobolev spaces with norm $\|\cdot\|_{H^k}$ and write $\|\cdot\|_k := \|\cdot\|_{H^k}$ for convenience. Generally, we use L^p $(1 \leq p \leq \infty)$ to denote the usual $L^p(\mathbb{R}^3)$ spaces with norm $\|\cdot\|_{L^p}$. The notation $a \lesssim b$ means that $a \leq Cb$ for a universal positive constant which is independent of time t. For simplicity, we write $\|(A, B)\|_X := \|A\|_X + \|B\|_X$. For a radial function $\phi \in C_0^\infty(\mathbb{R}_\xi^3)$ such that $\phi(\xi) = 1$ when $|\xi| \leq 1$ and $\phi(\xi) = 0$ when $|\xi| \geq 2$, we define the low–frequency part and the high-frequency part of f as follows

$$f^L = \mathfrak{F}^{-1}[\phi(\xi)\widehat{f}], \quad \text{and} \quad f^H = \mathfrak{F}^{-1}[(1 - \phi(\xi))\widehat{f}]$$

Before stating our main results, let us recall the following a priori estimates for the Cauchy problem (1.1) in [15].

Proposition 1.1. For $T > 0$, let $(\rho - \tilde{\rho}, u, \phi - \tilde{\phi})(x, t)$ be a solution of (1.1) in $[0, T]$ and introduce $E(T) = \sup_{0 \leq t \leq T} \|(\rho - \tilde{\rho}, u)(\cdot, t)\|_{H^2}^2$. Then there exists $\delta > 0$ such that if

$$E(T) + \epsilon_1 \leq \delta, \tag{1.6}$$

then the following a-priori estimate holds

$$\|(\rho - \tilde{\rho}, u, \nabla\phi - \nabla\tilde{\phi})(\cdot, t)\|_{H^2}^2 + \int_0^t \|(\rho - \tilde{\rho}, \nabla u, \nabla^2\phi - \nabla^2\tilde{\phi})(\cdot, s)\|_{H^2}^2 ds \tag{1.7}$$
$$\leq C\|(\rho_0 - \tilde{\rho}, u_0)\|_{H^2}^2,$$

for any $t \in [0, T]$, where C is a positive constant independent of t.

Now, we are in a position to state our main results.

Theorem 1.2. Let $(\rho_0 - \bar{\rho}, u_0)(x) \in H^2(\mathbb{R}^3) \cap L^1(\mathbb{R}^3)$, there exists $0 < \delta_0 < \epsilon_1$ such that if

$$\|(\rho_0 - \bar{\rho}, u_0)\|_{H^2} + \|\Delta\psi\|_{H^2} + \sum_{k=0}^{1} \|(1 + |x|)\nabla^k\Delta\psi\|_{L^2} \leq \delta_0, \tag{1.8}$$

then the solution $(\rho, u, \phi)(x, t)$ of the Cauchy problem (1.1) has time decay estimate

$$\|\nabla(\rho - \tilde{\rho}, u, \nabla\phi - \nabla\tilde{\phi})(t)\|_{H^1} \leq C(1 + t)^{-5/4}, \tag{1.9}$$

for any $t \in [0, \infty)$.

Theorem 1.3. Suppose that all the hypotheses of Theorem 1.2 are satisfied, and the Fourier transform $(\widehat{\rho_0 - \rho}, \widehat{u_0}, \widehat{\nabla\phi_0})$ satisfies

$$|\widehat{\rho_0 - \tilde{\rho}}(\xi)| \geq C\delta_0^{3/2}, \quad |\widehat{u_0}(\xi)| = 0, \quad |\widehat{\nabla\phi_0}(\xi)| \geq C\delta_0^{3/2}, \quad \text{for } 0 \leq |\xi| \ll 1,$$

where c_0 is a positive constant. Then if

$$U_0 = \|(\rho_0 - \bar{\rho}, u_0, \nabla\phi_0)\|_{L^1} \leq \delta_0, \tag{1.10}$$

then for any $t \in [0, \infty)$, it holds that

$$\min\left\{\|\nabla(\rho - \tilde{\rho})(t)\|_{L^2}, \|\nabla u(t)\|_{L^2}, \|\nabla(\nabla\phi - \nabla\tilde{\phi})(t)\|_{L^2}\right\} \geq C\delta_0^{3/2}(1+t)^{-5/4}. \tag{1.11}$$

Remark 1.4. Compared to the decay rate (1.5) of [15], the decay rate (1.9) implies that the first and second order spatial derivatives of the solution converge to zero at the L^2–rate $(1+t)^{-5/4}$, which is faster than the L^2–rate $(1+t)^{-3/4}$ in (1.5). Furthermore, the decay rate (1.11) also gives the lower optimal decay rate of the first order spatial derivative of the solution. So, our decay rates are optimal in this sense.

Now, let us sketch the strategy of proving Theorem 1.2-Theorem 1.3 and explain some main difficulties and techniques involved in the process. Roughly speaking, we will make full use of the benefit of a pure energy method, the low-frequency and high-frequency decomposition $f = f^L + f^H$, where f^L and f^H stand for the low-frequency part and high-frequency part of f, respectively. Our strategy can be outlined as follows.

For the proof of Theorem 1.2, we hope to establish the optimal decay rate for the first order spatial derivative of solution to the 3D compressible Navier-Stokes-Poisson equations (1.1). Our strategy mainly involves the following three steps. Firstly, we deduce the first order low frequency decay estimates including the linear decay estimates part and the nonlinear energy estimates part. Fortunately, we can obtain the optimal decay rates on $\|\nabla^k(\bar{n}^L, \bar{u}^L, \nabla\bar{\Phi}^L)\|_{L^2}$ for the corresponding linearized NSP system with an external force from Chen-Wu-Zhang [6] directly. Unfortunately, when the nonlinear terms in (2.2)-(2.5) of equations (2.1) are estimated, the main difficulty comes from the term involving $(\tilde{\rho} - \bar{\rho})$, which however has no specific time decay rate. Our key method to get over this difficulty is to introduce a time-weighted energy functional $M(t) = \sup_{0 \leq s \leq t}(1+s)^{5/2}\|\nabla(n, u, \nabla\Phi)(s)\|_{H^1}^2$, thus we can deduce the first order low frequency decay estimates as follows (see the proof of Lemma 3.2 for details).

$$\|\nabla(n^L, u^L, (\nabla\Phi)^L)(t)\|_{L^2}^2 \leq C(1+t)^{-5/2}(\|U_0\|_{L^1}^2 + \delta_0^2 M(t)). \tag{1.12}$$

Secondly, we deduce the first order and the second order high frequency decay estimates. In this step, we establish the relevant energy estimates as follows:

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(\frac{P'(\bar{\rho})}{\bar{\rho}}\|\nabla^l n^H\|_{L^2}^2 + \bar{\rho}\|\nabla^l u^H\|_{L^2}^2 + \|\nabla^l(\nabla\Phi)^H\|_{L^2}^2\right) \\
&+ \mu\|\nabla^{l+1}u^H\|_{L^2}^2 + (\mu+\nu)\|\nabla^l\nabla\cdot u^H\|_{L^2}^2 \\
&\lesssim (1+t)^{-5/2} + (\delta_0 + (1+t)^{-1/2})(\|\nabla u\|_{H^2}^2 + \|\nabla n\|_{H^1}^2),
\end{aligned} \tag{1.13}$$

for $l = 1, 2$. Note that the energy inequality (1.13) only gives the dissipative estimate for u^H. To look for the dissipative estimates of n^H and $\nabla\Phi^H$, we will use the benefit of the low-frequency and high-frequency decomposition to employ the new interactive energy functional to get (see the proof of Lemma 3.4 for details):

$$\begin{aligned}
&\frac{d}{dt}\langle\nabla^{l-1}u^H, \nabla^l n^H\rangle + \frac{P'(\bar{\rho})}{2\bar{\rho}}\|\nabla^l n^H\|_{L^2}^2 + \|\nabla^l(\nabla\Phi)^H\|_{L^2}^2 \\
&\lesssim (1+t)^{-5/2} + (\delta_0 + (1+t)^{-1/2})(\|\nabla u\|_{H^2}^2 + \|\nabla n\|_{H^1}^2) + C_1\|\nabla^2 u^H\|_{H^1}^2,
\end{aligned} \tag{1.14}$$

for $l = 1, 2$. Thirdly, we prove the upper optimal decay rates of the solutions. We choose two sufficiently large positive constants D_0 and T_0, and define the temporal energy functional

$$\mathfrak{E}(t) = D_0 \|\nabla(n^H, u^H, \nabla\Phi^H)\|_{H^1}^2 + \sum_{l=1}^{2} \langle \nabla^{l-1} u^H, \nabla^l n^H \rangle, \qquad (1.15)$$

which is equivalent to $\|\nabla(n^H, u^H, \nabla\Phi^H)\|_{H^1}^2$. From (1.13) and (1.14), we can obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathfrak{E}(t) + C_3\mathfrak{E}(t) \lesssim (1+t)^{-5/2} + (\delta_0 + (1+t)^{-1/2})(\|\nabla u^L\|_{L^2}^2 + \|\nabla n^L\|_{L^2}^2). \quad (1.16)$$

By using Gronwall's inequality and noticing that T_0 is large enough and δ_0 is sufficiently small, we can deduce that

$$\mathfrak{E}(t) \leq C(1+t)^{-5/2}(\mathfrak{E}(0) + \|U_0\|_{L^1}^2 + \delta_0^2 M(t)), \qquad (1.17)$$

which together with low frequency decay rate in (1.12), the definition of $M(t)$ and the smallness of δ_0 implies the key uniform time-independent bound on $M(t)$. Therefore, this proves (1.9) and completes the proof of Theorem 1.2.

For the proof of Theorem 1.3, we will employ Duhamel's principle, the lower decay rates of the linear system in (3.1) and (1.15), and Theorem 1.2 to obtain the lower optimal decay rate of the solution.

2. Reformulation of original problem

In this section, we reformulate the Cauchy problem (1.1). Let $(\rho, u, \phi) = (n + \tilde{\rho}, u, \Phi + \tilde{\phi})$. Then (1.1) is equivalent to

$$\partial_t n + \bar{\rho} \nabla \cdot u = f_{11} + f_{12},$$

$$\partial_t u - \mu_1 \Delta u - \mu_2 \nabla(\nabla \cdot u) + \frac{P'(\bar{\rho})}{\bar{\rho}} \nabla n - \nabla\Phi = f_{21} + f_{22},$$

$$\Delta\Phi = n, \quad \lim_{|x|\to\infty} \Phi(x, t) = 0, \qquad (2.1)$$

$$(n, u)(x, 0) = (n_0, u_0)(x) = (\rho_0 - \tilde{\rho}, u_0)(x),$$

where $\mu_1 = \mu/\bar{\rho}$, $\mu_2 = (\mu + \nu)/\bar{\rho}$,

$$f_{11} = -\nabla \cdot ((\tilde{\rho} - \bar{\rho})u), \qquad (2.2)$$

$$f_{12} = -\nabla \cdot (nu), \qquad (2.3)$$

$$f_{21} = -\Big(\frac{P'(n+\tilde{\rho})}{n+\tilde{\rho}} - \frac{P'(\tilde{\rho})}{\tilde{\rho}}\Big)\nabla\tilde{\rho} - \Big(\frac{P'(\tilde{\rho})}{\tilde{\rho}} - \frac{P'(\bar{\rho})}{\bar{\rho}}\Big)\nabla n$$
$$+ (\tilde{\rho} - \bar{\rho})(\frac{\mu_1}{\tilde{\rho}}\Delta u + \frac{\mu_2}{\tilde{\rho}}\nabla \operatorname{div} u), \qquad (2.4)$$

$$f_{22} = -(u \cdot \nabla)u - \Big(\frac{P'(n+\tilde{\rho})}{n+\tilde{\rho}} - \frac{P'(\tilde{\rho})}{\tilde{\rho}}\Big)\nabla n$$
$$+ \Big(\frac{1}{n+\tilde{\rho}} - \frac{1}{\tilde{\rho}}\Big)(\mu\Delta u + (\mu + \nu)\nabla(\nabla \cdot u)), \qquad (2.5)$$

Notice that

$$\frac{P'(n+\tilde{\rho})}{n+\tilde{\rho}} - \frac{P'(\tilde{\rho})}{\tilde{\rho}} \sim n, \quad \frac{P'(\tilde{\rho})}{\tilde{\rho}} - \frac{P'(\bar{\rho})}{\bar{\rho}} \sim \tilde{\rho} - \bar{\rho}, \quad \frac{1}{n+\tilde{\rho}} - \frac{1}{\tilde{\rho}} \sim n.$$

3. Proof of Theorem 1.2

In this section we shall prove Theorem (1.2) by using good properties of the low-frequency and high-frequency decomposition, and delicate energy estimates. First, we recall the L^2 time decay rates on the linearized system of (2.1).

Lemma 3.1. *Let k be an integer. Assume that $(\bar{n}, \bar{u}, \nabla\bar{\Phi})$ is the solution of the linearized system of (2.1) with the initial data $(\bar{n}_0, \bar{u}_0, \nabla\bar{\Phi}_0) \in H^2(\mathbb{R}^3) \cap L^1(\mathbb{R}^3)$, then for $1 \leq k \leq 2,$, it holds that*

$$\|\nabla^k(\bar{n}^L, \bar{u}^L, \nabla\bar{\Phi}^L)\|_{L^2} \lesssim (1+t)^{-\frac{3}{4}-\frac{k}{2}}\|(\bar{n}_0, \bar{u}_0, \nabla\bar{\Phi}_0)\|_{L^1}. \tag{3.1}$$

For a proof of the above lemma, see [6]. Before deriving the L^2 time decay rates on the nonlinear system (2.1), let us define the time-weighted energy functional

$$M(t) = \sup_{0 \leq s \leq t} (1+s)^{5/2}\|\nabla(n, u, \nabla\Phi)(s)\|_{H^1}^2. \tag{3.2}$$

Notice that $M(t)$ is non-decreasing, and we will deduce the L^2 time decay rates on the low-frequent part of the solution to the nonlinear system (2.1) as follows.

Lemma 3.2. *Under the assumptions in Theorem 1.2, the solution $U = (n, u, \nabla\Phi)$ of the nonlinear system (2.1) satisfies the decay estimate*

$$\|\nabla(n^L, u^L, (\nabla\Phi)^L)(t)\|_{L^2}^2 \lesssim (1+t)^{-5/2}(\|U_0\|_{L^1}^2 + \delta_0^2 M(t)). \tag{3.3}$$

Proof. To derive the decay on $(\nabla n^L, \nabla u^L, \nabla(\nabla\Phi)^L)$, we need to estimate the non-linear terms $S(t) := (f_{11}, f_{12}, f_{21}, f_{22})^t$ as follows. By virtue of (1.4), (1.7) , (1.8), (2.2) -(2.5) ,(3.2), Lemma 5.2, Lemma 5.3, Hölder's inequality and Hardy inequality, we have

$$\begin{aligned}
\|S(t)^L\|_{L^1} \lesssim &\|\nabla \cdot [(\tilde{\rho} - \bar{\rho})u]\|_{L^1} + \|\nabla \cdot (nu)\|_{L^1} + \|n\nabla(\tilde{\rho} - \bar{\rho})\|_{L^1} \\
&+ \|((\tilde{\rho} - \bar{\rho})\nabla n, (\tilde{\rho} - \bar{\rho})\Delta u, (\tilde{\rho} - \bar{\rho})\nabla(\nabla \cdot u))\|_{L^1} \\
&+ \|(u \cdot \nabla u, n\nabla n, n\Delta u, n\nabla(\nabla \cdot u))\|_{L^1} \\
\lesssim &\|(1+|x|)\nabla(\tilde{\rho} - \bar{\rho})\|_{L^2}\left\|\frac{u}{1+|x|}\right\|_{L^2} + \|\tilde{\rho} - \bar{\rho}\|_{L^2}\|\nabla u\|_{L^2} \\
&+ \left\|\frac{n}{1+|x|}\right\|_{L^2}\|(1+|x|)\nabla(\tilde{\rho} - \bar{\rho})\|_{L^2} + \|\tilde{\rho} - \bar{\rho}\|_{L^2}\|(\nabla n, \Delta u)\|_{L^2} \\
&+ \|(u, n)\|_{L^2}\|(\nabla u, \nabla n, \Delta u)\|_{L^2} \\
\lesssim &\delta_0\|(\nabla u, \nabla n, \Delta u)\|_{L^2} + \|(u, n)\|_{L^2}\|(\nabla u, \nabla n, \Delta u)\|_{L^2} \\
\lesssim &\delta_0(1+t)^{-5/4}\sqrt{M(t)}.
\end{aligned} \tag{3.4}$$

By using equation $(2.1)_1$-$(2.1)_2$, Lemma 3.1, Duhamel's principle, and Hölder's inequality, we have

$$\begin{aligned}
&\|\nabla(n^L, u^L, (\nabla\Phi)^L)(t)\|_{L^2} \\
&\leq C(1+t)^{-5/4}\|(n_0, u_0, \nabla\Phi_0)\|_{L^1} + \int_0^t (1+t-s)^{-5/4}\|S(s)\|_{L^1}\mathrm{d}s \\
&\leq C(1+t)^{-5/4}\|U_0\|_{L^1} + C\delta_0\sqrt{M(t)}\int_0^t (1+t-s)^{-5/4}(1+s)^{-5/4}\mathrm{d}s \\
&\leq C(1+t)^{-5/4}(\|U_0\|_{L^1} + \delta_0\sqrt{M(t)}),
\end{aligned} \tag{3.5}$$

which implies (3.3). $\square$

Next, we turn to derive the first and second order high-frequency decay estimates of the solution, which are stated in the following Lemma.

Lemma 3.3. *Under the assumption of Theorem 1.2, for $l = 1, 2$, it holds that*

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\Big(\frac{P'(\bar{\rho})}{\bar{\rho}}\|\nabla^l n^H\|_{L^2}^2 + \bar{\rho}\|\nabla^l u^H\|_{L^2}^2 + \|\nabla^l(\nabla\Phi)^H\|_{L^2}^2\Big)$$
$$+ \mu\|\nabla^{l+1} u^H\|_{L^2}^2 + (\mu+\nu)\|\nabla^l\nabla\cdot u^H\|_{L^2}^2 \tag{3.6}$$
$$\lesssim (1+t)^{-5/2} + (\delta_0 + (1+t)^{-1/2})(\|\nabla u\|_{H^2}^2 + \|\nabla n\|_{H^1}^2),$$

for any $t \in [0, \infty)$.

Proof. For $l = 1, 2$, by taking

$$\langle \nabla^l \mathfrak{F}^{-1}[(1-\phi(\xi))\mathfrak{F}(2.1)_1], \frac{P'(\bar{\rho})}{\bar{\rho}}\nabla^l n^H\rangle + \langle \nabla^l\mathfrak{F}^{-1}[(1-\phi(\xi))\mathfrak{F}(2.1)_2], \bar{\rho}\nabla^l u^H\rangle,$$

and using integration by parts, we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\Big(\frac{P'(\bar{\rho})}{\bar{\rho}}\|\nabla^l n^H\|_{L^2}^2 + \bar{\rho}\|\nabla^l u^H\|_{L^2}^2 + \|\nabla^l(\nabla\Phi)^H\|_{L^2}^2\Big)$$
$$+ \mu\|\nabla^{l+1} u^H\|_{L^2}^2 + (\mu+\nu)\|\nabla^l\nabla\cdot u^H\|_{L^2}^2$$
$$= \frac{P'(\bar{\rho})}{\bar{\rho}}\langle\nabla^l f_{11}^H + \nabla^l f_{12}^H, \nabla^l n^H\rangle + \bar{\rho}\langle\nabla^l f_{21}^H + \nabla^l f_{22}^H, \nabla^l u^H\rangle \tag{3.7}$$
$$- \langle\nabla^l f_{11}^H + \nabla^l f_{12}^H, \nabla^l \Phi^H\rangle := \sum_{i=1}^{6} I_i.$$

Next, we shall estimate each term at the right-hand side of (3.7). Firstly, for term I_1, by using (1.4), (2.2), Lemmas 5.2–5.4, Hölder's and Young inequalities, we obtain

$$|I_1| = |\frac{P'(\bar{\rho})}{\bar{\rho}}\langle\nabla^l\nabla\cdot[(\tilde{\rho}-\bar{\rho})u]^H, \nabla^l n^H\rangle|$$
$$\lesssim \|\nabla^{l+1}[(\tilde{\rho}-\bar{\rho})u]\|_{L^2}\|\nabla^l n^H\|_{L^2}$$
$$\lesssim (\|\tilde{\rho}-\bar{\rho}\|_{L^\infty}\|\nabla^{l+1}u\|_{L^2} + \|\nabla^{l+1}(\tilde{\rho}-\bar{\rho})\|_{L^2}\|u\|_{L^\infty})\|\nabla^l n^H\|_{L^2} \tag{3.8}$$
$$\lesssim (\|\nabla(\tilde{\rho}-\bar{\rho})\|_{H^1}\|\nabla^{l+1}u\|_{L^2} + \|\nabla^{l+1}(\tilde{\rho}-\bar{\rho})\|_{L^2}\|\nabla u\|_{H^1})\|\nabla^l n^H\|_{L^2}$$
$$\lesssim \delta_0(\|\nabla u\|_{H^2}^2 + \|\nabla n^H\|_{H^1}^2).$$

For the terms I_2 , from (2.3), we have

$$I_2 = -\frac{P'(\bar{\rho})}{\bar{\rho}}\langle\nabla^l[\nabla\cdot(nu)]^H, \nabla^l n^H\rangle$$
$$= -\frac{P'(\bar{\rho})}{\bar{\rho}}\langle\nabla^l(n\nabla\cdot u)^H + \nabla^l(u\cdot\nabla n)^H, \nabla^l n^H\rangle \tag{3.9}$$
$$:= I_{2,1} + I_{2,2}.$$

For the term $I_{2,1}$, it follows from the decay rate of (1.5), Lemmas 5.1 and 5.4, Hölder's and Young inequalities that

$$
\begin{aligned}
&|I_{2,1}| \\
&\lesssim \|\nabla^l(n\nabla\cdot u)\|_{L^2}\|\nabla^l n^H\|_{L^2} \\
&\lesssim (\|n\|_{L^\infty}\|\nabla^{l+1}u\|_{L^2} + \|\nabla u\|_{L^\infty}\|\nabla^l n\|_{L^2})\|\nabla^l n^H\|_{L^2} \\
&\lesssim \|\nabla n\|_{L^2}^{1/2}\|\nabla^2 n\|_{L^2}^{1/2}\|\nabla^{l+1}u\|_{L^2}\|\nabla^l n\|_{L^2} \\
&\quad + \|\nabla^2 u\|_{L^2}^{1/2}\|\nabla^l n\|_{L^2}^{1/2}\|\nabla^l n\|_{L^2}^{3/2}\|\nabla^3 u\|_{L^2}^{1/2} \\
&\lesssim (1+t)^{-\frac{3}{2}}\|\nabla^{l+1}u\|_{L^2} + (1+t)^{-3/4}\|\nabla^l n\|_{L^2}^{3/2}\|\nabla^3 u\|_{L^2}^{1/2} \\
&\lesssim (1+t)^{-5/2} + (1+t)^{-1/2}\|\nabla^{l+1}u\|_{L^2}^2 + (1+t)^{-3/4}(\|\nabla^l n\|_{L^2}^2 + \|\nabla^3 u\|_{L^2}^2) \\
&\lesssim (1+t)^{-5/2} + (1+t)^{-1/2}(\|\nabla n\|_{H^1}^2 + \|\nabla u\|_{H^2}^2).
\end{aligned}
\tag{3.10}
$$

For the term $I_{2,2}$, we first rewrite it as follows

$$
\begin{aligned}
I_{2,2} &= \langle \nabla^l(u\cdot\nabla n)^H, \nabla^l n^H\rangle \\
&= \langle \nabla^l(u\cdot\nabla n) - \nabla^l(u\cdot\nabla n)^L, \nabla^l n^H\rangle \\
&= \langle \nabla^l(u\cdot\nabla n^H) + \nabla^l(u\cdot\nabla n^L) - \nabla^l(u\cdot\nabla n)^L, \nabla^l n^H\rangle \\
&:= \sum_{i=1}^{3} I_{2,2,i}.
\end{aligned}
\tag{3.11}
$$

For the term $I_{2,2,1}$, if $l = 1$, we have

$$
\begin{aligned}
|I_{2,2,1}| &= |\langle \nabla(u\cdot\nabla n^H), \nabla n^H\rangle| \\
&\lesssim (\|u\|_{L^\infty}\|\nabla^2 n^H\|_{L^2} + \|\nabla n^H\|_{L^6}\|\nabla u\|_{L^3})\|\nabla n^H\|_{L^2} \\
&\lesssim \|\nabla u\|_{L^2}^{1/2}\|\nabla^2 u\|_{L^2}^{1/2}\|\nabla^2 n^H\|_{L^2}\|\nabla n^H\|_{L^2} \\
&\lesssim (1+t)^{-\frac{3}{2}}\|\nabla n^H\|_{L^2} \\
&\lesssim (1+t)^{-5/2} + (1+t)^{-1/2}\|\nabla n\|_{L^2}^2;
\end{aligned}
\tag{3.12}
$$

if $l = 2$, it is easy to see that

$$
\begin{aligned}
I_{2,2,1} &= \langle \nabla^2(u\cdot\nabla n^H), \nabla^2 n^H\rangle \\
&= \langle u\cdot\nabla(\nabla^2 n^H) + 2\nabla u\cdot\nabla^2 n^H + \nabla^2 u\cdot\nabla n^H, \nabla^2 n^H\rangle.
\end{aligned}
$$

By using integration by parts, we have

$$
\int_{\mathbb{R}^3} u\cdot\nabla(\nabla^2 n^H)\cdot\nabla^2 n^H \mathrm{d}x = -\frac{1}{2}\int_{\mathbb{R}^3} \mathrm{div} u|\nabla^2 n^H|^2 \mathrm{d}x,
$$

and hence we obtain

$$
\begin{aligned}
&|I_{2,2,1}| \\
&\lesssim (\|\nabla u\|_{L^\infty}\|\nabla^2 n^H\|_{L^2} + \|\nabla n^H\|_{L^3}\|\nabla^2 u\|_{L^6})\|\nabla^2 n^H\|_{L^2} \\
&\lesssim (\|\nabla^2 u\|_{L^2}^{1/2}\|\nabla^3 u\|_{L^2}^{1/2}\|\nabla^2 n\|_{L^2} + \|\nabla n^H\|_{L^2}^{1/2}\|\nabla^2 n^H\|_{L^2}^{1/2}\|\nabla^3 u\|_{L^2})\|\nabla^2 n\|_{L^2} \\
&\lesssim \|\nabla^2 u\|_{L^2}^{1/2}\|\nabla^2 n\|_{L^2}^{1/2}\|\nabla^3 u\|_{L^2}^{1/2}\|\nabla^2 n\|_{L^2}^{3/2} + \|\nabla n\|_{L^2}^{1/2}\|\nabla^2 n\|_{L^2}^{3/2}\|\nabla^3 u\|_{L^2} \\
&\lesssim (1+t)^{-3/4}\|\nabla^3 u\|_{L^2}^{1/2}\|\nabla^2 n\|_{L^2}^{3/2} + (1+t)^{-\frac{3}{2}}\|\nabla^3 u\|_{L^2} \\
&\lesssim (1+t)^{-5/2} + (1+t)^{-1/2}(\|\nabla^2 n\|_{L^2}^2 + \|\nabla^3 u\|_{L^2}^2).
\end{aligned}
\tag{3.13}
$$

Consequently, (3.12) together with (3.13) implies that for $1 \le l \le 2$,

$$
|I_{2,2,1}| \lesssim (1+t)^{-5/2} + (1+t)^{-1/2}(\|\nabla^l n\|_{L^2}^2 + \|\nabla^3 u\|_{L^2}^2). \tag{3.14}
$$

The term $I_{2,2,2}$ can be estimated as follows

$$
\begin{aligned}
&|I_{2,2,2}| \\
&\lesssim \|\nabla^l(u \cdot \nabla n^L)\|_{L^2}\|\nabla^l n^H\|_{L^2} \\
&\lesssim (\|u\|_{L^\infty}\|\nabla^{l+1} n^L\|_{L^2} + \|\nabla n^L\|_{L^\infty}\|\nabla^l u\|_{L^2})\|\nabla^l n^H\|_{L^2} \\
&\lesssim (\|\nabla u\|_{L^2}^{1/2}\|\nabla^2 u\|_{L^2}^{1/2}\|\nabla^l n\|_{L^2} + \|\nabla^2 n^L\|_{L^2}^{1/2}\|\nabla^3 n^L\|_{L^2}^{1/2}\|\nabla^l u\|_{L^2})\|\nabla^l n^H\|_{L^2} \\
&\lesssim (\|\nabla u\|_{L^2}^{1/2}\|\nabla^2 u\|_{L^2}^{1/2}\|\nabla^l n\|_{L^2} + \|\nabla n\|_{L^2}^{1/2}\|\nabla^2 n\|_{L^2}^{1/2}\|\nabla^l u\|_{L^2})\|\nabla^l n^H\|_{L^2} \\
&\lesssim (1+t)^{-\frac{3}{2}}\|\nabla^l n^H\|_{L^2} \\
&\lesssim (1+t)^{-5/2} + (1+t)^{-1/2}\|\nabla^l n^H\|_{L^2}^2,
\end{aligned}
\tag{3.15}
$$

where we have used (5.2). Similar to the proof of (3.12), we also have

$$
\begin{aligned}
&|I_{2,2,3}| \\
&\lesssim \|\nabla^l(u \cdot \nabla n)^L\|_{L^2}\|\nabla^l n^H\|_{L^2} \\
&\lesssim \|\nabla^{l-1}(u \cdot \nabla n)\|_{L^2}\|\nabla^l n^H\|_{L^2} \\
&\lesssim (\|u\|_{L^\infty}\|\nabla^l n\|_{L^2} + \|\nabla^{l-1} u\|_{L^6}\|\nabla n\|_{L^3})\|\nabla^l n^H\|_{L^2} \\
&\lesssim (\|\nabla u\|_{L^2}^{1/2}\|\nabla^2 u\|_{L^2}^{1/2}\|\nabla^l n\|_{L^2} + \|\nabla n\|_{L^2}^{1/2}\|\nabla^2 n\|_{L^2}^{1/2}\|\nabla^l u\|_{L^2})\|\nabla^l n\|_{L^2} \\
&\lesssim (1+t)^{-\frac{3}{2}}\|\nabla^l n\|_{L^2} \\
&\lesssim (1+t)^{-5/2} + (1+t)^{-1/2}\|\nabla^l n\|_{L^2}^2.
\end{aligned}
\tag{3.16}
$$

Substituting (3.14)–(3.16) into (3.11), we arrive at

$$
|I_{2,2}| \lesssim (1+t)^{-5/2} + (1+t)^{-1/2}(\|\nabla n\|_{H^1}^2 + \|\nabla^3 u\|_{L^2}^2), \tag{3.17}
$$

Thus, combining (3.10) with (3.17) gives rise to

$$
|I_2| \lesssim (1+t)^{-5/2} + (1+t)^{-1/2}(\|\nabla n\|_{H^1}^2 + \|\nabla u\|_{H^2}^2). \tag{3.18}
$$

For the term I_3, from (2.4), one has

$$
\begin{aligned}
&I_3 \\
&= \bar{\rho}\langle \nabla^l f_{21}^H, \nabla^l u^H\rangle \\
&= \bar{\rho}\Big\langle \nabla^l\Big[-\Big(\frac{P'(n+\tilde{\rho})}{n+\tilde{\rho}} - \frac{P'(\tilde{\rho})}{\tilde{\rho}}\Big)\nabla(\tilde{\rho}-\bar{\rho})\Big]^H, \nabla^l u^H\Big\rangle \\
&\quad + \bar{\rho}\Big\langle \nabla^l\Big[-\Big(\frac{P'(\tilde{\rho})}{\tilde{\rho}} - \frac{P'(\bar{\rho})}{\bar{\rho}}\Big)\nabla n\Big]^H, \nabla^l u^H\Big\rangle \\
&\quad + \bar{\rho}\Big\langle \nabla^l\Big[\frac{\mu_1}{\tilde{\rho}}(\tilde{\rho}-\bar{\rho})\Delta u\Big]^H, \nabla^l u^H\Big\rangle + \bar{\rho}\Big\langle \nabla^l\Big[\frac{\mu_2}{\tilde{\rho}}(\tilde{\rho}-\bar{\rho})\nabla\operatorname{div}u\Big]^H, \nabla^l u^H\Big\rangle \\
&:= \sum_{i=1}^4 I_{3,i}.
\end{aligned}
\tag{3.19}
$$

By employing a similar argument as in the proof of (3.8), we have

$$
\begin{aligned}
|I_{3,1}| &\lesssim \|\nabla^l[n\nabla(\tilde{\rho}-\bar{\rho})]\|_{L^2}\|\nabla^l u^H\|_{L^2} \\
&\lesssim (\|n\|_{L^\infty}\|\nabla^{l+1}(\tilde{\rho}-\bar{\rho})\|_{L^2} + \|\nabla^l n\|_{L^2}\|\nabla(\tilde{\rho}-\bar{\rho})\|_{L^\infty})\|\nabla^l u^H\|_{L^2} \\
&\lesssim (\delta_0\|\nabla n\|_{H^1} + \delta_0\|\nabla^l n\|_{L^2})\|\nabla^l u^H\|_{L^2} \\
&\lesssim \delta_0(\|\nabla^{l+1}u^H\|_{L^2}^2 + \|\nabla n\|_{H^1}^2).
\end{aligned}
\tag{3.20}
$$

For the term $I_{3,2}$, by using integration by parts, we have

$$
\begin{aligned}
|I_{3,2}| &\lesssim |\langle\nabla^l[(\tilde{\rho}-\bar{\rho})\nabla n]^H, \nabla^l u^H\rangle| \\
&\lesssim |\langle\nabla^{l-1}[(\tilde{\rho}-\bar{\rho})\nabla n]^H, \nabla^l\nabla\cdot u^H\rangle| \\
&\lesssim (\|\nabla^{l-1}(\tilde{\rho}-\bar{\rho})\|_{L^\infty}\|\nabla n\|_{L^2} + \|\tilde{\rho}-\bar{\rho}|_{L^\infty}\|\nabla^l n\|_{L^2})\|\nabla^l\nabla\cdot u^H\|_{L^2} \\
&\lesssim (\|\nabla^l(\tilde{\rho}-\bar{\rho})\|_{H^1}\|\nabla n\|_{L^2} + \|\nabla(\tilde{\rho}-\bar{\rho})\|_{H^1}\|\nabla^l n\|_{L^2})\|\nabla^l\nabla\cdot u^H\|_{L^2} \\
&\lesssim \delta_0(\|\nabla^{l+1}u^H\|_{L^2}^2 + \|\nabla n\|_{H^1}^2).
\end{aligned}
\tag{3.21}
$$

Similarly, for the term $I_{3,3}$, we have

$$
\begin{aligned}
|I_{3,3}| &\lesssim |\langle\nabla^l[(\tilde{\rho}-\bar{\rho})\Delta u]^H, \nabla^l\nabla u^H\rangle| \\
&\lesssim |\langle\nabla^{l-1}[(\tilde{\rho}-\bar{\rho})\Delta u]^H, \nabla^l\nabla\cdot u^H\rangle| \\
&\lesssim (\|\nabla^{l-1}(\tilde{\rho}-\bar{\rho})\|_{L^\infty}\|\Delta u\|_{L^2} + \|\tilde{\rho}-\bar{\rho}\|_{L^\infty}\|\nabla^{l+1}u\|_{L^2})\|\nabla^l\nabla\cdot u^H\|_{L^2} \\
&\lesssim (\|\nabla^l(\tilde{\rho}-\bar{\rho})\|_{H^1}\|\Delta u\|_{L^2} + \|\nabla(\tilde{\rho}-\bar{\rho})\|_{H^1}\|\nabla^{l+1}u\|_{L^2})\|\nabla^l\nabla\cdot u\|_{L^2} \\
&\lesssim \delta_0\|\nabla^2 u\|_{H^1}^2.
\end{aligned}
\tag{3.22}
$$

The term $I_{3,4}$ can be estimated in the same way, and it holds that

$$
|I_{3,4}| \lesssim \delta_0\|\nabla^2 u\|_{H^1}^2.
\tag{3.23}
$$

Putting the above inequalities (3.20)-(3.23) into (3.19), we have

$$
|I_3| \lesssim \delta_0(\|\nabla^2 u\|_{H^1}^2 + \|\nabla n\|_{H^1}^2).
\tag{3.24}
$$

For the term I_4, using (2.5) and integration by parts, it holds that

$$
\begin{aligned}
I_4 &= \bar{\rho}\langle \nabla^l f_{22}^H, \nabla^l u^H \rangle \\
&= \bar{\rho}\langle \nabla^{l-1} f_{22}^H, \nabla^l \nabla \cdot u^H \rangle \\
&= \bar{\rho}\langle \nabla^{l-1}(-u \cdot \nabla u)^H, \nabla^l \nabla \cdot u^H \rangle \\
&\quad - \bar{\rho}\Big\langle \nabla^{l-1}\Big[\Big(\frac{P'(n+\tilde{\rho})}{n+\tilde{\rho}} - \frac{P'(\tilde{\rho})}{\tilde{\rho}}\Big)\nabla n\Big]^H, \nabla^l \nabla \cdot u^H \Big\rangle \\
&\quad + \bar{\rho}\Big\langle \nabla^{l-1}\Big[\mu\Big(\frac{1}{n+\tilde{\rho}} - \frac{1}{\tilde{\rho}}\Big)\Delta u\Big]^H, \nabla^l \nabla \cdot u^H \Big\rangle \\
&\quad + \bar{\rho}\Big\langle \nabla^{l-1}\Big[(\mu+\nu)\Big(\frac{1}{n+\tilde{\rho}} - \frac{1}{\tilde{\rho}}\Big)\Delta u\Big]^H, \nabla^l \nabla \cdot u^H \Big\rangle \\
&:= \sum_{i=1}^{4} I_{4,i}.
\end{aligned}
\tag{3.25}
$$

The four terms on the right-hand side of (3.25) can be estimated as follows. Similar to the proof of (3.12), we have

$$
\begin{aligned}
|I_{4,1}| &\lesssim \|\nabla^{l-1}(u\nabla u)\|_{L^2}\|\nabla^l \nabla \cdot u^H\|_{L^2} \\
&\lesssim (\|\nabla^{l-1}u\|_{L^6}\|\nabla u\|_{L^3} + \|u\|_{L^\infty}\|\nabla^l u\|_{L^2})\|\nabla^l \nabla \cdot u^H\|_{L^2} \\
&\lesssim \|\nabla u\|_{L^2}^{1/2}\|\nabla^2 u\|_{L^2}^{1/2}\|\nabla^l u\|_{L^2}\|\nabla^l \nabla \cdot u^H\|_{L^2} \\
&\lesssim (1+t)^{-\frac{3}{2}}\|\nabla^l \nabla \cdot u^H\|_{L^2} \\
&\lesssim (1+t)^{-5/2} + (1+t)^{-1/2}\|\nabla^l \nabla \cdot u^H\|_{L^2}^2
\end{aligned}
\tag{3.26}
$$

Similarly, for the term (3.26), we have

$$
|I_{4,2}| \lesssim (1+t)^{-5/2} + (1+t)^{-1/2}\|\nabla^l \nabla \cdot u^H\|_{L^2}^2.
\tag{3.27}
$$

and

$$
\begin{aligned}
|I_{4,3}| + |I_{4,4}| &\lesssim \|\nabla^{l-1}(n\Delta u)\|_{L^2}\|\nabla^l \nabla \cdot u^H\|_{L^2} \\
&\lesssim (\|n\|_{L^\infty}\|\nabla^{l+1}u\|_{L^2} + \|\nabla^{l-1}n\|_{L^6}\|\Delta u\|_{L^3})\|\nabla^l \nabla \cdot u^H\|_{L^2} \\
&\lesssim \|\nabla n\|_{H^1}\|\nabla^{l+1}u\|_{L^2}^2 + \|\nabla^l n\|_{L^2}\|\Delta u\|_{H^1}\|\nabla^l \nabla \cdot u^H\|_{L^2} \\
&\lesssim (1+t)^{-3/4}\|\nabla^{l+1}u\|_{L^2}^2 + (1+t)^{-3/4}\|\nabla \nabla \cdot u\|_{H^1}^2 \\
&\lesssim (1+t)^{-3/4}\|\nabla \nabla \cdot u\|_{H^1}^2.
\end{aligned}
\tag{3.28}
$$

Substituting (3.26) - (3.28) into (3.25) gives

$$
|I_4| \lesssim (1+t)^{-5/2} + (1+t)^{-1/2}\|\nabla \nabla \cdot u\|_{H^1}^2.
\tag{3.29}
$$

For the term I_5, we have

$$
\begin{aligned}
|I_5| &\lesssim |\langle \nabla^l \nabla \cdot [(\tilde{\rho}-\bar{\rho})u]^H, \nabla^l \Phi^H \rangle| \\
&\lesssim |\langle \nabla^l[(\tilde{\rho}-\bar{\rho})u]^H, \nabla^l \nabla \Phi^H \rangle| \\
&\lesssim (\|\nabla^l(\tilde{\rho}-\bar{\rho})\|_{L^2}\|u\|_{L^\infty} + \|\tilde{\rho}-\bar{\rho}\|_{L^3}\|\nabla^l u\|_{L^6})\|\nabla^{l-1}n^H\|_{L^2} \\
&\lesssim (\delta_0\|\nabla u\|_{H^1} + \delta_0\|\nabla^{l+1}u\|_{L^2})\|\nabla^l n^H\|_{L^2} \\
&\lesssim \delta_0(\|\nabla u\|_{H^2}^2 + \|\nabla^l n^H\|_{L^2}^2).
\end{aligned}
\tag{3.30}
$$

For the last term I_6, we obtain

$$
\begin{aligned}
|I_6| &\lesssim |\langle \nabla^l \nabla \cdot (nu)^H, \nabla^l \Phi^H \rangle| \\
&\lesssim |\langle \nabla^l (nu)^H, \nabla^l \nabla \Phi^H \rangle| \\
&\lesssim \|\nabla^l (nu)\|_{L^2} \|\nabla^{l-1} n^H\|_{L^2} \\
&\lesssim \|(n,u)\|_{L^\infty} \|\nabla^l (n,u)\|_{L^2} \|\nabla^l n^H\|_{L^2} \\
&\lesssim (\|\nabla (n,u)\|_{L^2}^{1/2} \|\nabla^2 (n,u)\|_{L^2}^{1/2} \|\nabla^l (n,u)\|_{L^2} \|\nabla^l n^H\|_{L^2} \\
&\lesssim (1+t)^{-\frac{3}{2}} \|\nabla^l n^H\|_{L^2} \\
&\lesssim (1+t)^{-5/2} + (1+t)^{-1/2} \|\nabla^l n^H\|_{L^2}^2.
\end{aligned}
\tag{3.31}
$$

Substituting (3.8),(3.18),(3.24) and (3.29)–(3.31) into (3.7), and using the smallness of δ_0, we obtain the estimate (3.6), and thus completes the proof. $\qquad\square$

Notice that the energy inequality (3.6) only gives the dissipative estimate for u^H. Next, we will establish dissipation estimates for n^H and $\nabla \Phi^H$.

Lemma 3.4. *Under the assumptions of Theorem 1.2, then for $l = 1, 2$, it holds that*

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t} \langle \nabla^{l-1} u^H, \nabla^l n^H \rangle + \frac{P'(\bar{\rho})}{2\bar{\rho}} \|\nabla^l n^H\|_{L^2}^2 + \|\nabla^l (\nabla \Phi)^H\|_{L^2}^2 \\
&\lesssim (1+t)^{-5/2} + (\delta_0 + (1+t)^{-1/2})(\|\nabla u\|_{H^2}^2 + \|\nabla n\|_{H^1}^2) + C_1 \|\nabla^2 u^H\|_{H^1}^2,
\end{aligned}
\tag{3.32}
$$

for any $t \in [0, \infty)$.

Proof. For $l = 1, 2$, by taking

$$
\langle \nabla^l \mathfrak{F}^{-1}[(1-\phi(\xi))\mathfrak{F}(2.1)_1], \nabla^{l-1} u^H \rangle + \langle \nabla^{l-1} \mathfrak{F}^{-1}[(1-\phi(\xi))\mathfrak{F}(2.1)_2], \nabla^l n^H \rangle,
$$

using integration by parts, Hölder's inequality and Young's inequality, we have

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t} \langle \nabla^{l-1} u^H, \nabla^l n^H \rangle + \frac{P'(\bar{\rho})}{\bar{\rho}} \|\nabla^l n^H\|_{L^2}^2 + \|\nabla^l (\nabla \Phi)^H\|_{L^2}^2 \\
&= \langle \nabla^l (-\bar{\rho} \nabla \cdot u^H + f_{11}^H + f_{12}^H), \nabla^{l-1} u^H \rangle \\
&\quad + \langle \nabla^{l-1}(\mu_1 \Delta u^H + \mu_2 \nabla \nabla \cdot u^H + f_{21}^H + f_{22}^H), \nabla^l n^H \rangle \\
&= \langle \bar{\rho} \nabla^l u^H, \nabla^l u^H \rangle + \langle \nabla^l [(\tilde{\rho} - \bar{\rho})u]^H + \nabla^l (nu)^H, \nabla^l u^H \rangle \\
&\quad + (\mu_1 + \mu_2)\langle \nabla^{l+1} u^H, \nabla^l n^H \rangle + \langle \nabla^{l-1}(f_{21}^H + f_{22}^H), \nabla^l n^H \rangle \\
&\leq C_1 \|\nabla^{l+1} u^H\|_{L^2}^2 + \frac{P'(\bar{\rho})}{2\bar{\rho}} \|\nabla^l n^H\|_{L^2}^2 + \langle \nabla^l [(\tilde{\rho} - \bar{\rho})u]^H, \nabla^l u^H \rangle \\
&\quad + \langle \nabla^l (nu)^H, \nabla^l u^H \rangle + \langle \nabla^{l-1}(f_{21}^H + f_{22}^H), \nabla^l n^H \rangle,
\end{aligned}
\tag{3.33}
$$

and for simplicity, we define

$$
\begin{aligned}
&\langle \nabla^l [(\tilde{\rho} - \bar{\rho})u]^H, \nabla^l u^H \rangle + \langle \nabla^l (nu)^H, \nabla^l u^H \rangle + \langle \nabla^{l-1}(f_{21}^H + f_{22}^H), \nabla^l n^H \rangle \\
&:= \sum_{i=1}^{4} J_i.
\end{aligned}
\tag{3.34}
$$

Next, we shall estimate the terms on the right-hand side of (3.34) one by one. For the terms J_1 and J_2, we have

$$
\begin{aligned}
|J_1| &= |\langle \nabla^l [(\tilde{\rho} - \bar{\rho})u]^H, \nabla^l u^H \rangle| \\
&\lesssim \|\nabla^l [(\tilde{\rho} - \bar{\rho})u]\|_{L^2} \|\nabla^l u^H\|_{L^2} \\
&\lesssim (\|\nabla^l (\tilde{\rho} - \bar{\rho})\|_{L^2} \|u\|_{L^\infty} + \|\tilde{\rho} - \bar{\rho}\|_{L^\infty} \|\nabla^l u\|_{L^2}) \|\nabla^l u^H\|_{L^2} \\
&\lesssim (\|\nabla^l (\tilde{\rho} - \bar{\rho})\|_{L^2} \|\nabla u\|_{H^1} + \|\nabla(\tilde{\rho} - \bar{\rho})\|_{H^1} \|\nabla^l u\|_{L^2}) \|\nabla^l u^H\|_{L^2} \\
&\lesssim \delta_0 \|\nabla u\|_{H^1}^2,
\end{aligned}
\tag{3.35}
$$

and

$$
\begin{aligned}
|J_2| &= |\langle \nabla^l (nu)^H, \nabla^l u^H \rangle| \\
&\lesssim \|\nabla^l (nu)\|_{L^2} \|\nabla^l n^H\|_{L^2} \\
&\lesssim \|(n,u)\|_{L^\infty} \|\nabla^l (n,u)\|_{L^2} \|\nabla^l n^H\|_{L^2} \\
&\lesssim (\|\nabla(n,u)\|_{L^2}^{1/2} \|\nabla^2 (n,u)\|_{L^2}^{1/2} \|\nabla^l (n,u)\|_{L^2} \|\nabla^l n^H\|_{L^2} \\
&\lesssim (1+t)^{-\frac{3}{2}} \|\nabla^l n^H\|_{L^2} \\
&\lesssim (1+t)^{-5/2} + (1+t)^{-1/2} \|\nabla^l n^H\|_{L^2}^2.
\end{aligned}
\tag{3.36}
$$

For the term J_3, it holds that

$$
\begin{aligned}
J_3 &= \langle \nabla^{l-1} f_{21}^H, \nabla^l n^H \rangle \\
&= \left\langle \nabla^{l-1} \Big[-\Big(\frac{P'(n+\tilde{\rho})}{n+\tilde{\rho}} - \frac{P'(\tilde{\rho})}{\tilde{\rho}} \Big) \nabla(\tilde{\rho} - \bar{\rho}) \Big]^H, \nabla^l n^H \right\rangle \\
&\quad + \left\langle \nabla^{l-1} \Big[-\Big(\frac{P'(\tilde{\rho})}{\tilde{\rho}} - \frac{P'(\bar{\rho})}{\bar{\rho}} \Big) \nabla n \Big]^H, \nabla^l n^H \right\rangle \\
&\quad + \left\langle \nabla^{l-1} \Big[\frac{\mu_1}{\tilde{\rho}} (\tilde{\rho} - \bar{\rho}) \Delta u \Big]^H, \nabla^l n^H \right\rangle \\
&\quad + \left\langle \nabla^{l-1} \Big[\frac{\mu_2}{\tilde{\rho}} (\tilde{\rho} - \bar{\rho}) \nabla \operatorname{div} u \Big]^H, \nabla^l n^H \right\rangle \\
&:= \sum_{i=1}^4 J_{3,i}.
\end{aligned}
\tag{3.37}
$$

For $J_{3,1}$ and $J_{3,2}$, we obtain

$$
\begin{aligned}
|J_{3,1}| &\lesssim |\langle \nabla^{l-1} [n \nabla(\tilde{\rho} - \bar{\rho})]^H, \nabla^l n^H \rangle| \\
&\lesssim \|\nabla^{l-1} [n \nabla(\tilde{\rho} - \bar{\rho})]\|_{L^2} \|\nabla^l n^H\|_{L^2} \\
&\lesssim (\|\nabla(\tilde{\rho} - \bar{\rho})\|_{L^3} \|\nabla^{l-1} n\|_{L^6} + \|\nabla^l (\tilde{\rho} - \bar{\rho})\|_{L^2} \|n\|_{L^\infty}) \|\nabla^l n^H\|_{L^2} \\
&\lesssim (\|\nabla(\tilde{\rho} - \bar{\rho})\|_{H^1} \|\nabla^2 n\|_{L^2} + \|\nabla^l (\tilde{\rho} - \bar{\rho})\|_{L^2} \|\nabla n\|_{H^1}) \|\nabla^l n^H\|_{L^2} \\
&\lesssim \delta_0 \|\nabla n\|_{H^1}^2.
\end{aligned}
\tag{3.38}
$$

$$
\begin{aligned}
|J_{3,2}| &\lesssim |\langle \nabla^{l-1} [(\tilde{\rho} - \bar{\rho}) \nabla n]^H, \nabla^l n^H \rangle| \\
&\lesssim \|\nabla^{l-1} [(\tilde{\rho} - \bar{\rho}) \nabla n]\|_{L^2} \|\nabla^l n^H\|_{L^2} \\
&\lesssim (\|\tilde{\rho} - \bar{\rho}\|_{L^\infty} \|\nabla^l n\|_{L^2} + \|\nabla^{l-1} (\tilde{\rho} - \bar{\rho})\|_{L^3} \|\nabla n\|_{L^6}) \|\nabla^l n^H\|_{L^2} \\
&\lesssim (\|\nabla(\tilde{\rho} - \bar{\rho})\|_{H^1} \|\nabla^l n\|_{L^2} + \|\nabla^{l-1} (\tilde{\rho} - \bar{\rho})\|_{H^1} \|\nabla^2 n\|_{L^2}) \|\nabla^l n^H\|_{L^2} \\
&\lesssim \delta_0 \|\nabla n\|_{H^1}^2.
\end{aligned}
\tag{3.39}
$$

Similarly, for the term $J_{3,3}$ and $J_{3,4}$, we have

$$
\begin{aligned}
&|J_{3,3}| + |J_{3,4}| \\
&\lesssim |\langle \nabla^{l-1}[(\tilde{\rho} - \bar{\rho})\Delta u]^H, \nabla^l n^H \rangle| \\
&\lesssim \|\nabla^{l-1}[(\tilde{\rho} - \bar{\rho})\Delta u]\|_{L^2} \|\nabla^l n^H\|_{L^2} \\
&\lesssim (\|\tilde{\rho} - \bar{\rho}\|_{L^\infty} \|\nabla^{l+1} u\|_{L^2} + \|\nabla^{l-1}(\tilde{\rho} - \bar{\rho})\|_{L^\infty} \|\Delta u\|_{L^2}) \|\nabla^l n^H\|_{L^2} \\
&\lesssim (\|\nabla(\tilde{\rho} - \bar{\rho})\|_{H^1} \|\nabla^{l+1} u\|_{L^2} + \|\nabla^l(\tilde{\rho} - \bar{\rho})\|_{H^1} \|\Delta u\|_{L^2}) \|\nabla^l n^H\|_{L^2} \\
&\lesssim \delta_0 (\|\nabla^l n^H\|_{L^2}^2 + \|\nabla^2 u\|_{H^1}^2).
\end{aligned}
\tag{3.40}
$$

Substituting (3.38)–(3.40) into (3.37) yields

$$
|J_3| \lesssim \delta_0 (\|\nabla n\|_{H^1}^2 + \|\nabla^2 u\|_{H^1}^2).
\tag{3.41}
$$

For the term J_4, we have

$$
\begin{aligned}
J_4 &= \langle \nabla^{l-1} f_{22}^H, \nabla^l n^H \rangle \\
&= \langle \nabla^{l-1}(-u \cdot \nabla u)^H, \nabla^l n^H \rangle - \left\langle \nabla^{l-1}\left[\left(\frac{P'(n + \tilde{\rho})}{n + \tilde{\rho}} - \frac{P'(\tilde{\rho})}{\tilde{\rho}}\right)\nabla n\right]^H, \nabla^l n^H \right\rangle \\
&\quad + \left\langle \nabla^{l-1}\left[\mu\left(\frac{1}{n + \tilde{\rho}} - \frac{1}{\tilde{\rho}}\right)\Delta u\right]^H, \nabla^l n^H \right\rangle \\
&\quad + \left\langle \nabla^{l-1}\left[(\mu + \nu)\left(\frac{1}{n + \tilde{\rho}} - \frac{1}{\tilde{\rho}}\right)\nabla(\nabla \cdot u)\right]^H, \nabla^l n^H \right\rangle \\
&:= \sum_{i=1}^4 J_{4,i}.
\end{aligned}
\tag{3.42}
$$

The four terms on the right-hand side of (3.42) can be estimated as follows. Firstly, similar to the proof of (3.26), we have

$$
\begin{aligned}
|J_{4,1}| &\lesssim \|\nabla^{l-1}(u \cdot \nabla u)\|_{L^2} \|\nabla^l n^H\|_{L^2} \\
&\lesssim (\|\nabla^{l-1} u\|_{L^6} \|\nabla u\|_{L^3} + \|u\|_{L^\infty} \|\nabla^l u\|_{L^2}) \|\nabla^l n^H\|_{L^2} \\
&\lesssim \|\nabla u\|_{L^2}^{1/2} \|\nabla^2 u\|_{L^2}^{1/2} \|\nabla^l u\|_{L^2} \|\nabla^l n^H\|_{L^2} \\
&\lesssim (1+t)^{-\frac{3}{2}} \|\nabla^l n^H\|_{L^2} \\
&\lesssim (1+t)^{-5/2} + (1+t)^{-1/2} \|\nabla^l n^H\|_{L^2}^2.
\end{aligned}
\tag{3.43}
$$

Similar to the proof of (3.43), we have

$$
\begin{aligned}
|J_{4,2}| &\lesssim \|\nabla^{l-1}(n \cdot \nabla n)\|_{L^2} \|\nabla^l n^H\|_{L^2} \\
&\lesssim (1+t)^{-5/2} + (1+t)^{-1/2} \|\nabla^l n^H\|_{L^2}^2,
\end{aligned}
\tag{3.44}
$$

and

$$
\begin{aligned}
|J_{4,3}| + |J_{4,4}| &\lesssim \|\nabla^{l-1}(n\Delta u)\|_{L^2} \|\nabla^l n^H\|_{L^2} \\
&\lesssim (\|n\|_{L^\infty} \|\nabla^{l+1} u\|_{L^2} + \|\nabla^{l-1} n\|_{L^6} \|\Delta u\|_{L^3}) \|\nabla^l n^H\|_{L^2} \\
&\lesssim \|\nabla n\|_{L^2}^{1/2} \|\nabla^2 n\|_{L^2}^{1/2} \|\nabla^l n\|_{L^2} \|\nabla^{l+1} u\|_{L^2} + \|\nabla^l n\|_{L^2}^2 \|\Delta u\|_{H^1} \\
&\lesssim (1+t)^{-\frac{3}{2}} \|\nabla^{l+1} u\|_{L^2} + (1+t)^{-\frac{3}{2}} \|\Delta u\|_{H^1} \\
&\lesssim (1+t)^{-5/2} + (1+t)^{-1/2} \|\Delta u\|_{H^1}^2.
\end{aligned}
\tag{3.45}
$$

Substituting (3.43)–(3.45) into (3.42) yields

$$|J_4| \lesssim (1+t)^{-5/2} + (1+t)^{-1/2}(\|\nabla^l n^H\|_{L^2}^2 + \|\Delta u\|_{H^1}^2). \qquad (3.46)$$

Substituting (3.35), (3.36), (3.41) and (3.46) into (3.34) and using (3.33), we can prove (3.32) and thus completes the proof. $\qquad \square$

Now, we are ready to prove Theorem 1.2. Now, multiplying (3.32) with some positive number $C_2 = \frac{2\mu+\nu}{2C_1}$, and summing (3.6) from $l = 1$ to 2 leads to

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big(\frac{P'(\bar{\rho})}{2\bar{\rho}}\|\nabla n^H\|_{H^1}^2 + \frac{\bar{\rho}}{2}\|\nabla u^H\|_{H^1}^2 + \frac{1}{2}\|\nabla(\nabla\Phi)^H\|_{H^1}^2 + \sum_{l=1}^{2}C_2\langle\nabla^{l-1}u^H, \nabla^l n^H\rangle\Big)$$

$$+ \frac{P'(\bar{\rho})}{2\bar{\rho}}C_2\|\nabla n^H\|_{H^1}^2 + \frac{2\mu+\nu}{2}\|\nabla^2 u^H\|_{H^1}^2 + C_2\|\nabla(\nabla\Phi)^H\|_{H^1}^2 \qquad (3.47)$$

$$\lesssim (1+t)^{-5/2} + (\delta_0 + (1+t)^{-1/2})(\|\nabla u\|_{H^2}^2 + \|\nabla n\|_{H^1}^2)).$$

Next, choosing sufficiently large time T_0 and positive constant D_0, for $t \geq T_0$, we define the temporary energy functional

$$\mathfrak{E}(t) = D_0\|\nabla(n^H, u^H, \nabla\Phi^H)\|_{H^1}^2 + \sum_{l=1}^{2}\langle\nabla^{l-1}u^H, \nabla^l n^H\rangle, \qquad (3.48)$$

which is equivalent to $\|\nabla(n^H, u^H, \nabla\Phi^H)\|_{H^1}^2$ since D_0 is large enough. Using the smallness of δ_0, (3.47) and Lemma 5.4, for $t \geq T_0$, it holds that

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathfrak{E}(t) + \|\nabla n^H\|_{H^1}^2 + \|\nabla^2 u^H\|_{H^1}^2 + \|\nabla(\nabla\Phi)^H\|_{H^1}^2$$

$$\lesssim (1+t)^{-5/2} + (\delta_0 + (1+t)^{-1/2})(\|\nabla u^L\|_{L^2}^2 + \|\nabla n^L\|_{L^2}^2), \qquad (3.49)$$

where we have used the fact that T_0 is large enough. On the other hand, it is clear that

$$\|\nabla n^H\|_{H^1}^2 + \|\nabla^2 u^H\|_{H^1}^2 + \|\nabla(\nabla\Phi)^H\|_{H^1}^2 \geq C_3\mathfrak{E}(t). \qquad (3.50)$$

Then by the Gronwall inequality and Lemma 3.2, we arrive at

$$\mathfrak{E}(t) \leq \mathfrak{E}(0)e^{-C_3 t} + C_4\int_0^t e^{-C_3(t-s)}[(1+s)^{-5/2} + (\delta_0 + (1+s)^{-1/2})$$

$$\times (\|\nabla u(s)^L\|_{L^2}^2 + \|\nabla n(s)^L\|_{L^L}^2)]\mathrm{d}s$$

$$\leq \mathfrak{E}(0)e^{-C_3 t} + C_4\int_0^t e^{-C_3(t-s)}[(1+s)^{-5/2} + (\delta_0 + (1+s)^{-1/2})] \qquad (3.51)$$

$$\times C(1+s)^{-5/2}(\|U_0\|_{L^1}^2 + \delta_0^2 M(s))\mathrm{d}s$$

$$\leq \mathfrak{E}(0)e^{-C_3 t} + C_5(\|U_0\|_{L^1}^2 + \delta_0^2 M(t))\int_0^t (1+t-s)^{-5/2}(1+s)^{-5/2}\mathrm{d}s$$

$$\leq C(1+t)^{-5/2}(\mathfrak{E}(0) + \|U_0\|_{L^1}^2 + \delta_0^2 M(t)).$$

Combining with Lemma 3.2 and (3.51), we deduce that

$$\|\nabla(n, u, \nabla\Phi)\|_{H^1}^2 \leq \|\nabla(n^H, u^H, \nabla\Phi^H)\|_{H^1}^2 + \|\nabla(n^L, u^L, \nabla\Phi^L)\|_{H^1}^2$$

$$\leq C(1+t)^{-5/2}(\mathfrak{E}(0) + \|U_0\|_{L^1}^2 + \delta_0^2 M(t)), \qquad (3.52)$$

which together with the definition (3.2) of $M(t)$ and the smallness of δ_0 implies that

$$M(t) \leq C(\mathfrak{E}(0) + \|U_0\|_{L^1}^2).$$

This gives (1.9) and thus completes the proof.

4. PROOF OF THEOREM 1.3

In this section, we devote ourselves to proving the lower decay estimates of the global solution of Cauchy problem (2.1). By virtue of Duhamel's principle, (1.9), (3.3), (3.4) and Lemma(5.4), we have

$$
\begin{aligned}
&\min\left\{\|\nabla n\|_{L^2}, \|\nabla u\|_{L^2}, \|\nabla \Phi\|_{L^2}\right\}\\
&\geq \min\left\{\|\nabla n^L\|_{L^2}, \|\nabla u^L\|_{L^2}, \|\nabla \Phi^L\|_{L^2}\right\}\\
&\geq C\delta_0^{3/2}(1+t)^{-5/4} - C\Big|\int_0^t (1+t-s)^{-\frac{5}{4}}\|S(s)^L\|_{L^1}\Big|\mathrm{d}s\\
&\geq C\delta_0^{3/2}(1+t)^{-5/4} - C\delta_0\sqrt{M(t)}\int_0^t (1+t-s)^{-\frac{5}{4}}(1+s)^{-5/4}\mathrm{d}s\\
&\geq (C\delta_0^{3/2} - C\delta_0(\mathfrak{E}(0) + \|U_0\|_{L^1}^2)^{\frac{1}{2}})(1+t)^{-5/4}.
\end{aligned}
\tag{4.1}
$$

This together with the fact that $\mathfrak{E}(0), \|U_0\|_{L^1} < \delta_0$ implies (1.11), and thus completes the proof of Theorem 1.3.

5. ANALYTIC TOOLS

Now, we recall the Sobolev interpolation of Gagliardo-Nirenberg's inequality.

Lemma 5.1 ([16]). *Given $2 \leq p \leq +\infty$ and $0 \leq k, m \leq \ell$, then for any $f \in H^\ell(\mathbb{R}^3)$, we have*
$$
\|\nabla^k f\|_{L^p} \lesssim \|\nabla^m f\|_{L^2}^\alpha \|\nabla^\ell f\|_{L^2}^{1-\alpha},
$$
where $\alpha \in [0,1]$ satisfies
$$
\frac{k}{3} - \frac{1}{p} = \left(\frac{m}{3} - \frac{1}{2}\right)\alpha + \left(\frac{\ell}{3} - \frac{1}{2}\right)(1-\alpha).
$$

Next, to estimate the L^2-norm of the spatial derivatives of the product of two functions, we shall record the following estimate.

Lemma 5.2 ([8]). *Let $k \geq 1$ be an integer, then it holds that*
$$
\|\nabla^k(fg)\|_{L^p} \lesssim \|f\|_{L^{p_1}}\|\nabla^k g\|_{L^{p_2}} + \|g\|_{L^{p_3}}\|\nabla^k f\|_{L^{p_4}},
$$
where $p, p_2, p_3 \in (1, +\infty)$ and
$$
\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p_3} + \frac{1}{p_4}.
$$

Next, we recall Hardy inequality and Sobolev embedding estimates.

Lemma 5.3 ([18]). *(i) If $u(x) \in H^1(\mathbb{R}^3)$, then the following inequalities hold:*
$$
\left\|\frac{u}{|x|}\right\|_{L^2} \leq C\|\nabla u\|_{L^2},
$$
$$
\|u\|_{L^6} \leq C\|\nabla u\|_{L^2},
$$
$$
\|u\|_{L^3} \leq C(\|u\|_{L^2} + \|u\|_{L^6}) \leq C\|u\|_{H^1}.
$$
(ii) If $u(x) \in H^2(\mathbb{R}^3)$, then $\|u\|_{L^\infty} \leq C\|\nabla u\|_{H^1}$

We also have the following lemma concerning the estimate for the low–frequent part and the high–frequent part of f.

Lemma 5.4. *If $f \in L^k(\mathbb{R}^3)$ for any $2 \le r \le \infty$, then*

$$\|\nabla^k f\| \le \|\nabla^k f^L\| + \|\nabla^k f^H\| \tag{5.1}$$

$$\|\nabla^k f^L\| \le \|\nabla^{k-1} f\|, \quad k \ge 1, \tag{5.2}$$

$$\|\nabla^k f^H\| \le \|\nabla^{k+1} f\|, \quad k \ge 1, \tag{5.3}$$

Lemma 5.5 ([22, 24]). *If $r_1, r_2 > 0$, then*

$$\int_0^t (1+t-\tau)^{-r_1}(1+\tau)^{-r_2} d\tau \le C(r_1, r_2)(1+t)^{-\min\{r_1, r_2, r_1+r_2-1-\eta\}}, \tag{5.4}$$

for an arbitrarily small $\eta > 0$.

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