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FIXED POINT THEOREM FOR MIXED MONOTONE NEARLY ASYMPTOTICALLY NONEXPANSIVE MAPPINGS AND APPLICATIONS TO INTEGRAL EQUATIONS

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Abstract. This work concerns the existence of a fixed point for mixed monotone nearly asymptotically nonexpansive mappings. We extend and generalize some well-known results concerning nearly asymptotically nonexpansive mappings in a uniformly convex hyperbolic metric space. As application of our results, we study the existence of solutions for an integral equation.

1. Introduction

The mixed monotone operators were introduced by Guo and Lakshmikantham in 1987, and have been applied in many areas since then; see, e.g., [8, 17, 22, 23]. In recent years, new generalizations of nonexpansive mappings have been discovered and their fixed point theory has been studied by many authors. Some interesting generalizations of nonexpansive mappings can be found in [3, 7, 12, 13, 14, 15, 19]. Recall that the theory of fixed point of nonexpansive mapping extends the classical theory of successive approximations for strict contractions.

Recently, a new direction has been developed when the nonexpansive mapping is monotone and defined in partially ordered hyperbolic metric spaces. One can see for example some various work containing those new results in [1, 6, 18] and the references therein.

Our goal in the present work is twofold. First, we introduce a new class of mixed monotone defined as mixed monotone nearly asymptotically nonexpansive mappings and we extend to it the fixed point results for monotone nearly asymptotically nonexpansive mappings, obtained by Agarwal et al. in [3]. Second, we apply our result to prove the existence of solutions for a nonlinear integral equation.

This article is organized as follows. In section 2, we give definitions and basic results which will be used later. In Section 3, we study the existence of fixed point for mixed monotone mapping that is nearly asymptotically nonexpansive. In section 4, we illustrate our results with an application.

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2. Definitions and preliminaries

The following notation will be used. $\mathbb{R}$ is the set of real numbers, $\mathbb{R}_+$ is the set of nonnegative real numbers and $\mathbb{N}$ is the set of nonnegative integers. Let (Ω, d) be a metric space endowed with a partial order $\preceq$. We will say that $(\Omega, d, \preceq)$ is a partial ordered metric space and we will say that $x, y \in \Omega$ are comparable whenever $x \preceq y$ or $y \preceq x$. When referring to the partial order on the product space $\Omega \times \Omega$, we understand the following partial order

$$(x,y),(u,v) \in \Omega \times \Omega, \quad (x,y) \precsim (u,v) \;\Leftrightarrow\; x \preceq u,\; y \succeq v.$$

A metric space $\big(\Omega, d\big)$ is a convex metric space if any two points $x, y \in \Omega$ are endpoints of a unique metric segment $[x, y]$, we shall denote by $w = \alpha x \oplus (1-\alpha) y$ the unique point of $[x, y]$ which satisfies

$$d(x,w) = (1-\alpha) d(x,y) \text{ and } d(y,w) = \alpha d(x,y), \quad \text{for } \alpha \in [0,1].$$

From the definition of convex metric space, we have (i) $0x \oplus 1y = y$, (ii) $1x \oplus 0y = x$, (iii) $\alpha x \oplus (1-\alpha) x = x$.

A convex metric space $\big(\Omega, d\big)$ is hyperbolic if

$$d(\alpha x \oplus (1-\alpha) y, \alpha u \oplus (1-\alpha) v) \le \alpha d(x,u) + (1-\alpha) d(y,v),$$

for all $x, y, u, v \in \Omega$ and $\alpha \in [0,1]$.

Let (Ω, d) be a convex metric space. Consider the following metric in $\Omega \times \Omega$

$$\partial\big((x,y),(u,v)\big) = \big(d^2(x,u) + d^2(y,v)\big)^{1/2}, \quad \text{for every } (x,y),(u,v) \in \Omega \times \Omega.$$

And for all $(x,y),(u,v) \in \Omega \times \Omega$, we have

$$\begin{aligned}
&\partial\Big(\big(\alpha x \oplus (1-\alpha) u, \alpha y \oplus (1-\alpha) v\big), (x,y)\Big) \\
&= \Big(d^2\big(\alpha x \oplus (1-\alpha) u, x\big) + d^2\big(\alpha y \oplus (1-\alpha) v, y\big)\Big)^{1/2} \\
&= \Big((1-\alpha)^2 d^2\big(u,x\big) + (1-\alpha)^2 d^2\big(v,y\big)\Big)^{1/2} \\
&= (1-\alpha)\partial\big((x,y),(u,v)\big).
\end{aligned}$$

Similarly, we have $\partial\big(\big(\alpha x \oplus (1-\alpha) u, \alpha y \oplus (1-\alpha) v\big), (u,v)\big) = \alpha \partial\big((x,y),(u,v)\big)$.

Thus, $\Omega \times \Omega$ is a convex metric space, and for every $(x,y),(u,v) \in \Omega \times \Omega$

$$\alpha(x,y) \oplus (1-\alpha)(u,v)\big) = \big(\alpha x \oplus (1-\alpha) u, \alpha y \oplus (1-\alpha) v\big).$$

Definition 2.1 ([10])**.** Let $\big(\Omega, d\big)$ be a hyperbolic metric space. If for any $z \in \Omega$, $r > 0$ and $\varepsilon > 0$,

$$\delta(r,\varepsilon) = \inf\big\{1 - \frac{1}{r} d\big(z, \frac{1}{2} x \oplus \frac{1}{2} y\big) : d(x,z) \le r,\; d(y,z) \le r,\; d(x,y) \ge r\varepsilon\big\} > 0,$$

then $\big(\Omega, d\big)$ is said to be uniformly convex hyperbolic metric space.

Definition 2.2. Let $\big(\Omega, d, \preceq\ \big)$ be a partial order metric space. A mapping $A : \Omega \times \Omega \to \Omega$ is said to be mixed monotone if A is nondecreasing in the first argument and nonincreasing in the second argument, i.e.,

$$\begin{gathered}
x_1, x_2, y \in \Omega;\; x_1 \preceq x_2 \Longrightarrow A(x_1, y) \preceq A(x_2, y), \\
x, y_1, y_2 \in \Omega;\; y_1 \preceq y_2 \Longrightarrow A(x, y_2) \preceq A(x, y_1).
\end{gathered}$$

The following theorem is a metric version of the parallelogram identity.

Theorem 2.3 ([10])**.** *Let (Ω, d) be a hyperbolic uniformly convex metric space. For any $a \in \Omega$, for each $r > 0$ and for each $\varepsilon > 0$, set*

$$\Psi(r,\varepsilon) = \inf\big\{\frac{1}{2}d^2(a,x) + \frac{1}{2}d^2(a,y) - d^2(a, \frac{1}{2}x \oplus \frac{1}{2}y)\big\}, \tag{2.1}$$

where the infimum is taken over all $x, y \in \Omega$ such that $d(a,x) \le r$, $d(a,y) \le r$ and $d(x,y) \ge r\varepsilon$. Then $\Psi(r,\varepsilon) > 0$ for any $r > 0$ and each $\varepsilon > 0$. Moreover, for a fixed $r > 0$, we have

- (i) $\Psi(r, 0) = 0$;
- (ii) $\Psi(r,\varepsilon)$ *is a nondecreasing function of* ε;
- (iii) *If* $\lim_{n\to+\infty} \Psi(r, t_n) = 0$, *then* $\lim_{n\to+\infty} t_n = 0$.

Recall that the result below is one of the important properties of complete uniformly convex hyperbolic metric space and known as property (R).

Proposition 2.4 ([10])**.** *Let $\big(\Omega, d\big)$ be a uniformly convex hyperbolic metric space and $\{C_n\}_{n\in\mathbb{N}}$ be a decreasing sequence of nonempty, closed, bounded and convex subsets of Ω. Then, $\cap_{n\in\mathbb{N}} C_n \neq \emptyset$.*

Through this article, we will assume that all order intervals are closed and convex. It is worth noting that an order interval is any of the subsets

$$[a,\rightarrow) = \{x \in \Omega : a \preceq x\}, \quad (\leftarrow, b] = \{x \in \Omega : x \preceq b\}, \quad [a,b] = [a,\rightarrow) \cap (\leftarrow, b],$$

for every $a, b \in \Omega$.

3. Fixed point theorem

In this section, we give sufficient conditions so that a mixed monotone nearly asymptotically nonexpansive mapping has fixed point. Similarly to the definition of monotone nearly asymptotically nonexpansive mapping [2], we introduce the following definition for mixed monotone mapping.

Definition 3.1. Let $(\Omega, d, \preceq)$ be a partially ordered metric space, $A : \Omega \times \Omega \to \Omega$ be a map and $\{a_n\}$ be a fixed sequence in $[0, +\infty)$ with $a_n \to 0$. Then the map A is said to be mixed monotone nearly Lipschitzian mapping with respect to a_n if A is mixed monotone and for each $n \in \mathbb{N}^*$, there exists a constant $K_n \ge 0$, such that

$$d\left(A^n(x,y), A^n(u,v)\right) \le K_n\Big(\frac{1}{2}d(x,u) + \frac{1}{2}d(y,v) + a_n\Big), \tag{3.1}$$

for every comparable elements $(x, y), (u, v) \in \Omega \times \Omega$, for any $n \in \mathbb{N}$ $A^0(x, y) = x$ and $A^{n+1}(x,y) = A\big(A^n(x,y), A^n(y,x)\big)$ for all $x, y \in \Omega$. The infimum of constants K_n for which the last inequality (3.1) holds is denoted by $\eta(A^n)$ and called the nearly Lipschitz constant. The mixed monotone nearly Lipschitzian mapping A with sequence $\{a_n, \eta(A^n)\}$ is said to be mixed monotone nearly asymptotically nonexpansive if

- (i) $\eta(A^n) \ge 1$ for all $n \in \mathbb{N}^*$ and
- (ii) $\lim_{n\to\infty} \eta(A^n) = 1$.

Recall that a map $T : \Omega \to \Omega$ is said to be monotone nearly asymptotically nonexpansive if it is nondecreasing and for any $n \ge 0$ there exists $K_n \ge 1$ and $a_n \ge 0$, such that

- (i) $\lim_{n\to\infty} K_n = 1$ and
- (ii) $\lim_{n\to\infty} a_n = 0$.

And for every comparable element $x, y \in \Omega$, for any $n \geq 0$, we have

$$d\left(T^n(x), T^n(y)\right) \leq K_n\left(d(x,y) + a_n\right). \tag{3.2}$$

Proposition 3.2. *Let (Ω, d) be a uniformly convex hyperbolic metric space. If $\{C_n\}$ is a decreasing sequence of bounded, nonempty, closed and convex subsets of $\big(\Omega \times \Omega, \partial\big)$, then $\cap_{n\in\mathbb{N}} C_n \neq \emptyset$.*

Proof. Let $P_1 : \Omega \times \Omega \to \Omega$ defined by $P_1(x,y) = x$ for every $(x,y) \in \Omega \times \Omega$. Since for each $n \geq 0$, C_n is a convex subset in $\Omega \times \Omega$, it follows that for every $(x,y), (u,v) \in C_n$, for each $\alpha \in [0,1]$ and $n \geq 0$,

$$\begin{aligned} \alpha P_1(x,y) \oplus (1-\alpha) P_1(u,v) &= \alpha x \oplus (1-\alpha) u \\ &= P_1\big(\alpha x \oplus (1-\alpha)u, \alpha y \oplus (1-\alpha) v\big) \\ &= P_1\big(\alpha(x,y) \oplus (1-\alpha)(u,v)\big), \end{aligned}$$

which implies that $\alpha P_1(x,y) \oplus (1-\alpha)P_1(u,v) \in P_1(C_n)$. Thus $P_1(C_n)$ is convex. Moreover, it is clear that if $\{C_n\}$ is a decreasing sequence of bounded, nonempty and closed subsets of $\Omega \times \Omega$ then $\{P_1(C_n)\}$ is a decreasing sequence of bounded, nonempty and closed subsets of Ω. Hence, $\cap_{n\geq 0} P_1(C_n) \neq \emptyset$, therefore for any $n \in \mathbb{N}$ there exist $y_n \in \Omega$ such that $(x, y_n) \in C_n$, since $\{C_n\}$ is decreasing, closed and convex subsets, then for any $n \in \mathbb{N}$, $\{x\} \times \overline{\text{conv}}\{y_k : k \geq n\} \subset C_n$, with $\text{conv}\{y_k : \ k \geq n\}$ is the convex envelope of $\{y_k : \ k \geq n\}$. Thus for any $n \in \mathbb{N}$, $\{x\} \times \cap_{n\geq 0} \overline{\text{conv}}\{y_k : k \geq n\} \subset \cap_{n\in\mathbb{N}} C_n$. Since $\{\overline{\text{conv}}\{y_k : k \geq n\}\}$ is a decreasing sequence of bounded, nonempty, closed and convex subsets of Ω, then $\cap_{n\geq 0} \overline{\text{conv}}\{y_k : k \geq n\} \neq \emptyset$, therefore $\cap_{n\in\mathbb{N}} C_n \neq \emptyset$. □

Lemma 3.3. *Let C and D be a nonempty, closed and convex subsets of a uniformly convex hyperbolic metric space (Ω, d). Let $\tau : C \times D \to [0, +\infty)$ be a type function, i.e., there exist bounded sequences $\{x_n\}, \{y_n\} \in \Omega$ such that*

$$\tau(x,y) = \limsup_{n\to\infty} [d^2(x_n, x) + d^2(y_n, y)]^{1/2}, \quad (x,y) \in C \times D. \tag{3.3}$$

Then τ is continuous, and since Ω is hyperbolic, τ is convex, i.e., the subset $\{(x,y) \in C \times D : \tau(x,y) \leq r\}$ is convex for any $r \geq 0$. Moreover, there exists a unique minimum point $(a,b) \in C \times D$ such that

$$\tau(a,b) = \inf\{\tau(x,y); (x,y) \in C \times D\} = \tau_0.$$

In addition, if $\{(a_n, b_n)\}$ is a minimizing sequence in $C \times D$, i.e., $\lim_{n\to\infty} \tau(a_n, b_n) = \tau(a,b)$, then $\{a_n\}$ converges to a and $\{b_n\}$ converges to b.

Proof. The continuity of τ is obvious. Let us prove the convexity of τ. Since (Ω, d) is hyperbolic, and using the discrete Minkowski inequality, we have for any $\alpha \in [0,1]$ and all $(x,y), (u,v) \in C \times D$,

$$\begin{aligned} &\tau\big(\alpha(x,y) \oplus (1-\alpha)(u,v)\big) \\ &= \tau(\alpha x \oplus (1-\alpha)u, \alpha y \oplus (1-\alpha)v) \\ &= \limsup_{n\to\infty} [d^2(x_n, \alpha x \oplus (1-\alpha)u) + d^2(y_n, \alpha y \oplus (1-\alpha)v)]^{1/2} \\ &\leq \limsup_{n\to\infty} [(\alpha d(x_n, x) + (1-\alpha) d(x_n, u))^2 + (\alpha d(y_n, y) + (1-\alpha) d(y_n, v))^2]^{1/2} \\ &\leq \limsup_{n\to\infty} \Big([(\alpha^2 d^2(x_n, x) + (\alpha^2 d^2(y_n, y)]^{1/2} \end{aligned}$$

$$
\begin{aligned}
&+ [(1-\alpha)^2 d^2(x_n,u) + (1-\alpha)^2 d^2(y_n,v)]^{1/2}\Big) \\
&\le \alpha\tau(x,y) + (1-\alpha)\tau(u,v),
\end{aligned}
$$

which implies that τ is convex.

Now, denote $\tau_0 = \inf\{\tau(x,y) : (x,y) \in C \times D\}$. Then for any $n \ge 1$, the subset $C_n = \{(x,y) \in C \times D : \tau(x,y) \le \tau_0 + \frac{1}{n}\}$ is nonempty, closed, bounded and convex subset of $C \times D$. From proposition 3.2, we have $C_\infty = \cap_{n\ge 1} C_n \neq \emptyset$. Obviously we have

$$
C_\infty = \{(u,v) \in C \times D : \tau(u,v) = \tau_0\}.
$$

To prove that C_∞ is reduced to one point (a,b), let (a_1,b_1) and (a_2,b_2) be in C_∞. If we assume that $(a_1,b_1) \neq (a_2,b_2)$, then we must have $\tau_0 > 0$. From the definition of τ, we obtain that for any $\alpha \in (0,\tau_0)$, there exists $n_0 \ge 1$ such that for any $n \ge n_0$,

$$
d(x_n,a_1) \le \tau_0 + \alpha \quad \text{and } d(x_n,a_2) \le \tau_0 + \alpha.
$$

Since $d(a_1,a_2) \ge (\tau_0+\alpha)\dfrac{d(a_1,a_2)}{2\tau_0}$, by Theorem 2.3 we have

$$
d^2(x_n, \frac{1}{2}a_1 \oplus \frac{1}{2}a_2) \le \frac{1}{2}d^2(x_n,a_1) + \frac{1}{2}d^2(x_n,a_2) - \Psi(\tau_0+\alpha, \frac{1}{2\tau_0}d(a_1,a_2)).
$$

Analogously, we obtain

$$
d^2(y_n, \frac{1}{2}b_1 \oplus \frac{1}{2}b_2) \le \frac{1}{2}d^2(y_n,b_1) + \frac{1}{2}d^2(y_n,b_2) - \Psi(\tau_0+\alpha, \frac{1}{2\tau_0}d(b_1,b_2)).
$$

It follows that,

$$
\begin{aligned}
&d^2(x_n, \frac{1}{2}a_1 \oplus \frac{1}{2}a_2) + d^2(y_n, \frac{1}{2}b_1 \oplus \frac{1}{2}b_2) \\
&\le \frac{1}{2}[d^2(x_n,a_1) + d^2(y_n,b_1)] + \frac{1}{2}[d^2(x_n,a_2) + d^2(y_n,b_2)] \\
&\quad - [\Psi(\tau_0+\alpha, \frac{1}{2\tau_0}d(a_1,a_2)) + \Psi(\tau_0+\alpha, \frac{1}{2\tau_0}d(b_1,b_2))].
\end{aligned}
$$

Hence,

$$
\tau_0^2 \le \tau_0^2 - [\Psi(\tau_0+\alpha, \frac{1}{2\tau_0}d(a_1,a_2)) + \Psi(\tau_0+\alpha, \frac{1}{2\tau_0}d(b_1,b_2))].
$$

Therefore, $[\Psi(\tau_0+\alpha, \frac{1}{2\tau_0}d(a_1,a_2)) + \Psi(\tau_0+\alpha, \frac{1}{2\tau_0}d(b_1,b_2))] = 0$. Thus, by the property (iii) in Theorem 2.3 we obtain $a_1 = a_2$ and $b_1 = b_2$.

Next, let $\{(a_n,b_n)\} \subset C \times D$ be a minimizing sequence of τ. Since $\{x_n\}$ and $\{y_n\}$ are bounded sequences, then $\{a_n\}$ and $\{b_n\}$ are also bounded. Therefore, there exists $R > 0$ such that

$$
\max\{d(x_n,a_k), d(y_n,b_k), d(x_n,a), d(y_n,b)\} \le R,
$$

for every $n,k \in \mathbb{N}$.

Once again, by Theorem 2.3 we have

$$
\begin{aligned}
d^2(x_n, \frac{1}{2}a_k \oplus \frac{1}{2}a) &\le \frac{1}{2}d^2(x_n,a_k) + \frac{1}{2}d^2(x_n,a) - \Psi(R, \frac{1}{R}d(a_k,a)), \\
d^2(y_n, \frac{1}{2}b_k \oplus \frac{1}{2}b) &\le \frac{1}{2}d^2(y_n,b_k) + \frac{1}{2}d^2(y_n,b) - \Psi(R, \frac{1}{R}d(b_k,b)).
\end{aligned}
$$

When n goes to infinity, taking the limit-sup, we obtain

$$
\tau^2\Big(\frac{1}{2}a_k \oplus \frac{1}{2}a, \frac{1}{2}b_k \oplus \frac{1}{2}b\Big)
$$

$$\leq \frac{1}{2}\Big(\tau^2(a_k,b_k)+\tau^2(a,b)\Big)-\Psi(R,\frac{1}{R}d(a_k,a))-\Psi(R,\frac{1}{R}d(b_k,b)),$$

for every $k\in\mathbb{N}$, which implies

$$\Psi(R,\frac{1}{R}d(a_k,a))+\Psi(R,\frac{1}{R}d(b_k,b))\leq \frac{1}{2}(\tau^2(a_k,b_k)+\tau^2(a,b))-\tau_0^2.$$

Consequently, $\lim_{k\to\infty}\Psi(R,\frac{1}{R}d(a_k,a))=0$ and $\lim_{k\to\infty}\Psi(R,\frac{1}{R}d(b_k,b))=0$. The property (iii) in Theorem 2.3 gives $\lim_{k\to\infty}d(a_k,a)=\lim_{k\to\infty}d(b_k,b)=0$. $\square$

The proof of the following lemma is similar to the previous proof.

Lemma 3.4. *Let C be a nonempty, closed and convex subset of a uniformly convex hyperbolic metric space (Ω,d). Let $\tau:C\to[0,+\infty)$ be a pseudo-type function, i.e., there exist bounded sequences $\{x_n\},\{y_n\}\in\Omega$ such that*

$$\tau(x)=\limsup_{n\to\infty}[d^2(x_n,x)+d^2(y_n,x)]^{1/2},\quad x\in C. \tag{3.4}$$

Then τ is continuous, and since Ω is hyperbolic, τ is convex, i.e., the subset $\{x\in C:\tau(x)\leq r\}$ is convex for any $r\geq 0$. Moreover, there exists a unique minimum point $a\in C$ such that

$$\tau(a)=\inf\{\tau(x);x\in C\}=\tau_0.$$

In addition, if $\{a_n\}$ is a minimizing sequence in C, i.e., $\lim_{n\to\infty}\tau(a_n)=\tau(a)$, then $\{a_n\}$ converges to a.

Theorem 3.5. *Let $(\Omega,d,\preceq)$ be a complete uniformly convex partially ordered hyperbolic metric space, and C be a nonempty, convex, bounded and closed subset of Ω. Let $T:C\times C\to C\times C$ be a continuous monotone nearly asymptotically nonexpansive mapping. If there exist $(x_0,x^0)\in C\times C$ such that*

$$(x_0,x^0)\precsim T(x_0,x^0), \tag{3.5}$$

then T has a fixed point $(x^*,y^*)\in C^2$, that is, $T(x^*,y^*)=(x^*,y^*)$.

Proof. From (3.5) and the monotonicity of T we have $T^n(x_0,x^0)\precsim T^{n+1}(x_0,x^0)$, for all $n\in\mathbb{N}$. By the convexity of C and proposition 3.2, we conclude that

$$C_\infty=\cap\Big([T^n(x_0,x^0),\to)\cap C\times C\Big)\neq\emptyset.$$

Let $(x,y)\in C_\infty$, then $T^{n+1}(x_0,x^0)\precsim T(x,y)$, for all $n\geq 1$. It follows that $T^n(C_\infty)\subset C_\infty$. Consider the type function $\tau:C_\infty\to[0,+\infty)$ generated by the sequences $\{T_1^n(x_0,x^0)\}$ and $\{T_2^n(x_0,x^0)\}$, with $T_i^n(x_0,x^0)=P_i\big(T^n(x_0,x^0)\big)$. Using Lemma 3.3, there exists a unique minimum point $(x^*,y^*)\in C_\infty$, that is, $\tau(x^*,y^*)=\inf\{\tau(x,y);(x,y)\in C_\infty\}$. In addition, we have $T^p(x^*,y^*)\in C_\infty$ for all $p\in\mathbb{N}$.

The same reasoning as in the proof of [2, Theorem 3.1] gives $T(x^*,y^*)=(x^*,y^*)$. $\square$

Note that the previous theorem is a generalization to $\Omega\times\Omega$ of [2, theorem 3-1]. In addition the next consequence prove the existence of a coupled fixed point of a continuous mixed monotone nearly asymptotically nonexpansive mapping.

Corollary 3.6. *Let $(\Omega,d,\preceq)$ be a complete uniformly convex partially ordered hyperbolic metric space, and C be a nonempty, convex, bounded and closed subset of*

Ω. Let $A : C\times C \to C$ be a continuous mixed monotone nearly asymptotically nonexpansive mapping. If there exist $x_0, x^0 \in C$ such that

$$x_0 \preceq A(x_0,x^0) \quad \text{and} \quad A(x^0,x_0) \preceq x^0, \tag{3.6}$$

then A has a coupled fixed point $(x^*,y^*)\in C^2$, that is, $A(x^*,y^*) = x^*$ and $A(y^*,x^*) = y^*$.

Proof. We shall prove that the hypotheses of the previous theorem are verified with the operator T defined by $T(x,y) = (A(x,y),A(y,x))$, for all $(x,y)\in C\times C$.

Let $(x,y) \precsim (u,v)$ in $C\times C$ i.e., $x\preceq u$ and $v\preceq y$. Using the mixed monotone property of A, we show easily that $T(x,y)\precsim T(u,v)$.

Now, for any two comparable elements (x,y) , (u,v) in $C\times C$, we have

$$\begin{aligned}
&\partial\big(T^n(x,y),T^n(u,v)\big)\\
&= \Big(d^2\big(A^n(x,y),A^n(u,v)\big) + d^2\big(A^n(y,x),A^n(v,u)\big)\Big)^{1/2}\\
&\le \Big(K_n^2\Big(\frac12 d(x,u)+\frac12 d(y,v)+a_n\Big)^2 + K_n^2\Big(\frac12 d(x,u)+\frac12 d(y,v)+a_n\Big)^2\Big)^{1/2}\\
&\le K_n\Big(2\Big(\frac12 d(x,u)+\frac12 d(y,v)+a_n\Big)^2\Big)^{1/2}\\
&\le K_n\Big(\big(d^2(x,u)+d^2(y,v)\big)^{1/2}+\sqrt2 a_n\Big).
\end{aligned}$$

Since A is continuous, $\sqrt2 a_n \to 0$ and $K_n\to 1$, when $n\to+\infty$. Thus, T is a continuous monotone nearly asymptotically nonexpansive mapping with respect to $\sqrt2 a_n$. By Theorem 3.5, T has a fixed point $(x^*,y^*)\in C\times C$, i.e., $T(x^*,y^*) = \big(A(x^*,y^*),A(y^*,x^*)\big) = (x^*,y^*)$, that is, $A(x^*,y^*) = x^*$ and $A(y^*,x^*) = y^*$. ☐

Theorem 3.7. *Assume that all hypotheses of corollary 3.6 hold. If*

$$x_0 \preceq A(x_0,x^0)\preceq A(x^0,x_0)\preceq x^0, \tag{3.7}$$

then A has a fixed point x^* in C with $x_0\preceq x^*\preceq x^0$.

Proof. We construct successively the iterative sequences

$$x_n = A^n(x_0,x^0) \quad \text{and} \quad x^n = A^n(x^0,x_0).$$

Since $x_0\preceq x^0$, according to the inequality (3.7) and the mixed monotonicity of A, it is easy to show by induction that for any $n\in\mathbb{N}$

$$x_0\preceq x_1\preceq\dots\preceq x_n\preceq\dots\preceq x^n\preceq\dots\preceq x^1\preceq x^0.$$

By the convexity of C and Proposition 2.4, we conclude that

$$C_0 = \cap_{n\ge0}[x_n,x^n]\ne\emptyset.$$

Let $x\in C_0$, i.e., for all $n\in\mathbb{N}$, $x_n\preceq x\preceq x^n$, then $A(x_n,x^n)\preceq A(x,x)\preceq A(x^n,x_n)$, and hence $x_{n+1}\preceq A(x,x)\preceq x^{n+1}$, for all $n\ge0$. It follows that $A^p(x,x)\in C_0$ for all $x\in C_0$ and for any $p\in\mathbb{N}$. Now we consider the pseudo-type function $\tau : C_0\to[0,+\infty)$ generated by the sequences $\{A^n(x_0,x^0)\}$ and $\{A^n(x^0,x_0)\}$.

Using Lemma 3.4, there exists a unique minimum point $x^*\in C_0$, that is, $\tau(x^*) = \inf\{\tau(x); x\in C_0\}$. By Definition 3.1 and the discrete Minkowski inequality, we have

$$\begin{aligned}
&\tau\left(A^p(x^*,x^*)\right)\\
&= \limsup_{n\to\infty}\Big[d^2\Big(A^n(x_0,x^0),A^p(x^*,x^*)\Big) + d^2\Big(A^n(x^0,x_0),A^p(x^*,x^*)\Big)\Big]^{1/2}
\end{aligned}$$

$$
\begin{aligned}
&\le \eta(A^p)\limsup_{n\to\infty}\Big[\Big(\frac{1}{2}d\Big(A^n(x_0,x^0),x^*\Big)+\frac{1}{2}d\Big(A^n(x^0,x_0),x^*\Big)+a_p\Big)^2\\
&\quad+\Big(\frac{1}{2}d\Big(A^n(x^0,x_0),x^*\Big)+\frac{1}{2}d\Big(A^n(x_0,x^0),x^*\Big)+a_p\Big)^2\Big]^{1/2}\\
&\le \eta(A^p)\limsup_{n\to\infty}\Big[d^2\Big(A^n(x^0,x_0),x^*\Big)+d^2\Big(A^n(x_0,x^0),x^*\Big)\Big]^{1/2}+\sqrt{2}\eta(A^p)a_p.
\end{aligned}
$$

Since $\tau(x^*)$ is minimum,

$$
\tau(x^*)\le\tau\left(A^p(x^*,x^*)\right)\le\eta(A^p)\tau(x^*)+\sqrt{2}\eta(A^p)a_p.
$$

Also, from Definition 3.1, we obtain

$$
\lim_{p\to\infty}\tau\left(A^p(x^*,x^*)\right)=\tau(x^*),
$$

which gives by Lemma 3.4 that $\lim_{p\to\infty}A^p(x^*,x^*)=x^*$. Using the continuity of A,

$$
\tau(x^*)=\tau\Big(A\Big(\lim_{p\to\infty}A^{p-1}(x^*,x^*),\lim_{p\to\infty}A^{p-1}(x^*,x^*)\Big)\Big)=\tau\Big(A(x^*,x^*)\Big).
$$

By the uniqueness of minimum point, we obtain $A(x^*,x^*)=x^*$. Thus, x^* is fixed point of A. Obviously $x^*\in C_0\subset[x_0,x^0]$, hence $x_0\preceq x^*\preceq x^0$. $\square$

As a consequence of the previous theorem, we obtain in the following the existence of a fixed point for a continuous nonincreasing nearly asymptotically nonexpansive mapping.

Corollary 3.8. *Let $(\Omega,d,\preceq)$ be a complete uniformly convex partially ordered hyperbolic metric space and C be a nonempty convex, bounded and closed subset of Ω. Let $T:C\to C$ be a continuous nonincreasing nearly asymptotically nonexpansive mapping. If there exists x_0,x^0 in C, such that $x_0\preceq T(x^0)\preceq T(x_0)\preceq x^0$, then T has a fixed point $x^*\in C$.*

Proof. Let $A:C\times C\to C$ be the mapping defined by $A(x,y)=T(y)$, for every $x,y\in C$. Then we have

$$
x_0\preceq T(x^0)=A(x_0,x^0)\preceq A(x^0,x_0)=T(x_0)\preceq x^0.
$$

Moreover A is a mixed monotone in C, since T is continuous and nearly asymptotically nonexpansive, then according to theorem 3.7, T has a fixed point $x^*\in C$. $\square$

Example 3.9. Let $C=\{x=(x_1,x_2,x_3,\dots)\in\ell^\infty : x_i\in[-2,2]\text{ for all } i\ge 1\}$. The order relation $x=(x_1,x_2,x_3,\dots)\preceq y=(y_1,y_2,y_3,\dots)$ is defined by $x_i\le y_i$ for all $i\ge1$. Let $\{k_n\}$ be a nonincreasing sequences, such that $k_n\ge1$ and $\lim_{n\to\infty}k_n=1$. Consider the mapping $T:C\times C\to C$ defined by

$$
T(x,y)=\Big(0,\frac{k_1}{2}(x_1-y_1),\frac{k_2}{2k_1}(x_2-y_2),\frac{k_3}{2k_2}(x_3-y_3),\dots\Big),\quad x,y\in C.
$$

Obviously, T is a continuous mixed monotone mapping, and

$$
T^n(x,y)=\Big(0,\dots,0,\frac{k_n}{2}(x_1-y_1),\frac{k_{n+1}}{2k_1}(x_2-y_2),\frac{k_{n+2}}{2k_2}(x_3-y_3),\dots\Big),
$$

for $n\ge2$, where 0 appears n times on the right side. Then, we have for all $x,y,u,v\in C$

$$
\|T^n(x,y)-T^n(u,v)\|\le k_n\Big(\frac{1}{2}\|x-u\|+\frac{1}{2}\|y-v\|+a_n\Big).
$$

For any sequences $\{a_n\} \subset [0, +\infty)$, satisfying $\lim_{n\to\infty} a_n = 0$. Thus, T is mixed monotone nearly asymptotically nonexpansive with $(0, 0, \dots, 0)$ fixed point.

4. Applications to integral equations

In this section, we will apply Theorem 3.7 to study the existence of solutions to the integral equation

$$x(t) = \int_0^1 a(t,s) f(s, x(s)) g(x(s)) ds. \tag{4.1}$$

Firstly, we make the following assumptions, with $I =]0, 1[$.

- (H1) (i) $a(\cdot,\cdot) \in L^2\big(I \times I\big)$, $\partial_1 a(\cdot,\cdot) \in L^2\big(I \times I\big)$ (where $\partial_1 a(t,s) = \frac{\partial}{\partial t} a(t,s)$), and there exist $\delta > 0$ such that $a(t,s) > \delta$ almost everywhere $t, s \in I$ (or, a.e. $t, s \in I$ for short).
  - (ii) $f : [0,1] \times \mathbb{R}_+ \to \mathbb{R}_+$ and $g : \mathbb{R}_+ \to \mathbb{R}_+$, with for every $s \in I$, $f(s, \cdot)$ is nondecreasing and g is nonincreasing.
- (H2) There exist $f_1, f_2 : \mathbb{R}_+ \to \mathbb{R}_+$, such that $f_1 \in L^2(I)$ and f_2 with $f_2 a$ and $f_2 \partial_1 a$ in $L^2(I^2)$. Furthermore, there exist $\eta > 0$, $M > 0$, such that
  - (i) For all $x \in [0, \eta]$, $f_1(s) x \le f(s, x)$, a.e. $s \in I$,
  - (ii) For all $x \in \mathbb{R}_+$, $f(s, x) \le f_2(s)$, a.e. $s \in I$,
  - (iii) There exist $\mu > 0$ and $\nu > 0$, such that $\mu \le g(x) \le \nu$, for all $x \in \mathbb{R}_+$.
- (H3) There exists $\lambda > 0$ and $f_3 : \mathbb{R}_+ \to \mathbb{R}_+$ with $f_3 a$ and $f_3 \partial_1 a$ in $L^2(I^2)$, such that
$$|f(s,x) - f(s,y)| \le f_3(s)|x - y| \quad \text{and} \quad |g(x) - g(y)| \le \lambda |x - y|,$$
for all $x, y \in \mathbb{R}_+$ and a.e. $s \in I$.

Next, we recall some notion and results about positive cone. Let E be a real Banach space. A closed convex set P in E is called a convex cone if the following conditions are satisfied.

1. If $x \in P$, then $\lambda x \in P$ for all $\lambda \in \mathbb{R}^+$,
2. If $x \in P$, and $-x \in P$, then $x = 0$.

A cone P induces a partial ordering $\preceq$ in E by $x \preceq y$ if and only if $y - x \in P$. We denote by $\mathring{P}$ the interior set of P. A cone P is called a solid cone if $\mathring{P} \ne \emptyset$.

Recall that $H^1(I)$ is the space of all equivalence classes of functions $x \in L^2(I)$ having derivatives in the sense of distributions $x' \in L^2(I)$. Set

$$P = \{x \in H^1(I) : x(s) \ge 0 \text{ a.e. } s \in I\}. \tag{4.2}$$

Thus, P is a positive cone in $H^1(I)$.

Theorem 4.1 ([5]). *The embedding $i : H^1(I) \to C([0,1])$ is compact.*

Using the above theorem, one can prove easily the following lemma.

Lemma 4.2. *The cone P defined in (4.2) is solid and $P = i^{-1}(K)$, where K is the positive cone of $C([0,1])$ given by $K = \{x \in C([0,1]) : x(s) \ge 0,\ s \in [0,1]\}$.*

Lemma 4.3. *Let P be the positive cone of $H^1(I)$ given by (4.2). Then, the topological interior of P is defined by*

$$\mathring{P} = \{x \in P : \exists \alpha > 0 \text{ with } x(s) \ge \alpha,\ \text{a.e. } s \in I\}.$$

Proof. Suppose that there exists $x \in \mathring{P}$ while for any $\varepsilon > 0$ there exists $\mathcal{J}_\varepsilon \subset I$ such that $\operatorname{meas} \mathcal{J}_\varepsilon > 0$ and $x(s) < \varepsilon$ for each $s \in \mathcal{J}_\varepsilon$ (here $\operatorname{meas} \mathcal{J}_\varepsilon$ is the Lebesgue measure of $\mathcal{J}_\varepsilon$). It follows that there exists $r > 0$ such that any $y \in H^1(I)$ satisfying $\|x - y\|_{H^1} < r$, we have $y \in P$.

Choose $r/2 < \delta < r$, then $z = x - \delta \in H^1(I)$ and $\|x - z\|_{H^1} = \delta < r$. However, $z(s) = x(s) - \delta < x(s) - \frac{r}{2} < 0$ for each $s \in \mathcal{J}_{\frac{r}{2}}$, which means that $z \notin P$, which is a contradiction.

Conversely, let $x \in P$ such that $x(s) \geq \alpha$ almost everywhere $s \in I$ for some $\alpha > 0$. Then, by theorem 4.1 we have $i(x) \geq \alpha > 0$. Hence $i(x) \in \mathring{K}$, which implies that $x \in i^{-1}\big(\mathring{K}\big) \subset \widehat{i^{-1}(K)}$. Thus, $x \in \mathring{P}$. □

Lemma 4.4. *Suppose $a(\cdot,\cdot) \in L^2\big(I \times I\big)$, $\partial_1 a(\cdot,\cdot) \in L^2\big(I \times I\big)$, and there exists $\delta > 0$ such that $a(t,s) > \delta$ a.e. $t, s \in I$. Then, the operator $A : H^1(I) \to H^1(I)$ defined by*

$$A(x)(t) = \int_0^1 a(t,s)x(s)ds,$$

is linear, compact and strongly positive.

Proof. It is clear that A is linear. Let $x \in H^1(I)$, from the inequality of Holder and from Fubini's theorem we prove that $\int_0^1 a(t,s)x(s)ds$ and $\int_0^1 \partial_1 a(t,s)x(s)ds$ are in $L^2(I)$.

And also we apply Fubini's theorem and integration by parts on the integral

$$\int_0^1 \Big(\int_0^1 a(t,s)x(s)ds \Big) \varphi'(t)dt, \quad \varphi \in C_0^\infty(I).$$

With $C_0^\infty(I)$ is the class of test functions. Moreover from the definition of the derivative in the sense of distributions, we obtain

$$\int_0^1 \partial_1 a(t,s)x(s)ds = \Big(\int_0^1 a(t,s)x(s)ds \Big)'. \tag{4.3}$$

Therefore, $A\big(H^1(I)\big) \subset H^1(I)$.

Let A_1 and A_2 defined from $L^2(I)$ to $L^2(I)$ by

$$A_1(x) = \int_0^1 a(\cdot,s)x(s)ds, \quad A_2(x) = \int_0^1 \partial_1 a(\cdot,s)x(s)ds\,.$$

from Holder's inequality and Fubini's theorem it follows that A_1, A_2 are continuous; thus there exist $M_1, M_2 \geq 0$ such that $\|A_1(x)\|_{L^2(I)} \leq M_1 \|x\|_{L^2(I)}$ and $\|A_2(x)\|_{L^2(I)} \leq M_2 \|x\|_{L^2(I)}$. Hence

$$\begin{aligned}
\|A(x)\|_{H^1(I)} &= \big(\|A_1(x)\|^2_{L^2(I)} + \|A_2(x)\|^2_{L^2(I)}\big)^{1/2} \\
&\leq \big(M_1^2 + M_2^2\big)^{1/2} \|x\|_{L^2(I)} \\
&\leq \big(M_1^2 + M_2^2\big)^{1/2} \|x\|_{H^1(I)}.
\end{aligned}$$

Therefore A is continuous.

For proving that A is compact, let $\{x_n\}$ be a bounded sequence in $H^1(I)$. It follows that $\{x_n\}$ is bounded in $L^2(I)$. Hence, by [9, p. 188], we obtain that

$A(x) = \int_0^1 a(\cdot,s)x(s)ds$ and $B(x) = \int_0^1 \partial_1 a(\cdot,s)x(s)ds$ are compact operators in $L^2(I)$. Therefore, there exist a subsequence $\{x_k\}$ and $x, y \in L^2(I)$ such that

$$\int_0^1 a(\cdot,s)x_k(s)ds \to x, \quad \int_0^1 \partial_1 a(\cdot,s)x_k(s)ds \to y \quad \text{in } L^2(I)(\text{ as } k\to\infty).$$

We shall prove that $x \in H^1(I)$ and $x' = y$. Let $\varphi \in C_0^\infty(I)$. Then

$$\int_0^1 \Big(\int_0^1 a(t,s)x_k(s)ds\Big)\varphi'(t)dt \to \int_0^1 x(t)\varphi'(t)dt,$$
$$\int_0^1 \Big(\int_0^1 \partial_1 a(t,s)x_k(s)ds\Big)\varphi(t)dt \to \int_0^1 y(t)\varphi(t)dt,$$

as $k\to\infty$. By equality (4.3), we obtain

$$\int_0^1 \Big(\int_0^1 \partial_1 a(t,s)x_k(s)ds\Big)\varphi(t)dt = -\int_0^1 \Big(\int_0^1 a(t,s)x_k(s)ds\Big)\varphi'(t)dt.$$

Thus

$$\int_0^1 x(t)\varphi'(t)dt = -\int_0^1 y(t)\varphi(t)dt,$$

which implies that $x' = y$. Also, we have

$$\int_0^1 a(\cdot,s)x_k(s)ds \to x \quad \text{and} \quad \Big(\int_0^1 a(\cdot,s)x_k(s)ds\Big)' \to x'$$

as $k\to\infty$. This means that $\int_0^1 a(\cdot,s)x_k(s)ds \to x$ in $H^1(I)$ (as $k\to\infty$). Thus A is a compact operator from $H^1(I)$ to $H^1(I)$.

Finally, we show that A is strongly positive. Let $x \in P\setminus\{0\}$, i.e., there exists $J\subset I$ with $\big(measJ > 0\big)$ such that $x(s) > 0$ for each $s\in J$. By theorem 4.1 $i(x)(s) > 0$ a.e. $s \in J$. Then there exists $s_0 \in J$ such that $i(x)(s_0) = x(s_0) > 0$, since $i(x)$ is continuous. Thus there exists $\varepsilon > 0$ and $\alpha > 0$ such that $i(x)(s) \ge \alpha$, for all $]s_0-\varepsilon, s_0+\varepsilon[$. Since $a(t,s) > \delta$ a.e. $t,s\in I$, $a(t,s)i(x)(s) > \alpha\delta$ a.e. $(t,s) \in I\times]s_0-\varepsilon, s_0+\varepsilon[$. Therefore, $A(x)(t) = \int_0^1 a(t,s)x(s)ds = \int_0^1 a(t,s)i(x)(s)ds \ge 2\varepsilon\alpha\delta > 0$ a.e. $t\in I$. Thus, Lemma 4.3 gives $A(x) \in \mathring{P}$, and consequently A is strongly positive. □

Let us now define the operators

$$L_1(x)(t) = \mu\int_0^1 a(t,s)f_1(s)x(s)ds \quad\text{and}\quad L_2(x)(t) = \nu\int_0^1 a(t,s)f_2(s)x(s)ds.$$

Theorem 4.5. *Assume* (H1)–(H3) *and*

(H4)
$$r(L_1) \ge 1 \quad \textit{and} \quad r\big(L_2\big) \le 1,$$

where $r(L_i) = \lim_{n\to\infty}\|L_i^n\|^{\frac1n}$ $(i = 1,2)$ *is the spectral radius of the linear operator* L_i*. Then equation* (4.1) *has a positive solution* $x^* \in P$.

Proof. First the cone P defined in (4.2) induces a partial ordering $\preceq$ in $H^1(I)$ by, $x \preceq y$ if and only if $y - x\in P$, and since $H^1(I)$ is a Hilbert space, we have that $H^1(I)$ is a complete uniformly convex partially ordered hyperbolic metric space.

From (H1) and Lemma 4.4, we obtain that L_i is a linear compact and strongly positive operator from $H^1(I)$ to $H^1(I)$, for $i = 1,2$. Moreover, by (H4) there exist $m_1\le 1$ and $m_2 \ge 1$ such that $r(m_iL_i) = 1$. Thus, by Krein-Rutman's theorem, there exist $x_i \in \mathring{P}$ verifying $m_i L_i(x_i) = x_i$. It follows that $x_1 \preceq L_1(x_1)$ and $L_2(x_2) \preceq x_2$.

Using Theorem 4.1, there exists $\alpha > 0$ such that $\alpha x_1 \preceq \eta$, and by Lemma 4.3, there exists $\beta > 0$ such that $1 \preceq \beta x_2$. Let $x_0 = \alpha x_1$, $x^0 = \beta x_2$. Then, by (H2) it is easy to show that

$$x_0 \preceq L_1(x_0) \preceq A(x_0, x^0), \quad A(x^0, x_0) \preceq L_2(x^0) \preceq x^0.$$

With

$$A(x,y)(t) = \int_0^1 a(t,s) f(s, x(s)) g(y(s))\, ds. \tag{4.4}$$

We impose on ν and λ the conditions

$$\begin{aligned} 2\nu^2\big(\|f_3.a\|^2_{L^2(I^2)} + \|f_3.\partial_1 a\|^2_{L^2(I^2)}\big) &\le \frac{1}{4}, \\ 2\lambda^2\big(\|f_2.a\|^2_{L^2(I^2)} + \|f_2.\partial_1 a\|^2_{L^2(I^2)}\big) &\le \frac{1}{4}. \end{aligned} \tag{4.5}$$

Let $C = \{x \in P : \|x\|_{H^1} \le \rho\}$, such that

$$\rho = \sup\big\{\|x_0\|_{H^1} : \|x^0\|_{H^1},\ \nu\Big(\|f_2.a\|^2_{L^2(I^2)} + \|f_2.\partial_1 a\|^2_{L^2(I^2)}\Big)^{1/2}\big\}.$$

Now, we shall prove that A defined in 4.4 is a well-defined operator from $C \times C$ to C.

Recall that if $x \in L^2(I)$ then $\int_0^1 a(\cdot, s)x(s)\, ds \in L^2(I)$ and $\big(\int_0^1 a(\cdot,s)x(s)\, ds\big)' \in L^2(I)$. Since for all $x, y \in C$, we have $f(s, x(s))g(y(s)) \le \nu f_2(s)$ a.e. $s \in I$, therefore $f(., x(.))g(y(.))$ in $L^2(I)$, then $A(x, y) \in H^1(I)$. Moreover, f and g are non-negative functions, and $a(t, s) > \delta$ a.e. $t, s \in I$, hence $A(x, y) \in P$. On the other hand, letting $x, y \in C$,

$$\begin{aligned} \|A(x,y)\|^2_{H^1} &= \int_0^1 \Big(\int_0^1 a(t,s) f(s,x(s)) g(y(s))\, ds\Big)^2 dt \\ &\quad + \int_0^1 \Big(\int_0^1 \partial_1 a(t,s) f(s,x(s)) g(y(s))\, ds\Big)^2 dt \\ &\le \nu^2 \int_0^1 \Big(\int_0^1 a(t,s) f_2(s)\, ds\Big)^2 dt + \nu^2 \int_0^1 \Big(\int_0^1 \partial_1 a(t,s) f_2(s)\, ds\Big)^2 dt \\ &\le \nu^2 \Big(\|f_2 a\|^2_{L^2(I^2)} + \|f_2 \partial_1 a\|^2_{L^2(I^2)}\Big) \le \rho^2. \end{aligned}$$

Thus $A(C \times C) \subset C$. And from the monotonicity of f and g, it is easy to show that A is a mixed monotone operator.

Next, for all $x, y, u, v \in C$, we have

$$\begin{aligned} &\|A(x,y) - A(u,v)\|^2_{H^1} \\ &= \int_0^1 \Big(\int_0^1 a(t,s)\Big(f(s,x(s))g(y(s)) - f(s,u(s))g(v(s))\Big)\, ds\Big)^2 dt \\ &\quad + \int_0^1 \Big(\int_0^1 \partial_1 a(t,s)\Big(f(s,x(s))g(y(s)) - f(s,u(s))g(v(s))\Big)\, ds\Big)^2 dt \\ &\le \int_0^1 \Big(\int_0^1 a(t,s) f(s,x(s))\Big(g(y(s)) - g(v(s))\Big)\, ds \\ &\quad + \int_0^1 a(t,s) g(v(s))\Big(f(s,x(s)) - f(s,u(s))\Big)\, ds\Big)^2 dt \end{aligned}$$

$$
\begin{aligned}
&+ \int_0^1 \Big(\int_0^1 \partial_1 a(t,s) f(s,x(s)) \Big(g(y(s)) - g(v(s)) \Big)\, ds \\
&+ \int_0^1 \partial_1 a(t,s) g(v(s) \Big(f(s,x(s)) - f(s,u(s)) \Big)\, ds \Big)^2 dt \\
&\le 2 \int_0^1 \Big(\int_0^1 a(t,s) f_2(s) \lambda \big| y(s) - v(s) \big|\, ds \Big)^2 dt \\
&+ 2 \int_0^1 \Big(\int_0^1 \nu a(t,s) f_3(s) \big| x(s) - u(s) \big|\, ds \Big)^2 dt \\
&+ 2 \int_0^1 \Big(\int_0^1 \partial_1 a(t,s) f_2(s) \lambda \big| y(s) - v(s) \big|\, ds \Big)^2 dt \\
&+ 2 \int_0^1 \Big(\int_0^1 \nu \partial_1 a(t,s) f_3(s) \big| x(s) - u(s) \big|\, ds \Big)^2 dt \\
&\le 2\lambda^2 \|y-v\|_{L^2}^2 \|f_2.a\|_{L^2(I^2)}^2 + 2\nu^2 \|x-u\|_{L^2}^2 \|f_3 a\|_{L^2(I^2)}^2 \\
&+ 2\lambda^2 \|y-v\|_{L^2}^2 \|f_2 \partial_1 a\|_{L^2(I^2)}^2 + 2\nu^2 \|x-u\|_{L^2}^2 \|f_3 .\partial_1 a\|_{L^2(I^2)}^2 \\
&\le 2\lambda^2 \|y-v\|_{H^1}^2 \Big(\|f_2 a\|_{L^2(I^2)}^2 + \|f_2 \partial_1 a\|_{L^2(I^2)}^2 \Big) \\
&+ 2\nu^2 \|x-u\|_{H^1}^2 \Big(\|f_3 a\|_{L^2(I^2)}^2 + \|f_3 \partial_1 a\|_{L^2(I^2)}^2 \Big). \\
&\le \Big(\frac{1}{2} \|x-u\|_{H^1} \Big)^2 + \Big(\frac{1}{2} \|y-v\|_{H^1} \Big)^2.
\end{aligned}
$$

Th last inequality above follows from (4.5). Hence, for all $x, y, u, v \in C$

$$
\|A(x,y) - A(u,v)\|_{H^1} \le \frac{1}{2} \|x-u\|_{H^1} + \frac{1}{2} \|y-v\|_{H^1}.
$$

It follows that A is continuous. By induction we show that

$$
\|A^n(x,y) - A^n(u,v)\|_{H^1} \le \frac{1}{2} \|x-u\|_{H^1} + \frac{1}{2} \|y-v\|_{H^1}
$$

for all comparable elements $(x,y), (u,v) \in C \times C$, and each $n \in \mathbb{N}^*$. Consequently, all the hypotheses of theorem 3.7 are satisfied for the operator $A : C \times C \to C$. Therefore, there exist $x^* \in P$, such that

$$
x^*(t) = \int_0^1 a(t,s) f(s, x^*(s)) g(x^*(s)) ds.
$$

□

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