Electronic Journal of Differential Equations, Vol. 2022 (2022), No. 68, pp. 1–12.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu

SOLUTION ESTIMATES AND STABILITY TESTS FOR NONLINEAR DELAY INTEGRO-DIFFERENTIAL EQUATIONS

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Abstract. In this article, we examine various qualitative features of solutions of a nonlinear delay integro-differential equation. We prove three new theorems which include sufficient conditions on asymptotic stability (AS), integrability, and boundedness of solutions, using a suitable Lyapunov-Krasovskii functional. We present examples that show applications of our results.

1. Introduction

According to the literature, Volterra's work [39] on elasticity was a starting point of theory on delay integro-differential equations (DIDEs). It was found that for some substances, the magnetic or electric polarization depends not only on the electromagnetic field at that moment, but also on the electromagnetic state of the matter at earlier instants. This and other scientific and engineering problems been modeled with DIDEs. For example, population dynamics, biological applications, genetics, noise term phenomenon, competition between tumor cells and immune system, artificial neural networks. and RLC circuits have been modeled as IDEs in [4, 6, 5, 10, 18, 19, 20, 22, 25, 39, 42].

In the previous five decades, qualitative properties of solutions of first order IDEs and functional DEs have been discussed, see for example the references in this article. However, there are only a few works on second order IDEs, see [1, 7, 9, 15, 16, 29, 30, 33, 45].

In this work, we consider the second order DIDE

$$\begin{aligned} &\ddot{x} + \sum_{i=1}^{m} f_i(t, x, \dot{x}) + \sum_{i=1}^{n} g_i(x(t-\tau_i)) + g(x, \dot{x}) + h(x) \\ &= p(t, x, \dot{x}) + \sum_{i=1}^{l} \int_{t-\tau_i}^{t} C_i(t, s) q_i(s, \dot{x}(s))\, \mathrm{d}s\,. \end{aligned} \tag{1.1}$$

2020 *Mathematics Subject Classification.* 34D05, 34K20, 45J05.

Key words and phrases. Lyapunov- Krasovskii functions; delay integro-differential equation; qualitative properties.

Submitted August 1, 2022. Published October 5, 2022.

As a next step, we transform (1.1) into the system

$$
\begin{aligned}
\frac{\mathrm{d}x}{\mathrm{d}t} &= y,\\
\frac{\mathrm{d}y}{\mathrm{d}t} &= -\sum_{i=1}^{m} f_i(t,x,y) - \sum_{i=1}^{n} g_i(x) - g(x,y) - h(x)\\
&\quad + \sum_{i=1}^{n}\int_{t-\tau_i}^{t} g_i'(x(s))y(s)\,\mathrm{d}s + p(t,x,y) + \sum_{i=1}^{l}\int_{t-\tau_i}^{t} C_i(t,s)q_i(s,y(s))\,\mathrm{d}s,
\end{aligned}
\tag{1.2}
$$

where $x \in \mathbb{R}$, $t \in [-\tau,\infty)$, $\tau_i > 0$ are constant delays, $\tau = \max\{\tau_1,\dots,\tau_n\}$, $l \le n$, l, $m,n \in \mathbb{N}$, $f_i, p \in C(\mathbb{R}^+ \times \mathbb{R}^2,\mathbb{R})$, $f_i(t,x,0) = 0$, $g \in C(\mathbb{R}^2,\mathbb{R})$, $g(x,0) = 0$, $g_i \in C^1(\mathbb{R},\mathbb{R})$, $g_i(0) = 0$, $h \in C(\mathbb{R},\mathbb{R})$, $h(0) = 0$, $C_i \in C([-\tau,\infty) \times [-\tau,\infty),\mathbb{R})$, $q_i \in C([-\tau,\infty) \times \mathbb{R},\mathbb{R})$ and $q_i(s,0) = 0$, $(i = 1,\dots,l)$. This continuity condition allows the existence of solutions to (1.1). In addition, through this paper, it is assumed the existence and continuity of the derivatives $g_i'(x) = \frac{dg_i}{\mathrm{d}x}$, $i = 1,2,\dots,n$. Throughout this article x and y denote $x(t)$ and $y(t)$, respectively.

It is seen that nonlinear system (1.2) has multiple kernels and delays. In particular, the mathematical models given as (1.1) and its modified version are useful for researchers working on ecology problems, population dynamics, artificial neural networks, and so forth.

Berezansky et al. [9] studied the following qualitative properties of solutions to second order functional differential equations (FDEs): existence of solutions, oscillation and non-oscillation, exponential stability, and instability. These equations include delay differential equations, integro-differential equations and equations with distributed delay. In particular, Berezansky et al. [9] considered the following linear FDEs with variable delays:

$$
\ddot{x}(t) + \sum_{i=1}^{2m} p_i(t)x(t-\tau_i(t)) = f(t),
$$

$$
\ddot{x}(t) + \sum_{i=1}^{2m} p_i(t)x(t-\tau_i(t)) + \sum_{j=1}^{n} q_j(t)x(t-\theta_j(t)) = f(t),
$$

$$
\ddot{x}(t) + \sum_{i=1}^{m} a_i(t)x(t-\tau_i(t)) - \sum_{i=1}^{m} b_i(t)x(t-\theta_i(t)) = f(t).
$$

Next we outline the contributions of this article. To the best of our information, the movements of orbits to (1.1) have not investigated in the literature; therefore, we present a novel work. Second order DIDEs with multiple kernels and delays have many applications in engineering [10, 18, 19, 20, 22, 23, 25], but fundamental properties of their solutions are rarely investigated. Therefore, investigating second order DIDEs is also a desirable feature of our work Finally, the results of this article have suitable conditions for applications, because of functional w defined by (2.1) below. The techniques and results here are different from those in [9].

The rest of this article is arranged as follows: Section 2 presents two theorems about stability and integrability results. Section 3 includes a numerical example as applications of the stability and integrability results in Section 2. Section 4 includes Theorem 4.1 which addresses the boundedness of solutions. Section 5 includes a numerical example as an application of the boundedness result of Section 5. Section 6 presents the conclusions from this article.

2. Stability and integrability results

We use the following assumptions for proving the results of this article.

(A1)
$$\begin{gathered} g_i(0)=0, \frac{g_i(x)}{x} \ge b_i, \quad x \neq 0, \\ h(0)=0, \frac{h(x)}{x} \ge h_0, \quad x \neq 0, \\ |g_i'(x)| \le \alpha_i \quad \text{for all } x \in \mathbb{R}, \end{gathered}$$

where, $b_i > 0$, $h_0 > 0$, $\alpha_i > 0$, $b_i, h_0, \alpha_i \in \mathbb{R}$, for $i = 1, 2, \dots, n$;

(A2)
$$\begin{gathered} f_i(t,x,0)=0, y f_i(t,x,y) \ge f_{i0} y^2, \quad y \neq 0,\ i=1,2,\dots,m, \\ g(x,0)=0, y g(x,y) \ge g_0 y^2, y \neq 0, \quad \text{for all } x, y \in \mathbb{R}, \\ |q_i(s,y(s))| \le r_i |y(s)|, \quad |C_i(t,s)| \le d_i, \end{gathered}$$

where $r_i > 0$, $d_i > 0$, $r_i, q_i \in \mathbb{R}$ for $i = 1, 2, \dots, l$; and the positive constants $f_{i0}, g_0, \alpha_i, d_i, r_i$ satisfy

$$\sum_{i=1}^{m} f_{i0} + g_0 - 2^{-1}\tau\Big(\sum_{i=1}^{n}\alpha_i + \sum_{i=1}^{l}(\alpha_i + 2d_i r_i) + \sum_{i=l+1}^{n}\alpha_i\Big) \ge \sigma,$$

where $\sigma > 0$, $\sigma \in \mathbb{R}$;

(A3)
$$\begin{gathered} |p(t,x,y)| \le |p_0(t)||y|, \quad \text{for all } t \in \mathbb{R}^+,\ x, y \in \mathbb{R}, \\ \int_0^\infty |p_0(t)| \,\mathrm{d}t < \infty. \end{gathered}$$

Theorem 2.1. *If* (A1) *and* (A2) *hold and* $p(t,x,y) \equiv 0$*, then the trivial solution of* (1.2) *is asymptotically stable.*

Proof. As an auxiliary tool to prove this theorem, we define the Lyapunov-Krasovskii functional

$$\begin{aligned} W(\cdot) &= W(t, x_t, y_t) \\ &= 2\sum_{i=1}^{n}\int_0^x g_i(s)\,\mathrm{d}s + 2\int_0^x h(s)\,\mathrm{d}s + y^2 \\ &\quad + \sum_{i=1}^{n}\gamma_i \int_{-\tau_i}^{0}\int_{t+s}^{t} y^2(\theta)\,\mathrm{d}\theta\,\mathrm{d}s, \end{aligned} \tag{2.1}$$

where $\gamma_1, \dots, \gamma_n$ are positive constants to be determined later. We have

$$\begin{aligned} W(t,x_t,y_t) = 2\int_0^x \frac{g_1(s)}{s} s\,\mathrm{d}s + \dots + 2\int_0^x \frac{g_n(s)}{s} s\,\mathrm{d}s + 2\int_0^x \frac{h(s)}{s} s\,\mathrm{d}s + y^2 \\ + \sum_{i=1}^{n}\gamma_i \int_{-\tau_i}^{0}\int_{t+s}^{t} y^2(\theta)\,\mathrm{d}\theta\,\mathrm{d}s. \end{aligned}$$

Using condition (A1), we obtain

$$W(t,x_t,y_t) \geq (b_1+b_2+\ldots+b_n+h_0)x^2+y^2. \tag{2.2}$$

The derivative of $W(t,x_t,y_t)$ along the trajectories of (1.2) gives

$$\begin{aligned}
W'(\cdot) &= 2g_1(x)y+2g_2(x)y+\ldots+2g_n(x)y+2h(x)y+2yy'\\
&\quad+\sum_{i=1}^{n}(\gamma_i\tau_i)y^2-\sum_{i=1}^{n}(\gamma_i\int_{t-\tau_i}^{t}y^2(s))\,\mathrm{d}s\\
&= 2y\sum_{i=1}^{n}g_i(x)+2y\Big[-\sum_{i=1}^{m}f_i(t,x,y)-\sum_{i=1}^{n}g_i(x)-g(x,y)-h(x)\Big]\\
&\quad+2h(x)y+2y\sum_{i=1}^{n}\int_{t-\tau_i}^{t}g_i'(x(s))y(s)\,\mathrm{d}s\\
&\quad+2y\sum_{i=1}^{l}\int_{t-\tau_i}^{t}C_i(t,s)q_i(s,y(s))\,\mathrm{d}s+\sum_{i=1}^{n}(\gamma_i\tau_i)y^2\\
&\quad-\sum_{i=1}^{n}(\gamma_i\int_{t-\tau_i}^{t}y^2(s))\,\mathrm{d}s\\
&= -2y\sum_{i=1}^{m}f_i(t,x,y)-2yg(x,y)+2y\sum_{i=1}^{n}\int_{t-\tau_i}^{t}g_i'(x(s))y(s)\,\mathrm{d}s\\
&\quad+2y\sum_{i=1}^{l}\int_{t-\tau_i}^{t}C_i(t,s)q_i(s,y(s))\,\mathrm{d}s+\sum_{i=1}^{n}(\gamma_i\tau_i)y^2\\
&\quad-\sum_{i=1}^{n}\gamma_i\int_{t-\tau_i}^{t}y^2(s)\,\mathrm{d}s.
\end{aligned}$$

Using conditions (A1), (A2) and doing elementary calculations, we obtain

$$\begin{aligned}
2y\int_{t-\tau_i}^{t}g_i'(x(s))y(s)\,\mathrm{d}s &\leq 2|y(t)|\int_{t-\tau_i}^{t}|g_i'(x(s))||y(s)|\,\mathrm{d}s\\
&\leq \alpha_i\int_{t-\tau_i}^{t}(y^2(t)+y^2(s))\,\mathrm{d}s\\
&= \alpha_i\tau_i y^2+\alpha_i\int_{t-\tau_i}^{t}y^2(s)\,\mathrm{d}s,
\end{aligned}$$

for $i=1,2,\ldots n$; and

$$\begin{aligned}
2y\int_{t-\tau_i}^{t}C_i(t,s)q_i(s,y(s))\,\mathrm{d}s &\leq 2|y|\int_{t-\tau_i}^{t}|C_i(t,s)||q_i(s,y(s))|\,\mathrm{d}s\\
&\leq 2d_ir_i|y|\int_{t-\tau_i}^{t}|y(s)|\,\mathrm{d}s\\
&\leq d_ir_i\int_{t-\tau_i}^{t}(y^2(t)+y^2(s))\,\mathrm{d}s\\
&= d_ir_i\tau_iy^2+d_ir_i\int_{t-\tau_i}^{t}y^2(s)\,\mathrm{d}s,
\end{aligned}$$

for $i = 1, 2, \dots, l$. Hence,

$$
\begin{aligned}
W'(\cdot) \le -2y\sum_{i=1}^{m} f_i(t,x,y) - 2yg(x,y) + \Big[\sum_{i=1}^{n}(\alpha_i\tau_i) + \sum_{i=1}^{l}(d_i r_i \tau_i) + \sum_{i=1}^{n}(\gamma_i\tau_i)\Big]y^2 \\
+ (\alpha_1 + d_1 r_1 - \gamma_1)\int_{t-\tau_1}^{t} y^2(s)\,\mathrm{d}s + (\alpha_2 + d_2 r_2 - \gamma_2)\int_{t-\tau_2}^{t} y^2(s)\,\mathrm{d}s \\
+ (\alpha_l + d_l r_l - \gamma_l)\int_{t-\tau_l}^{t} y^2(s)\,\mathrm{d}s + \dots + (\alpha_n - \gamma_n)\int_{t-\tau_n}^{t} y^2(s)\,\mathrm{d}s.
\end{aligned}
$$

Let $\gamma_1 = \alpha_1 + d_1 r_1$, $\gamma_2 = \alpha_2 + d_2 r_2$, $\dots$, $\gamma_l = \alpha_l + d_l r_l$, $\dots$, $\gamma_n = \alpha_n$. Then using condition (A2), we obtain

$$
\begin{aligned}
W'(\cdot) &\le -2y\sum_{i=1}^{m} f_i(t,x,y) - 2yg(x,y) \\
&\quad + \Big[\sum_{i=1}^{n}(\alpha_i\tau_i) + \sum_{i=1}^{l}(d_i r_i \tau_i) + \sum_{i=1}^{l}(\alpha_i + d_i r_i)\,\tau_i + \sum_{i=l+1}^{n}(\alpha_i\tau_i)\Big]y^2 \\
&\le -2y^2\sum_{i=1}^{m} f_{i0} - 2g_0 y^2 \\
&\quad + \Big[\sum_{i=1}^{n}(\alpha_i\tau_i) + \sum_{i=1}^{l}(d_i r_i \tau_i) + \sum_{i=1}^{l}(\alpha_i + d_i r_i)\,\tau_i + \sum_{i=l+1}^{n}(\alpha_i\tau_i)\Big]y^2.
\end{aligned}
$$

Let $\tau = \max\{\tau_1, \tau_2, \dots, \tau_n\}$. Then

$$
\begin{aligned}
W'(\cdot) &\le -2\Big[\sum_{i=1}^{m} f_{i0} + g_0 - 2^{-1}\tau\Big(\sum_{i=1}^{n}\alpha_i + \sum_{i=1}^{l}(\alpha_i + 2d_i r_i) + \sum_{i=l+1}^{n}\alpha_i\Big)\Big]y^2 \\
&\le -\sigma y^2 < 0, \quad y \ne 0,
\end{aligned}
$$

provided that

$$
\tau < \frac{2\sum_{i=1}^{m} f_{i0} + 2g_0}{\sum_{i=1}^{n}\alpha_i + \sum_{i=1}^{l}(\alpha_i + 2d_i r_i) + \sum_{i=l+1}^{n}\alpha_i} = \sigma.
$$

In addition, it can be shown that the only invariant set in $W'(\cdot) = 0$ is $\{0, 0\}$ (see, Hale [18]). Then, the trivial solution of system of(1.2) is asymptotically stable. $\square$

Theorem 2.2. *If* (A1), (A2) *hold and* $p(t,x,y) \equiv 0$*, then the squares of the derivative of solutions* $x(t)$ *of* (1.2) *are Lebesgue integrable.*

Proof. From Theorem 2.1, we have that

$$
W'(t, x_t, y_t) \le -\sigma y^2 < 0, \quad y \ne 0.
$$

Integrating we obtain

$$
W(t, x_t, y_t) - W(t_0, \phi(t_0), \psi(t_0)) \le -\sigma\int_{t_0}^{t} y^2(s)\,\mathrm{d}s.
$$

Then

$$
\int_{t_0}^{\infty} y^2(s)\,\mathrm{d}s \le \sigma^{-1}W(t_0, \phi(t_0), \psi(t_0)) - \sigma^{-1}W(t, x_t, y_t) \le K,
$$

where $K = \sigma^{-1}W(t_0, \phi(t_0), \psi(t_0))$. $\square$

3. Numerical applications of stability and integrability results

Example 3.1. Let $p(\cdot) \equiv 0$. As a particular case of (1.1), we consider the nonlinear second order DIDE with multiple kernels and delays,

$$\begin{aligned} &\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + \Big(t + x^2 + (\frac{\mathrm{d}x}{\mathrm{d}t})^2 + 25\Big)\frac{\mathrm{d}x}{\mathrm{d}t} + 17\frac{\mathrm{d}x}{\mathrm{d}t} + 2x \\ &+ x^7 + 2x(t-4^{-1}) + 2x(t-8^{-1}) \\ &= \int_{t-\frac{1}{4}}^{t} \frac{1}{1+t^4+s^2} \frac{x'(s)}{[1+(x'(s))^2][1+\exp(s^2)]}\,\mathrm{d}s \\ &\quad + \int_{t-\frac{1}{8}}^{t} \frac{1}{1+t^6+s^2} \frac{x'(s)}{[1+(x'(s))^2][1+\exp(s^4)]}\,\mathrm{d}s. \end{aligned} \tag{3.1}$$

This equation can be transformed into the system

$$\begin{aligned} \frac{\mathrm{d}x}{\mathrm{d}t} &= y, \\ \frac{\mathrm{d}y}{\mathrm{d}t} &= -\big(t + x^2 + y^2 + 25\big)y - 17y - 6x - x^7 \\ &\quad + 2\int_{t-\frac{1}{4}}^{t} y(s)\,\mathrm{d}s + 2\int_{t-\frac{1}{8}}^{t} y(s)\,\mathrm{d}s \\ &= \int_{t-\frac{1}{4}}^{t} \frac{1}{1+t^4+s^2} \frac{y(s)}{[1+y^2(s)][1+\exp(s^2)]}\,\mathrm{d}s \\ &\quad + \int_{t-\frac{1}{8}}^{t} \frac{1}{1+t^6+s^2} \frac{y(s)}{[1+y^2(s)][1+\exp(s^4)]}\,\mathrm{d}s, t \ge \frac{1}{8}. \end{aligned} \tag{3.2}$$

Hence, comparing (1.2) and (3.2) gives the relations

$$f_1(t,x,y) = \big(t + x^2 + y^2 + 25\big)\, y, \quad f_1(t,x,0) = 0,$$
$$f_1(t,x,y)y = \big(t + x^2 + y^2 + 25\big)\, y^2 \ge 25y^2, \quad f_{10} = 25, \quad y \ne 0;$$
$$g(x,y) = 17y, \quad g(x,0) = 0,$$
$$g(x,y)y = 17y^2 \ge 16y^2, \quad g_0 = 16, y \ne 0;$$
$$g_1(x) = 2x, \quad g_1(0) = 0,$$
$$\frac{g_1(x)}{x} = 2 > 1 = b_1, \quad x \ne 0,$$
$$g_1'(x) = 2, \quad |g_1'(x)| = 2 < 3 = \alpha_1;$$
$$g_2(x) = 2x, \quad g_2(0) = 0,$$
$$\frac{g_2(x)}{x} = 2 > 1 = b_2, \quad x \ne 0;$$
$$g_2'(x) = 2, \quad |g_2'(x)| = 2 < 3 = \alpha_2;$$
$$h(x) = 2x + x^7, \quad h(0) = 0,$$
$$\frac{h(x)}{x} = 2 + x^6 \ge 2 = h_0, \quad x \ne 0,$$
$$\int_{t-\tau_1}^{t} C_1(t,s)q_1(s,y(s))\,\mathrm{d}s = \int_{t-\frac{1}{4}}^{t} \frac{1}{1+t^4+s^2} \frac{y(s)}{[1+y^2(s)][1+\exp(s^2)]}\,\mathrm{d}s,$$

$$
\begin{gathered}
C_1(t,s)=\frac{1}{1+t^4+s^2},\quad |C_1(t,s)|=\frac{1}{1+t^4+s^2}\le 1=d_1,\\
q_1(s,y(s))=\frac{y(s)}{[1+y^2(s)][1+\exp(s^2)]},\\
|q_1(s,y(s))|=\frac{|y(s)|}{[1+y^2(s)][1+\exp(s^2)]}\le |y(s)|,\quad r_1=1;\\
\int_{t-\tau_2}^t C_2(t,s)q_2(s,y(s))\,\mathrm{d}s=\int_{t-\frac{1}{8}}^t \frac{1}{1+t^6+s^2}\frac{y(s)}{[1+y^2(s)][1+\exp(s^4)]}\,\mathrm{d}s,\\
C_2(t,s)=\frac{1}{1+t^6+s^2},|C_2(t,s)|=\frac{1}{1+t^6+s^2}\le 1=d_2,\\
q_2(s,y(s))=\frac{y(s)}{[1+y^2(s)][1+\exp(s^4)]},\\
|q_2(s,y(s))|=\frac{|y(s)|}{[1+y^2(s)][1+\exp(s^4)]}\le |y(s)|,\quad r_2=1;\\
\tau=\max\{4^{-1},8^{-1}\}=4^{-1};
\end{gathered}
$$

$$
\begin{aligned}
[f_{10}+g_0-2^{-1}\tau(\alpha_1+\alpha_2+d_1r_1+d_2r_2)]&=[25+16-8^{-1}(3+3+1+1)]\\
&=40>39=\sigma>0.
\end{aligned}
$$

Hence, when $p(t,x,y)\equiv 0$, the conditions of Theorems 2.1 and 2.2 are fulfilled. Therefore their results hold.

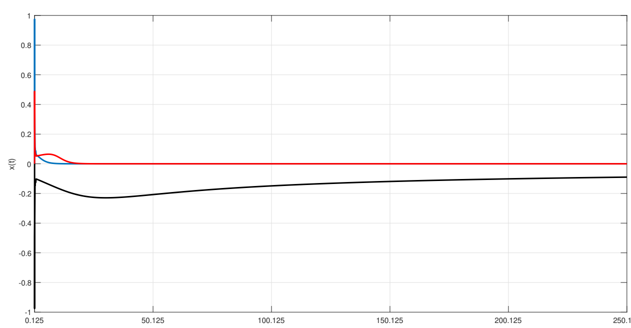

FIGURE 1. Trajectories of the solution x of (3.1), which shows the asymptotic stability and integrability of the solutions depending on various values of initial function.

4. Boundedness result

Theorem 4.1. *If* (A1)–(A3) *hold, then the solution* $(x(t),y(t))$ *of system* (1.2) *are bounded.*

Proof. From (A1)–(A3) and some calculations, we obtain

$$
\begin{aligned}
W'(t,x_t,y_t)&\le 2yp(t,x,y)\\
&\le 2|y|\,|p(t,x,y)|
\end{aligned}
$$

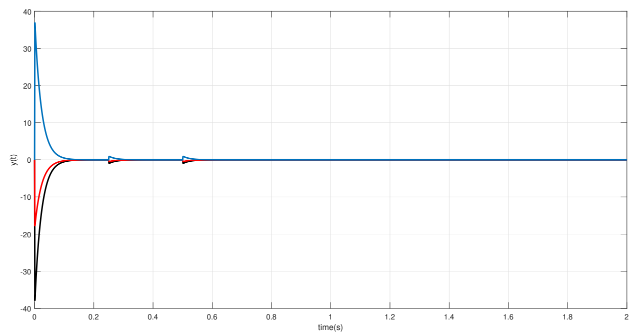

FIGURE 2. Trajectories of the solution y of (3.1), which shows the asymptotic stability and integrability of the solutions depending on various values of initial function.

$$\begin{aligned} &\le 2|p_0(t)|y^2 \\ &\le 2|p_0(t)|W(t, x_t, y_t). \end{aligned}$$

Hence,

$$\frac{W'(t, x_t, y_t)}{W(t, x_t, y_t)} \le 2|p_0(t)|.$$

Integrating this inequality, we obtain

$$\begin{aligned} W(t, x_t, y_t) &\le W(t_0, \phi_{t_0}, \psi_{t_0}) \exp(2 \int_{t_0}^{t} |p_0(s)| \, \mathrm{d}s) \\ &\le W(t_0, \phi_{t_0}, \psi_{t_0}) \exp(2 \int_{t_0}^{\infty} |p_0(s)| \, \mathrm{d}s) \\ &\le M_0. \end{aligned}$$

Hence, in view of (2.2) and the last inequality above, we derive that

$$(b_1 + b_2 + \dots + b_n + h_0)x^2 + y^2 \le W(t, x_t, y_t) \le M_0.$$

Then $(b_1 + b_2 + \dots + b_n + h_0)x^2 + y^2 \le M_0$. Thus,

$$|x(t)| \le \Big(\frac{M_0}{\sum_{i=1}^{n} b_i + h_0}\Big)^{1/2}, \quad |y(t)| \le \sqrt{M_0} \quad \text{for all } t \ge t_0.$$

These inequalities verify that the solution $(x(t), y(t))$ of (1.2) are bounded. $\square$

5. Numerical application of the bounded result

Example 5.1. Let $p(\cdot) \neq 0$. As a particular case of (1.1), we consider the nonlinear second order DIDE with multiple kernels and delays,

$$\begin{aligned}
&\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} + \Big(t + x^2 + (\frac{\mathrm{d}x}{\mathrm{d}t})^2 + 25\Big)\frac{\mathrm{d}x}{\mathrm{d}t} + 17\frac{\mathrm{d}x}{\mathrm{d}t} + 2x \\
&+ x^7 + 2x(t - 4^{-1}) + 2x(t - 8^{-1}) \\
&= \int_{t-\frac{1}{4}}^{t} \frac{1}{1 + t^4 + s^2}\,\frac{x'(s)}{[1 + (x'(s))^2][1 + \exp(s^2)]}\,\mathrm{d}s \\
&\quad + \int_{t-\frac{1}{8}}^{t} \frac{1}{1 + t^6 + s^2}\,\frac{x'(s)}{[1 + (x'(s))^2][1 + \exp(s^4)]}\,\mathrm{d}s \\
&\quad + \frac{x' \exp(t)}{1 + \exp(2t) + \exp(x^2 + (x')^2)}.
\end{aligned} \tag{5.1}$$

This equation can be transformed into the system

$$\begin{aligned}
\frac{\mathrm{d}x}{\mathrm{d}t} &= y, \\
\frac{\mathrm{d}y}{\mathrm{d}t} &= -\big(t + x^2 + y^2 + 25\big)\, y - 17y - 6x - x^7 \\
&\quad + 2\int_{t-\frac{1}{4}}^{t} y(s)\,\mathrm{d}s + 2\int_{t-\frac{1}{8}}^{t} y(s)\,\mathrm{d}s \\
&= \int_{t-\frac{1}{4}}^{t} \frac{1}{1 + t^4 + s^2}\,\frac{y(s)}{[1 + y^2(s)][1 + \exp(s^2)]}\,\mathrm{d}s \\
&\quad + \int_{t-\frac{1}{8}}^{t} \frac{1}{1 + t^6 + s^2}\,\frac{y(s)}{[1 + y^2(s)][1 + \exp(s^4)]}\,\mathrm{d}s \\
&\quad + \frac{y \exp(t)}{1 + \exp(2t) + \exp(x^2 + y^2)}.
\end{aligned} \tag{5.2}$$

All the data in Example 3.1 hold for (5.2). We need only to consider the function $p(t, x, y)$. Hence, we derive that

$$\begin{gathered}
|p(t, x, y)| = \frac{|y| \exp(t)}{1 + \exp(2t) + \exp(x^2 + y^2)} \leq \frac{|y| \exp(t)}{1 + \exp(2t)} \leq |p_0(t)||y|, \\
|p_0(t)| = \frac{\exp(t)}{1 + \exp(2t)}, \\
\int_0^\infty |p_0(t)|\,\mathrm{d}t = \int_0^\infty \frac{\exp(t)}{1 + \exp(2t)}\,\mathrm{d}t = \frac{\pi}{4} < \infty.
\end{gathered}$$

Thus, the conditions of Theorem 4.1 hold. Then, all solutions of (5.2) are bounded.

6. Conclusion

In this article, a class of nonlinear DIDEs of second order with multiple kernels and delays has been considered. Three new results have been given on the behaviors of solutions of considered equations. New numerical applications related to the obtained results have been given. The aim of this paper is to do the new contributions to the theory of DIDEs of higher order.

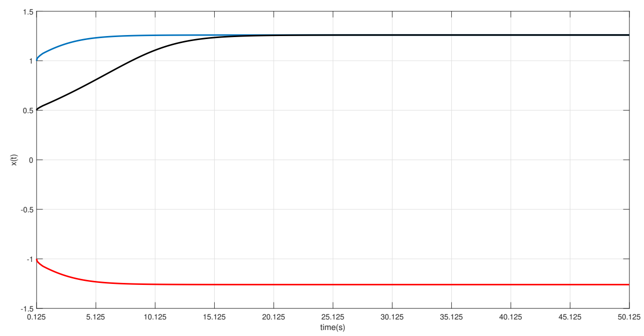

FIGURE 3. Trajectories of the solution $x(t)$ of (5.1), which shows the boundedness of the solutions depending on various values of initial function.

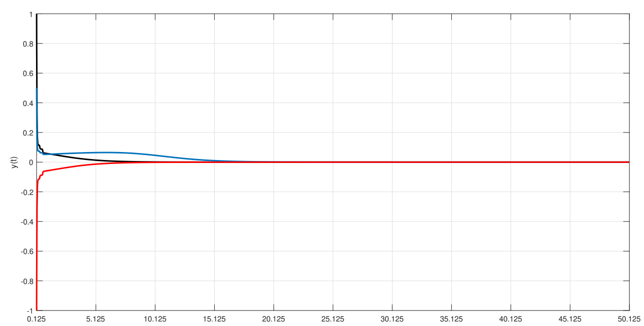

FIGURE 4. Trajectories of the solution $y(t)$ of (5.1), which shows the boundedness of the solutions depending on various values of initial function.

Acknowledgements. The authors would like to thank the anonymous referee and the handling editor for many useful comments and suggestions, leading to a substantial improvement of the presentation of this article.

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