A KAM THEOREM FOR HIGHER DIMENSIONAL REVERSIBLE NONLINEAR SCHRÖDINGER EQUATIONS

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ABSTRACT. In this article we prove an abstract Kolmogorov-Arnold-Moser (KAM) theorem for infinite dimensional reversible systems. Using this theorem, we obtain the existence of quasi-periodic solutions for a class of reversible (non-Hamiltonian) coupled nonlinear Schrödinger systems on a d-torus.

1. INTRODUCTION AND MAIN RESULT

Among various techniques for studying the existence of quasi-periodic solutions of nonlinear partial differential equations (PDEs), the Kolmogorov-Arnold-Moser (KAM) theory is one of the most powerful tools. Kuksin [12] and Wayne [18] first developed a Newtonian scheme to investigate quasi-periodic solutions of Hamiltonian PDEs in one dimensional space. The general idea is that Hamiltonian function is thought of as a normal form plus a real analytic perturbation, then constructing an infinite symplectic transformation sequences to make the perturbation smaller and smaller and construct a converged local normal form. The normal form is helpful to understand the dynamics around the quasi–periodic solutions, for example, one sees the linear stability and zero Lyapunov exponents.

The feasibility of the KAM method in one dimensional Hamiltonian PDEs, however, depends crucially on the second Melnikov condition. Because of the multiple eigenvalues of the linear operator, such condition is not naturally available in higher dimensional case, and the KAM method is in general not easy to apply. In 1998, Bourgain [3] first made a breakthrough. He used multi-scale analysis method to avoid the cumbersome second Melnikov condition and thus obtained small-amplitude quasi-periodic solutions of two dimensional nonlinear Schrödinger equations (NLS). Later, he improved his method and studied quasi-periodic solutions of NLS and nonlinear wave equations in any dimensional space. Following the idea and methods in [3], many works [4, 2, 17] have been done.

There are strong hopes to develop the KAM theory for higher dimensional PDEs, because multi-scale analysis can not help us to understand the dynamics around quasi-periodic solutions. Geng and You [9] first built a KAM theorem for higher dimensional beam equation and nonlocal smooth NLS. They used momentum conservation condition which means the nonlinearity is independent of spatial variable x to avoid the difficulty of multiple eigenvalues. In 2010, Eliasson and Kuksin

[6] studied a class of higher dimensional NLS with convolution type potential and nonlinearity containing spatial variable x. They used the block diagonal normal form structure to deal with multiple eigenvalues of linear operator. Also, they introduced Lipschitz domain property of perturbation to handle infinitely many resonances at each KAM step. By developing Töplitz-Lipschitz property of perturbation and constructing appropriate tangential sites on $\mathbb{Z}^2$, Geng, Xu and You [8] obtained quasi-periodic solutions of two-dimensional completely resonant NLS. Later on, Geng and You [11] simplified the proof of [6] via momentum conservation condition. Procesi and Procesi [15] extended the result in [8] to the d-dimensional case by a very ingenious choice of tangential sites. See [5, 16, 19, 10] for further studies.

Recently, KAM theory for Hamiltonian PDEs has been generalized to reversible ones in one dimensional space [20, 1]. In fact, reversible PDEs are a class of physically important PDEs as well as Hamiltonian ones. For example, the following coupled NLS system arising from nonlinear optics (see [13]):

$$iu_t - \Delta u + M_\xi u + \partial_{\bar{u}} G_1(|u|^2, |v|^2) = 0,$$
$$iv_t - \Delta v + M_{\widetilde{\xi}} v + \partial_{\bar{v}} G_2(|u|^2, |v|^2) = 0, \quad x \in \mathbb{T}^d := \mathbb{R}^d/2\pi\mathbb{Z}^d, \tag{1.1}$$

where M_ξ and $M_{\widetilde{\xi}}$ are real Fourier multiplier, $G_i = o(|u|^3 + |v|^3)$, $i = 1, 2$ are real analytic functions near $(u, v) = (0, 0)$. When $G_1 = G_2$, equation (1.1) is not only reversible (with respect to the involution $S_0(u(x), v(x)) = (\bar{u}(-x), \bar{v}(-x))$) but also Hamiltonian. When $G_1 \neq G_2$, equation (1.1) is no longer Hamiltonian but still reversible. This motives us to develop reversible KAM theory for equation (1.1).

As in the Hamiltonian case, the major difficulty in constructing KAM scheme for (1.1) is to deal with infinitely many resonances. In this article, by introducing the class of Töplitz-Lipschitz vector fields (inspired by [6, 8, 1]) and momentum conservation condition, the difficulty can be overcome. Töplitz-Lipschitz vector field plays the most essential role and it reduces infinitely many resonances to only finitely many ones. Momentum conservation condition can simplify the proof. We mention that Töplitz-Lipschitz vector field introduced here is the generalization of Töplitz-Lipschitz functions in [8].

Following [6], we could study more general equation (1.1) with nonlinearities G_i containing the spatial variable x explicitly, but the proof would be more complicated since we have to deal with block diagonal normal form. This article is working on NLS with the external parameters, and the completely resonant case (i.e. no Fourier multiplier in equation (1.1)) is in [7]. As in the Hamiltonian case (see [8, 15]), the construction of Birkhoff normal form will be a new challenge.

1.1. **Main result.** Let

$$\mathcal{I}_1 = \{i^{(1)}, i^{(2)}, \ldots, i^{(n)}\} \subset \mathbb{Z}^d,$$
$$\mathcal{I}_2 = \{\tilde{i}^{(1)}, \tilde{i}^{(2)}, \ldots, \tilde{i}^{(m)}\} \subset \mathbb{Z}^d$$

be two sets of distinguished sites of Fourier modes. For technical convenience, we suppose $0 \in \mathcal{I}_1 \cap \mathcal{I}_2$. We denote by λ_i, $i \in \mathbb{Z}^d$ (resp. $\widetilde{\lambda}_i$) the eigenvalues of $-\Delta + M_\xi$ (resp. $-\Delta + M_{\widetilde{\xi}}$) under periodic boundary conditions:

$$\lambda_{i^{(j)}} = \omega_j = |i^{(j)}|^2 + \xi_j, \quad 1 \le j \le n,$$
$$\lambda_i = |i|^2, \quad i \notin \mathcal{I}_1,$$

$$\widetilde{\lambda}_{\tilde{i}^{(j)}} = \widetilde{\omega}_j = |\tilde{i}^{(j)}|^2 + \widetilde{\xi}_j, \quad 1 \le j \le m,$$

$$\widetilde{\lambda}_i = |i|^2, \quad i \notin \mathcal{I}_2,$$

and the corresponding eigenfunctions $\phi_i(x) = \frac{1}{\sqrt{(2\pi)^d}} e^{i\langle i,x \rangle}$. Assume the parameters $(\xi, \tilde{\xi}) \in \mathcal{O} := [0,1]^n \times [0,1]^m \subset \mathbb{R}^{n+m}$. Then we have the following main result.

Theorem 1.1. *For any $0 < \gamma \ll 1$, there exists a Cantor subset $\mathcal{O}_\gamma \subset \mathcal{O}$ with $\mathrm{meas}(\mathcal{O} \setminus \mathcal{O}_\gamma) = O(\gamma^{1/4})$, such that for any $(\xi, \tilde{\xi}) \in \mathcal{O}_\gamma$, equation (1.1) with reversible perturbation*

$$G_1 = |u|^4|v|^2 + o((|u|^2 + |v|^2)^3) \ne G_2 = |u|^2|v|^2 + o((|u|^2 + |v|^2)^2)$$

possesses a small amplitude quasi-periodic solution of the form

$$\begin{aligned}
u(t,x) &= \sum_{i \in \mathbb{Z}^d} u_i(\acute{\omega}_1 t, \ldots, \acute{\omega}_n t)\phi_i(x), \\
v(t,x) &= \sum_{i \in \mathbb{Z}^d} v_i(\acute{\tilde{\omega}}_1 t, \ldots, \acute{\tilde{\omega}}_m t)\phi_i(x),
\end{aligned} \tag{1.2}$$

where $u_i : \mathbb{T}^n \to \mathbb{R}$ (resp. $v_i : \mathbb{T}^m \to \mathbb{R}$) and $\acute{\omega}_1, \ldots, \acute{\omega}_n$ (resp. $\acute{\tilde{\omega}}_1, \ldots, \acute{\tilde{\omega}}_m$) are close to the unperturbed frequencies $\omega_1, \ldots, \omega_n$ (resp. $\tilde{\omega}_1, \ldots, \tilde{\omega}_m$). Moreover, the quasi-periodic solutions are real analytic and may not be linear stable.

Remark 1.2. In the following proof, we set the nonlinearities $G_1 = |u|^4|v|^2 + o((|u|^2 + |v|^2)^3)$ and $G_2 = |u|^2|v|^2 + o((|u|^2 + |v|^2)^2)$ to simplify notations. Our method also applies to the general cases $G_i = (|u|^2 + |v|^2)^{n_i} + o((|u|^2 + |v|^2)^{n_i})$, $n_i \in \mathbb{Z}$, $n_i \ge 2, i = 1, 2$.

We point out that the reversible coupled nonlinearities may lead to the linear instability of KAM tori. Note that because $G_1 \ne G_2$ in equation (1.1), the eigenvalues of

$$M_{j,\infty} = \begin{pmatrix} \Omega_{j,\infty} & A_{j,\infty} \\ \tilde{A}_{j,\infty} & \tilde{\Omega}_{j,\infty} \end{pmatrix}, \quad j \in \mathbb{Z}_1^d \cap \mathbb{Z}_2^d$$

after KAM iteration (see (5.16)) may not be real numbers. Thus we can not obtain the linear stability of quasi-periodic solution (1.2). But for the case of Hamiltonian coupled nonlinearities $G_1 = G_2$, $A_{j,\infty} = \tilde{A}_{j,\infty}$ (i.e., $M_{j,\infty}$ is symmetric), this means the eigenvalues of $M_{j,\infty}$ are real numbers and thus their result is linearly stable. For the single (non-coupled) reversible PDEs in [1], $M_{j,\infty}$ is the scalar normal frequency $\Omega_{j,\infty}$, thus their result is also linearly stable.

The rest of this article is organized as follows. In Section 2, we give the definitions of weighted norms for functions and vector fields. An abstract KAM theorem (Theorem 3.2) for infinite dimensional reversible systems is presented in Section 3. In Section 4, we use the KAM theorem to prove Theorem 1.1. The proof of Theorem 3.2 is given in Section 5. Some properties of reversible system and technical lemmas are listed in the Appendix.

2. Preliminaries

For the sake of completeness, we first introduce some definitions and notation. Let $\mathcal{I} \subset \mathbb{Z}^d$ be a finite subset and $\rho > 0$. We introduce the Banach space $\ell_{\mathcal{I}}^\rho$ of all

complex sequences $z = (z_j)_{j \in \mathbb{Z}^d \setminus \mathcal{I}}$ with

$$\|z\|_\rho = \sum_{j \in \mathbb{Z}^d \setminus \mathcal{I}} e^{|j|\rho} |z_j| < \infty,$$

where $|j| = \sqrt{|j_1|^2 + \cdots + |j_d|^2}$.

Given two subsets of $\mathbb{Z}^d$: $\mathcal{I}_1 = \{i^{(1)}, i^{(2)}, \ldots, i^{(n)}\}$ and $\mathcal{I}_2 = \{\tilde{i}^{(1)}, \tilde{i}^{(2)}, \ldots, \tilde{i}^{(m)}\}$, we denote $\mathbb{Z}_l^d := \mathbb{Z}^d \setminus \mathcal{I}_l$ and $\ell_l^\rho := \ell_{\mathcal{I}_l}^\rho$, $(l = 1, 2)$. We consider the phase space

$$\mathscr{P}_\rho := \mathbb{T}^n \times \mathbb{T}^m \times \mathbb{R}^n \times \mathbb{R}^m \times \ell_1^\rho \times \ell_2^\rho \times \ell_1^\rho \times \ell_2^\rho \ni y := (\theta, \varphi, I, J, z, w, \bar{z}, \bar{w}).$$

We introduce a complex neighborhood

$$D_\rho(r, s) = \big\{ y : |\operatorname{Im} \theta| < r, |\operatorname{Im} \varphi| < r, |I| < s, |J| < s,$$
$$\|z\|_\rho < s, \|w\|_\rho < s, \|\bar{z}\|_\rho < s, \|\bar{w}\|_\rho < s \big\}$$

of $\mathcal{T}_0^{n+m} := \mathbb{T}^n \times \mathbb{T}^m \times \{I = 0\} \times \{J = 0\} \times \{z = 0\} \times \{w = 0\} \times \{\bar{z} = 0\} \times \{\bar{w} = 0\}$, where $|\cdot|$ is the sup-norm.

Suppose $\mathcal{O} \subset \mathbb{R}^{n+m}$ is a compact parameter subset. A function $f : D_\rho(r, s) \times \mathcal{O} \to \mathbb{C}$ is real analytic in y and C_W^4 (i.e., C^4-smooth in the sense of Whitney) in $\zeta \in \mathcal{O}$ and has Taylor-Fourier series expansion

$$f(y; \zeta) = \sum_{\substack{k \in \mathbb{Z}^n, l \in \mathbb{N}^n, \alpha, \beta \in \mathbb{N}^{\mathbb{Z}_1^d}, \\ \tilde{k} \in \mathbb{Z}^m, \tilde{l} \in \mathbb{N}^m, \tilde{\alpha}, \tilde{\beta} \in \mathbb{N}^{\mathbb{Z}_2^d}}} f_{kl\alpha\beta, \tilde{k}\tilde{l}\tilde{\alpha}\tilde{\beta}}(\zeta) e^{i(\langle k, \theta \rangle + \langle \tilde{k}, \varphi \rangle)} I^l J^{\tilde{l}} z^\alpha \bar{z}^\beta w^{\tilde{\alpha}} \bar{w}^{\tilde{\beta}},$$

where $\langle k, \theta \rangle = \sum_{i=1}^n k_i \theta_i$, $I^l = \prod_{i=1}^n I_i^{l_i}$, and $z^\alpha \bar{z}^\beta = \prod_{j \in \mathbb{Z}_1^d} z_j^{\alpha_j} \bar{z}_j^{\beta_j}$. α, β have only finitely many nonzero components, and similarly for the other indexes. We define the weighted norm of f as

$$\|f\|_{D_\rho(r,s) \times \mathcal{O}} = \sup_{\substack{|z|_\rho < s, |\bar{z}|_\rho < s \\ |w|_\rho < s, |\bar{w}|_\rho < s}} \sum_{\substack{k, l, \alpha, \beta, \\ \tilde{k}, \tilde{l}, \tilde{\alpha}, \tilde{\beta}}} |f_{kl\alpha\beta, \tilde{k}\tilde{l}\tilde{\alpha}\tilde{\beta}}|_{\mathcal{O}} e^{(|k| + |\tilde{k}|)r} s^{(|l| + |\tilde{l}|)} |z^\alpha| |\bar{z}^\beta| |w^{\tilde{\alpha}}| |\bar{w}^{\tilde{\beta}}|.$$

where

$$|f_{kl\alpha\beta, \tilde{k}\tilde{l}\tilde{\alpha}\tilde{\beta}}|_{\mathcal{O}} = \sup_{\zeta \in \mathcal{O}} \sum_{0 \leq b \leq 4} |\partial_\zeta^b f_{kl\alpha\beta, \tilde{k}\tilde{l}\tilde{\alpha}\tilde{\beta}}|.$$

Let

$$z_j^\varrho = \begin{cases} z_j, & \varrho = +, \\ \bar{z}_j, & \varrho = -, \end{cases}$$

and similarly for $z^\varrho = (z_j^\varrho)_{j \in \mathbb{Z}_1^d}$, w_j^ϱ, and w^ϱ.

We consider a vector field $X(y), y \in D_\rho(r, s)$:

$$X(y) = X^{(\theta)}(y) \frac{\partial}{\partial \theta} + X^{(\varphi)}(y) \frac{\partial}{\partial \varphi} + X^{(I)}(y) \frac{\partial}{\partial I} + X^{(J)}(y) \frac{\partial}{\partial J}$$
$$+ X^{(z)}(y) \frac{\partial}{\partial z} + X^{(w)}(y) \frac{\partial}{\partial w} + X^{(\bar{z})}(y) \frac{\partial}{\partial \bar{z}} + X^{(\bar{w})}(y) \frac{\partial}{\partial \bar{w}} \qquad (2.1)$$
$$= \sum_{\mathsf{v} \in \mathscr{V}} X^{(\mathsf{v})}(y) \frac{\partial}{\partial \mathsf{v}},$$

where $\mathscr{V} = \{\theta_a, \varphi_b, z_i, w_j, \bar{z}_i, \bar{w}_j : a = 1, \ldots, n; b = 1, \ldots, m; i \in \mathbb{Z}_1^d; j \in \mathbb{Z}_2^d\}$. Suppose X is real analytic in y and depends C_W^4 smoothly on parameters $\zeta \in \mathcal{O}$,

we define the weighted norm of X as follows

$$\|X\|_{s;D_\rho(r,s)\times\mathcal{O}} = \|X^{(\theta)}\|_{D_\rho(r,s)\times\mathcal{O}} + \|X^{(\varphi)}\|_{D_\rho(r,s)\times\mathcal{O}}$$
$$+ \frac{1}{s}\|X^{(I)}\|_{D_\rho(r,s)\times\mathcal{O}} + \frac{1}{s}\|X^{(J)}\|_{D_\rho(r,s)\times\mathcal{O}}$$
$$+ \frac{1}{s}\sum_{\sigma=\pm}(\sum_{i\in\mathbb{Z}_1^d}\mathrm{e}^{|i|\rho}\|X^{(z_i^\sigma)}\|_{D_\rho(r,s)\times\mathcal{O}} + \sum_{i\in\mathbb{Z}_2^d}\mathrm{e}^{|i|\rho}\|X^{(w_i^\sigma)}\|_{D_\rho(r,s)\times\mathcal{O}}).$$

The norm of vector valued function $G : D_\rho(r,s)\times\mathcal{O}\to\mathbb{C}^n$, $n<\infty$, is defined as $\|G\|_{D_\rho(r,s)\times\mathcal{O}} = \sum_{b=1}^n\|G_b\|_{D_\rho(r,s)\times\mathcal{O}}$.

The Lie bracket of two vector fields X and Y is defined as $[X,Y] = YX - XY$.

3. A KAM THEOREM FOR INFINITE DIMENSIONAL REVERSIBLE SYSTEMS

In this section, we give an abstract KAM theorem for infinite dimensional reversible systems. The definition and properties of reversible system are listed in the appendix.

Given an involution $S : (\theta,\varphi,I,J,z,w,\bar{z},\bar{w}) \mapsto (-\theta,-\varphi,I,J,\bar{z},\bar{w},z,w)$. We consider a family of S-reversible vector fields

$$X^0(y;\zeta) = N(y;\zeta) + \mathcal{A}(y;\zeta),$$

where

$$N = \omega(\zeta)\frac{\partial}{\partial\theta} + \tilde{\omega}(\zeta)\frac{\partial}{\partial\varphi} + i\Omega(\zeta)z\frac{\partial}{\partial z} + i\widetilde{\Omega}(\zeta)w\frac{\partial}{\partial w} - i\Omega(\zeta)\bar{z}\frac{\partial}{\partial\bar{z}} - i\widetilde{\Omega}(\zeta)\bar{w}\frac{\partial}{\partial\bar{w}}$$

$$= \sum_{b=1}^n\omega_b(\zeta)\frac{\partial}{\partial\theta_b} + \sum_{b=1}^m\tilde{\omega}_b(\zeta)\frac{\partial}{\partial\varphi_b}$$

$$+ \sum_{j\in\mathbb{Z}_1^d}(i\Omega_j(\zeta)z_j\frac{\partial}{\partial z_j} - i\Omega_j(\zeta)\bar{z}_j\frac{\partial}{\partial\bar{z}_j}) \qquad (3.1)$$

$$+ \sum_{j\in\mathbb{Z}_2^d}(i\widetilde{\Omega}_j(\zeta)w_j\frac{\partial}{\partial w_j} - i\widetilde{\Omega}_j(\zeta)\bar{w}_j\frac{\partial}{\partial\bar{w}_j}),$$

and

$$\mathcal{A} = iA(\zeta)w\frac{\partial}{\partial z} + i\widetilde{A}(\zeta)z\frac{\partial}{\partial w} - iA(\zeta)\bar{w}\frac{\partial}{\partial\bar{z}} - i\widetilde{A}(\zeta)\bar{z}\frac{\partial}{\partial\bar{w}}$$

$$= \sum_{j\in\mathbb{Z}_1^d\cap\mathbb{Z}_2^d}(iA_j(\zeta)w_j\frac{\partial}{\partial z_j} + i\tilde{A}_j(\zeta)z_j\frac{\partial}{\partial w_j} - iA_j(\zeta)\bar{w}_j\frac{\partial}{\partial\bar{z}_j} - i\tilde{A}_j(\zeta)\bar{z}_j\frac{\partial}{\partial\bar{w}_j}), \qquad (3.2)$$

where $\omega_b,\tilde{\omega}_b,\Omega_j,\tilde{\Omega}_j,A_j,\tilde{A}_j\in\mathbb{R}$ and $A_j = 0, (j\in\mathbb{Z}_1^d\setminus\mathbb{Z}_2^d)$, $\tilde{A}_j = 0$ $(j\in\mathbb{Z}_2^d\setminus\mathbb{Z}_1^d)$.

For each $\zeta\in\mathcal{O}$, the motion equation governed by the vector field X^0 is

$$\dot{\theta} = \omega, \quad \dot{\varphi} = \tilde{\omega},$$

$$\dot{I} = 0, \quad \dot{J} = 0,$$

$$\dot{z}_j^\sigma = \sigma i\Omega_j z_j^\sigma, \quad \sigma = \pm, j\in\mathbb{Z}_1^d\setminus\mathbb{Z}_2^d,$$

$$\dot{w}_j^\sigma = \sigma i\tilde{\Omega}_j w_j^\sigma, \quad j\in\mathbb{Z}_2^d\setminus\mathbb{Z}_1^d,$$

$$\begin{pmatrix}\dot{z}_j^\sigma \\ \dot{w}_j^\sigma\end{pmatrix} = \sigma i\begin{pmatrix}\Omega_j & A_j \\ \tilde{A}_j & \tilde{\Omega}_j\end{pmatrix}\begin{pmatrix}z_j^\sigma \\ w_j^\sigma\end{pmatrix}, \quad j\in\mathbb{Z}_1^d\cap\mathbb{Z}_2^d.$$

Obviously, $\{(\theta + \omega t, \varphi + \tilde{\omega} t, 0, 0, 0, 0, 0, 0) : t \in \mathbb{R}\}$ forms an invariant torus of the above system.

We consider now the perturbed S-reversible vector field

$$X = X^0 + P = N + \mathcal{A} + P(y; \zeta). \tag{3.3}$$

We will prove that, for typical (in the sense of Lebesgue measure) $\zeta \in \mathcal{O}$, the vector fields (3.3) still admit invariant tori for sufficiently small P. For this purpose, we need the following six assumptions:

(A1) Non-degeneracy: The map $\zeta \mapsto (\omega(\zeta), \tilde{\omega}(\zeta))$ is a C_W^4 diffeomorphism between $\mathcal{O}$ and its image.

(A2) Asymptotics of normal frequencies:

$$\Omega_j = |j|^2 + \Omega_j^0, \quad j \in \mathbb{Z}_1^d, \quad \tilde{\Omega}_j = |j|^2 + \tilde{\Omega}_j^0, \quad j \in \mathbb{Z}_2^d, \tag{3.4}$$

where $\Omega_j^0, \tilde{\Omega}_j^0 \in C_W^4(\mathcal{O})$ with C_W^4-norm bounded by a small positive constant L.

(A3) Non-resonance conditions: We denote

$$M_j = \begin{pmatrix} \Omega_j & A_j \\ \tilde{A}_j & \tilde{\Omega}_j \end{pmatrix}, \quad j \in \mathbb{Z}_1^d \cap \mathbb{Z}_2^d.$$

Suppose $A_j, \tilde{A}_j \in C_W^4(\mathcal{O})$ and there exist $\gamma, \tau > 0$, such that

$$|\langle k, \omega \rangle + \langle \tilde{k}, \tilde{\omega} \rangle| \geq \frac{\gamma}{(|k| + |\tilde{k}|)^\tau}, \quad (k, \tilde{k}) \neq 0, \tag{3.5}$$

$$|\langle k, \omega \rangle + \langle \tilde{k}, \tilde{\omega} \rangle + \Omega_i| \geq \frac{\gamma}{(|k| + |\tilde{k}|)^\tau}, \quad i \in \mathbb{Z}_1^d \setminus \mathbb{Z}_2^d, \tag{3.6}$$

$$|\langle k, \omega \rangle + \langle \tilde{k}, \tilde{\omega} \rangle + \tilde{\Omega}_i| \geq \frac{\gamma}{(|k| + |\tilde{k}|)^\tau}, \quad i \in \mathbb{Z}_2^d \setminus \mathbb{Z}_1^d, \tag{3.7}$$

$$|\langle k, \omega \rangle + \langle \tilde{k}, \tilde{\omega} \rangle + \Omega_i \pm \Omega_j| \geq \frac{\gamma}{(|k| + |\tilde{k}|)^\tau}, \quad (k, \tilde{k}) \neq 0, i, j \in \mathbb{Z}_1^d \setminus \mathbb{Z}_2^d, \tag{3.8}$$

$$|\langle k, \omega \rangle + \langle \tilde{k}, \tilde{\omega} \rangle + \Omega_i \pm \tilde{\Omega}_j| \geq \frac{\gamma}{(|k| + |\tilde{k}|)^\tau}, \quad i \in \mathbb{Z}_1^d \setminus \mathbb{Z}_2^d, j \in \mathbb{Z}_2^d \setminus \mathbb{Z}_1^d, \tag{3.9}$$

$$|\langle k, \omega \rangle + \langle \tilde{k}, \tilde{\omega} \rangle + \tilde{\Omega}_i \pm \tilde{\Omega}_j| \geq \frac{\gamma}{(|k| + |\tilde{k}|)^\tau}, \quad (k, \tilde{k}) \neq 0, i, j \in \mathbb{Z}_2^d \setminus \mathbb{Z}_1^d, \tag{3.10}$$

$$|\det((\langle k, \omega \rangle + \langle \tilde{k}, \tilde{\omega} \rangle) I_2 \pm M_i)| \geq \frac{\gamma}{(|k| + |\tilde{k}|)^\tau}, \quad i \in \mathbb{Z}_1^d \cap \mathbb{Z}_2^d, \tag{3.11}$$

$$\begin{aligned} &|\det((\langle k, \omega \rangle + \langle \tilde{k}, \tilde{\omega} \rangle + \Omega_i) I_2 \pm M_j)| \\ &\geq \frac{\gamma}{(|k| + |\tilde{k}|)^\tau}, \quad (i, j) \text{ or } (j, i) \in (\mathbb{Z}_1^d \cap \mathbb{Z}_2^d) \times (\mathbb{Z}_1^d \setminus \mathbb{Z}_2^d), \end{aligned} \tag{3.12}$$

$$\begin{aligned} &|\det((\langle k, \omega \rangle + \langle \tilde{k}, \tilde{\omega} \rangle + \tilde{\Omega}_i) I_2 \pm M_j)| \\ &\geq \frac{\gamma}{(|k| + |\tilde{k}|)^\tau}, \quad (i, j) \text{ or } (j, i) \in (\mathbb{Z}_1^d \cap \mathbb{Z}_2^d) \times (\mathbb{Z}_2^d \setminus \mathbb{Z}_1^d), \end{aligned} \tag{3.13}$$

$$\begin{aligned} &|\det((\langle k, \omega \rangle + \langle \tilde{k}, \tilde{\omega} \rangle) I_4 + M_i \otimes I_2 \pm I_2 \otimes M_j^T)| \\ &\geq \frac{\gamma}{(|k| + |\tilde{k}|)^\tau}, \quad (k, \tilde{k}) \neq 0, \ i, j \in \mathbb{Z}_1^d \cap \mathbb{Z}_2^d. \end{aligned} \tag{3.14}$$

where I_b is $b \times b$ identity matrix, $\det(\cdot)$ is the determinant, $\otimes$ is the the tensor product, and $(\cdot)^T$ is the transpose of matrices.

(A4) Regularity: $\mathcal{A} + P$ is real analytic in y and C_W^4-smooth in ζ. Moreover, $\|\mathcal{A}\|_{s;D(r,s)\times\mathcal{O}} < 1$, $\varepsilon_0 := \|P\|_{s;D(r,s)\times\mathcal{O}} < \infty$.

(A5) Momentum conservation: The perturbation P satisfies $[P, \mathbb{M}_l] = 0$ ($l = 1, \ldots, d$), where

$$\mathbb{M}_l = \sum_{b=1}^{n} i_l^{(b)} \frac{\partial}{\partial \theta_b} + \sum_{b=1}^{m} \tilde{i}_l^{(b)} \frac{\partial}{\partial \varphi_b} + \sum_{\varrho=\pm} \sum_{j \in \mathbb{Z}_1^d} \varrho i j_l z_j^\varrho \frac{\partial}{\partial z_j^\varrho} + \sum_{\varrho=\pm} \sum_{j \in \mathbb{Z}_2^d} \varrho i j_l w_j^\varrho \frac{\partial}{\partial w_j^\varrho}.$$

(A6) Töplitz-Lipschitz property: Let

$$\Lambda = \sum_{\sigma=\pm} \sigma i \big(\sum_{j \in \mathbb{Z}_1^d} \Omega_j^0(\zeta) z_j^\sigma \frac{\partial}{\partial z_j^\sigma} + \sum_{j \in \mathbb{Z}_2^d} \widetilde{\Omega}_j^0(\zeta) w_j^\sigma \frac{\partial}{\partial w_j^\sigma} \big).$$

For fixed $i, j \in \mathbb{Z}^d$, $c \in \mathbb{Z}^d \setminus \{0\}$, the following limits exist and satisfy:

$$\| \lim_{t\to\infty} \frac{\partial P^{(x)}}{\partial u_{i+tc}^\sigma} \|_{s;D_\rho(r,s)\times\mathcal{O}} \le \varepsilon_0, \quad x = \theta_b, \varphi_b, I_b, J_b, u = z, w; \tag{3.15}$$

$$\| \lim_{t\to\infty} \frac{\partial(\Lambda + P)^{(u_{i+tc}^\sigma)}}{\partial u_{j\pm tc}^{\pm\sigma}} \|_{s;D_\rho(r,s)\times\mathcal{O}} \le \varepsilon_0 e^{-|i\mp j|\rho}, \quad u = z, w; \tag{3.16}$$

$$\| \lim_{t\to\infty} \frac{\partial(\mathcal{A} + P)^{(u_{i+tc}^\sigma)}}{\partial v_{j\pm tc}^{\pm\sigma}} \|_{s;D_\rho(r,s)\times\mathcal{O}} \le \varepsilon_0 e^{-|i\mp j|\rho}, \quad (u,v) = (z,w), (w,z). \tag{3.17}$$

Furthermore, there exists $K > 0$ such that when $|t| > K$, the following estimates hold.

$$\begin{aligned} &\| \frac{\partial P^{(x)}}{\partial u_{i+tc}^\sigma} - \lim_{t\to\infty} \frac{\partial P^{(x)}}{\partial u_{i+tc}^\sigma} \|_{s;D_\rho(r,s)\times\mathcal{O}} \\ &\le \varepsilon_0, \quad x = \theta_b, \varphi_b, I_b, J_b; u = z, w; \end{aligned} \tag{3.18}$$

$$\begin{aligned} &\| \frac{\partial(\Lambda + P)^{(u_{i+tc}^\sigma)}}{\partial u_{j\pm tc}^{\pm\sigma}} - \lim_{t\to\infty} \frac{\partial(\Lambda + P)^{(u_{i+tc}^\sigma)}}{\partial u_{j\pm tc}^{\pm\sigma}} \|_{s;D_\rho(r,s)\times\mathcal{O}} \\ &\le \frac{\varepsilon_0}{|t|} e^{-|i\mp j|\rho}, \quad u = z, w; \end{aligned} \tag{3.19}$$

$$\begin{aligned} &\| \frac{\partial(\mathcal{A} + P)^{(u_{i+tc}^\sigma)}}{\partial v_{j\pm tc}^{\pm\sigma}} - \lim_{t\to\infty} \frac{\partial(\mathcal{A} + P)^{(u_{i+tc}^\sigma)}}{\partial v_{j\pm tc}^{\pm\sigma}} \|_{s;D_\rho(r,s)\times\mathcal{O}} \\ &\le \frac{\varepsilon_0}{|t|} e^{-|i\mp j|\rho}, \quad (u,v) = (z,w), (w,z). \end{aligned} \tag{3.20}$$

Remark 3.1. In (A6), the conditions (3.16)-(3.17) and (3.19)-(3.20) are the most important for measure estimates. The role played by the conditions (3.15) and (3.18) is to preserve Töplitz-Lipschitz property after the KAM iteration (see Lemmas 5.5 and 5.6 below).

Now we state our KAM theorem.

Theorem 3.2. *Suppose the S-reversible vector field $X = N + \mathcal{A} + P$ in (3.3) satisfies* (A1)–(A6). *$\gamma > 0$ is small enough. Then there exists a positive ε depending only on n, m, L, K, τ, r, s and ρ such that if $\|P\|_{s;D(r,s)\times\mathcal{O}} \le \varepsilon$, the following holds: There exist (1) a Cantor subset $\mathcal{O}_\gamma \subset \mathcal{O}$ with Lebesgue measure $\mathrm{meas}(\mathcal{O} \setminus \mathcal{O}_\gamma) = O(\gamma^{1/4})$; (2) a C_W^4-smooth family of real analytic torus embeddings*

$$\Psi : \mathbb{T}^{n+m} \times \mathcal{O}_\gamma \to D_\rho(r,s)$$

which is $\frac{\varepsilon}{\gamma^{20}}$-close to the trivial embedding $\Psi_0 : \mathbb{T}^{n+m} \times \mathcal{O} \to \mathcal{T}_0^{n+m}$;

(3) a C_W^4-smooth map $\phi : \mathcal{O}_\gamma \to \mathbb{R}^{n+m}$ which is ε−close to the unperturbed frequency $(\omega, \tilde{\omega})$ such that for every $\zeta \in \mathcal{O}_\gamma$ and $(\theta, \varphi) \in \mathbb{T}^{n+m}$ the curve $t \mapsto \Psi((\theta, \varphi) + \phi(\zeta)t; \zeta)$ is a quasi-periodic solution of the equation governed by the vector field $X = N + \mathcal{A} + P$.

4. APPLICATION TO THE COUPLED NLS

4.1. **Lattice form of equation** (1.1). Let $\mathcal{I}_1 = \{i^{(1)}, i^{(2)}, \ldots, i^{(n)}\} \subset \mathbb{Z}^d$ and $\mathcal{I}_2 = \{\tilde{i}^{(1)}, \tilde{i}^{(2)}, \ldots, \tilde{i}^{(m)}\} \subset \mathbb{Z}^d$ and $0 \in \mathcal{I}_1 \cap \mathcal{I}_2$. Under periodic boundary conditions, we denote the eigenvalues of $-\Delta + M_\xi$ and $-\Delta + M_{\tilde{\xi}}$ by λ_i, $i \in \mathbb{Z}^d$ and $\tilde{\lambda}_i$, $i \in \mathbb{Z}^d$, respectively, satisfying

$$\omega_j = \lambda_{i^{(j)}} = |i^{(j)}|^2 + \xi_j, \quad 1 \le j \le n,$$
$$\Omega_i = \lambda_i = |i|^2, \quad i \notin \mathcal{I}_1,$$
$$\tilde{\omega}_j = \tilde{\lambda}_{\tilde{i}^{(j)}} = |\tilde{i}^{(j)}|^2 + \tilde{\xi}_j, \quad 1 \le j \le m,$$
$$\tilde{\Omega}_i = \tilde{\lambda}_i = |i|^2, \quad i \notin \mathcal{I}_2,$$

and the corresponding eigenfunctions $\phi_i(x) = (2\pi)^{-d/2} e^{i\langle i, x\rangle}$.

Without loss of generality, we consider (1.1) when $G_1 = |u|^4 |v|^2$, $G_2 = |u|^2 |v|^2$ since the higher order terms of nonlinearities will not make any difference.

Let $u(t, x) = \sum_{h \in \mathbb{Z}^d} q_h(t) \phi_h(x)$, $v(t, x) = \sum_{h \in \mathbb{Z}^d} p_h(t) \phi_h(x)$, then we obtain the equivalent lattice reversible equations

$$\dot{q}_h = i\lambda_h q_h + Q^{(q_h)}(q, p, \bar{q}, \bar{p}),$$
$$\dot{p}_h = i\tilde{\lambda}_h p_h + \tilde{Q}^{(p_h)}(q, p, \bar{q}, \bar{p})$$
$$\dot{\bar{q}}_h = -i\lambda_h \bar{q}_h + Q^{(\bar{q}_h)}(q, p, \bar{q}, \bar{p}),$$
$$\dot{\bar{p}}_h = -i\tilde{\lambda}_h \bar{p}_h + \tilde{Q}^{(\bar{p}_h)}(q, p, \bar{q}, \bar{p}), \quad h \in \mathbb{Z}^d, \tag{4.1}$$

which is reversible with respect to $S(q, p, \bar{q}, \bar{p}) = (\bar{q}, \bar{p}, q, p)$, where

$$Q^{(q_h)} = \sum_{i,j,k,l,m \in \mathbb{Z}^d} Q_{ijklm}^{(q_h)} q_i q_j p_k \bar{q}_l \bar{p}_m, \quad Q^{(\bar{q}_h)} = \overline{Q^{(q_h)}}, \tag{4.2}$$

$$\tilde{Q}^{(p_h)} = \sum_{i,j,k \in \mathbb{Z}^d} \tilde{Q}_{ijk}^{(p_h)} q_i p_j \bar{q}_k, \quad \tilde{Q}^{(\bar{p}_h)} = \overline{\tilde{Q}^{(p_h)}} \tag{4.3}$$

with

$$Q_{ijklm}^{(q_h)} = 2i \int_{\mathbb{T}^d} \phi_i \phi_j \phi_k \bar{\phi}_l \bar{\phi}_m \bar{\phi}_h dx$$
$$= \begin{cases} \frac{2i}{(2\pi)^{2d}}, & i + j + k - l - m - h = 0, \\ 0, & i + j + k - l - m - h \neq 0, \end{cases} \tag{4.4}$$

and

$$\tilde{Q}_{ijk}^{(p_h)} = i \int_{\mathbb{T}^d} \phi_i \phi_j \bar{\phi}_k \bar{\phi}_h dx = \begin{cases} \frac{i}{(2\pi)^d}, & i + j - k - h = 0, \\ 0 & i + j - k - h \neq 0, \end{cases} \tag{4.5}$$

By direct computation, one can verify that the perturbations $Q^{(q)} = (Q^{(q_h)})_{h \in \mathbb{Z}_1^d}$ and $\tilde{Q}^{(p)} = (\tilde{Q}^{(p_h)})_{h \in \mathbb{Z}_2^d}$ have the following regularity properties.

Lemma 4.1. *For each fixed $\rho > 0$, $Q^{(q)}$ (resp. $\tilde{Q}^{(p)}$) is real analytic as a map in a neighborhood of the origin with*

$$\|Q^{(q)}\|_\rho \le c\|(q,p)\|_\rho^5, \quad (resp. \ \|\tilde{Q}\|_\rho^{(p)} \le c\|(q,p)\|_\rho^3).$$

Let

$$P^0 = \sum_{\varrho = \pm}(Q^{(q^\varrho)}\frac{\partial}{\partial q^\varrho} + \tilde{Q}^{(p^\varrho)}\frac{\partial}{\partial p^\varrho}).$$

Lemma 4.2. (1) $[P^0, \mathbb{M}_l^0] = 0$, *for $l = 1, \ldots, d$, where*

$$\mathbb{M}_l^0 = \sum_{\varrho = \pm}\sum_{j \in \mathbb{Z}^d}\varrho i j_l q_j^\varrho \frac{\partial}{\partial q_j^\varrho} + \sum_{\varrho = \pm}\sum_{j \in \mathbb{Z}^d}\varrho i j_l p_j^\varrho \frac{\partial}{\partial p_j^\varrho}; \tag{4.6}$$

(2) P^0 *satisfies Töplitz-Lipschitz property.*

Proof. (1) If we write

$$Q^{(q_h^\varrho)} = \sum_{\alpha,\beta,\tilde{\alpha},\tilde{\beta}} Q_{\alpha\beta\tilde{\alpha}\tilde{\beta}}^{(q_h^\varrho)} q^\alpha \bar{q}^\beta p^{\tilde{\alpha}} \bar{p}^{\tilde{\beta}}, \quad \tilde{Q}^{(p_h^\varrho)} = \sum_{\alpha,\beta,\tilde{\alpha},\tilde{\beta}} \tilde{Q}_{\alpha\beta\tilde{\alpha}\tilde{\beta}}^{(p_h^\varrho)} q^\alpha \bar{q}^\beta p^{\tilde{\alpha}} \bar{p}^{\tilde{\beta}}, \tag{4.7}$$

then by (4.4) and (4.5), we have $Q_{\alpha\beta\tilde{\alpha}\tilde{\beta}}^{(q_h^\varrho)} = 0$ and $\tilde{Q}_{\alpha\beta\tilde{\alpha}\tilde{\beta}}^{(p_h^\varrho)} = 0$ when $\pi_l(\alpha\beta, \tilde{\alpha}\tilde{\beta}; v) \neq 0, v = q_h^\varrho, p_h^\varrho$. where

$$\pi_l(\alpha\beta, \tilde{\alpha}\tilde{\beta}; v) = \sum_{j \in \mathbb{Z}^d}(\alpha_j - \beta_j)j_l + \sum_{j \in \mathbb{Z}^d}(\tilde{\alpha}_j - \tilde{\beta}_j)j_l - \varrho h_l.$$

Note that by elementary computations, we have

$$[q^\alpha \bar{q}^\beta p^{\tilde{\alpha}} \bar{p}^{\tilde{\beta}}, \mathbb{M}_l^0] = i\pi_l(\alpha\beta, \tilde{\alpha}\tilde{\beta}; v)q^\alpha \bar{q}^\beta p^{\tilde{\alpha}} \bar{p}^{\tilde{\beta}},$$

which implies $[P^0, \mathbb{M}_l^0] = 0$.

(2) We only consider $\lim_{t \to \infty} \frac{\partial Q^{(q_{i+tc})}}{\partial q_{j+tc}}$. It follows from (4.2) and (4.4) that

$$\frac{\partial Q^{(q_i)}}{\partial q_j} = \sum_{n+k+l-m=i-j} \frac{4i}{(2\pi)^{2d}} q_n p_k \bar{q}_l \bar{p}_m,$$

then

$$\frac{\partial Q^{(q_{i+tc})}}{\partial q_{j+tc}} = \frac{\partial Q^{(q_i)}}{\partial q_j} = \lim_{t \to \infty} \frac{\partial Q^{(q_{i+tc})}}{\partial q_{j+tc}}. \qquad \square$$

4.2. Verification of assumptions (A1)–(A6). We introduce the action-angle variables (θ, φ, I, J) and normal coordinates $(z, w, \bar{z}, \bar{w})$ by the following transformation Ψ on some $D(r, s)$, $(r, s > 0)$:

$$q_{i(j)} = \sqrt{I_j + I_j^0}\mathrm{e}^{i\theta_j}, \quad \bar{q}_{i(j)} = \sqrt{I_j + I_j^0}\mathrm{e}^{-i\theta_j}, \quad j = 1, \ldots, n,$$

$$p_{\tilde{i}(j)} = \sqrt{J_j + J_j^0}\mathrm{e}^{i\varphi_j}, \quad \bar{p}_{\tilde{i}(j)} = \sqrt{J_j + J_j^0}\mathrm{e}^{-i\varphi_j}, \quad j = 1, \ldots, m, \tag{4.8}$$

$$q_h = z_h, \quad \bar{q}_h = \bar{z}_h, \quad h \notin \mathcal{I}_1,$$

$$p_h = w_h, \quad \bar{p}_h = \bar{w}_h, \quad h \notin \mathcal{I}_2,$$

where the I_j^0 and J_j^0 are fixed numbers satisfying $0 < s < I_j^0$ and $J_j^0 < 2s$.

We obtain a new vector field

$$\begin{aligned} X(\theta, &\varphi, I, J, z, \bar{z}, w, \bar{w}) \\ &= N(\theta, \varphi, I, J, z, \bar{z}, w, \bar{w}) + P(\theta, \varphi, I, J, z, \bar{z}, w, \bar{w}), \end{aligned} \tag{4.9}$$

where

$$N = \omega\frac{\partial}{\partial\theta} + \tilde{\omega}\frac{\partial}{\partial\varphi} + i\Omega z\frac{\partial}{\partial z} + i\widetilde{\Omega}w\frac{\partial}{\partial w} - i\Omega\bar{z}\frac{\partial}{\partial\bar{z}} - i\widetilde{\Omega}\bar{w}\frac{\partial}{\partial\bar{w}}$$

and

$$P = \sum_{\mathsf{w}\in\{\theta,\varphi,I,J,z,\bar{z},w,\bar{w}\}} P^{(\mathsf{w})}(\theta,\varphi,I,J,z,\bar{z},w,\bar{w};\xi,\widetilde{\xi})\frac{\partial}{\partial\mathsf{w}} \tag{4.10}$$

with

$$\omega_b = |i^{(b)}|^2 + \xi_b, \quad 1 \le b \le n,$$
$$\widetilde{\omega}_b = |\tilde{i}^{(b)}|^2 + \widetilde{\xi}_b, \quad 1 \le b \le m,$$
$$\Omega_h = |h|^2, \quad h \notin \mathcal{I}_1,$$
$$\widetilde{\Omega}_h = |h|^2, \quad h \notin \mathcal{I}_2;$$

and

$$P^{(\theta_b)} = \frac{1}{2iq_{i^{(b)}}}Q^{(q_{i^{(b)}})}\circ\Psi - \frac{1}{2i\bar{q}_{i^{(b)}}}Q^{(\bar{q}_{i^{(b)}})}\circ\Psi. \tag{4.11}$$

$$P^{(I_b)} = \bar{q}_{i^{(b)}}Q^{(q_{i^{(b)}})}\circ\Psi + q_{i^{(b)}}Q^{(\bar{q}_{i^{(b)}})}\circ\Psi, \tag{4.12}$$

$$P^{(z_h^\sigma)} = Q^{(q_h^\sigma)}\circ\Psi, \quad \sigma = \pm; \tag{4.13}$$

$$P^{(\varphi_b)} = \frac{1}{2ip_{\tilde{i}^{(b)}}}\tilde{Q}^{(p_{\tilde{i}^{(b)}})}\circ\Psi - \frac{1}{2i\bar{p}_{\tilde{i}^{(b)}}}\tilde{Q}^{(\bar{p}_{\tilde{i}^{(b)}})}\circ\Psi; \tag{4.14}$$

$$P^{(J_b)} = \bar{p}_{\tilde{i}^{(b)}}\tilde{Q}^{(p_{\tilde{i}^{(b)}})}\circ\Psi + p_{\tilde{i}^{(b)}}\tilde{Q}^{(\bar{p}_{\tilde{i}^{(b)}})}\circ\Psi; \tag{4.15}$$

$$P^{(w_h^\sigma)} = \tilde{Q}^{(p_h^\sigma)}\circ\Psi, \quad \sigma = \pm. \tag{4.16}$$

Here X is reversible with respect to the involution

$$S(\theta,\varphi,I,J,z,w,\bar{z},\bar{w}) = (-\theta,-\varphi,I,J,\bar{z},\bar{w},z,w).$$

Now we verify assumptions (A1)–(A6) for (4.9):

(A1) Set $\zeta = (\xi,\tilde{\xi})$, it is obvious as the Jacobian matrix

$$\frac{\partial(\omega,\tilde{\omega})}{\partial\zeta} = \frac{\partial(\omega,\tilde{\omega})}{\partial(\xi,\tilde{\xi})} = I_{n+m}.$$

(A2) It is also obvious.

(A3) One can refer to [8, Section 3.2], their proof is similar.

(A4) Suppose vector field (4.9) is defined on the domain $D(r,s)$ with $0 < r < 1, s = \varepsilon^2$. It follows from (4.11)-(4.16) that

$$P^{(\theta_b)} = O(s^2), \quad P^{(I_b)} = O(s^3), \quad P^{(z_h^\sigma)} = O(s^{\frac{5}{2}})\mathrm{e}^{-\rho|h|},$$
$$P^{(\varphi_b)} = O(s), \quad P^{(J_b)} = O(s^2), \quad P^{(w_h^\sigma)} = O(s^{\frac{3}{2}})\mathrm{e}^{-\rho|h|}.$$

Then

$$\|P\|_{s;D_\rho(r,s)\times\mathcal{O}} \le cs^{1/2} \le c\varepsilon.$$

(A5) Through the transformation Ψ in (4.8), the vector fields $\mathbb{M}_l^0$ in (4.6) are transformed into $\mathbb{M}_l = \Psi^*\mathbb{M}_l^0$, then

$$[P,\mathbb{M}_l] = [\Psi^*P^0,\Psi^*\mathbb{M}_l^0] = \Psi^*[P^0,\mathbb{M}_l^0] = 0.$$

(A6) We only consider $\frac{\partial P^{(\theta_b)}}{\partial z_{j+tc}}$ and $\frac{\partial P^{(z_{i+tc})}}{\partial z_{j+tc}}$ and the others can be verified similarly.

$$\frac{\partial P^{(\theta_b)}}{\partial z_{j+tc}} = O(s^2)e^{-\rho|j+tc|} \to 0, \quad t \to \infty.$$

then

$$\|\frac{\partial P^{(\theta_b)}}{\partial z_{j+tc}} - \lim_{t\to\infty} \frac{\partial P^{(\theta_b)}}{\partial z_{j+tc}}\|_{s;D_\rho(s,r)\times\mathcal{O}} \le \varepsilon.$$

It follows from (4.13) and (2) in Lemma 4.2 that $\frac{\partial P^{(z_i)}}{\partial z_j} = O(s^{5/2})e^{-\rho|i-j|}$ and

$$\frac{\partial P^{(z_{i+tc})}}{\partial z_{j+tc}} = \frac{\partial P^{(z_i)}}{\partial z_j} = \lim_{t\to\infty} \frac{\partial P^{(z_{i+tc})}}{\partial z_{j+tc}}.$$

Then

$$\|\frac{\partial P^{(z_{i+tc})}}{\partial z_{j+tc}} - \lim_{t\to\infty} \frac{\partial P^{(z_{i+tc})}}{\partial z_{j+tc}}\|_{s;D_\rho(s,r)\times\mathcal{O}} \le \frac{\varepsilon}{|t|}e^{-\rho|i-j|}.$$

5. Proof of Theorem 3.2

At the νth step of the KAM iteration, we consider an S-reversible vector field on $D_{\rho_\nu}(r_\nu, s_\nu) \times \mathcal{O}_\nu$:

$$X_\nu = N_\nu + \mathcal{A}_\nu + P_\nu$$

satisfying (A1)–(A6), where N_ν and $\mathcal{A}_\nu$ have the same form as N and $\mathcal{A}$ in (3.1) and (3.2).

We shall construct an S-invariant transformation

$$\Phi_\nu : D_{\rho_{\nu+1}}(r_{\nu+1}, s_{\nu+1}) \times \mathcal{O}_\nu \to D_{\rho_\nu}(r_\nu, s_\nu) \times \mathcal{O}_\nu$$

such that $\Phi_\nu^* X_\nu := (D\Phi_\nu)^{-1} \cdot X_\nu \circ \Phi_\nu = N_{\nu+1} + \mathcal{A}_{\nu+1} + P_{\nu+1}$ with a new normal form $N_{\nu+1}$, $\mathcal{A}_{\nu+1}$ and a much smaller perturbation term $P_{\nu+1}$ and still satisfying (A1)–(A6).

In the sequel, for simplicity, we drop the subscript ν and write the symbol '+' for '$\nu + 1$'. Then we have the vector field

$$X = N + \mathcal{A} + P \tag{5.1}$$

with

$$N = \sum_{b=1}^n \omega_b(\zeta)\frac{\partial}{\partial\theta_b} + \sum_{b=1}^m \tilde{\omega}_b(\zeta)\frac{\partial}{\partial\varphi_b}$$

$$+ \sum_{\varrho=\pm}\Big(\sum_{j\in\mathbb{Z}_1^d} \varrho i\Omega_j(\zeta)z_j^\varrho\frac{\partial}{\partial z_j^\varrho} + \sum_{j\in\mathbb{Z}_2^d} \varrho i\widetilde{\Omega}_j(\zeta)w_j^\varrho\frac{\partial}{\partial w_j^\varrho}\Big),$$

$$\mathcal{A} = \sum_{\varrho=\pm}\sum_{j\in\mathbb{Z}_1^d\cap\mathbb{Z}_2^d} \Big(\varrho iA_j(\zeta)w_j^\varrho\frac{\partial}{\partial z_j^\varrho} + \varrho i\tilde{A}_j(\zeta)z_j^\varrho\frac{\partial}{\partial w_j^\varrho}\Big),$$

Let $0 < r_+ < r$ and

$$s_+ = \frac{1}{4}s\varepsilon^{1/3}, \quad \varepsilon_+ = c\gamma^{-5}(2r-2r_+)^{-1}K^{5\tau+19}\varepsilon^{5/3} + \varepsilon^{7/6},$$

where c is some suitable (possible different) constant independent of the iterative steps. Then our goal is to find a set $\mathcal{O}_+ \subset \mathcal{O}$ and an S-invariant transformation

$\Phi : D_\rho(r_+, s_+) \times \mathcal{O}_+ \to D_\rho(r, s) \times \mathcal{O}$ such that it transforms the above X in (5.1) into

$$X_+ = N_+ + \mathcal{A}_+ + P_+$$

with

$$N_+ = \sum_{b=1}^n \omega_{+,b}(\zeta) \frac{\partial}{\partial \theta_b} + \sum_{b=1}^m \tilde{\omega}_{+,b}(\zeta) \frac{\partial}{\partial \varphi_b}$$
$$+ \sum_{\varrho=\pm} \Big(\sum_{j \in \mathbb{Z}_1^d} \varrho i \Omega_{+,j}(\zeta) z_j^\varrho \frac{\partial}{\partial z_j^\varrho} + \sum_{j \in \mathbb{Z}_2^d} \varrho i \widetilde{\Omega}_{+,j}(\zeta) w_j^\varrho \frac{\partial}{\partial w_j^\varrho} \Big),$$
$$\mathcal{A}_+ = \sum_{\varrho=\pm} \sum_{j \in \mathbb{Z}_1^d \cap \mathbb{Z}_2^d} \Big(\varrho i A_{+,j}(\zeta) w_j^\varrho \frac{\partial}{\partial z_j^\varrho} + \varrho i \tilde{A}_{+,j}(\zeta) z_j^\varrho \frac{\partial}{\partial w_j^\varrho} \Big).$$

5.1. **Solving the homological equations.** For $K > 0$, we define the truncation operator $\mathcal{T}_K$ as follows: for f on $D(r) = \{(\theta, \varphi) \in \mathbb{C}^n \times \mathbb{C}^m : |\operatorname{Im} \theta| < r, |\operatorname{Im} \varphi| < r\}$,

$$\mathcal{T}_K f(\theta, \varphi) := \sum_{(k, \tilde{k}) \mathbb{Z}^n \times \mathbb{Z}^m, |k|+|\tilde{k}| \le K} f_{k, \tilde{k}} e^{i(\langle k, \theta \rangle + \langle \tilde{k}, \varphi \rangle)}.$$

The average of f with respect to (θ, φ) is defined as

$$[f] = f_{0,0} = \frac{1}{(2\pi)^{n+m}} \int_{\mathbb{T}^{n+m}} f(\theta, \varphi) \, d\theta \, d\varphi$$

We write the reversible vector field P as Taylor-Fourier series

$$P(y; \zeta) = \sum_{\mathsf{v} \in \mathscr{V}} \sum_{k, \tilde{k}, l, \tilde{l}, \alpha, \beta, \tilde{\alpha}, \tilde{\beta}} P^{(\mathsf{v})}_{kl\alpha\beta, \tilde{k}\tilde{l}\tilde{\alpha}\tilde{\beta}}(\zeta) e^{i(\langle k, \theta \rangle + \langle \tilde{k}, \varphi \rangle)} I^l J^{\tilde{l}} z^\alpha \bar{z}^\beta w^{\tilde{\alpha}} \bar{w}^{\tilde{\beta}} \frac{\partial}{\partial \mathsf{v}},$$

Let $R = \sum_{\mathsf{v} \in \mathscr{V}} R^{(\mathsf{v})}(y; \zeta) \frac{\partial}{\partial \mathsf{v}}$ be the truncation of P with

$$R^{(\mathsf{v})} = \sum_{|l|+|\tilde{l}|+|\alpha|+|\beta|+|\tilde{\alpha}|+|\tilde{\beta}| \le 1} \mathcal{T}_K P^{(\mathsf{v})}_{l\alpha\beta, \tilde{l}\tilde{\alpha}\tilde{\beta}}(\theta, \varphi) I^l J^{\tilde{l}} z^\alpha \bar{z}^\beta w^{\tilde{\alpha}} \bar{w}^{\tilde{\beta}}$$

for $\mathsf{v} \in \{\theta_b, \varphi_b, I_b, J_b, z_j, \bar{z}_j, w_j, \bar{w}_j\}$.

We rewrite $R^{(\mathsf{v})}$ as follows: for $\mathsf{v} \in \{\theta_b, \varphi_b, I_b, J_b, z_j, \bar{z}_j, w_j, \bar{w}_j\}$,

$$R^{(\mathsf{v})} = R^{\mathsf{v}}(\theta, \varphi) + \sum_{\mathsf{u} \in \{I_a, J_a, z_j, \bar{z}_j, w_j, \bar{w}_j\}} R^{\mathsf{vu}}(\theta, \varphi) \mathsf{u}.$$

Remark 5.1. For the usual KAM procedure, we only need to eliminate terms R^θ, R^φ in $R^{(\theta)}, R^{(\varphi)}$. In this paper, we need to control the first derivatives of perturbation vector field $P - R$ in lemma 5.6. So we must eliminate all the linear terms $R^{\theta \mathsf{u}} \mathsf{u}, R^{\varphi \mathsf{u}} \mathsf{u}$ in $R^{(\theta)}, R^{(\varphi)}$.

We define the normal form of R as

$$[R] = \sum_{b=1}^n [R^{\theta_b}] \frac{\partial}{\partial \theta_b} + \sum_{b=1}^m [R^{\varphi_b}] \frac{\partial}{\partial \varphi_b}$$
$$+ \sum_{j \in \mathbb{Z}_1^d} [R^{z_j z_j}] z_j \frac{\partial}{\partial z_j} + \sum_{j \in \mathbb{Z}_1^d \cap \mathbb{Z}_2^d} [R^{z_j w_j}] w_j \frac{\partial}{\partial z_j}$$
$$+ \sum_{j \in \mathbb{Z}_1^d} [R^{\bar{z}_j \bar{z}_j}] \bar{z}_j \frac{\partial}{\partial \bar{z}_j} + \sum_{j \in \mathbb{Z}_1^d \cap \mathbb{Z}_2^d} [R^{\bar{z}_j \bar{w}_j}] \bar{w}_j \frac{\partial}{\partial \bar{z}_j}$$

$$+ \sum_{j \in \mathbb{Z}_2^d} [R^{w_j w_j}] w_j \frac{\partial}{\partial w_j} + \sum_{j \in \mathbb{Z}_1^d \cap \mathbb{Z}_2^d} [R^{w_j z_j}] z_j \frac{\partial}{\partial w_j}$$

$$+ \sum_{j \in \mathbb{Z}_2^d} [R^{\bar{w}_j \bar{w}_j}] \bar{w}_j \frac{\partial}{\partial \bar{w}_j} + \sum_{j \in \mathbb{Z}_1^d \cap \mathbb{Z}_2^d} [R^{\bar{w}_j \bar{z}_j}] \bar{z}_j \frac{\partial}{\partial \bar{w}_j}$$

In the sequel, we denote by ϕ_X^t the flow generated by vector field X and $\phi_X^1 = \phi_X^t|_{t=1}$. Suppose vector field F has the same form as R, $\Phi^* X = (\phi_F^1)^* X$,

$$\Phi^* X = (\phi_F^1)^*(N + \mathcal{A}) + (\phi_F^1)^* R + (\phi_F^1)^*(P - R)$$

$$= N + \mathcal{A} + [N + \mathcal{A}, F] + \int_0^1 (1-t)(\phi_F^t)^*[[N + \mathcal{A}, F], F] dt$$

$$+ R + \int_0^1 (\phi_F^t)^*[R, F] dt + (\phi_F^1)^*(P - R).$$

We solve the homological equation

$$[N + \mathcal{A}, F] + R = [R]. \tag{5.2}$$

We denote $\partial_{(\omega, \tilde{\omega})} f(\theta, \varphi) := \partial_\omega f(\theta, \varphi) + \partial_{\tilde{\omega}} f(\theta, \varphi)$. By the definition of Lie bracket, the homological equation (5.2) is equivalent to the following scalar forms (5.3)–(5.6):

$$\partial_{(\omega, \tilde{\omega})} f = g \tag{5.3}$$

where

$$(f, g) = (F^u, R^u - [R^u]), (F^v, R^v),$$

$$u \in \{\theta_a, \varphi_a\}; v \in \{I_a, J_a, \theta_a I_b, \theta_a J_b, \varphi_a I_b, \varphi_a J_b, I_a I_b, I_a J_b, J_a I_b, J_a J_b\}.$$

$$\partial_{(\omega, \tilde{\omega})} f - i\lambda f = g, \tag{5.4}$$

where

$$(f, g; \lambda) \in \{(F^{u_l}, R^{u_l}; \lambda_l) : l = 1, \ldots, 6\},$$

with

$$u_1 \in \{\theta_a z_i^{-\varrho}, \varphi_a z_i^{-\varrho}, I_a z_i^{-\varrho}, J_a z_i^{-\varrho}, z_i^\varrho, z_i^\varrho I_a, z_i^\varrho J_a : i \in \mathbb{Z}_1^d \setminus \mathbb{Z}_2^d\}, \quad \lambda_1 = \varrho \Omega_i,$$

$$u_2 \in \{\theta_a w_j^{-\varrho}, \varphi_a w_j^{-\varrho}, I_a w_j^{-\varrho}, J_a w_j^{-\varrho}, w_j^\varrho, w_j^\varrho I_a, w_j^\varrho J_a : j \in \mathbb{Z}_2^d \setminus \mathbb{Z}_1^d\}, \quad \lambda_2 = \varrho \tilde{\Omega}_j,$$

$$u_3 = z_i^\varrho z_j^\sigma, \quad \lambda_3 = \varrho \Omega_i - \sigma \Omega_j, \quad i, j \in \mathbb{Z}_1^d \setminus \mathbb{Z}_2^d, \quad i \neq j, \varrho \neq \sigma,$$

$$u_4 = z_i^\varrho w_j^\sigma, \quad \lambda_4 = \varrho \Omega_i - \sigma \tilde{\Omega}_j, \quad i \in \mathbb{Z}_1^d \setminus \mathbb{Z}_2^d, \quad j \in \mathbb{Z}_2^d \setminus \mathbb{Z}_1^d,$$

$$u_5 = w_i^\varrho z_j^\sigma, \quad \lambda_5 = \varrho \tilde{\Omega}_i - \sigma \Omega_j, \quad i \in \mathbb{Z}_2^d \setminus \mathbb{Z}_1^d, \quad j \in \mathbb{Z}_1^d \setminus \mathbb{Z}_2^d,$$

$$u_6 = w_i^\varrho w_j^\sigma, \quad \lambda_6 = \varrho \tilde{\Omega}_i - \sigma \tilde{\Omega}_j, \quad i, j \in \mathbb{Z}_2^d \setminus \mathbb{Z}_1^d, \quad i \neq j, \varrho \neq \sigma.$$

$$\partial_{(\omega, \tilde{\omega})} \mathcal{F} + i\mathcal{M}\mathcal{F} = \mathcal{G}, \tag{5.5}$$

where

$$(\mathcal{F}, \mathcal{G}; \mathcal{M}) \in \left\{ \left(\begin{pmatrix} F^{u_l} \\ F^{v_l} \end{pmatrix}, \begin{pmatrix} R^{u_l} \\ R^{v_l} \end{pmatrix}; \mathcal{M}_l \right) : l = 1, \ldots, 6 \right\},$$

with

$$(u_1, v_1) \in \{(\mathsf{u} z_j^\varrho, \mathsf{u} w_j^\varrho) : \mathsf{u} \in \{\theta_a, \varphi_a, I_a, J_a\}, \quad j \in \mathbb{Z}_1^d \cap \mathbb{Z}_2^d\}, \quad \mathcal{M}_1 = \varrho M_j^T,$$

$$(u_2, v_2) \in \{(z_i^\varrho, w_i^\varrho), (z_i^\varrho I_a, w_i^\varrho I_a), (z_i^\varrho J_a, w_i^\varrho J_a) : i \in \mathbb{Z}_1^d \cap \mathbb{Z}_2^d\}, \quad \mathcal{M}_2 = -\varrho M_i,$$

$$(u_3, v_3) = (z_i^\varrho z_j^\sigma, z_i^\varrho w_j^\sigma), \quad \mathcal{M}_3 = -\varrho \Omega_i I_2 + \sigma M_j^T, \quad i \in \mathbb{Z}_1^d \setminus \mathbb{Z}_2^d, \quad j \in \mathbb{Z}_1^d \cap \mathbb{Z}_2^d,$$

$$(u_4, v_4) = (w_i^\varrho z_j^\sigma, w_i^\varrho w_j^\sigma), \quad \mathcal{M}_4 = -\varrho \tilde{\Omega}_i I_2 + \sigma M_j^T, \quad i \in \mathbb{Z}_2^d \setminus \mathbb{Z}_1^d, \quad j \in \mathbb{Z}_1^d \cap \mathbb{Z}_2^d,$$

$$(u_5, v_5) = (z_i^\varrho z_j^\sigma, w_i^\varrho z_j^\sigma), \quad \mathcal{M}_5 = \sigma\Omega_j I_2 - \varrho M_i, \quad i \in \mathbb{Z}_1^d \cap \mathbb{Z}_2^d, \quad j \in \mathbb{Z}_1^d \setminus \mathbb{Z}_2^d,$$
$$(u_6, v_6) = (z_i^\varrho w_j^\sigma, w_i^\varrho w_j^\sigma), \quad \mathcal{M}_6 = \sigma\tilde\Omega_j I_2 - \varrho M_i, \quad i \in \mathbb{Z}_1^d \cap \mathbb{Z}_2^d, \quad j \in \mathbb{Z}_2^d \setminus \mathbb{Z}_1^d.$$

$$\begin{aligned}
&\partial_{(\omega,\tilde\omega)} F^{z_i^\varrho z_j^\sigma} - \varrho i\Omega_i F^{z_i^\varrho z_j^\sigma} + \sigma i F^{z_i^\varrho z_j^\sigma} \Omega_j - \varrho i A_i F^{w_i^\varrho z_j^\sigma} + \sigma i F^{z_i^\varrho w_j^\sigma} \tilde A_j \\
&\quad = R^{z_i^\varrho z_j^\sigma} - \delta_{\varrho\sigma}\delta_{ij}[R^{z_i^\varrho z_j^\sigma}], \\
&\partial_{(\omega,\tilde\omega)} F^{z_i^\varrho w_j^\sigma} - \varrho i\Omega_i F^{z_i^\varrho w_j^\sigma} + \sigma i F^{z_i^\varrho w_j^\sigma} \tilde\Omega_j - \varrho i A_i F^{w_i^\varrho w_j^\sigma} + \sigma i F^{z_i^\varrho z_j^\sigma} A_j \\
&\quad = R^{z_i^\varrho w_j^\sigma} - \delta_{\varrho\sigma}\delta_{ij}[R^{z_i^\varrho w_j^\sigma}], \\
&\partial_{(\omega,\tilde\omega)} F^{w_i^\varrho z_j^\sigma} - \varrho i\tilde\Omega_i F^{w_i^\varrho z_j^\sigma} + \sigma i F^{w_i^\varrho z_j^\sigma} \Omega_j - \varrho i \tilde A_i F^{z_i^\varrho z_j^\sigma} + \sigma i F^{w_i^\varrho w_j^\sigma} \tilde A_j \\
&\quad = R^{w_i^\varrho z_j^\sigma} - \delta_{\varrho\sigma}\delta_{ij}[R^{w_i^\varrho z_j^\sigma}], \\
&\partial_{(\omega,\tilde\omega)} F^{w_i^\varrho w_j^\sigma} - \varrho i\tilde\Omega_i F^{w_i^\varrho w_j^\sigma} + \sigma i F^{w_i^\varrho w_j^\sigma} \tilde\Omega_j - \varrho i \tilde A_i F^{z_i^\varrho w_j^\sigma} + \sigma i F^{w_i^\varrho z_j^\sigma} A_j \\
&\quad = R^{w_i^\varrho w_j^\sigma} - \delta_{\varrho\sigma}\delta_{ij}[R^{w_i^\varrho w_j^\sigma}], i, j \in \mathbb{Z}_1^d \cap \mathbb{Z}_2^d.
\end{aligned} \tag{5.6}$$

here $\delta_{\mu\nu} = 1$ if $\mu = \nu$, and 0 otherwise.

Suppose that for $\zeta \in \mathcal{O}$, $|k| + |\tilde k| \le K$,

$$|\langle k, \omega\rangle + \langle \tilde k, \tilde\omega\rangle| \ge \frac{\gamma}{K^\tau}, \quad (k, \tilde k) \ne 0,$$

$$|\langle k, \omega\rangle + \langle \tilde k, \tilde\omega\rangle + \Omega_i| \ge \frac{\gamma}{K^\tau}, \quad i \in \mathbb{Z}_1^d \setminus \mathbb{Z}_2^d,$$

$$|\langle k, \omega\rangle + \langle \tilde k, \tilde\omega\rangle + \tilde\Omega_i| \ge \frac{\gamma}{K^\tau}, \quad i \in \mathbb{Z}_2^d \setminus \mathbb{Z}_1^d,$$

$$|\langle k, \omega\rangle + \langle \tilde k, \tilde\omega\rangle + \Omega_i \pm \Omega_j| \ge \frac{\gamma}{K^\tau}, \quad (k, \tilde k) \ne 0, \ i, j \in \mathbb{Z}_1^d \setminus \mathbb{Z}_2^d,$$

$$|\langle k, \omega\rangle + \langle \tilde k, \tilde\omega\rangle + \Omega_i \pm \tilde\Omega_j| \ge \frac{\gamma}{K^\tau}, \quad i \in \mathbb{Z}_1^d \setminus \mathbb{Z}_2^d, \ j \in \mathbb{Z}_2^d \setminus \mathbb{Z}_1^d,$$

$$|\langle k, \omega\rangle + \langle \tilde k, \tilde\omega\rangle + \tilde\Omega_i \pm \tilde\Omega_j| \ge \frac{\gamma}{K^\tau}, \quad (k, \tilde k) \ne 0, \ i, j \in \mathbb{Z}_2^d \setminus \mathbb{Z}_1^d,$$

$$|\det((\langle k, \omega\rangle + \langle \tilde k, \tilde\omega\rangle) I_2 \pm M_i)| \ge \frac{\gamma}{K^\tau}, \quad i \in \mathbb{Z}_1^d \cap \mathbb{Z}_2^d,$$

$$|\det((\langle k, \omega\rangle + \langle \tilde k, \tilde\omega\rangle + \Omega_i) I_2 \pm M_j)| \ge \frac{\gamma}{K^\tau}, \quad (i, j) \text{ or } (j, i) \in (\mathbb{Z}_1^d \cap \mathbb{Z}_2^d) \times (\mathbb{Z}_1^d \setminus \mathbb{Z}_2^d),$$

$$|\det((\langle k, \omega\rangle + \langle \tilde k, \tilde\omega\rangle + \tilde\Omega_i) I_2 \pm M_j)| \ge \frac{\gamma}{K^\tau}, \quad (i, j) \text{ or } (j, i) \in (\mathbb{Z}_1^d \cap \mathbb{Z}_2^d) \times (\mathbb{Z}_2^d \setminus \mathbb{Z}_1^d),$$

$$|\det((\langle k, \omega\rangle + \langle \tilde k, \tilde\omega\rangle) I_4 + M_i \otimes I_2 \pm I_2 \otimes M_j^T)| \ge \frac{\gamma}{K^\tau}, \quad (k, \tilde k) \ne 0, \ i, j \in \mathbb{Z}_1^d \cap \mathbb{Z}_2^d. \tag{5.7}$$

Lemma 5.2. *Suppose that on $\mathcal{O}$, the non-resonance conditions in (5.7) hold uniformly. Then there exists a positive $c = c(n, m, \tau)$ such that (5.2) has a unique solution F with $[F] = 0$, which is regular on $D(r, s) \times \mathcal{O}$. Moreover,*

(1) $\|F\|_{s;D_\rho(r,s)\times\mathcal{O}} \le c\gamma^{-5} K^{5\tau+19}\varepsilon$;

(2) $F \circ S = DS \cdot F$;

(3) $[F, \mathbb{M}_l] = 0$ *for* $l = 1, \ldots, d$;

(4) F *satisfies* (A6) *with* $\varepsilon^{2/3}$ *in place of* ε *on* $D(r - \delta, s/2)$, *where* $0 < \delta < r/2$.

Proof. (1) As we have mentioned above, (5.2) is equivalent to (5.3)–(5.6). Below we only consider the most difficult equation (5.6) with $\varrho = \sigma, i \ne j$ since the other ones can be solved similarly.

By Fourier expansion, we have

$$
(((\langle k,\omega\rangle + \langle \tilde{k},\tilde{\omega}\rangle)I_4 - \varrho M_i \otimes I_2 + \varrho I_2 \otimes M_j^T)
\begin{pmatrix}
iF_{k,\tilde{k}}^{z_i^\varrho z_j^\varrho} \\
iF_{k,\tilde{k}}^{z_i^\varrho w_j^\varrho} \\
iF_{k,\tilde{k}}^{w_i^\varrho z_j^\varrho} \\
iF_{k,\tilde{k}}^{w_i^\varrho w_j^\varrho}
\end{pmatrix}
=
\begin{pmatrix}
R_{k,\tilde{k}}^{z_i^\varrho z_j^\varrho} \\
R_{k,\tilde{k}}^{z_i^\varrho w_j^\varrho} \\
R_{k,\tilde{k}}^{w_i^\varrho z_j^\varrho} \\
R_{k,\tilde{k}}^{w_i^\varrho w_j^\varrho}
\end{pmatrix},
$$

where

$$
(\langle k,\omega\rangle + \langle \tilde{k},\tilde{\omega}\rangle)I_4 - \varrho M_i \otimes I_2 + \varrho I_2 \otimes M_j^T
$$
$$
= (\langle k,\omega\rangle + \langle \tilde{k},\tilde{\omega}\rangle)I_4
$$
$$
+
\begin{pmatrix}
-\varrho\Omega_i + \varrho\Omega_j & \varrho\tilde{A}_j & -\varrho A_i & 0 \\
\varrho A_j & -\varrho\Omega_i + \varrho\tilde{\Omega}_j & 0 & -\varrho A_i \\
-\varrho\tilde{A}_i & 0 & -\varrho\tilde{\Omega}_i + \varrho\Omega_j & \varrho\tilde{A}_j \\
0 & -\varrho\tilde{A}_i & \varrho A_j & -\varrho\tilde{\Omega}_i + \varrho\tilde{\Omega}_j
\end{pmatrix}.
$$

Using the non-resonance conditions in (5.7), we obtain that for $|k| + |\tilde{k}| \le K$,

$$
|F_{k,\tilde{k}}^{z_i^\varrho z_j^\varrho}|_\mathcal{O}, \quad |F_{k,\tilde{k}}^{z_i^\varrho w_j^\varrho}|_\mathcal{O}, \quad |F_{k,\tilde{k}}^{w_i^\varrho z_j^\varrho}|_\mathcal{O}, \quad |F_{k,\tilde{k}}^{w_i^\varrho w_j^\varrho}|_\mathcal{O}
$$
$$
\le c\gamma^{-5}K^{5\tau+19}\Big(|R_{k,\tilde{k}}^{z_i^\varrho z_j^\varrho}|_\mathcal{O} + |R_{k,\tilde{k}}^{z_i^\varrho w_j^\varrho}|_\mathcal{O} + |R_{k,\tilde{k}}^{w_i^\varrho z_j^\varrho}|_\mathcal{O} + |R_{k,\tilde{k}}^{w_i^\varrho w_j^\varrho}|_\mathcal{O}\Big).
$$

Then according to the definition of vector fields, we have

$$
\|F\|_{s;D_\rho(r,s)\times\mathcal{O}} \le c\gamma^{-5}K^{5\tau+19}\|R\|_{s;D_\rho(r,s)\times\mathcal{O}} \le c\gamma^{-5}K^{5\tau+19}\varepsilon.
$$

(2) $F \circ S = DS \cdot F$ can be implied by the uniqueness of solutions of homological equation.

(3) We verify that $[F,\mathbb{M}_l] = 0$. For $l = 1, \ldots, d$, consider

$$
\pi_l(k\alpha\beta; \tilde{k}\tilde{\alpha}\tilde{\beta}; \mathsf{v}) =
\begin{cases}
\pi_l(k\alpha\beta; \tilde{k}\tilde{\alpha}\tilde{\beta}), & \mathsf{v} = \theta_b, \varphi_b, I_b, J_b, \\
\pi_l(k\alpha\beta; \tilde{k}\tilde{\alpha}\tilde{\beta}) - \varrho j_l, & \mathsf{v} = z_j^\varrho, w_j^\varrho,
\end{cases}
\tag{5.8}
$$

where

$$
\pi_l(k\alpha\beta; \tilde{k}\tilde{\alpha}\tilde{\beta}) = \sum_{b=1}^{n} i_l^{(b)}k_l + \sum_{b=1}^{m} \tilde{i}_l^{(b)}\tilde{k}_l + \sum_{j\in\mathbb{Z}_1^d}(\alpha_j - \beta_j)j_l + \sum_{j\in\mathbb{Z}_2^d}(\tilde{\alpha}_j - \tilde{\beta}_j)j_l.
$$

As in the proof of Lemma (4.2), one can verify that a vector field X satisfying $[X,\mathbb{M}_l] = 0$ is equivalent to $X_{kl\alpha\beta,\tilde{k}\tilde{l}\tilde{\alpha}\tilde{\beta}}^{(\mathsf{v})} = 0$, if $\pi_l(k\alpha\beta; \tilde{k}\tilde{\alpha}\tilde{\beta}; \mathsf{v}) \ne 0$. Thus to prove $[F,\mathbb{M}_l] = 0$, it suffices to verify that $F_{kl\alpha\beta,\tilde{k}\tilde{l}\tilde{\alpha}\tilde{\beta}}^{(\mathsf{v})} = 0$, if $\pi_l(k\alpha\beta; \tilde{k}\tilde{\alpha}\tilde{\beta}; \mathsf{v}) \ne 0$. This can be implied by $[P,\mathbb{M}] = 0$ since F is determined by R.

(4) Fr the proof we can follow that of [8, Lemma 4.3] since there is no essential difference. □

5.2. Estimates on the coordinate transformation.

Lemma 5.3. *If $\varepsilon \ll \delta\gamma^5 K^{-5\tau-19}$, then for every $-1 \le t \le 1$, we have*

$$
\phi_F^t : D_\rho(r - 2\delta, s/4) \to D_\rho(r - \delta, s/2),
$$
$$
\|\phi_F^t - id\|_{s;D_\rho(r-2\delta,s/4)\times\mathcal{O}} \le c\gamma^{-5}K^{5\tau+19}\varepsilon,
$$
$$
\|D\phi_F^t - Id\|_{s;D_\rho(r-2\delta,s/4)\times\mathcal{O}} \le c\gamma^{-5}\delta^{-1}K^{5\tau+19}\varepsilon.
$$

Proof. Using Cauchy's inequality, we obtain

$$\|DF\|_{s;D_\rho(r-\delta,s/2)\times\mathcal{O}} \le \frac{c}{\delta}\|F\|_{s;D_\rho(r,s)\times\mathcal{O}} \le \frac{c}{\delta}\gamma^{-5}K^{5\tau+19}\varepsilon,$$

then if $\varepsilon \ll \delta\gamma^5 K^{-5\tau-19}$, for every $-1 \le t \le 1$,

$$\phi_F^t : D_\rho(r-2\delta,s/4) \to D_\rho(r-\delta,s/2)$$

is well-defined. Thus by Gronwall's inequality and the estimate for DF, we have

$$\|\phi_F^t - id\|_{s;D(r-2\delta,s/4)\times\mathcal{O}} \le c\|F\|_{s;D(r,s)\times\mathcal{O}} \le c\gamma^{-5}K^{5\tau+19}\varepsilon,$$
$$\|D\phi_F^t - Id\|_{s;D(r-2\delta,s/4)\times\mathcal{O}} \le c\|DF\|_{s;D(r-\delta,s/2)\times\mathcal{O}} \le c\delta^{-1}\gamma^{-5}K^{5\tau+19}\varepsilon. \qquad \square$$

5.3. **New normal form.** Through the time-1 map $\Phi = \phi_F^1$ defined above, the vector field X is transformed into $X_+ = \Phi^* X = N_+ + \mathcal{A}_+ + P_+$ with new normal form $N_+, \mathcal{A}_+$ and new perturbation P_+. In this subsection, we consider the new normal form

$$N_+ = N + \hat{N}, \quad \mathcal{A}_+ = \mathcal{A} + \hat{\mathcal{A}},$$

where

$$\begin{aligned}
\hat{N} = &\sum_{b=1}^n [R^{\theta_b}]\frac{\partial}{\partial\theta_b} + \sum_{b=1}^m [R^{\varphi_b}]\frac{\partial}{\partial\varphi_b} \\
&+ \sum_{j\in\mathbb{Z}_1^d}([R^{z_j z_j}]z_j\frac{\partial}{\partial z_j} + [R^{\bar{z}_j\bar{z}_j}]\bar{z}_j\frac{\partial}{\partial\bar{z}_j}) \\
&+ \sum_{j\in\mathbb{Z}_2^d}([R^{w_j w_j}]w_j\frac{\partial}{\partial w_j} + [R^{\bar{w}_j\bar{w}_j}]\bar{w}_j\frac{\partial}{\partial\bar{w}_j}),
\end{aligned} \tag{5.9}$$

$$\hat{\mathcal{A}} = \sum_{j\in\mathbb{Z}_1^d\cap\mathbb{Z}_2^d}([R^{z_j w_j}]w_j\frac{\partial}{\partial z_j} + [R^{\bar{z}_j\bar{w}_j}]\bar{w}_j\frac{\partial}{\partial\bar{z}_j} + [R^{w_j z_j}]z_j\frac{\partial}{\partial w_j} + [R^{\bar{w}_j\bar{z}_j}]\bar{z}_j\frac{\partial}{\partial\bar{w}_j})$$

and

$$\begin{aligned}
\omega_{+,b} &= \omega_b + [R^{\theta_b}], \quad (b=1,\dots,n), \\
\tilde{\omega}_{+,b} &= \tilde{\omega}_b + [R^{\varphi_b}], \quad (b=1,\dots,m), \\
\Omega_{+,i} &= \Omega_i - i[R^{z_i z_i}], \quad (i\in\mathbb{Z}_1^d), \\
\tilde{\Omega}_{+,i} &= \tilde{\Omega}_i - i[R^{w_i w_i}], \quad (i\in\mathbb{Z}_2^d), \\
A_{+,i} = A_i - i[R^{z_i w_i}], \quad &\tilde{A}_{+,i} = \tilde{A}_i - i[R^{w_i z_i}], \quad (i\in\mathbb{Z}_1^d\cap\mathbb{Z}_2^d).
\end{aligned}$$

It follows form (5.9) that $\|\hat{N}\|_{s;D_\rho(r,s)\times\mathcal{O}} \le \|R\|_{s;D_\rho(r,s)\times\mathcal{O}}$. Then $|[R^{z_i^\sigma z_i^\sigma}]|_\mathcal{O} \le \|R\|_{s;D_\rho(r,s)\times\mathcal{O}} \le \varepsilon$; thus

$$|\Omega_{+,i} - \Omega_i|_\mathcal{O} = |[R^{z_i z_i}]|_\mathcal{O} \le \varepsilon, \quad \text{for } i\in\mathbb{Z}_1^d.$$

Similarly,

$$|\tilde{\Omega}_{+,i} - \tilde{\Omega}_i|_\mathcal{O}, |A_{+,i} - A_i|_\mathcal{O}, \quad |\tilde{A}_{+,i} - \tilde{A}_i|_\mathcal{O}, \quad |\omega_{+,b} - \omega_b|_\mathcal{O}, \quad |\tilde{\omega}_{+,b} - \tilde{\omega}_b|_\mathcal{O} \le \varepsilon.$$

5.4. **New perturbation.** The new perturbation is

$$P_+ = \int_0^1 (\phi_F^t)^*[R(t), F]dt + (\phi_F^1)^*(P - R),$$

with $R(t) = (1 - t)[R] + tR$.

Let $\eta = \varepsilon^{1/3}$. We now give the estimate of $\|P_+\|_{\eta s; D(r-2\delta, \eta s/4) \times \mathcal{O}}$.

$$\|(\phi_F^t)^*[R(t), F]\|_{\eta s; D(r-2\delta, s/4) \times \mathcal{O}} \le c\delta^{-1}\eta^{-1}\|R\|_{s; D(r,s) \times \mathcal{O}}\|F\|_{s; D(r,s) \times \mathcal{O}}$$
$$\le c\delta^{-1}\eta^{-1}\gamma^{-5}K^{5\tau+19}\varepsilon^2.$$

Consider the estimate for $\|(\phi_F^1)^*(P - R)\|_{\eta s; D(r-2\delta, \eta s/4) \times \mathcal{O}}$. Rewrite $P - R$ as $P - R = P_{(1)} + P_{(2)}$, where

$$P_{(1)} = \sum_{\mathsf{v} \in \{\theta_b, \varphi_b, I_b, J_b, z_j^e, w_j^e\}} \sum_{\substack{|l|+|\tilde{l}|+|\alpha| \\ +|\beta|+|\tilde{\alpha}|+|\tilde{\beta}| \le 1}} (1 - \mathcal{T}_K) P_{l\alpha\beta, \tilde{l}\tilde{\alpha}\tilde{\beta}}^{(\mathsf{v})}(\theta, \varphi) I^l J^{\tilde{l}} z^\alpha \bar{z}^\beta w^{\tilde{\alpha}} \bar{w}^{\tilde{\beta}} \frac{\partial}{\partial \mathsf{v}},$$

$$P_{(2)} = \sum_{\mathsf{v} \in \{I_b, J_b, z_j^e, w_j^e\}} \sum_{\substack{|l|+|\tilde{l}|+|\alpha| \\ +|\beta|+|\tilde{\alpha}|+|\tilde{\beta}| \ge 2}} P_{l\alpha\beta, \tilde{l}\tilde{\alpha}\tilde{\beta}}^{(\mathsf{v})}(\theta, \varphi) I^l J^{\tilde{l}} z^\alpha \bar{z}^\beta w^{\tilde{\alpha}} \bar{w}^{\tilde{\beta}} \frac{\partial}{\partial \mathsf{v}}.$$

Then

$$\|P_{(1)}\|_{\eta s; D(r-\delta, \eta s/2) \times \mathcal{O}} \le \eta^{-1} e^{-K\delta}\|P\|_{s; D(r,s) \times \mathcal{O}} \le \eta^{-1} e^{-K\delta}\varepsilon,$$
$$\|P_{(2)}\|_{\eta s; D(r-\delta, \eta s/2) \times \mathcal{O}} \le c\eta\|P\|_{s; D(r,s) \times \mathcal{O}} \le c\eta\varepsilon.$$

This implies that

$$\|P - R\|_{\eta s; D(r-\delta, \eta s/2) \times \mathcal{O}} \le e^{-K\delta}\|P\|_{\eta s; D(r, \eta s/2) \times \mathcal{O}} + c\eta\|P\|_{s; D(r,s) \times \mathcal{O}}$$
$$\le \eta^{-1} e^{-K\delta}\varepsilon + c\eta\varepsilon.$$

Therefore,

$$\|P_+\|_{\eta s; D(r-2\delta, \eta s/4) \times \mathcal{O}}$$
$$\le \|(\phi_F^t)^*[R(t), F]\|_{\eta s; D(r-2\delta, \eta s/4) \times \mathcal{O}} + \|(\phi_F^1)^*(P - R)\|_{\eta s; D(r-2\delta, \eta s/4) \times \mathcal{O}}$$
$$\le c\delta^{-1}\eta^{-1}\gamma^{-5}K^{5\tau+19}\varepsilon^2 + \eta^{-1} e^{-K\delta}\varepsilon + c\eta\varepsilon$$
$$= c\delta^{-1}\gamma^{-5}K^{5\tau+19}\varepsilon^{5/3} + e^{-K\delta}\varepsilon^{2/3} + c\varepsilon^{\frac{4}{3}} \le \varepsilon_+.$$

The following lemma ensures that the new perturbation P_+ satisfies reversibility and momentum conservation condition.

Lemma 5.4. (1) P_+ is S-reversible;
(2) $[P_+, \mathbb{M}_l] = 0$, $l = 1, \ldots, d$;

Proof. (1) We conclude from (2) in Lemma 5.2 that $\Phi \circ S = S \circ \Phi$, which implies $X_+ \circ S = -DS \cdot X_+$. It is obvious that $N_+ \circ S = -DS \cdot N_+$, thus $P_+ \circ S = -DS \cdot P_+$.

(2) We know that $[N, \mathbb{M}_l] = 0$ and $[P, \mathbb{M}_l] = 0$. From Lemma 5.2, we obtain $[F, \mathbb{M}_l] = 0$. This together with

$$P_+ = P - R + [P, F] + \frac{1}{2!}[[N, F], F] + \frac{1}{2!}[[P, F], F]$$
$$+ \cdots + \frac{1}{i!}[\ldots [N, \underbrace{F] \ldots, F}_{i}] + \frac{1}{i!}[\ldots [P, \underbrace{F] \ldots, F}_{i}] + \ldots \tag{5.10}$$

implies that $[P_+, \mathbb{M}_l] = 0$. $\qquad \square$

Lemma 5.5. *Suppose P satisfies* (A6), *and F satisfies* (A6) *with $\varepsilon^{2/3}$ in place of ε. Also assume that for $\sigma = \pm$, $|j| > K$,*

$$\frac{\partial F^{(I_b)}}{\partial z_j^\sigma} = 0, \quad \frac{\partial F^{(I_b)}}{\partial w_j^\sigma} = 0, \quad \frac{\partial F^{(J_b)}}{\partial z_j^\sigma} = 0, \quad \frac{\partial F^{(J_b)}}{\partial w_j^\sigma} = 0,$$

$$\frac{\partial F^{(\theta_b)}}{\partial z_j^\sigma} = 0, \quad \frac{\partial F^{(\theta_b)}}{\partial w_j^\sigma} = 0, \quad \frac{\partial F^{(\varphi_b)}}{\partial z_j^\sigma} = 0, \quad \frac{\partial F^{(\varphi_b)}}{\partial w_j^\sigma} = 0,$$

and for $|i \mp j| > K$,

$$\frac{\partial F^{(z_i^\sigma)}}{\partial z_j^{\pm\sigma}} = 0, \quad \frac{\partial F^{(w_i^\sigma)}}{\partial w_j^{\pm\sigma}} = 0, \quad \frac{\partial F^{(z_i^\sigma)}}{\partial w_j^{\pm\sigma}} = 0, \quad \frac{\partial F^{(w_i^\sigma)}}{\partial z_j^{\pm\sigma}} = 0.$$

Then $[P, F]$ also satisfies (A6) *with ε_+ in place of ε.*

Proof. By the definition of Lie bracket, the z_i-component of $[P, F]$ is

$$[P, F]^{(z_i)} = \sum_{\mathsf{u} \in \mathscr{V}} \left(\frac{\partial P^{(z_i)}}{\partial \mathsf{u}} F^{(\mathsf{u})} - \frac{\partial F^{(z_i)}}{\partial \mathsf{u}} P^{(\mathsf{u})} \right),$$

where $\mathscr{V} = \{\theta_a, \varphi_b, z_i, w_j, \bar{z}_i, \bar{w}_j : a = 1, \ldots, n; b = 1, \ldots, m; i \in \mathbb{Z}_1^d; j \in \mathbb{Z}_2^d\}$.

To verify that $[P, F]$ satisfies (A6), we only consider $\frac{\partial}{\partial z_j}[P, F]^{(z_i)}$ and the derivatives with respect to the other components are similarly analyzed.

It suffices to consider $\sum_h \frac{\partial^2 P^{(z_i)}}{\partial z_h \partial z_j} F^{(z_h)}$ and $\sum_h \frac{\partial P^{(z_i)}}{\partial z_h} \frac{\partial F^{(z_h)}}{\partial z_j}$ in $\frac{\partial}{\partial z_j}[P, F]^{(z_i)}$ since the other terms can be similarly studied.

Let $p_{ij}^{zz} = \lim_{t \to \infty} \frac{\partial P^{(z_{i+tc})}}{\partial z_{j+tc}}$ and $f_{ij}^{zz} = \lim_{t \to \infty} \frac{\partial F^{(z_{i+tc})}}{\partial z_{j+tc}}$. Then

$$\left\| \sum_h \left(\frac{\partial^2 P^{(z_{i+tc})}}{\partial z_h \partial z_{j+tc}} F^{(z_h)} - \lim_{t \to \infty} \frac{\partial^2 P^{(z_{i+tc})}}{\partial z_h \partial z_{j+tc}} F^{(z_h)} \right) \right\|_{s; D_\rho(r-\delta, s/2)}$$

$$\leq \|F\|_{s; D_\rho(r,s)} \left\| \frac{\partial P^{(z_{i+tc})}}{\partial z_{j+tc}} - p_{ij}^{zz} \right\|_{s; D_\rho(r,s)} \tag{5.11}$$

$$\leq c \frac{\varepsilon^{5/3}}{|t|} \mathrm{e}^{-\rho|i-j|} \leq \frac{\varepsilon_+}{30|t|} \mathrm{e}^{-\rho_+|i-j|}$$

and

$$\left\| \sum_h \left(\frac{\partial P^{(z_{i+tc})}}{\partial z_{h+tc}} \frac{\partial F^{(z_{h+tc})}}{\partial z_{j+tc}} - p_{ih}^{zz} f_{hj}^{zz} \right) \right\|_{s; D_\rho(r-\delta, s/2)}$$

$$\leq \sum_h \|f_{hj}^{zz}\|_{s; D_\rho(r-\delta, s/2)} \left\| \frac{\partial P^{(z_{i+tc})}}{\partial z_{h+tc}} - p_{ih}^{zz} \right\|_{s; D_\rho(r-\delta, s/2)}$$

$$+ \sum_h \|p_{ih}^{zz}\|_{s; D_\rho(r-\delta, s/2)} \left\| \frac{\partial F^{(z_{h+tc})}}{\partial z_{j+tc}} - f_{hj}^{zz} \right\|_{s; D_\rho(r-\delta, s/2)} \tag{5.12}$$

$$+ \sum_h \left\| \frac{\partial P^{(z_{i+tc})}}{\partial z_{h+tc}} - p_{ih}^{zz} \right\|_{s; D_\rho(r-\delta, s/2)} \left\| \frac{\partial F^{(z_{h+tc})}}{\partial z_{j+tc}} - f_{hj}^{zz} \right\|_{s; D_\rho(r-\delta, s/2)}$$

$$\leq c K^d \frac{\varepsilon^{5/3}}{|t|} \mathrm{e}^{-\rho|i-j|} + c K^d \frac{\varepsilon^{5/3}}{t^2} \mathrm{e}^{-\rho|i-j|}$$

$$\leq \frac{\varepsilon_+}{30|t|} \mathrm{e}^{-\rho_+|i-j|}.$$

Note that h is bounded by cK^d in the above inequality since $|i - h| \leq K$ and $|j - h| \leq K$.   □

The following lemma follows from (5.10) and Lemma 5.5.

Lemma 5.6. P_+ *satisfies* (A6) *with* $K_+, \varepsilon_+, \rho_+$ *in place of* K, ε, ρ.

5.5. **Iteration and convergence.** For given $r > 0, s > 0, L > 0$ and $0 < \gamma, \varepsilon < 1$, consider c a positive constant depending only on n, m, τ. For $\nu \geq 0$, we define the iterative sequences:

$$\delta_\nu = \frac{r}{2^{\nu+3}}, \quad r_{\nu+1} = r_\nu - 2\delta_\nu, \quad r_0 = r,$$

$$\varepsilon_{\nu+1} = c\gamma^{-5}\delta_\nu^{-1}K_\nu^{5\tau+19}\varepsilon_\nu^{5/3} + \varepsilon_\nu^{7/6}, \quad \varepsilon_0 = \varepsilon,$$

$$\mathrm{e}^{-K_\nu\delta_\nu} = \varepsilon_\nu^{1/2},$$

$$\eta_\nu = \varepsilon_\nu^{1/3}, \quad s_{\nu+1} = \frac{1}{4}\eta_\nu s_\nu, \quad s_0 = s,$$

$$L_{\nu+1} = L_\nu + \varepsilon_\nu, \quad L_0 = L,$$

$$\rho_\nu = \rho(1 - \sum_{i=2}^{\nu+1} 2^{-i}).$$

5.5.1. *Iteration lemma.* According to the preceding analysis, we obtain the following lemma.

Lemma 5.7. *Let* ε *be small enough and* $\nu \geq 0$. *Suppose that*

(1) *The normal form*

$$N_\nu + \mathcal{A}_\nu = \omega_\nu(\zeta)\frac{\partial}{\partial\theta} + \tilde{\omega}_\nu(\zeta)\frac{\partial}{\partial\varphi} + \sum_{\sigma=\pm}\sigma i(\Omega_\nu(\zeta)z^\sigma\frac{\partial}{\partial z^\sigma} + \widetilde{\Omega}_\nu(\zeta)w^\sigma\frac{\partial}{\partial w^\sigma}) + \mathcal{A}_\nu,$$

with $\zeta \in \mathcal{O}_\nu$ *satisfies* (5.7) *with* $\omega_\nu, \tilde{\omega}_\nu, \Omega_\nu, \tilde{\Omega}_\nu, A_\nu, \tilde{A}_\nu$ *and* K_ν;
(2) $\omega_\nu, \tilde{\omega}_\nu, \Omega_{\nu,j}, \tilde{\Omega}_{\nu,j}$ *are* C_W^4 *smooth in* ζ *and satisfy*

$$|\omega_\nu - \omega_{\nu-1}|_\mathcal{O}, |\tilde{\omega}_\nu - \tilde{\omega}_{\nu-1}|_\mathcal{O}, |\Omega_{\nu,j} - \Omega_{\nu-1,j}|_\mathcal{O}, |\tilde{\Omega}_{\nu,j} - \tilde{\Omega}_{\nu-1,j}|_\mathcal{O} \leq \varepsilon_{\nu-1};$$

(3) $N_\nu + \mathcal{A}_\nu + P_\nu$ *satisfies* (A5) *and* (A6) *with* $K_\nu, \varepsilon_\nu, \rho_\nu$ *and*

$$\|P_\nu\|_{s_\nu; D_{\rho_\nu}(r_\nu, s_\nu) \times \mathcal{O}_\nu} \leq \varepsilon_\nu.$$

Then there exists a real analytic, S-invariant transformation

$$\Phi_\nu : D_{\rho_\nu}(r_{\nu+1}, s_{\nu+1}) \times \mathcal{O}_\nu \to D_{\rho_\nu}(r_\nu, s_\nu)$$

satisfying

$$\|\Phi_\nu - id\|_{s_{\nu+1}; D_{\rho_\nu}(r_{\nu+1}, s_{\nu+1}) \times \mathcal{O}_\nu} \leq c\varepsilon_\nu^{1/2}, \qquad (5.13)$$

$$\|D\Phi_\nu - Id\|_{s_{\nu+1}; D_{\rho_\nu}(r_{\nu+1}, s_{\nu+1}) \times \mathcal{O}_\nu} \leq c\varepsilon_\nu^{1/2}, \qquad (5.14)$$

and a closed subset

$$\mathcal{O}_{\nu+1} = \mathcal{O}_\nu \setminus \cup_{K_\nu < |k| + |\tilde{k}| \leq K_{\nu+1}} \mathcal{R}_{k\tilde{k}}^{\nu+1}(\gamma), \qquad (5.15)$$

where $\mathcal{R}_{k\tilde{k}}^{\nu+1}(\gamma)$ *is defined in* (5.18), *such that* $X_{\nu+1} = (\Phi_\nu)^*X_\nu = N_{\nu+1} + \mathcal{A}_{\nu+1} + P_{\nu+1}$ *satisfies the same assumptions as* X_ν *with* '$\nu + 1$' *in place of* 'ν'.

5.5.2. *Convergence.* We now complete the proof of Theorem 3.2. Let

$$X_0 = N_0 + \mathcal{A}_0 + P_0 = N + \mathcal{A} + P$$

be an initial S-reversible vector field and satisfies the assumptions of Theorem 3.2. Recall that $\varepsilon_0 = \varepsilon$, $r_0 = r$, $s_0 = s$, $\rho_0 = \rho$, $L_0 = L$. Suppose $\mathcal{O}$ is a compact set of positive Lebesgue measure and all the conditions in the iterative lemma with $\nu = 0$ hold. Then we inductively obtain the following sequences

$$\mathcal{O}_{\nu+1} \subset \mathcal{O}_\nu,$$
$$\Psi^\nu = \Phi_0 \circ \Phi_1 \circ \cdots \circ \Phi_\nu : D_{\rho_\nu}(r_{\nu+1}, s_{\nu+1}) \times \mathcal{O}_\nu \to D_{\rho_0}(r_0, s_0),$$
$$X_{\nu+1} = (\Psi^\nu)^* X = N_{\nu+1} + \mathcal{A}_{\nu+1} + P_{\nu+1}.$$

Let $\widetilde{\mathcal{O}} = \cap_{\nu=0}^\infty \mathcal{O}_\nu$. Using (5.13), (5.14) and following from [14], we obtain that $N_\nu + A_\nu, \Psi^\nu, D\Psi^\nu$ converge uniformly on $D_{\frac{\varrho}{2}}(\frac{r}{2}, 0) \times \widetilde{\mathcal{O}}$ with

$$N_\infty + \mathcal{A}_\infty = \omega_\infty \frac{\partial}{\partial \theta} + \tilde{\omega}_\infty \frac{\partial}{\partial \varphi} \sum_{\sigma = \pm} \sigma i (\Omega_\infty z^\sigma \frac{\partial}{\partial z^\sigma} + \widetilde{\Omega}_\infty w^\sigma \frac{\partial}{\partial w^\sigma}) + \mathcal{A}_\infty.$$

which corresponds to a motion equation,

$$\begin{aligned} \dot{\theta} &= \omega_\infty, \quad \dot{\varphi} = \tilde{\omega}_\infty, \\ \dot{I} &= 0, \quad \dot{J} = 0, \\ \dot{z}_j^\sigma &= \sigma i \Omega_{\infty,j} z_j^\sigma, \quad \sigma = \pm, \ j \in \mathbb{Z}_1^d \setminus \mathbb{Z}_2^d, \\ \dot{w}_j^\sigma &= \sigma i \tilde{\Omega}_{\infty,j} w_j^\sigma, \quad j \in \mathbb{Z}_2^d \setminus \mathbb{Z}_1^d, \\ \begin{pmatrix} \dot{z}_j^\sigma \\ \dot{w}_j^\sigma \end{pmatrix} &= \sigma i \begin{pmatrix} \Omega_{\infty,j} & A_{\infty,j} \\ \tilde{A}_{\infty,j} & \tilde{\Omega}_{\infty,j} \end{pmatrix} \begin{pmatrix} z_j^\sigma \\ w_j^\sigma \end{pmatrix}, \quad j \in \mathbb{Z}_1^d \cap \mathbb{Z}_2^d. \end{aligned} \tag{5.16}$$

By the choice of ε_ν and K_ν, we have $\varepsilon_{\nu+1} = O(\varepsilon_\nu^{7/6})$, thus $\varepsilon_\nu \to 0, \nu \to \infty$. And we also have $\sum_{\nu=0}^\infty \varepsilon_\nu \le 2\varepsilon$. Consider the flow ϕ_X^t of X. It follows from $X_{\nu+1} = (\Psi^\nu)^* X$ that

$$\phi_X^t \circ \Psi^\nu = \Psi^\nu \circ \phi_{X_{\nu+1}}^t. \tag{5.17}$$

Thanks to the uniform converge of X_ν, Ψ^ν and $D\Psi^\nu$, we can take limits on both sides of (5.17). Therefore, on $D_{\frac{\varrho}{2}}(\frac{r}{2}, 0) \times \widetilde{\mathcal{O}}$, we have

$$\phi_X^t \circ \Psi^\infty = \Psi^\infty \circ \phi_{X_\infty}^t,$$
$$\Psi^\infty : D_{\frac{\varrho}{2}}(\frac{r}{2}, 0) \times \widetilde{\mathcal{O}} \to D_\rho(r, s) \times \mathcal{O}.$$

It follows that for each $\zeta \in \widetilde{\mathcal{O}}$, the set $\Psi^\infty(\mathbb{T}^{n+m} \times \{\zeta\})$ is and embedded torus which is invariant for the original perturbed reversible system at $\zeta \in \widetilde{\mathcal{O}}$.

5.6. **Measure estimate.** Let $\mathcal{O}_{-1} = \mathcal{O}$ and $K_{-1} = 0$. At the νth step of the KAM iteration, the resonant set $\mathcal{R}^\nu \subset \mathcal{O}_{\nu-1}$ need to be excluded.

$$\mathcal{R}^\nu = \cup_{K_{\nu-1} < |k| + |\tilde{k}| \le K_\nu} \mathcal{R}_{k\tilde{k}}^\nu, \tag{5.18}$$

with

$$\mathcal{R}_{k\tilde{k}}^\nu = \mathcal{R}_{k\tilde{k}}^{0\nu} \cup (\cup_i \mathcal{R}_{k\tilde{k},i}^{1\nu}) \cup (\cup_i \mathcal{R}_{k\tilde{k},i}^{2\nu}) \cup (\cup_{ij} \mathcal{R}_{k\tilde{k},ij}^{11,\pm\nu}) \cup (\cup_{ij} \mathcal{R}_{k\tilde{k},ij}^{12,\pm\nu})$$
$$(\cup_{ij} \mathcal{R}_{k\tilde{k},ij}^{22,\pm\nu}) \cup (\cup_i \mathcal{R}_{k\tilde{k},i}^{3\nu}) \cup (\cup_{ij} \mathcal{R}_{k\tilde{k},ij}^{13,\pm\nu}) \cup (\cup_{ij} \mathcal{R}_{k\tilde{k},ij}^{23,\pm\nu}) \cup (\cup_{ij} \mathcal{R}_{k\tilde{k},ij}^{34,\pm\nu}),$$

where

$$\mathcal{R}_{k\tilde{k}}^{0\nu} = \{\zeta \in \mathcal{O}_{\nu-1} : |\langle k, \omega_\nu(\zeta)\rangle + \langle \tilde{k}, \tilde{\omega}_\nu(\zeta)\rangle| < \frac{\gamma}{K_\nu^\tau}\}; \tag{5.19}$$

$$\mathcal{R}_{k\tilde{k},i}^{1\nu} = \{\zeta : |\langle k, \omega_\nu\rangle + \langle \tilde{k}, \tilde{\omega}_\nu\rangle + \Omega_{\nu,i}| < \frac{\gamma}{K_\nu^\tau}\}, \quad i \in \mathbb{Z}_1^d; \tag{5.20}$$

$$\mathcal{R}_{k\tilde{k},i}^{2\nu} = \{\zeta : |\langle k, \omega_\nu\rangle + \langle \tilde{k}, \tilde{\omega}_\nu\rangle + \tilde{\Omega}_{\nu,i}| < \frac{\gamma}{K_\nu^\tau}\}, \quad i \in \mathbb{Z}_2^d; \tag{5.21}$$

$$\mathcal{R}_{k\tilde{k},ij}^{11,\pm\nu} = \{\zeta : |\langle k, \omega_\nu\rangle + \langle \tilde{k}, \tilde{\omega}_\nu\rangle + \Omega_{\nu,i} \pm \Omega_{\nu,j}| < \frac{\gamma}{K_\nu^\tau}\}, \quad i,j \in \mathbb{Z}_1^d \setminus \mathbb{Z}_2^d; \tag{5.22}$$

$$\mathcal{R}_{k\tilde{k},ij}^{12,\pm\nu} = \{\zeta : |\langle k, \omega_\nu\rangle + \langle \tilde{k}, \tilde{\omega}_\nu\rangle + \Omega_{\nu,i} \pm \tilde{\Omega}_{\nu,j}| < \frac{\gamma}{K_\nu^\tau}\},$$
$$i \in \mathbb{Z}_1^d \setminus \mathbb{Z}_2^d, j \in \mathbb{Z}_2^d \setminus \mathbb{Z}_1^d; \tag{5.23}$$

$$\mathcal{R}_{k\tilde{k},ij}^{22,\pm\nu} = \{\zeta : |\langle k, \omega_\nu\rangle + \langle \tilde{k}, \tilde{\omega}_\nu\rangle + \tilde{\Omega}_{\nu,i} \pm \tilde{\Omega}_{\nu,j}| < \frac{\gamma}{K_\nu^\tau}\}, \quad i,j \in \mathbb{Z}_2^d \setminus \mathbb{Z}_1^d; \tag{5.24}$$

$$\mathcal{R}_{k\tilde{k},i}^{3\nu} = \{\zeta : |\det((\langle k, \omega_\nu\rangle + \langle \tilde{k}, \tilde{\omega}_\nu\rangle)I_2 + M_{\nu,i})| < \frac{\gamma}{K_\nu^\tau}\}, \quad i \in \mathbb{Z}_1^d \cap \mathbb{Z}_2^d; \tag{5.25}$$

$$\mathcal{R}_{k\tilde{k},ij}^{13,\pm\nu} = \{\zeta : |\det((\langle k, \omega_\nu\rangle + \langle \tilde{k}, \tilde{\omega}_\nu\rangle + \Omega_{\nu,i})I_2 \pm M_{\nu,j})| < \frac{\gamma}{K_\nu^\tau}\},$$
$$(i,j) \text{ or } (i,j) \in (\mathbb{Z}_1^d \cap \mathbb{Z}_2^d) \times (\mathbb{Z}_1^d \setminus \mathbb{Z}_2^d); \tag{5.26}$$

$$\mathcal{R}_{k\tilde{k},ij}^{23,\pm\nu} = \{\zeta : |\det((\langle k, \omega_\nu\rangle + \langle \tilde{k}, \tilde{\omega}_\nu\rangle + \tilde{\Omega}_{\nu,i})I_2 \pm M_{\nu,j})| < \frac{\gamma}{K_\nu^\tau}\},$$
$$(i,j) \text{ or } (i,j) \in (\mathbb{Z}_1^d \cap \mathbb{Z}_2^d) \times (\mathbb{Z}_2^d \setminus \mathbb{Z}_1^d); \tag{5.27}$$

$$\mathcal{R}_{k\tilde{k},ij}^{34,\pm\nu} = \{\zeta : |\det((\langle k, \omega_\nu\rangle + \langle \tilde{k}, \tilde{\omega}_\nu\rangle)I_4 + M_{\nu,i} \otimes I_2 \pm I_2 \otimes M_{\nu,j}^T)| < \frac{\gamma}{K_\nu^\tau}\},$$
$$i,j \in \mathbb{Z}_1^d \cap \mathbb{Z}_2^d. \tag{5.28}$$

Note that $\mathcal{R}_{k\tilde{k},ij}^{11,-\nu}$, $\mathcal{R}_{k\tilde{k},ij}^{22,-\nu}$, and $\mathcal{R}_{k\tilde{k},ij}^{34,-\nu}$ are the most complicated three case, and the former two have been studied in [11]; thus it suffices to consider the last case. We denote

$$D_\nu = (\langle k, \omega_\nu\rangle + \langle \tilde{k}, \tilde{\omega}_\nu\rangle)I_4 + M_{\nu,i} \otimes I_2 - I_2 \otimes M_{\nu,j}^T.$$

Lemma 5.8. *For any given $i,j \in \mathbb{Z}_1^d \cap \mathbb{Z}_2^d$ with $|i-j| \leq K_\nu$, either*

$$|\det(D_\nu)| \geq 1$$

or there are $i_0, j_0, c_1, \ldots, c_{d-1} \in \mathbb{Z}^d$ with $|i_0|, |j_0|, |c_1|, \ldots, |c_{d-1}| \leq 3K_\nu^2$ and $t_1, \ldots, t_{d-1} \in \mathbb{Z}$ such that $i = i_0 + t_1 c_1 + \cdots + t_{d-1} c_{d-1}, j = j_0 + t_1 c_1 + \cdots + t_{d-1} c_{d-1}$.

For a proof of the above lemma see [11]. In the following, for convenience of notation, let

$$\mathbf{t} := (t_1, t_2, \ldots, t_{d-1}), \quad \mathbf{c} := (c_1, c_2, \ldots, c_{d-1})$$

and $\mathbf{t} \cdot \mathbf{c} := t_1 c_1 + t_2 c_2 + \cdots + t_{d-1} c_{d-1}$. By Lemma 5.8, we have the following result.

Lemma 5.9.

$$\cup_{i,j \in \mathbb{Z}_1^d \cap \mathbb{Z}_2^d} \mathcal{R}_{k\tilde{k},ij}^{34,-\nu} \subset \cup_{\substack{i_0,j_0,c_1,c_2,\ldots,c_{d-1}\in\mathbb{Z}^d; \\ t_1,t_2,\ldots,t_{d-1}\in\mathbb{Z}}} \mathcal{R}_{k\tilde{k},i_0+\mathbf{t}\cdot\mathbf{c},j_0+\mathbf{t}\cdot\mathbf{c}}^{34,-\nu},$$

where $|i_0|, |j_0|, |c_1|, |c_2|, \ldots, |c_{d-1}| \leq 3K_\nu^2$.

Lemma 5.10. *Let $\tau > \frac{4(d-1)(d+1)!}{(d-1)!(d+1)-1}$. Then for fixed $k, \tilde{k}, i_0, j_0, c_1, \ldots, c_{d-1}$,*

$$\text{meas}\left(\cup_{t_1, t_2, \ldots, t_{d-1} \in \mathbb{Z}} \mathcal{R}^{34, -\nu}_{k\tilde{k}, i_0 + \mathbf{t} \cdot \mathbf{c}, j_0 + \mathbf{t} \cdot \mathbf{c}}\right) \leq c \frac{\gamma^{1/4}}{K_\nu^{\frac{\tau}{(d+1)!}}}.$$

Proof. Without loss of generality, we assume $|t_1| \leq |t_2| \leq \cdots \leq |t_{d-1}|$. Let $\Omega_{\nu,j} = |j|^2 + \Omega^0_{\nu,j}$, $\tilde{\Omega}_{\nu,j} = |j|^2 + \tilde{\Omega}^0_{\nu,j}$, and $D_\nu(\mathbf{t}) = (\langle k, \omega_\nu \rangle + \langle \tilde{k}, \tilde{\omega}_\nu \rangle) I_4 + M_{\nu, i_0 + \mathbf{t} \cdot \mathbf{c}} \otimes I_2 - I_2 \otimes M^T_{\nu, j_0 + \mathbf{t} \cdot \mathbf{c}}$.

Using Töplitz-Lipschitz property of $\mathcal{A}_\nu + P_\nu$, for $l = i_0, j_0, 1 \leq j \leq d-1$, we have

$$|\Omega^0_{\nu, l + \mathbf{t} \cdot \mathbf{c}} - \lim_{t_j \to \infty} \Omega^0_{\nu, l + \mathbf{t} \cdot \mathbf{c}}| < \frac{\varepsilon}{|t_j|},$$

$$|\tilde{\Omega}^0_{\nu, l + \mathbf{t} \cdot \mathbf{c}} - \lim_{t_j \to \infty} \tilde{\Omega}^0_{\nu, l + \mathbf{t} \cdot \mathbf{c}}| < \frac{\varepsilon}{|t_j|},$$

$$|A_{\nu, l + \mathbf{t} \cdot \mathbf{c}} - \lim_{t_j \to \infty} A_{\nu, l + \mathbf{t} \cdot \mathbf{c}}| < \frac{\varepsilon}{|t_j|},$$

$$|\tilde{A}_{\nu, l + \mathbf{t} \cdot \mathbf{c}} - \lim_{t_j \to \infty} \tilde{A}_{\nu, l + \mathbf{t} \cdot \mathbf{c}}| < \frac{\varepsilon}{|t_j|}.$$

Then we have

$$|\det(D_\nu(\mathbf{t})) - \lim_{t_j \to \infty} \det(D_\nu(\mathbf{t}))| < \frac{\varepsilon K_\nu^4}{|t_j|}.$$

We consider the resonant set

$$\mathcal{R}^{34, -\nu}_{k\tilde{k}, i_0 j_0 \mathbf{c} \infty^{d-1}} = \left\{\zeta \in \mathcal{O}_{\nu-1} : |\lim_{t_1 \to \infty} (\lim_{t_2, \ldots, t_{d-1} \to \infty} \det(D_\nu(\mathbf{t})))| < \frac{\gamma}{K_\nu^{\frac{\tau}{d!}}}\right\}.$$

For fixed $k, \tilde{k}, i_0, j_0, \mathbf{c}$, its Lebesgue measure satisfies

$$\text{meas}(\mathcal{R}^{34, -\nu}_{k\tilde{k}, i_0 j_0 \mathbf{c} \infty^{d-1}}) \leq \frac{\gamma^{1/4}}{K_\nu^{\frac{\tau}{d!}}},$$

and for $\zeta \in \mathcal{O}_{\nu-1} \setminus \mathcal{R}^{34, -\nu}_{k\tilde{k}, i_0 j_0 \mathbf{c} \infty^{d-1}}$,

$$|\lim_{t_1 \to \infty} (\lim_{t_2, \ldots, t_{d-1} \to \infty} \det(D_\nu(\mathbf{t})))| \geq \frac{\gamma}{K_\nu^{\frac{\tau}{d!}}}.$$

Below we consider the following cases:

Case 1: $|t_1| > K_\nu^{\frac{\tau}{d!}+4}$. For $\zeta \in \mathcal{O}_{\nu-1} \setminus \mathcal{R}^{34, -\nu}_{k\tilde{k}, i_0 j_0 \mathbf{c} \infty^{d-1}}$, we have

$$|\det(D_\nu(\mathbf{t}))| \geq |\lim_{t_1 \to \infty} (\lim_{t_2, \ldots, t_{d-1} \to \infty} \det(D_\nu(\mathbf{t})))| - \sum_{j=1}^{d-1} \frac{\varepsilon K_\nu^4}{|t_j|}$$

$$\geq \frac{\gamma}{K_\nu^{\frac{\tau}{d!}}} - (d-1) \frac{\varepsilon}{K_\nu^{\frac{\tau}{d!}}}$$

$$\geq \frac{\gamma}{2K_\nu^{\frac{\tau}{d!}}} \geq \frac{\gamma}{K_\nu^\tau}.$$

Case l: This is the general case $2 \leq l \leq d-1$. We consider $|t_1| \leq K_\nu^{\frac{\tau}{d!}+4}$, $|t_2| \leq K_\nu^{\frac{2\tau}{d!}+4}, \ldots, |t_{l-1}| \leq K_\nu^{\frac{(l-1)!\tau}{d!}+4}$, $|t_l| > K_\nu^{\frac{l!\tau}{d!}+4}$. We define the resonant set

$$\mathcal{R}^{34, -\nu}_{k\tilde{k}, i_0 j_0 \mathbf{c} t_1 t_2 \ldots t_{l-1} \infty^{d-l}} = \left\{\zeta \in \mathcal{O}_{\nu-1} : |\lim_{t_l, \ldots, t_{d-1} \to \infty} \det(D_\nu(\mathbf{t}))| < \frac{\gamma}{K_\nu^{\frac{l!\tau}{d!}}}\right\}.$$

Then for fixed $k, \tilde{k}, i_0, j_0, \mathbf{c}, t_1, t_2, \ldots, t_{l-1}$, its Lebesgue measure satisfies

$$\text{meas}(\mathcal{R}^{34,-\nu}_{k\tilde{k},i_0j_0\mathbf{c}t_1\ldots t_{l-1}\infty^{d-l}}) \leq \frac{\gamma^{1/4}}{K_\nu^{\frac{l!\tau}{d!}}},$$

$$\text{meas}\left(\cup_{|t_1|,\ldots,|t_{l-1}|\leq K_\nu^{\frac{(l-1)!\tau}{d!}+4}} \mathcal{R}^{34,-\nu}_{k\tilde{k},i_0j_0\mathbf{c}t_1\ldots t_{l-1}\infty^{d-l}} \right)$$

$$\leq 2^{l-1}K_\nu^{\frac{(l-1)!(l-1)\tau}{d!}+4(l-1)}\frac{\gamma^{1/4}}{K_\nu^{\frac{l!\tau}{d!}}}$$

$$\leq \frac{2^{l-1}\gamma^{1/4}}{K_\nu^{\frac{(l-1)!\tau}{d!}-4(l-1)}}.$$

Thus for $|t_1| \leq K_\nu^{\frac{\tau}{d!}+4}$, $|t_2| \leq K_\nu^{\frac{2\tau}{d!}+4}$, $\ldots$, $|t_{l-1}| \leq K_\nu^{\frac{(l-1)!\tau}{d!}+4}$, $|t_l| > K_\nu^{\frac{l!\tau}{d!}+4}$, $\zeta \in \mathcal{O}_{\nu-1} \setminus \mathcal{R}^{34,-\nu}_{k\tilde{k},i_0j_0\mathbf{c}t_1\ldots t_{l-1}\infty^{d-l}}$, we have

$$|\det(D_\nu(\mathbf{t}))| \geq \big|\lim_{t_l,\ldots,t_{d-1}\to\infty} \det(D_\nu(\mathbf{t}))\big| - \sum_{j=l}^{d-1}\frac{\varepsilon K_\nu^4}{|t_j|}$$

$$\geq \frac{\gamma}{K_\nu^{\frac{l!\tau}{d!}}} - (d-l)\frac{\varepsilon}{K_\nu^{\frac{l!\tau}{d!}}}$$

$$\geq \frac{\gamma}{2K_\nu^{\frac{l!\tau}{d!}}} \geq \frac{\gamma}{K_\nu^\tau}.$$

Case d: $|t_1| \leq K_\nu^{\frac{\tau}{d!}+4}$, $|t_2| \leq K_\nu^{\frac{2\tau}{d!}+4}$, $\ldots$, $|t_{d-1}| \leq K_\nu^{\frac{(d-1)\tau}{d!}+4}$. We define the resonant set

$$\mathcal{R}^{34,-\nu}_{k\tilde{k},i_0j_0\mathbf{c}t_1t_2\ldots t_{d-1}} = \left\{\zeta \in \mathcal{O}_{\nu-1} : |\det(D_\nu(\mathbf{t}))| < \frac{\gamma}{K_\nu^\tau}\right\}.$$

For fixed $k, \tilde{k}, i_0, j_0, \mathbf{c}, t_1, t_2, \ldots, t_{d-1}$, its Lebesgue measure satisfies

$$\text{meas}(\mathcal{R}^{34,-\nu}_{k\tilde{k},i_0j_0\mathbf{c}t_1t_2\ldots t_{d-1}}) \leq \frac{\gamma^{1/4}}{K_\nu^\tau},$$

$$\text{meas}\left(\cup_{|t_1|,\ldots,|t_{d-1}|\leq K_\nu^{\frac{(d-1)!\tau}{d!}+4}} \mathcal{R}^{34,-\nu}_{k\tilde{k},i_0j_0\mathbf{c}t_1t_2\ldots t_{d-1}} \right)$$

$$\leq 2^{d-1}K_\nu^{\frac{(d-1)!(d-1)\tau}{d!}+4(d-1)}\frac{\gamma^{1/4}}{K_\nu^\tau}$$

$$\leq \frac{2^{d-1}\gamma^{1/4}}{K_\nu^{\frac{\tau}{d}-4(d-1)}}.$$

Therefore, if $\tau > \frac{4(d-1)(d+1)!}{(d-1)!(d+1)-1}$, we obtain

$$\text{meas}(\cup_{t_1,t_2,\ldots,t_{d-1}\in\mathbb{Z}}\mathcal{R}^{34,-\nu}_{k\tilde{k},i_0+\mathbf{t}\cdot\mathbf{c},j_0+\mathbf{t}\cdot\mathbf{c}}) \leq c\frac{\gamma^{1/4}}{K_\nu^{\frac{\tau}{(d+1)!}}}.$$

$\square$

According to the above analysis, we obtain the following lemma.

Lemma 5.11. *Let* $\tau > d!(2d(d+1) + n + m + 1) + \frac{4(d-1)(d+1)!}{(d-1)!(d+1)-1}$. *Then the total measure of resonant set should be excluded during the KAM iteration is*

$$\text{meas}(\cup_{\nu \geq 0} \mathcal{R}^\nu) = O(\gamma^{1/4}).$$

6. Appendix

Suppose the vector field $X(\theta, I, z, \bar{z})$ is defined on $D_\rho(r, s) = \{y = (\theta, I, z, \bar{z}) : |\operatorname{Im}\theta| < r, |I| < s, \|z\|_\rho < s, \|\bar{z}\|_\rho < s\}$.

Definition 6.1. Suppose S is an involution map: $S^2 = id$. Vector field X is called reversible with respect to S (or S-reversible), if

$$DS \cdot X = -X \circ S,$$

i.e.,

$$(DS(y))X(y) = -X(S(y)), y \in D_\rho(r, s),$$

where DS is the tangent map of S.

Definition 6.2. *Suppose S is an involution map: $S^2 = id$. Vector field X is called invariant with respect to S (or S-invariant), if $DS \cdot X = X \circ S$.*

Definition 6.3. A transformation Φ is called invariant with respect to above involution S (or S-invariant), if $\Phi \circ S = S \circ \Phi$.

Lemma 6.4. (1) *If X and Y are both S-reversible (or S-invariant), then $[X, Y]$ is S-invariant.*
(2) *If X is S-reversible, Y is S-invariant and the transformation Φ is S-invariant, then $[X, Y]$ and $\Phi^* X$ are both S-reversible. In particular, the flow ϕ_Y^t of Y are S-invariant, thus $(\phi_Y^t)^* X$ is S-reversible.*

Lemma 6.5 (Cauchy's inequality, [11]). *Let $0 < \delta < r$. For an analytic function $f(\theta, I, z, \bar{z})$ on $D_\rho(r, s)$, it holds*

$$\|\frac{\partial f}{\partial \theta_b}\|_{s;D_\rho(r-\delta,s)} \leq \frac{c}{\delta}\|f\|_{s;D_\rho(r,s)},$$

$$\|\frac{\partial f}{\partial I_b}\|_{s;D_\rho(r,s/2)} \leq \frac{c}{s}\|f\|_{s;D_\rho(r,s)},$$

$$\|\frac{\partial f}{\partial z_i^\sigma}\|_{s;D_\rho(r,s/2)} \leq \frac{c}{s}\|f\|_{s;D_\rho(r,s)}\mathrm{e}^{\rho|i|}, \ \sigma = \pm.$$

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