TWO SOLUTIONS FOR NONHOMOGENEOUS KLEIN-GORDON EQUATIONS COUPLED WITH BORN-INFELD TYPE EQUATIONS

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ABSTRACT. This article concerns the nonhomogeneous Klein-Gordon equation coupled with a Born-Infeld type equation,

$$-\Delta u + V(x)u - (2\omega + \phi)\phi u = f(x, u) + h(x), \quad x \in \mathbb{R}^3,$$
$$\Delta\phi + \beta\Delta_4\phi = 4\pi(\omega + \phi)u^2, \quad x \in \mathbb{R}^3,$$

where ω is a positive constant. We obtain the existence of two solutions using the Mountain Pass Theorem, and the Ekeland's variational principle in critical point theory.

1. INTRODUCTION AND MAIN RESULTS

We consider a nonhomogeneous Klein-Gordon equation coupled with a Born-Infeld type equation,

$$-\Delta u + V(x)u - (2\omega + \phi)\phi u = f(x, u) + h(x), \quad x \in \mathbb{R}^3,$$
$$\Delta\phi + \beta\Delta_4\phi = 4\pi(\omega + \phi)u^2, \quad x \in \mathbb{R}^3, \tag{1.1}$$

where ω is a positive constant. Under certain assumptions on V, f and h, we show the existence of two solutions by using the Mountain Pass Theorem and the Ekeland's variational principle in critical point theory.

It is well known that the Klein-Gordon equation (1.1) can be used in the theory of electrically charged fields (see [16]), and that the Born-Infeld theory was proposed by Born [6, 7, 8] to overcome the infinite energy problem associated with a point-charge source in the original Maxwell theory. The presence of the nonlinear term f in (1.1) can model the interaction between many particles, or the external nonlinear perturbations. For more details and physical aspects of (1.1), we refer the readers to [4, 9, 17, 21, 28].

In recent years, the Born-Infeld nonlinear electromagnetism has become more important since its relevance in the theory of superstring and membranes. By using variational methods, several existence results for problem (1.1) have been found with constant potential $V(x) = m^2 - \omega^2$. We recall some of them.

The case of $h \equiv 0$, that is the homogeneous case, has been widely studied. In 2002, D'Avenia et al. [14] considered for the following Klein-Gordon equation with

an Born-Infeld equation on $\mathbb{R}^3$,

$$
\begin{aligned}
-\Delta u + [m^2 - (\omega + \phi)^2]\phi u &= f(x, u), \quad x \in \mathbb{R}^3, \\
\Delta \phi + \beta \Delta_4 \phi &= 4\pi(\omega + \phi)u^2, \quad x \in \mathbb{R}^3,
\end{aligned}
\tag{1.2}
$$

for the power nonlinearity $f(x, u) = |u|^{p-2}u$, where ω and m are constants. By using the mountain pass theorem, they proved that(1.2) has infinitely many radially symmetric solutions under $|m| > |\omega|$ and $4 < p < 6$. Mugnai [21] studied the case $2 < p \leq 4$ assuming $((p-2)/2)^{1/2}|m| > \omega > 0$. Later, Teng et al. [22] obtained a nontrivial solution for (1.2) with $f(x, u) = |u|^{p-2}u + |u|^{2^*-2}u$ under the conditions $4 \leq p < 6$ and $m > \omega$. He et al. [20] improved the existence results from [22] and studied the existence of ground state solution for the problem(1.2) with $f(x, u) = |u|^{p-2}u + |u|^{2^*-2}u$.

Recently, for general potential $V(x)$, Chen and Song [13] obtained the existence of multiple nontrivial solutions for (1.2) with the nonlinearity $f(x, u) = \lambda k(x)|u|^{q-2}u + g(x)|u|^{p-2}u$, that is, the Klein-Gordon equation with concave and convex nonlinearities coupled with Born-Infeld equations on $\mathbb{R}^3$. Other related results about homogeneous Klein-Gordon equation with Born-Infeld equations can be found in [1, 23, 24, 25, 29]. For the nonhomogeneous case, $h \not\equiv 0$, Chen and Li [10] proved that (1.1) has two nontrivial radially symmetric solutions if $f(x, u) = |u|^{p-2}u$ and $h(x)$ is radially symmetric.

Motivated by above works, we consider (1.1) with general assumptions on f, and without any radially symmetric assumptions on f and h. More precisely, we assume the following:

(H1) $V \in C(\mathbb{R}^3, \mathbb{R})$ satisfies $V_0 = \inf_{x \in \mathbb{R}^3} V(x) > 0$. For every $M > 0$, and $\text{meas}\{x \in \mathbb{R}^3 : V(x) \leq M\} < +\infty$, where meas denotes the Lebesgue measures in $\mathbb{R}^3$;

(H2) $f \in C(\mathbb{R}^3 \times \mathbb{R}, \mathbb{R})$ and there exist $C_1 > 0$ and $p \in (2, 6)$ such that

$$
|f(x, t)| \leq C_1(|t| + |t|^{p-1});
$$

(H3) $f(x, t) = o(t)$ uniformly in x as $|t| \to 0$, uniformly for $x \in \mathbb{R}^3$;

(H4) There exist $\theta > 2$ and $D_1, D_2 > 0$ such that $F(x, t) \geq D_1|t|^\theta - D_2$, for a.e. $x \in \mathbb{R}^3$ and every t sufficiently large;

(H5) There exist C_2, r_0 are two positive constants and $\mu > 2$ such that

$$
\mathcal{F}(x, t) := \frac{1}{\mu}f(x, t)t - F(x, t) \geq -C_2|t|^2, \quad |t| \geq r_0;
$$

(H6) $h \in L^2(\mathbb{R}^3), h(x) \geq 0, h(x) \not\equiv 0$.

Before stating our main results, we introduce some notation. Let $H^1(\mathbb{R}^3)$ be the usual Sobolev space endowed with the standard scalar product and norm

$$
(u, v)_H = \int_{\mathbb{R}^3} (\nabla u \nabla v + uv)\, dx, \quad \|u\|_H = \left(\int_{\mathbb{R}^3} (|\nabla u|^2 + |u|^2)\, dx\right)^{1/2}.
$$

The space $D^{1,2}(\mathbb{R}^3)$ is the completion of $C_0^\infty(\mathbb{R}^3)$ with respect to the norm

$$
\|u\|_{D^{1,2}(\mathbb{R}^3)} = \left(\int_{\mathbb{R}^3} |\nabla u|^2\, dx\right)^{1/2}.
$$

The norm on $L^s = L^s(\mathbb{R}^3)$ with $1 < s < \infty$ is given by $|u|_s^s = \int_{\mathbb{R}^3} |u|^s \, dx$. $D(\mathbb{R}^3)$ is the completion of $C_0^\infty(\mathbb{R}^3)$ with respect to the norm

$$\|u\|_D := |\nabla u|_2 + |\nabla u|_4.$$

$D(\mathbb{R}^3)$ is continuously embedded in $D^{1,2}(\mathbb{R}^3)$. By the Sobolev inequality, we know that $D^{1,2}(\mathbb{R}^3)$ is continuously embedded in $L^6 = L^6(\mathbb{R}^3)$ and $D(\mathbb{R}^3)$ is continuously embedded in $D(\mathbb{R}^3)$ is continuously embedded in $L^\infty = L^\infty(\mathbb{R}^3)$.

Under condition (H1), we define a new Hilbert space

$$E := \left\{ u \in H^1(\mathbb{R}^3) : \int_{\mathbb{R}^3} (|\nabla u|^2 + V(x)u^2) \, dx < \infty \right\}$$

with the inner product

$$\langle u, v \rangle = \int_{\mathbb{R}^3} (\nabla u \cdot \nabla v + V(x)uv) \, dx$$

and the norm $\|u\| = \langle u, u \rangle^{1/2}$. Obviously, the embedding $E \hookrightarrow L^s(\mathbb{R}^3)$ is continuous, for $2 \le s \le 6$. Consequently, for $2 \le s \le 6$, there exists a constant $d_s > 0$ such that

$$|u|_s \le d_s \|u\|, \quad \forall u \in E. \tag{1.3}$$

Furthermore, it follows from (H1) that the embedding $E \hookrightarrow L^s(\mathbb{R}^3)$ is compact for any $s \in [2, 6)$ (See [3]).

System (1.1) has a variational structure. In fact, we consider the functional $\mathcal{J} : E \times D(\mathbb{R}^3) \to \mathbb{R}$ defined by

$$\mathcal{J}(u, \phi) = \frac{1}{2} \int_{\mathbb{R}^3} (|\nabla u|^2 + V(x)u^2 - (2\omega + \phi)\phi u^2) \, dx - \frac{1}{8\pi} \int_{\mathbb{R}^3} |\nabla \phi|^2 \, dx$$
$$- \frac{\beta}{16\pi} \int_{\mathbb{R}^3} |\nabla \phi|^4 \, dx - \int_{\mathbb{R}^3} F(x, u) \, dx - \int_{\mathbb{R}^3} h(x)u \, dx.$$

The solutions $(u, \phi) \in E \times D(\mathbb{R}^3)$ of system (1.1) are the critical points of $\mathcal{J}$. As it is pointed in [13], the functional $\mathcal{J}$ is strongly indefinite and is difficult to investigate. By the reduction method described in [5], we are led to the study of a new functional $I : E \to \mathbb{R}$ defined by $I(u) = \mathcal{J}(u, \phi_u)$. By Proposition 2.1 below, $I(u)$ is defined as follows which does not present such strongly indefinite nature. We can obtain a C^1 functional $I : E \to \mathbb{R}$ given by

$$I(u) = \frac{1}{2} \int_{\mathbb{R}^3} [|\nabla u|^2 + V(x)u^2 - (2\omega + \phi_u)\phi_u u^2] \, dx$$
$$- \frac{1}{8\pi} \int_{\mathbb{R}^3} |\nabla \phi_u|^2 \, dx - \frac{\beta}{16\pi} \int_{\mathbb{R}^3} |\nabla \phi_u|^4 \, dx - \int_{\mathbb{R}^3} F(x, u) \, dx - \int_{\mathbb{R}^3} h(x)u \, dx$$
$$= \frac{1}{2} \int_{\mathbb{R}^3} (|\nabla u|^2 + V(x)u^2 + \phi_u^2 u^2) \, dx$$
$$+ \frac{1}{8\pi} \int_{\mathbb{R}^3} |\nabla \phi_u|^2 \, dx + \frac{3\beta}{16\pi} \int_{\mathbb{R}^3} |\nabla \phi_u|^4 \, dx - \int_{\mathbb{R}^3} F(x, u) \, dx - \int_{\mathbb{R}^3} h(x)u \, dx$$
$$= \frac{1}{2} \int_{\mathbb{R}^3} (|\nabla u|^2 + V(x)u^2 - \omega \phi_u u^2) \, dx$$
$$+ \frac{\beta}{16\pi} \int_{\mathbb{R}^3} |\nabla \phi_u|^4 \, dx - \int_{\mathbb{R}^3} F(x, u) \, dx - \int_{\mathbb{R}^3} h(x)u \, dx. \tag{1.4}$$

We consider the map $\Phi : E \to D$, $u \mapsto \phi_u$. Then by standard arguments, $\Phi \in C^1(E, D)$. The Gateaux derivative of I is

$$\langle I'(u), v \rangle = \int_{\mathbb{R}^3} \left(\nabla u \cdot \nabla v + V(x)uv - (2\omega + \phi_u)\phi_u uv \right) dx$$
$$- \int_{\mathbb{R}^3} f(x, u)v \, dx - \int_{\mathbb{R}^3} h(x)v \, dx$$

for all $u, v \in E$. Our main result reads as follows.

Theorem 1.1. *Suppose* (H1)–(H6) *hold. Then there exists a positive constant m_0 such that system* (1.1) *admits at least two different solutions $u_0, \widetilde{u}_0$ in E satisfying $I(u_0) < 0$ and $I(\widetilde{u}_0) > 0$ if $|h|_2 < m_0$.*

Remark 1.2. Chen and Song [11] proved that Klein-Gordon-Maxwell equation had two nontrivial solutions if $f(x, t)$ satisfies the local (AR) condition:

(LAR) There exist $\mu > 2$ and $r_0 > 0$ such that $\mathcal{F}(x, t) := \frac{1}{\mu} f(x, t)t - F(x, t) \geq 0$
for every $x \in \mathbb{R}^3$ and $|t| \geq r_0$, where $F(x, t) = \int_0^t f(x, s) ds$.

This condition is employed not only to prove that the Euler-Lagrange function associated has a mountain pass geometry, but also to guarantee that the Palais-Smale sequences, or Cerami sequences are bounded. Under condition (LAR) in this article, $\mathcal{F}(x, t)$ may have negative values.

Another widely used condition was introduced by Jeanjean [18],

- There exists $\theta \geq 1$ such that $\theta \mathcal{F}_1(x, t) \geq \mathcal{F}_1(x, st)$ for all $s \in [0, 1]$ and $t \in \mathbb{R}$, where $\mathcal{F}_1(x, t) := \frac{1}{4} f(x, t)t - F(x, t)$.

We can observe that if $s = 0$, then $\mathcal{F}_1(x, t) \geq 0$, but in our condition (H5), $\mathcal{F}(x, t)$ may assume negative values.

In [2, 12], the authors studied the Schrödinger-Poisson equation by assuming the following global condition to replace the (AR) condition:

- There exists $0 \leq \beta < \alpha$ such that $tf(t) - 4F(t) \geq -\beta t^2$, for all $t \in \mathbb{R}$, where α is a positive constant such that $\alpha \leq V(x)$.

Notice that we only need the local condition (H5) to obtain nontrivial solutions.

Li and Tang [19] used the following condition to obtain infinitely many solutions for homogenous KGM systems:

- There exist two positive constants D_3 and r_0 such that $\frac{1}{4} f(x, t)t - F(x, t) \geq -D_3 |t|^2$, if $|t| \geq r_0$.

Obviously, our condition (H5) is weaker than this condition. Therefore, it is interesting to consider the nonhomogeneous system (1.1) under the conditions (H4) and (H5).

To the best of our knowledge, Theorem 1.1 is the first result about the existence of two solutions for the nonhomogeneous Klein-Gordon equation coupled with Born-Infeld equations on $\mathbb{R}^3$ with general nonlinearity f.

Throughout this article, letters $C_i, d_i, L_i, M_i, i = 1, 2, 3 \ldots$ will denote various positive constants which may vary from line to line and are not essential to the problem. We denote the weak convergence by "$\rightharpoonup$" and the strong convergence by "$\to$". Also if we take a subsequence of a sequence $\{u_n\}$, we shall denote it again by $\{u_n\}$.

The paper is organized as follows. In Section 2, we introduce the variational setting for the problem and give some related preliminaries. In Section 3, we prove our main result.

2. Variational setting and compactness condition

From [5], we know that the signs of ω is not relevant for the existence of solutions, so we can assume that $\omega > 0$. Evidently, the properties of ϕ_u plays an important role in the study of $\mathcal{J}$. So we need the following technical results.

Proposition 2.1. *For any $u \in H^1(\mathbb{R}^3)$, there exists a unique $\phi = \phi_u \in D(\mathbb{R}^3)$ which satisfies*
$$\Delta\phi + \beta\Delta_4\phi = 4\pi(\phi + \omega)u^2 \quad in \ \mathbb{R}^3.$$
Moreover, the map $\Phi : u \in H^1(\mathbb{R}^3) \mapsto \phi_u \in D(\mathbb{R}^3)$ is continuously differentiable, and

$$-\omega \leq \phi_u \leq 0 \quad on \ \{x \in \mathbb{R}^3 | u(x) \neq 0\}; \tag{2.1}$$

$$\int_{\mathbb{R}^3} (|\nabla\phi_u|^2 + \beta|\nabla\phi_u|^4) \, dx \leq 4\pi\omega^2 |u|_2^2. \tag{2.2}$$

Inequality (2.1) was proved in [13], and (2.1) was proved in [21]. By Proposition 2.1 and (1.1), if $u \in E$ is a critical point of I, then $(u, \phi_u) \in E \times D(\mathbb{R}^3)$ is a critical point of $\mathcal{J}$; that is, $(u, \phi_u) \in E \times D(\mathbb{R}^3)$ is a solution of system (1.1). Now we prove the function I has the mountain pass geometry.

Lemma 2.2. *Let $h \in L^2(\mathbb{R}^3)$ and assume that (H1)–(H3) hold. Then there are positive constants ρ, α, m_0 such that $I(u) \geq \alpha$ for all $u \in E$ satisfying $\|u\| = \rho$ and h satisfying $|h|_2 < m_0$.*

Proof. By (H3), for any $\varepsilon > 0$, there exists $\delta > 0$ such that $|f(x,t)| \leq \varepsilon|t|$ for all $x \in \mathbb{R}^3$ and $|t| \leq \delta$. By (H2), we obtain

$$\begin{aligned}
|f(x,t)| &\leq C_1(|t| + |t|^{p-1}) \\
&\leq C_1\Big(|t|\frac{t}{\delta}|^{p-2} + |t|^{p-1}\Big) \\
&= C_1\Big(\frac{1}{\delta^{p-2}} + 1\Big)|t|^{p-1}, \quad \text{for } |t| \geq \delta, \text{ a.e. } x \in \mathbb{R}^3.
\end{aligned}$$

Then for all $t \in \mathbb{R}$ and a.e. $x \in \mathbb{R}^3$ we have

$$|f(x,t)| \leq \varepsilon|t| + C_1\Big(\frac{1}{\delta^{p-2}} + 1\Big)|t|^{p-1} := \varepsilon|t| + C_\varepsilon|t|^{p-1}$$

and

$$|F(x,t)| \leq \frac{\varepsilon}{2}|t|^2 + \frac{C_\varepsilon}{p}|t|^p. \tag{2.3}$$

Therefore, by (2.3), Proposition 2.1 and Hölder's inequality, we have

$$\begin{aligned}
I(u) &\geq \frac{1}{2}\|u\|^2 - \frac{\varepsilon}{2}\int_{\mathbb{R}^3} |u|^2 \, dx - \frac{C_\varepsilon}{p}\int_{\mathbb{R}^3} |u|^p \, dx - |h|_2|u|_2 \\
&\geq \frac{1}{2}\|u\|^2 - \frac{\varepsilon}{2}d_2^2\|u\|^2 - \frac{C_\varepsilon}{p}d_p^p\|u\|^p - d_2|h|_2\|u\| \\
&= \|u\|\Big\{\Big(\frac{1}{2} - \frac{\varepsilon}{2}d_2^2\Big)\|u\| - \frac{C_\varepsilon}{p}d_p^p\|u\|^{p-1} - d_2|h|_2\Big\}.
\end{aligned}$$

Let $\varepsilon = 1/2d_2^2$ and $g(t) = \frac{t}{4} - \frac{C_\varepsilon}{p} d_p^p t^{p-1}$ for $t \geq 0$. Because $2 < p < 6$, there exists a positive constant ρ such that $\tilde{m}_0 := g(\rho) = \max_{t \geq 0} g(t) > 0$. Taking $m_0 := \tilde{m}_0 / 2d_2^2$, it follows that there exists a positive constant α such that $I(u)|_{\|u\|=\rho} \geq \alpha$ for all h satisfying $|h|_2 < m_0$. The proof is complete. $\qquad \square$

Lemma 2.3. *Assume that* (H1)–(H5) *are satisfied. Then there exists a function* $u_0 \in E$ *with* $\|u_0\| > \rho$ *such that* $I(u_0) < 0$, *where* ρ *is given in Lemma 2.2.*

Proof. By (H4), there exist $L_1 > 0$ large enough and $M_1 > 0$, such that

$$F(x,t) \geq M_1|t|^\theta, \quad \text{for } |t| \geq L_1. \tag{2.4}$$

By (2.3), we obtain

$$|F(x,t)| \leq C_3(1 + |t|^{p-2})|t|^2, \quad \text{where } C_3 = \max\{\frac{\varepsilon}{2}, \frac{C_\varepsilon}{p}\}, \tag{2.5}$$

and then

$$|F(x,t)| \leq C_3(1 + L_1^{p-2})|t|^2, \quad \text{when } |t| \leq L_1. \tag{2.6}$$

By (2.4) and (2.6), we have

$$F(x,t) \geq M_1|t|^\theta - M_2|t|^2, \quad \forall t \in \mathbb{R}, \tag{2.7}$$

where $M_2 = M_1 L_1^{\theta-2} + C_3(1 + L_1^{p-2})$.

Thus, by Proposition 2.1, taking $u \in E, u \neq 0$ and $\theta > 2$ we have

$$
\begin{aligned}
I(tu) &= \frac{t^2}{2}\|u\|^2 - \frac{t^2}{2}\int_{\mathbb{R}^3}(2\omega\phi_{tu}u^2 + \phi_{tu}^2 u^2)\,dx - \frac{1}{8\pi}\int_{\mathbb{R}^3}|\nabla\phi_{tu}|^2\,dx \\
&\quad - \frac{\beta}{16\pi}\int_{\mathbb{R}^3}|\nabla\phi_{tu}|^4\,dx - \int_{\mathbb{R}^3}F(x,tu)\,dx - t\int_{\mathbb{R}^3}h(x)u\,dx \\
&\leq \frac{t^2}{2}\|u\|^2 + \frac{t^2}{2}\int_{\mathbb{R}^3}\omega^2 u^2\,dx - M_1 t^\theta\int_{\mathbb{R}^3}|u|^\theta\,dx \\
&\quad + M_2 t^2\int_{\mathbb{R}^3}u^2\,dx - t\int_{\mathbb{R}^3}h(x)u\,dx,
\end{aligned}
$$

thus $I(tu) \to -\infty$ as $t \to +\infty$. The lemma is proved by taking $u_0 = t_0 u$ with $t_0 > 0$ large enough and $u \neq 0$. $\qquad \square$

Lemma 2.4. *Under assumptions* (H1)–(H6), *every sequence* $\{u_n\} \subset E$ *satisfying*

$$I(u_n) \to c > 0, \quad \langle I'(u_n), u_n \rangle \to 0$$

is bounded in E. *Moreover,* $\{u_n\}$ *has a strongly convergent subsequence in* E.

Proof. To prove the boundedness of $\{u_n\}$, we argue by contradiction. Suppose that, up to subsequences, we have $\|u_n\| \to +\infty$ as $n \to +\infty$. Let $v_n = \frac{u_n}{\|u_n\|}$, then $\{v_n\}$ is bounded. Going if necessary to a subsequence, for some $v \in E$, we obtain that

$$v_n \rightharpoonup v \quad \text{in } E,$$
$$v_n \to v \quad \text{in } L^s(\mathbb{R}^3),\ 2 \leq s < 6,$$
$$v_n(x) \to v(x) \quad \text{a.e. in } \mathbb{R}^3.$$

Let $\Lambda = \{x \in \mathbb{R}^3 : v(x) \neq 0\}$. Suppose that $meas(\Lambda) > 0$, then $|u_n(x)| \to +\infty$ as $n \to \infty$ for a.e. $x \in \Lambda$. By (1.3) and (2.7), we obtain

$$\begin{aligned}
\int_{\mathbb{R}^3} \frac{F(x, u_n)}{\|u_n\|^\theta} \, dx &\geq M_1 \int_{\mathbb{R}^3} |v_n|^\theta \, dx - M_2 \frac{|u_n|^2}{\|u_n\|^\theta} \\
&\geq M_1 \int_{\mathbb{R}^3} |v_n|^\theta \, dx - \frac{M_2 d_2^2}{\|u_n\|^{\theta-2}} \\
&\to M_1 \int_\Lambda |v|^\theta \, dx > 0 \quad \text{as } n \to \infty.
\end{aligned}$$ (2.8)

By Proposition 2.1, from (1.3) and $\theta > 2$, so we obtain that

$$\Big| \int_{\mathbb{R}^3} \frac{\omega \phi_{u_n} u_n^2}{\|u_n\|^\theta} \, dx \Big| \leq \frac{\omega^2 |u_n|_2^2}{\|u_n\|^\theta} \leq \frac{\omega^2 d_2^2}{\|u_n\|^{\theta-2}} \to 0 \quad \text{as } n \to \infty.$$

$$\Big| \int_{\mathbb{R}^3} \frac{\beta |\nabla \phi_{u_n}|^4}{\|u_n\|^\theta} \, dx \Big| \leq \frac{4\pi \omega^2 |u_n|_2^2}{\|u_n\|^\theta} \leq \frac{4\pi \omega^2 d_2^2}{\|u_n\|^{\theta-2}} \to 0 \quad \text{as } n \to \infty.$$

Since $h \in L^2(\mathbb{R}^3)$, we obtain that

$$\Big| \int_{\mathbb{R}^3} \frac{h(x) u_n}{\|u_n\|^\theta} \, dx \Big| \leq \frac{|h|_2 |u_n|_2}{\|u_n\|^\theta} \leq \frac{|h|_2 d_2}{\|u_n\|^{\theta-1}} \to 0 \quad \text{as } n \to \infty.$$

By the definition of I, (2.7), and the above inequalities, we have

$$\begin{aligned}
0 &= \lim_{n \to +\infty} \frac{I(u_n)}{\|u_n\|^\theta} \\
&= \lim_{n \to +\infty} \Big[\frac{1}{2\|u_n\|^{\theta-2}} - \int_{\mathbb{R}^3} \frac{\omega \phi_{u_n} u_n^2}{2\|u_n\|^\theta} \, dx + \frac{\beta}{16\pi} \int_{\mathbb{R}^3} \frac{|\nabla \phi_{u_n}|^4}{\|u_n\|^\theta} \, dx \\
&\quad - \int_{\mathbb{R}^3} \frac{F(x, u_n)}{\|u_n\|^\theta} \, dx - \int_{\mathbb{R}^3} \frac{h(x) u_n}{\|u_n\|^\theta} \, dx \Big] < 0,
\end{aligned}$$

which is a contradiction. Therefore, $meas(\Lambda) = 0$, which implies $v(x) = 0$ for almost everywhere $x \in \mathbb{R}^3$. By (H2) and (2.5), we have that for all $x \in \mathbb{R}^3$ and $|t| \leq r_0$,

$$\begin{aligned}
|f(x, t)t - \mu F(x, t)| &\leq |f(x, t)t| + \mu |F(x, t)| \\
&\leq C_1(|t|^2 + |t|^p) + \mu C_3(1 + |t|^{p-2})t^2 \\
&\leq C_6(1 + |t|^{p-2})t^2 \\
&\leq C_6(1 + r_0^{p-2})t^2,
\end{aligned}$$

where $C_6 := 2 \max\{C_1, \mu C_3\}$. From this and (H5), we obtain

$$f(x, t)t - \mu F(x, t) \geq -C_7 t^2, \quad \forall (x, t) \in \mathbb{R}^3 \times \mathbb{R}.$$ (2.9)

Since $h \in L^2(\mathbb{R}^3)$, we can also obtain

$$\Big| \int_{\mathbb{R}^3} \frac{h(x) u_n}{\|u_n\|^2} \, dx \Big| \leq \frac{|h|_2 |u_n|_2}{\|u_n\|^2} \leq \frac{|h|_2 d_2}{\|u_n\|} \to 0 \quad \text{as } n \to \infty.$$ (2.10)

By (2.9), (2.10), Proposition 2.1, $v = 0$, and $\mu > 2$, we have

$$\begin{aligned}
&\frac{\mu I(u_n) - \langle I'(u_n), u_n \rangle}{\|u_n\|^2} \\
&= \big(\frac{\mu}{2} - 1 \big) + \int_{\mathbb{R}^3} \frac{f(x, u_n) u_n - \mu F(x, u_n)}{\|u_n\|^2} \, dx + \big(\frac{\mu}{2} + 1 \big) \int_{\mathbb{R}^3} \frac{\phi_{u_n} u_n^2}{\|u_n\|^2} \, dx
\end{aligned}$$

$$+ 2\omega \int_{\mathbb{R}^3} \frac{\phi_{u_n} u_n^2}{\|u_n\|^2} \, dx + \frac{\mu}{8\pi} \int_{\mathbb{R}^3} \frac{|\nabla \phi_{u_n}|^2}{\|u_n\|^2} \, dx + \frac{3\beta\mu}{16\pi} \int_{\mathbb{R}^3} \frac{|\nabla \phi_{u_n}|^4}{\|u_n\|^2} \, dx$$

$$+ (1 - \mu) \int_{\mathbb{R}^3} \frac{h(x) u_n}{\|u_n\|^2} \, dx$$

$$\geq \left(\frac{\mu}{2} - 1\right) - C_7 |v_n|_2^2 - 2\omega^2 |v_n|_2^2 + (1 - \mu) \int_{\mathbb{R}^3} \frac{h(x) u_n}{\|u_n\|^2} \, dx$$

$$\geq \left(\frac{\mu}{2} - 1\right) - (C_7 + 2\omega^2)|v_n|_2^2 + (1 - \mu) \int_{\mathbb{R}^3} \frac{h(x) u_n}{\|u_n\|^2} \, dx$$

$$\to \frac{\mu}{2} - 1 \quad \text{as } n \to \infty.$$

Then we have $0 \geq \frac{1}{2} - \frac{1}{\mu}$, which contradicts with $\mu > 2$. Therefore $\{u_n\}$ is a bounded in E.

Now we shall prove $\{u_n\}$ contains a convergent subsequence. Without loss of generality, passing to a subsequence if necessary, there exists $u \in E$ such that $u_n \rightharpoonup u$ in E. By using the embedding $E \hookrightarrow L^s(\mathbb{R}^3)$ are compact for any $s \in [2, 6)$, $u_n \to u$ in $L^s(\mathbb{R}^3)$ for $2 \leq s < 6$ and $u_n(x) \to u(x)$ a.e. $x \in \mathbb{R}^3$. By (1.4) and the Gateaux derivative of I, we obtain

$$\int_{\mathbb{R}^3} (|\nabla(u_n - u)|^2 + V(x)(u_n - u)^2) \, dx$$

$$= \langle I'(u_n) - I'(u), u_n - u \rangle + 2\omega \int_{\mathbb{R}^3} (\phi_{u_n} u_n - \phi_u u)(u_n - u) \, dx$$

$$+ \int_{\mathbb{R}^3} (f(x, u_n) - f(x, u))(u_n - u) \, dx + \int_{\mathbb{R}^3} (\phi_{u_n}^2 u_n - \phi_u^2 u)(u_n - u) \, dx.$$

Doing easy computations we have

$$\langle I'(u_n) - I'(u), u_n - u \rangle \to 0 \quad \text{as } n \to \infty$$

and

$$\int_{\mathbb{R}^3} [(2\omega + \phi_{u_n})\phi_{u_n} u_n - (2\omega + \phi_u)\phi_u u](u_n - u) \, dx$$

$$= 2\omega \int_{\mathbb{R}^3} [(\phi_{u_n} u_n - \phi_u u)(u_n - u) \, dx + \int_{\mathbb{R}^3} [(\phi_{u_n}^2 u_n - \phi_u^2 u)(u_n - u) \, dx \to 0$$

as $n \to +\infty$. Indeed, by the Hölder inequality, the Sobolev inequality, and Proposition 2.1, we have

$$\left| \int_{\mathbb{R}^3} (\phi_{u_n} - \phi_u)(u_n - u) u_n \, dx \right| \leq |(\phi_{u_n} - \phi_u)(u_n - u)|_2 |u_n|_2$$

$$\leq |\phi_{u_n} - \phi_u|_6 |u_n - u|_3 |u_n|_2$$

$$\leq C \|\phi_{u_n} - \phi_u\|_D |u_n - u|_3 |u_n|_2,$$

where C is a positive constant. Since $u_n \to u$ in $L^s(\mathbb{R}^3)$ for $2 \leq s < 6$, we have

$$\left| \int_{\mathbb{R}^3} (\phi_{u_n} - \phi_u)(u_n - u) u_n \, dx \right| \to 0 \quad \text{as } n \to +\infty,$$

$$\left| \int_{\mathbb{R}^3} \phi_u (u_n - u)(u_n - u) \, dx \right| \leq |\phi_u|_6 |u_n - u|_3 |u_n - u|_2 \to 0 \quad \text{as } n \to +\infty.$$

Thus we obtain

$$\int_{\mathbb{R}^3} [(\phi_{u_n} u_n - \phi_u u)(u_n - u) \, dx$$

$$= \int_{\mathbb{R}^3} (\phi_{u_n} - \phi_u)(u_n - u) u_n \, dx + \int_{\mathbb{R}^3} \phi_u(u_n - u)(u_n - u) \, dx \to 0$$

as $n \to +\infty$. In view of that the sequence $\{\phi_{u_n}^2 u_n\}$ is bounded in $L^{3/2}(\mathbb{R}^3)$, since

$$|\phi_{u_n}^2 u_n|_{3/2} \le |\phi_{u_n}|_6^2 |u_n|_3,$$

it follows that

$$\Big| \int_{\mathbb{R}^3} [(\phi_{u_n}^2 u_n - \phi_u^2 u)(u_n - u) \, dx \Big| \le |\phi_{u_n}^2 u_n - \phi_u^2 u|_{3/2} |u_n - u|_3$$

$$\le (|\phi_{u_n}^2 u_n|_{3/2} + |\phi_u^2 u|_{3/2})|u_n - u|_3 \to 0,$$

as $n \to +\infty$.

From the convergence $u_n \to u$ in $L^s(\mathbb{R}^3)$ for $2 \le s < 6$, we have

$$\int_{\mathbb{R}^3} (f(x, u_n) - f(x, u))(u_n - u) \, dx \to 0 \ as \ n \to +\infty.$$

Therefore,

$$\|u_n - u\|^2$$

$$= \langle I'(u_n) - I'(u), u_n - u \rangle + 2\omega \int_{\mathbb{R}^3} (\phi_{u_n} u_n - \phi_u u)(u_n - u) \, dx$$

$$+ \int_{\mathbb{R}^3} (f(x, u_n) - f(x, u))(u_n - u) \, dx + \int_{\mathbb{R}^3} (\phi_{u_n}^2 u_n - \phi_u^2 u)(u_n - u) \, dx \to 0$$

as $n \to +\infty$. Therefore $\|u_n - u\| \to 0$ in E as $n \to \infty$. The proof is complete. $\square$

3. Proof of main result

Proof of Theorem 1.1. Firstly, we prove that there exists a function $u_0 \in E$ such that $I'(u_0) = 0$ and $I(u_0) < 0$. Since $h \in L^2(\mathbb{R}^3)$, $h \ge 0$ and $h \not\equiv 0$, we can choose a function $\varphi \in E$ such that

$$\int_{\mathbb{R}^3} h(x)\varphi(x) \, dx > 0.$$

Hence, by Proposition 2.1, $\theta > 2$ and (2.7), we obtain that

$$I(t\varphi) \le \frac{t^2}{2}\|\varphi\|^2 + t^2 \int_{\mathbb{R}^3} \omega^2 \varphi^2 \, dx - M_1 t^\theta |\varphi|_\theta^\theta + M_2 t^2 |\varphi|_2^2 - t \int_{\mathbb{R}^3} h(x)\varphi \, dx < 0$$

for $t > 0$ small enough. Thus, we obtain

$$c_0 = \inf\{I(u) : u \in \overline{B}_\rho\} < 0,$$

where $\rho > 0$ is given by Lemma 2.2, $B_\rho = \{u \in E : \|u\| < \rho\}$. By the Ekeland's variational principle [15], there exists a sequence $\{u_n\} \subset \overline{B}_\rho$ such that

$$c_0 \le I(u_n) < c_0 + \frac{1}{n},$$

and

$$I(v) \ge I(u_n) - \frac{1}{n}\|v - u_n\|$$

for all $v \in \overline{B}_\rho$. Then by a standard procedure, we can prove that $\{u_n\}$ is a bounded (PS) sequence of I. Hence, by Lemma 2.4 we know that there exists a function $u_0 \in E$ such that $I'(u_0) = 0$ and $I(u_0) = c_0 < 0$.

Secondly, we prove that there exists a function $\widetilde{u}_0 \in E$ such that $I'(\widetilde{u}_0) = 0$ and $I(\widetilde{u}_0) > 0$.

By Lemmas 2.2 and 2.3 and the Mountain Pass Theorem [26, 27], there is a sequence $\{u_n\} \subset E$ such that

$$I(u_n) \to \widetilde{c}_0 > 0, \quad \text{and} \quad I'(u_n) \to 0.$$

In view of Lemma 2.4, we know that $\{u_n\}$ has a strongly convergent subsequence (still denoted by $\{u_n\}$) in E. So there exists a function $\widetilde{u}_0 \in E$ such that $\{u_n\} \to \widetilde{u}_0$ as $n \to \infty$ and $I'(\widetilde{u}_0) = 0$ and $I(\widetilde{u}_0) > 0$. The proof is complete. $\square$

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