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KIRCHHOFF SYSTEMS INVOLVING FRACTIONAL p-LAPLACIAN AND SINGULAR NONLINEARITY

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Abstract. In this work we consider the fractional Kirchhoff equations with singular nonlinearity,

$$\begin{aligned} &M\Big(\int_{\mathbb{R}^{2N}}\frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}}dxdy\Big)(-\Delta)^s_p u\\ &=\lambda a(x)|u|^{q-2}u+\frac{1-\alpha}{2-\alpha-\beta}c(x)|u|^{-\alpha}|v|^{1-\beta},\quad \text{in } \Omega,\\ &M\Big(\int_{\mathbb{R}^{2N}}\frac{|v(x)-v(y)|^p}{|x-y|^{N+sp}}dxdy\Big)(-\Delta)^s_p v\\ &=\mu b(x)|v|^{q-2}v+\frac{1-\beta}{2-\alpha-\beta}c(x)|u|^{1-\alpha}|v|^{-\beta},\quad \text{in } \Omega,\\ &u=v=0,\quad \text{in } \mathbb{R}^N\setminus\Omega, \end{aligned}$$

where Ω is a bounded domain in $\mathbb{R}^N$ with smooth boundary, $N>ps$, $s\in(0,1)$, $0<\alpha<1$, $0<\beta<1$, $2-\alpha-\beta<p\leq p\theta<q<p_s^*$, $p_s^*=\frac{2N}{N-2s}$ is the fractional Sobolev exponent, λ,μ are two parameters, $a,b,c\in C(\overline{\Omega})$ are non-negative weight functions, $M(t)=k+lt^{\theta-1}$ with $k>0,l,\theta\geq 1$, and $(-\Delta)^s_p$ is the fractional p-laplacian operator. We prove the existence of multiple non-negative solutions by studying the nature of the Nehari manifold with respect to the parameters λ and μ.

1. Introduction

Let Ω be a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$, $N>ps$, $s\in(0,1)$,and $p_s^*=\frac{Np}{N-ps}$. The purpose of this work is to study the existence of multiple solutions for the following Kirchhoff equations with fractional p-Laplacian operator and singular nonlinearity,

$$\begin{aligned} &M\Big(\int_{\mathbb{R}^{2N}}\frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}}dxdy\Big)(-\Delta)^s_p u\\ &=\lambda a(x)|u|^{q-2}u+\frac{1-\alpha}{2-\alpha-\beta}c(x)|u|^{-\alpha}|v|^{1-\beta},\quad \text{in } \Omega, \end{aligned}$$

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$$
\begin{aligned}
& M\Big(\int_{\mathbb{R}^{2N}}\frac{|v(x)-v(y)|^p}{|x-y|^{N+sp}}dxdy\Big)(-\Delta)^s_p v \\
& = \mu b(x)|v|^{q-2}v+\frac{1-\beta}{2-\alpha-\beta}c(x)|u|^{1-\alpha}|v|^{-\beta}, \quad \text{in } \Omega, \\
& u=v=0, \quad \text{in } \mathbb{R}^N\setminus\Omega,
\end{aligned} \tag{1.1}
$$

where $0<\alpha<1$, $0<\beta<1$, $2-\alpha-\beta<p\leq p\theta<q<p_s^*$, $p_s^*=\frac{N}{N-ps}$ is the fractional Sobolev exponent, λ,μ are two parameters, $a,b,c\in C(\overline{\Omega})$ are non-negative weight functions with compact support in Ω, $M(t)=k+lt^{\theta-1}$ with $k>0,l,\theta\geq 1$, and $(-\Delta)^s_p$ is the fractional p-Laplacian operator defined as

$$
(-\Delta)^s_p u(x)=2\lim_{\epsilon\searrow 0}\int_{\mathbb{R}^N\setminus B_\epsilon}\frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))}{|x-y|^{N+sp}}\,\mathrm{d}y, \quad x\in\mathbb{R}^N.
$$

Problems of this type describe diffusion processes in heterogeneous or complex medium (anomalous diffusion) caused by random displacements executed by jumpers that able to walk to neighboring and nearby sites and also excursions to remote sites by way of Lévy flights. They also can be used in modeling turbulence, chaotic dynamics, plasma physics and financial dynamics for more details see [1, 5] and references therein.

Recently, a great deal of attention has been focused on studying this kind of non-local problems. We refer the readers to [17, 18, 19, 20, 21] for Kirchhoff problems involving the laplace operator and a singular term. For fractional Kirchhoff problem involving a singular term of type $u^{-\gamma}$ has been studied in [10], by combining variational methods with an appropriate truncation argument. For further details on the fractional system, we refer the interested readers to [22, 32].

Problem (1.1) without a Kirchhoff coefficient has been studied extensively in recent years. In the case of the problem involving the fractional p-laplace existence results via Morse theory has been treated in Iannizzotto-Liu-Perera-Squassina [16]. The critical case is treated in Perera-Squassina-Yang [23] with additional new abstract result based on a pseudo-index related to the $\mathbb{Z}_2$-cohomological index. These restrictions are used to prove the existence of a range of the validity of the Palais-Smale condition. Note that, in this work, the bifurcation and multiplicity results is obtained for some restrictions on the parameter λ. Moreover, by Nehari manifold and fibering maps the multiplicity of solutions has been investigated in [3, 7, 11, 12, 28, 30]. In particular, in [3], the authors considered the problem

$$
\begin{gathered}
(-\Delta)^s_p u=\lambda|u|^{q-2}u+\frac{2\alpha}{\alpha+\beta}|u|^{\alpha-2}u|v|^\beta, \quad \text{in } \Omega, \\
(-\Delta)^s_p v=\mu|v|^{q-2}v+\frac{2\beta}{\alpha+\beta}|u|^\alpha|v|^{\beta-2}v, \quad \text{in } \Omega, \\
u=v=0, \quad \text{in } \mathbb{R}^N\setminus\Omega,
\end{gathered}
$$

where Ω is a bounded domain in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$, $N>sp$, $s\in(0,1)$, $p<\alpha+\beta<p^*$, $p_s^*=\frac{Np}{N-sp}$ is the fractional Sobolev exponent, λ,μ are two parameters. The authors considered the associated Nehari manifold using the fibering maps and showed the existence of solutions when the pair of parameters (λ,μ) satisfies certain conditions.

In the local setting ($s=1$), problem (1.1) without a Kirchhoff coefficient has an extensive literature. We refer the reader to the monographs of Ghergu-Radulescu

[13] for a more general presentation of these results and to the survey article of Crandall-Rabinowitz-Tartar [6]. After this, many authors have considered the problem above for laplacian, p-Laplacian, N-Laplacian operators, using the technique used in [6] or a combination of this approach with the Nehari's and Perron's methods. Among the references we like to mention [4, 8, 14, 15, 26, 29, 24].

Motivated by above results, we show the existence and multiplicity of nontrivial, non-negative solutions of the singular fractional p-Kirchhoff system (1.1). To state our result, we introduce some notation. Let

$$[u]_{s,p} = \Big(\int_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}} dxdy \Big)^{1/p}$$

be the Gagliardo seminorm of a measurable function $u : \mathbb{R}^{2N} \to \mathbb{R}$. Let

$$W^{s,p}(\mathbb{R}^N) := \{ u \in L^p(\mathbb{R}^N) : \ [u]_{s,p} < \infty \}$$

be the usual fractional Sobolev space endowed with the norm

$$\|u\|_{s,p} := \Big(\|u\|^p_{L^p(\Omega)} + [u]^p_{s,p} \Big)^{1/p}.$$

We denote $\mathcal{Q} = \mathbb{R}^{2N} \setminus \big((\mathbb{R}^N \setminus \Omega) \times (\mathbb{R}^N \setminus \Omega) \big)$ and define the space

$$X := \big\{ u : \mathbb{R}^N \to \mathbb{R} \text{ Lebesgue measurable} : u\backslash_{\Omega} \in L^p(\Omega) \text{ and } \int_{\mathcal{Q}} \frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}} \,\mathrm{d}x\,\mathrm{d}y < \infty \big\}$$

with the norm

$$\|u\|_X = \Big(\|u\|_{L^p(\Omega)} + \int_Q \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}} dxdy \Big)^{1/p}.$$

Through this paper we consider the space X_0 to be the completion of the space $C_0^\infty(\Omega)$ in X, which is can be defined with the norm

$$\|u\|_{X_0} = \Big(\int_Q \frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}} dxdy \Big)^{1/p}.$$

It is readily seen that $(X_0, \|.\|)$ is a uniformly convex Banach space and that the embedding $X_0 \hookrightarrow L^q(\Omega)$ is continuous for all $1 \le q \le p_s^*$, and compact for all $1 \le q < p_s^*$. We define the best constant S of the embedding as

$$S = \inf\{ \|u\|^p_{X_0} : u \in X_0, |u|^p_{p_s^*} = 1 \}. \tag{1.2}$$

The dual space of $(X_0, \|\cdot\|)$ is denoted by $(X^*, \|.\|_*)$, and $\langle \cdot, \cdot \rangle$ denotes the usual duality between X_0 and X^*. When $r + r' \in (p, p^*)$, then, for any $u \in X_0$, we obtain

$$\|u\|_{L^{r+r'}(\Omega)} \le S \|u\|_{X_0}. \tag{1.3}$$

Let $E = X_0 \times X_0$ be the Cartesian product of two Hilbert spaces, which is a reflexive Banach space endowed with the norm

$$\|(u,v)\| = \Big(\int_Q \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}} dxdy + \int_Q \frac{|v(x)-v(y)|^p}{|x-y|^{N+ps}} dxdy \Big)^{1/p}. \tag{1.4}$$

Definition 1.1. We say that $(u,v) \in E$ is a weak solution of problem (1.1) if $u, v > 0$ in Ω, one has

$$M(\|u\|_{X_0}) \int_Q \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))(\phi(x)-\phi(y))}{|x-y|^{N+sp}} \,dx\,dy$$

$$
\begin{aligned}
&+ M(\|v\|_{X_0}) \int_Q \frac{|v(x)-v(y)|^{p-2}(v(x)-v(y))(\psi(x)-\psi(y))}{|x-y|^{N+sp}}\,dx\,dy \\
&= \int_\Omega \big(\lambda a(x)|u|^{q-2}u\phi + \mu b(x)|v|^{q-2}v\phi\big)dx + \frac{1-\alpha}{2-\alpha-\beta}\int_\Omega c(x)u^{-\alpha}v^{1-\beta}\psi\,dx \\
&\quad + \frac{1-\beta}{2-\alpha-\beta}\int_\Omega c(x)u^{1-\alpha}v^{-\beta}\psi\,dx.
\end{aligned}
$$

for all $(\phi,\psi) \in E$.

We give below the precise statements of results that we will prove.

Theorem 1.2. *Let $s \in (0,1)$, $N > sp$ and Ω be a bounded domain in $\mathbb{R}^n$. If $0<\alpha<1,\ 0<\beta<1,\ 2-\alpha-\beta<p\le p\theta<q<p_s^*$, then, there exists a number*

$$
\Lambda_0 = \Big(\frac{q+\alpha+\beta-2}{\|c\|_\infty k(q-p)}\Big)^{\frac{p}{p+\alpha+\beta-2}} \Big(\frac{2-\alpha-\beta-q}{k(2-\alpha-\beta-p)}|\Omega|^{\frac{2_s^*-q}{2_s^*}}\Big)^{-\frac{p}{p-q}} S^{\frac{2-\alpha-\beta}{p+\alpha+\beta-2}},
$$

such that for $0 < (\lambda\|a\|_\infty)^{\frac{p}{p-q}} + (\mu\|b\|_\infty)^{\frac{p}{p-q}} < \Lambda_0$, problem (1.1) has at least two nontrivial positive solutions.

The rest of this article is organized as follows. Section 2 is devoted to proof some lemmas in preparation for the proof of our main result. While, existence of two solutions,Theorem 1.2, will be presented in Sections 3 and 4.

2. Nehari manifold and fibering map analysis

In this section, we collect some basic results on a Nehari manifold and we give the analysis of the fibering maps.

Associated to problem (1.1) we define the functional $E_{\lambda,\mu} : E \to \mathbb{R}$ given by

$$
\begin{aligned}
E_{\lambda,\mu}(u,v) &= \frac{k}{p}\|(u,v)\|^p + \frac{l}{p\theta}\|(u,v)\|^{p\theta} \\
&\quad - \frac{1}{q}\int_\Omega \big(\lambda a(x)|u|^q + \mu b(x)|v|^q\big)dx \\
&\quad - \frac{1}{2-\alpha-\beta}\int_\Omega c(x)(u^+)^{1-\alpha}(v^+)^{1-\beta}\,dx.
\end{aligned}
$$

As usual, $r^+ = \max\{r,0\}$ and $r^- = \max\{-r,0\}$ for $r \in \mathbb{R}$. Notice that $E_{\lambda,\mu}$ is not a C^1 functional in E, and hence classical variational methods are not applicable. Notice that (u,v) is a weak solution of problem (1.1), then $u,v>0$ in Ω and satisfy the equation

$$
\begin{aligned}
&k\|(u,v)\|^p + l\|(u,v)\|^{p\theta} - \lambda\int_\Omega a(x)|u|^q\,\mathrm{d}x \\
&- \mu\int_\Omega b(x)|v|^q\,\mathrm{d}x - \int_\Omega c(x)|u|^{1-\alpha}|v|^{1-\beta}\,\mathrm{d}x = 0.
\end{aligned}
\tag{2.1}
$$

One can easily verify that the energy functional $E_{\lambda,\mu}(u,v)$ is not bounded below on the space E. But we will show that $E_{\lambda,\mu}(u,v)$ is bounded below on the Nehari manifold defined below, and we will extract solutions by minimizing the functional on suitable subsets. The Nehari manifold is defined as

$$
\begin{aligned}
\mathcal{N}_{\lambda,\mu} = \Big\{&(u,v) \in E\setminus\{(0,0)\};\ \frac{k}{p}\|(u,v)\|^p + \frac{l}{p\theta}\|(u,v)\|^{p\theta} - \lambda\int_\Omega a(x)|u|^q\,\mathrm{d}x \\
&- \mu\int_\Omega b(x)|v|^q\,\mathrm{d}x - \int_\Omega c(x)|u|^{1-\alpha}|v|^{1-\beta}\,\mathrm{d}x = 0\Big\}.
\end{aligned}
$$

We note that $\mathcal{N}_{\lambda,\mu}$ contains every solution of problem (1.1).

Now as we know that the Nehari manifold is closely related to the behavior of the functions $\Phi_{u,v} : t \mapsto E_{\lambda,\mu}(tu, tv)$ for $t > 0$ defined by

$$\begin{aligned}\Phi_{u,v}(t) = \frac{kt^p}{p}\|(u,v)\|^p + \frac{lt^{p\theta}}{p\theta}\|(u,v)\|^{p\theta} - \frac{t^q}{q}\int_\Omega \big(\lambda a(x)|u|^q + \mu b(x)|v|^q\big)\,\mathrm{d}x \\ - \frac{t^{2-\alpha-\beta}}{2-\alpha-\beta}\int_\Omega c(x)|u|^{1-\alpha}|v|^{1-\beta}\,\mathrm{d}x,\end{aligned}$$

which gives

$$\begin{aligned}\Phi'_{u,v}(t) = kt^{p-1}\|(u,v)\|^p + lt^{p\theta-1}\|(u,v)\|^{p\theta} - t^{q-1}\int_\Omega \big(\lambda a(x)|u|^q + \mu b(x)|v|^q\big)\,\mathrm{d}x \\ - t^{1-\alpha-\beta}\int_\Omega c(x)|u|^{1-\alpha}|v|^{1-\beta}\,\mathrm{d}x,\end{aligned} \tag{2.2}$$

and

$$\begin{aligned}\Phi''_{u,v}(t) &= (p-1)kt^{p-2}\|(u,v)\|^p + l(p\theta-1)t^{p\theta-2}\|(u,v)\|^{p\theta} \\ &\quad - (q-1)t^{q-2}\int_\Omega \big(\lambda a(x)|u|^q + \mu b(x)|v|^q\big)\,\mathrm{d}x \\ &\quad - (1-\alpha-\beta)t^{-\alpha-\beta}\int_\Omega c(x)|u|^{1-\alpha}|v|^{1-\beta}\,\mathrm{d}x.\end{aligned} \tag{2.3}$$

Such maps are called fibering maps and were introduced by Drábek and Pohozaev in [9].

By Hölder's inequality and Sobolev inequalities, one has

$$\begin{aligned}&\int_\Omega (\lambda a(x)|u|^q + \mu b(x)|v|^q)\,dx \\ &\le |\Omega|^{\frac{p_s^*-q}{p_s^*}}\Big(\lambda\|a\|_\infty\|u\|_{p_s^*}^q + \mu\|b\|_\infty\|v\|_{p_s^*}^q\Big) \\ &\le |\Omega|^{\frac{p_s^*-q}{p_s^*}} S^{-\frac{q}{p}}\,(\lambda\|a\|_\infty\|u\|^q + \mu\|b\|_\infty\|v\|^q) \\ &\le |\Omega|^{\frac{p_s^*-q}{p_s^*}} S^{-\frac{q}{p}}\Big((\lambda\|a\|_\infty)^{\frac{p}{p-q}} + \mu\|b\|_\infty)^{\frac{p}{p-q}}\Big)^{\frac{p-q}{p}}\,(\|u\|^q + \|v\|^q) \\ &\le C|\Omega|^{\frac{p_s^*-q}{p_s^*}} S^{-\frac{q}{p}}\Big((\lambda\|a\|_\infty)^{\frac{p}{p-q}} + (\mu\|b\|_\infty)^{\frac{p}{p-q}}\Big)^{\frac{p-q}{p}}\,\|(u,v)\|^q\end{aligned} \tag{2.4}$$

and using Young's and Sobolev inequalities, we obtain

$$\begin{aligned}&\int_\Omega c(x)|u|^{1-\alpha}|v|^{1-\beta}\,\mathrm{d}x \\ &\le \|c\|_\infty\Big(\frac{1-\alpha}{2-\alpha-\beta}\int_\Omega |u|^{2-\alpha-\beta}\,\mathrm{d}x + \frac{1-\beta}{2-\alpha-\beta}\int_\Omega |v|^{2-\alpha-\beta}\,\mathrm{d}x\Big) \\ &\le \|c\|_\infty S^{-\frac{2-\alpha-\beta}{p}}\|(u,v)\|^{2-\alpha-\beta}.\end{aligned} \tag{2.5}$$

Lemma 2.1. *Let $(u,v) \in E\setminus\{(0,0)\}$. Then $(tu,tv) \in \mathcal{N}_{\lambda,\mu}$ if and only if $\Phi'_{u,v}(t) = 0$.*

Proof. The result is a consequence of the fact that

$$\begin{aligned}\Phi'_{u,v}(t) &= \langle E'_{\lambda,\mu}(u,v),(u,v)\rangle \\ &= kt^{p-1}\|(u,v)\|^p + lt^{p\theta-1}\|(u,v)\|^{p\theta}\end{aligned}$$

$$\begin{aligned}&- t^{q-1}\Big(\int_\Omega \lambda a(x)|u|^q\,\mathrm{d}x - \int_\Omega \mu b(x)|v|^q\,\mathrm{d}x\Big)\\&- t^{1-\alpha-\beta}\int_\Omega c(x)|u|^{1-\alpha}|v|^{1-\beta}\,\mathrm{d}x = 0\end{aligned}$$

if and only if $(tu,tv)\in\mathcal{N}_{\lambda,\mu}$. □

From lemma 2.1, we have that the elements in $\mathcal{N}_{\lambda,\mu}$ correspond to stationary points of the maps $\Phi_{u,v}(tu,tv)$ and in particular, $(u,v)\in\mathcal{N}_{\lambda,\mu}$ if and only if $\Phi'_{u,v}(1)=0$. Hence, it is natural to split $\mathcal{N}_{\lambda,\mu}$ into three parts corresponding to local minima, local maxima and points of inflection $\Phi_{u,v}(t)$ defined as follows:

$$\begin{aligned}\mathcal{N}^+_{\lambda,\mu} &= \{(u,v)\in\mathcal{N}_{\lambda,\mu} : \Phi''_{u,v}(1)>0\}\\&= \{(tu,tv)\in E\setminus\{0,0\} : \Phi'_{u,v}(t)=0, \Phi''_{u,v}(t)>0\},\\\mathcal{N}^-_{\lambda,\mu} &= \{(u,v)\in\mathcal{N}_{\lambda,\mu} : \Phi''_{u,v}(1)<0\}\\&= \{(tu,tv)\in E\setminus\{0,0\} : \Phi'_{u,v}(t)=0, \Phi''_{u}(t)<0\},\\\mathcal{N}^0_{\lambda,\mu} &= \{(u,v)\in\mathcal{N}_{\lambda,\mu} : \Phi''_{u,v}(1)=0\}\\&= \{(tu,tv)\in E\setminus\{0,0\} : \Phi'_{u,v}(t)=0, \Phi''_{u,v}(t)=0\}.\end{aligned}$$

Our first result is as follows.

Lemma 2.2. *If (u,v) is a minimizer of $E_{\lambda,\mu}$ on $\mathcal{N}_{\lambda,\mu}$ such that $(u,v)\notin\mathcal{N}^0_{\lambda,\mu}$. Then, (u,v) is a critical point for $E_{\lambda,\mu}$.*

For a proof of the above lemma see [31].

Lemma 2.3. *$E_{\lambda,\mu}$ is coercive and bounded below on $\mathcal{N}_{\lambda,\mu}$.*

Proof. Since $(u,v)\in\mathcal{N}_{\lambda,\mu}$, then using (2.1) and the embedding of X_0 in $L^{2-\alpha-\beta}(\Omega)$, we obtain

$$\begin{aligned}E_{\lambda,\mu}(u,v) &= k\big(\frac{1}{p}-\frac{1}{q}\big)\|(u,v)\|^p + l\big(\frac{1}{p\theta}-\frac{1}{q}\big)\|(u,v)\|^{p\theta}\\&\quad - \big(\frac{1}{2-\alpha-\beta}-\frac{1}{q}\big)\int_\Omega c(x)|u|^{1-\alpha}|v|^{1-\beta}\,\mathrm{d}x.\end{aligned}$$

Then by (2.5), we obtain

$$\begin{aligned}E_{\lambda,\mu}(u,v) &\ge k\big(\frac{1}{p}-\frac{1}{q}\big)\|(u,v)\|^p + l\big(\frac{1}{p\theta}-\frac{1}{q}\big)\|(u,v)\|^{p\theta}\\&\quad - \big(\frac{1}{2-\alpha-\beta}-\frac{1}{q}\big)\|c\|_\infty S^{-\frac{2-\alpha-\beta}{2}}\|(u,v)\|^{2-\alpha-\beta}.\end{aligned}$$

Since $2-\alpha-\beta<p\le p\theta$, it follows that $E_{\lambda,\mu}$ is coercive and bounded below on $\mathcal{N}_{\lambda,\mu}$. This completes the proof. □

Lemma 2.4. *For each $(u,v)\in\mathcal{N}^-_{\lambda,\mu}$ (respectively $\mathcal{N}^+_{\lambda,\mu}$) with $u,v\ge 0$, and all $(\phi,\psi)\in\mathcal{N}_{\lambda,\mu}$ with $(\phi,\psi)\ge 0$, there exist $\varepsilon>0$ and a continuous function $h=h(r)>0$ such that for all $r\in\mathbb{R}$ with $|r|<\varepsilon$ we have $h(0)=1$ and $h(r)(u+r\phi,v+r\psi)\in\mathcal{N}^-_{\lambda,\mu}$ (respectively $\mathcal{N}^+_{\lambda,\mu}$).*

Proof. We introduce the function $f:\mathbb{R}\times\mathbb{R}\longrightarrow\mathbb{R}$ defined by

$$f(t,r) = kt^{p+\alpha+\beta-2}\|(u+r\phi,v+r\psi)\|^p + lt^{p\theta+\alpha+\beta-2}\|(u+r\phi,v+r\psi)\|^{p\theta}$$

$$
\begin{aligned}
&- (q+\alpha+\beta-2)t^{q+\alpha+\beta-3}\int_\Omega (\lambda a(x)(u+r\phi)^q+\mu b(x)(v+r\psi)^q)\,dx\\
&-\int_\Omega c(x)(u+r\phi)^{1-\alpha}(v+r\psi)^{1-\beta}dx.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
f_t(t,r) &= k(p+\alpha+\beta-2)t^{p+\alpha+\beta-3}\|(u+r\phi,v+r\psi)\|^p\\
&\quad + l(p\theta+\alpha+\beta-2)t^{p\theta+\alpha+\beta-3}\|(u+r\phi,v+r\psi)\|^{p\theta}\\
&\quad - t^{q+\alpha+\beta-2}\int_\Omega (\lambda a(x)(u+r\phi)^q+\mu b(x)(v+r\psi)^q)\,dx.
\end{aligned}
$$

Then, f_t is continuous on $\mathbb{R}\times\mathbb{R}$. Now, since $(u,v)\in\mathcal{N}^-_{\lambda,\mu}\subset\mathcal{N}_{\lambda,\mu}$, we have $f(1,0)=0$, and

$$
\begin{aligned}
f_t(1,0) &= k(p+\alpha+\beta-2)\|(u,v)\|^p + l(p\theta+\alpha+\beta-2)\|(u,v)\|^{p\theta}\\
&\quad - (q+\alpha+\beta-2)\int_\Omega (\lambda a(x)u^q+\mu b(x)v^q)\,dx<0.
\end{aligned}
$$

Therefore, applying the implicit function theorem to the function f at the point $(1,0)$ we obtain a $\delta>0$ and a positive continuous function $h=h(r)>0$, $r\in\mathbb{R}$, $|r|<\delta$ satisfying $h(0)=1$ and $h(r)(u+r\phi,v+r\psi)\in\mathcal{N}_{\lambda,\mu}$, for all $r\in\mathbb{R}$, $|r|<\delta$. Hence, taking $\varepsilon>0$ possibly smaller enough ($\varepsilon<\delta$), we obtain

$$
h(r)(u+r\phi,v+r\psi)\in\mathcal{N}^-_{\lambda,\mu},\quad \forall r\in\mathbb{R},\ |r|<\varepsilon.
$$

The case $(u,v)\in\mathcal{N}^+_{\lambda,\mu}$ may be obtained in the same way. This completes the proof. □

Lemma 2.5. *There exists*

$$
\Lambda_0=\Big(\frac{q+\alpha+\beta-2}{\|c\|_\infty k(q-p)}\Big)^{\frac{p}{p+\alpha+\beta-2}}\Big(\frac{2-\alpha-\beta-q}{k(2-\alpha-\beta-p)}|\Omega|^{\frac{2^*_s-q}{2^*_s}}\Big)^{-\frac{p}{p-q}}S^{\frac{2-\alpha-\beta}{p+\alpha+\beta-2}},
$$

such that for $0<(\lambda\|a\|_\infty)^{\frac{p}{p-q}}+(\mu\|b\|_\infty)^{\frac{p}{p-q}}<\Lambda_0$ *the following holds:*

(1) *If* $\int_\Omega\big(\lambda a(x)|u|^q+\mu b(x)|v|^q\big)\,\mathrm{d}x>0$*, then, there exist a unique* $T_l>0$ *and unique* $t_0<T_l<t_1$ *such that*

$$
\begin{gathered}
\Phi_{u,v}(t_0)=\Phi_{u,v}(t_1),\\
\Phi'_{u,v}(t_0)<0<\Phi'_{u,v}(t_1);
\end{gathered}
$$

that is, $(t_0u,t_0v)\in\mathcal{N}^+_{\lambda,\mu}$, $(t_1u,t_1v)\in\mathcal{N}^-_{\lambda,\mu}$ *and*

$$
\begin{gathered}
E_{\lambda,\mu}(t_0u,t_0v)=\min_{0\le t\le t_1}E_{\lambda,\mu}(tu,tv),\\
E_{\lambda,\mu}(t_1u,t_1v)=\max_{t\ge T_l}E_{\lambda,\mu}(tu,tv).
\end{gathered}
$$

(2) *If* $\int_\Omega\big(\lambda a(x)|u|^q+\mu b(x)|v|^q\big)\,\mathrm{d}x<0$*, then there exist a unique* $T_l>0$ *such that* $(T_lu,T_lv)\in\mathcal{N}^-_{\lambda,\mu}$ *and* $E_{\lambda,\mu}(T_lu,T_lv)=\max_{t\ge0}E_{\lambda,\mu}(tu,tv)$.

Proof. (1) $\int_\Omega\big(\lambda a(x)|u|^q+\mu b(x)|v|^q\big)\,\mathrm{d}x>0$, We introduce the function $\psi_{u,v}:\mathbb{R}^+\longrightarrow\mathbb{R}$ define by

$$
\psi_{u,v}(t)=kt^{p-q}\|(u,v)\|^p+lt^{p\theta-q}\|(u,v)\|^{p\theta}-t^{2-\alpha-\beta-q}\int_\Omega c(x)|u|^{1-\alpha}|v|^{1-\beta}\,\mathrm{d}x.
$$

Note that $(tu, tv) \in \mathcal{N}_{\lambda,\mu}$ if and only if

$$\psi_{u,v}(t) = \int_\Omega \big(\lambda a(x)|u|^q + \mu b(x)|v|^q\big)\,\mathrm{d}x.$$

Now, the first derivative of the function ψ is

$$\begin{aligned}
\psi'_{u,v}(t) &= k(p-q)t^{p-q-1}\|(u,v)\|^p + (p\theta - q)lt^{p\theta-q-1}\|(u,v)\|^{p\theta} \\
&\quad - (2-\alpha-\beta-q)t^{1-\alpha-\beta-q}\int_\Omega c(x)|u|^{1-\alpha}|v|^{1-\beta}\,\mathrm{d}x \\
&= t^{-q-1}\Big(k(p-q)t^p\|(u,v)\|^p + (p\theta-q)lt^{p\theta}\|(u,v)\|^{p\theta} \\
&\quad - (2-\alpha-\beta-q)t^{-\alpha-\beta+2}\int_\Omega c(x)|u|^{1-\alpha}|v|^{1-\beta}\,\mathrm{d}x\Big)
\end{aligned} \tag{2.6}$$

It is clear that $\psi_{u,v}(t) \to -\infty$ as $t \to \infty$. Moreover, using (2.6), it is simple to see that $\lim_{t\to 0^+}\psi'_{u,v}(t) > 0$ and $\lim_{t\to\infty}\psi'_{u,v}(t) < 0$. That is there exists $T_l > 0$ such that $\psi_{u,v}(t)$ is increasing on $(0, T_l)$, decreasing on (T_l, ∞) and $\psi'_{u,v}(T_l) = 0$. So,

$$\psi_{u,v}(T_l) = kT_l^{p-q}\|(u,v)\|^p + lT_l^{p\theta-q}\|(u,v)\|^{p\theta} - T_l^{2-\alpha-\beta-q}\int_\Omega c(x)|u|^{1-\alpha}|v|^{1-\beta}\,\mathrm{d}x.$$

where T_l is the solution of

$$\begin{aligned}
&k(p-q)t^p\|(u,v)\|^p + (p\theta-q)lt^{p\theta}\|(u,v)\|^{p\theta} \\
&- (2-\alpha-\beta-q)t^{-\alpha-\beta+2}\int_\Omega c(x)|u|^{1-\alpha}|v|^{1-\beta}\,\mathrm{d}x = 0.
\end{aligned} \tag{2.7}$$

Then, using (2.7), we obtain

$$T_0 := \Big(\frac{(2-\alpha-\beta-q)\int_\Omega c(x)|u|^{1-\alpha}|v|^{1-\beta}\,\mathrm{d}x}{k(p-q)\|(u,v)\|^p}\Big)^{\frac{1}{p+\beta+\alpha-2}} \le T_l. \tag{2.8}$$

From inequality (2.8), we can find a constant $C = C(p, q, \alpha, \beta) > 0$ such that

$$\begin{aligned}
&\psi_{u,v}(T_l) \ge \psi_{u,v}(T_0) \\
&\ge kT_0^{p-q}\|(u,v)\|^p - T_0^{2-\alpha-\beta-q}\int_\Omega c(x)|u|^{1-\alpha}|v|^{1-\beta}\,\mathrm{d}x \\
&\ge k\Big(\frac{\alpha+\beta}{q+\alpha+\beta-2}\Big)\Big(\frac{q+\alpha+\beta-2}{k(q-2)}\Big)^{\frac{2-q}{\beta+\alpha}}\frac{\|(u,v)\|^{2\frac{q+\alpha+\beta-2}{\beta+\alpha}}}{\big(\int_\Omega c(x)|u|^{1-\alpha}|v|^{1-\beta}\,\mathrm{d}x\big)^{\frac{q-2}{\beta+\alpha}}} \\
&\quad - |\Omega|^{\frac{2_s^*-q}{2_s^*}}S^{-\frac{q}{2}}\Big((\lambda\|a\|_\infty)^{\frac{2}{2-q}} + (\mu\|b\|_\infty)^{\frac{2}{2-q}}\Big)^{\frac{2-q}{2}}\|(u,v)\|^q > 0
\end{aligned}$$

if and only if

$$\begin{aligned}
&(\lambda\|a\|_\infty)^{\frac{2}{2-q}} + (\mu\|b\|_\infty)^{\frac{2}{2-q}} \\
&< \Big(\frac{k(q-2)}{\|c\|_\infty(q+\alpha+\beta-2)}\Big)^{-\frac{2}{\alpha+\beta}}\Big(\frac{q+\alpha+\beta-2}{k(\alpha+\beta)}|\Omega|^{\frac{2_s^*-q}{2_s^*}}\Big)^{-\frac{2}{2-q}}S^{\frac{\alpha+\beta-2}{\alpha+\beta}+\frac{q}{2-q}} = \Lambda_0.
\end{aligned}$$

Then, there exists exactly two points $t_0 < T_l$ and $t_1 > T_l$ with

$$\psi'_{u,v}(t_0) = \int_\Omega \big(\lambda a(x)|u|^q + \mu b(x)|v|^q\big)\,\mathrm{d}x = \psi'_{u,v}(t_1).$$

Also, $\psi'_{u,v}(t_0) > 0$ and $\psi'_{u,v}(t_1) < 0$. That is, $(t_0u, t_0v) \in \mathcal{N}^+_{\lambda,\mu}$ and $(t_1u, t_1v) \in \mathcal{N}^-_{\lambda,\mu}$. Since

$$\Phi'_{u,v}(t) = t^q\Big(\psi_{u,v}(t) - \int_\Omega \big(\lambda a(x)|u|^q + \mu b(x)|v|^q\big)\,\mathrm{d}x\Big).$$

Thus, $\Phi'_{u,v}(t) < 0$ for all $t \in [0, t_0)$ and $\Phi'_{u,v}(t) > 0$ for all $t \in (t_0, t_1)$. Hence $E_{\lambda,\mu}(t_0u, t_0v) = \min_{0\le t\le t_1} E_{\lambda,\mu}(tu, tv)$. In the same way, $\Phi'_{u,v}(t) > 0$ for all $t \in (t_0, t_1)$, $\Phi'_{u,v}(t) = 0$ and $\Phi'_{u,v}(t) < 0$ for all $t \in (t_1, \infty)$ that is $E_{\lambda,\mu}(t_1u, t_1v) = \max\limits_{t\ge T_l} E_{\lambda,\mu}(tu, tv)$.

(2) $\int_\Omega \big(\lambda a(x)|u|^q + \mu b(x)|v|^q\big)\,\mathrm{d}x < 0$. So $\psi_{u,v}(t) \to -\infty$ as $t \to \infty$. Therefore, for all (λ, μ) there exists $T_l > 0$ such that $(T_lu, T_lv) \in \mathcal{N}^-_{\lambda,\mu}$ and $E_{\lambda,\mu}(T_lu, T_lv) = \max_{t\ge 0} E_{\lambda,\mu}(tu, tv)$. □

As a consequence of Lemma 2.5, we have the following result.

Lemma 2.6. *There exists*

$$\Lambda_0 = \Big(\frac{q+\alpha+\beta-2}{\|c\|_\infty k(q-p)}\Big)^{\frac{p}{p+\alpha+\beta-2}} \Big(\frac{2-\alpha-\beta-q}{k(2-\alpha-\beta-p)}|\Omega|^{\frac{p_s^*-q}{p_s^*}}\Big)^{-\frac{p}{p-q}} S^{\frac{2-\alpha-\beta}{p+\alpha+\beta-2}},$$

such that for $0 < (\lambda\|a\|_\infty)^{\frac{p}{p-q}} + (\mu\|b\|_\infty)^{\frac{p}{p-q}} < \Lambda_0$, *we have* $\mathcal{N}^\pm_{\lambda,\mu} \ne \emptyset$ *and* $\mathcal{N}^0_{\lambda,\mu} = \{0\}$.

Proof. Firstly, using Lemma 2.4, we conclude that $\mathcal{N}^\pm_{\lambda,\mu}$ are non-empty for all (λ, μ) with $0 < (\lambda\|a\|_\infty)^{\frac{p}{p-q}} + (\mu\|b\|_\infty)^{\frac{p}{p-q}} < \Lambda_0$. Now, we proceed by contradiction to prove that $\mathcal{N}^0_{\lambda,\mu} = \{0\}$ for all (λ, μ) with $0 < (\lambda\|a\|_\infty)^{\frac{p}{p-q}} + (\mu\|b\|_\infty)^{\frac{p}{p-q}} < \Lambda_0$. Let $(u, v) \in \mathcal{N}^0_{\lambda,\mu}$. Then, we have two cases.

Case 1 $(u, v) \in \mathcal{N}_{\lambda,\mu}$ and $\int_\Omega \big(\lambda a(x)|u|^q + \mu b(x)|v|^q\big)\,\mathrm{d}x = 0$. Using (2.2) and (2.3) with $t = 1$, it follows that

$$\begin{aligned}&(p-1)k\|(u,v)\|^p + l(p\theta-1)\|(u,v)\|^{p\theta} - (1-\alpha-\beta)\int_\Omega c(x)|u|^{1-\alpha}|v|^{1-\beta}\,\mathrm{d}x\\&= (p+\alpha+\beta-2)k\|(u,v)\|^p + l(p\theta+\alpha+\beta-2)\|(u,v)\|^{p\theta} > 0\end{aligned}$$

which is a contradiction.

Case 2 Let $(u, v) \in \mathcal{N}_{\lambda,\mu}$ and $\int_\Omega \big(\lambda a(x)|u|^q + \mu b(x)|v|^q\big)\,\mathrm{d}x = 0$. Using (2.2) and (2.3) with $t = 1$, it follows that

$$(p-q)k\|(u,v)\|^p + l(p\theta-q)\|(u,v)\|^{p\theta} \tag{2.9}$$

$$= -(q+\alpha+\beta)\int_\Omega c(x)|u|^{1-\alpha}|v|^{1-\beta}\,\mathrm{d}x, \tag{2.10}$$

$$\begin{aligned}&(2-\alpha-\beta-p)k\|(u,v)\|^p + l(2-\alpha-\beta-p\theta)\|(u,v)\|^{p\theta}\\&= (2-\alpha-\beta-q)\int_\Omega \big(\lambda a(x)|u|^q + \mu b(x)|v|^q\big)\,\mathrm{d}x.\end{aligned} \tag{2.11}$$

Now, we define $J_{\lambda,\mu} : \mathcal{N}_{\lambda,\mu} \to \mathbb{R}$ by

$$\begin{aligned}J_{\lambda,\mu}(u,v) = &\frac{2-\alpha-\beta-p}{2-\alpha-\beta-q}k\|(u,v)\|^p + \frac{2-\alpha-\beta-p\theta}{2-\alpha-\beta-q}l\|(u,v)\|^{p\theta}\\&- \int_\Omega \big(\lambda a(x)|u|^q + \mu b(x)|v|^q\big)\,\mathrm{d}x.\end{aligned}$$

Hence, from (2.11), $J_{\lambda,\mu}(u,v)=0$ for all $(u,v)\in\mathcal{N}^0_{\lambda,\mu}$. Moreover,

$$\begin{aligned}
J_{\lambda,\mu}(u,v) &\geq \frac{2-\alpha-\beta-p}{2-\alpha-\beta-q}k\|(u,v)\|^p - \int_\Omega \big(\lambda a(x)|u|^q+\mu b(x)|v|^q\big)\,\mathrm{d}x\\
&\geq \frac{2-\alpha-\beta-p}{2-\alpha-\beta-q}k\|(u,v)\|^p\\
&\quad - C|\Omega|^{\frac{p_s^*-q}{p_s^*}}S^{-\frac{q}{p}}\Big((\lambda\|a\|_\infty)^{\frac{p}{p-q}}+(\mu\|b\|_\infty)^{\frac{p}{p-q}}\Big)^{\frac{p-q}{p}}\|(u,v)\|^q\\
&\geq \|(u,v)\|^q\Big(\frac{2-\alpha-\beta-p}{2-\alpha-\beta-q}k\|(u,v)\|^{p-q}\\
&\quad - C|\Omega|^{\frac{p_s^*-q}{p_s^*}}S^{-\frac{q}{p}}\Big((\lambda\|a\|_\infty)^{\frac{p}{p-q}}+(\mu\|b\|_\infty)^{\frac{p}{p-q}}\Big)^{\frac{p-q}{p}}\Big).
\end{aligned}$$

Then, using (2.5) and (2.9), we obtain

$$\|(u,v)\|\geq \frac{1}{\|c\|_\infty}S^{-\frac{2-\alpha-\beta}{p(p+\alpha+\beta-2)}}\Big(\frac{k(p-q)}{2-\alpha-\beta-q}\Big)^{-\frac{1}{p+\alpha+\beta-2}}. \tag{2.12}$$

By (2.12) we obtain

$$\begin{aligned}
&J_{\lambda,\mu}(u,v)\\
&\geq \|(u,v)\|^q\Big(\frac{2-\alpha-\beta-p}{2-\alpha-\beta-q}k\Big(k(p-q)\|c\|_\infty S^{\frac{2-\alpha-\beta}{p(p+\alpha+\beta-2)}}\Big)\Big(\frac{k(p-q)}{2-\alpha-\beta-q}\Big)^{\frac{q-p}{p+\alpha+\beta-2}}\\
&\quad - C|\Omega|^{\frac{p_s^*-q}{p_s^*}}S^{-\frac{q}{p}}\Big((\lambda\|a\|_\infty)^{\frac{p}{p-q}}+(\mu\|b\|_\infty)^{\frac{p}{p-q}}\Big)^{\frac{p-q}{p}}\Big).
\end{aligned}$$

This implies that for $0<(\lambda\|a\|_\infty)^{\frac{p}{p-q}}+(\mu\|b\|_\infty)^{\frac{p}{p-q}}<\Lambda_0$, we have $J_{\lambda,\mu}(u,v)>0$, for all $(u,v)\in\mathcal{N}^0_{\lambda,\mu}$, which is a contradiction. The proof is complete. □

By Lemmas 2.3 and 2.4, for $0<(\lambda\|a\|_\infty)^{\frac{p}{p-q}}+(\mu\|b\|_\infty)^{\frac{p}{p-q}}<\Lambda_0$, we can write $\mathcal{N}_{\lambda,\mu}=\mathcal{N}^+_{\lambda,\mu}\cup\mathcal{N}^-_{\lambda,\mu}$ and define

$$c^+_{\lambda,\mu}=\inf_{(u,v)\in\mathcal{N}^+_{\lambda,\mu}}E_{\lambda,\mu}(u,v),\quad c^-_{\lambda,\mu}=\inf_{(u,v)\in\mathcal{N}^-_{\lambda,\mu}}E_{\lambda,\mu}(u,v).$$

3. Existence of a minimizer on $\mathcal{N}^+_{\lambda,\mu}$

In this section, we will show that the minimum of $E_{\lambda,\mu}$ is achieved in $\mathcal{N}^+_{\lambda,\mu}$. Also, we show that this minimizer is also a solution of problem (1.1).

Lemma 3.1. *If $0<(\lambda\|a\|_\infty)^{\frac{p}{p-q}}+(\mu\|b\|_\infty)^{\frac{p}{p-q}}<\Lambda_0$, then for all $(u,v)\in\mathcal{N}^+_{\lambda,\mu}$, $c^+_{\lambda,\mu}<0$.*

Proof. Let $(u_0^+,v_0^+)\in\mathcal{N}^+_{\lambda,\mu}$, then we have $\Phi''_{(u_0^+,v_0^+)}(1)>0$ which from (2.1) gives

$$\int_\Omega c(x)|u|^{1-\alpha}|v|^{1-\beta}\,\mathrm{d}x<\frac{k(p-q)}{2-\alpha-\beta-q}\|(u,v)\|^p+\frac{l(p\theta-q)}{2-\alpha-\beta-q}\|(u,v)\|^{p\theta}. \tag{3.1}$$

Hence, using (2.1) with (3.1), we have

$$\begin{aligned} E_{\lambda,\mu}(u,v) &\le k(\frac{1}{p}-\frac{1}{q})\|(u,v)\|^p + l(\frac{1}{p\theta}-\frac{1}{q})\|(u,v)\|^{p\theta} \\ &\quad - (\frac{1}{2-\alpha-\beta}-\frac{1}{q})\int_\Omega c(x)|u|^{1-\alpha}|v|^{1-\beta}\,\mathrm{d}x \\ &\le \big[k(\frac{1}{p}-\frac{1}{q}) - (\frac{1}{2-\alpha-\beta}-\frac{1}{q})\frac{k(p-q)}{2-\alpha-\beta-q}\big]\|(u,v)\|^p \\ &\quad + \big[l(\frac{1}{p\theta}-\frac{1}{q}) - (\frac{1}{2-\alpha-\beta}-\frac{1}{q})\frac{l(p\theta-q)}{2-\alpha-\beta-q}\big]\|(u,v)\|^{p\theta}. \end{aligned} \tag{3.2}$$

Thus, by (3.2), we obtain

$$\begin{aligned} E_{\lambda,\mu}(u,v) &< -\Big(\frac{k(q-p)(p+\alpha+\beta-2)}{pq(2-\alpha-\beta)}\|(u,v)\|^p \\ &\quad + \frac{l(q-p)(p+\alpha+\beta-2)}{pq(2-\alpha-\beta)}\|(u,v)\|^{p\theta}\Big) < 0. \end{aligned}$$

Therefore, $c^+_{\lambda,\mu} < 0$ follows from the definition $c^+_{\lambda,\mu}$. This completes the proof. $\square$

Theorem 3.2. *If $0 < (\lambda\|a\|_\infty)^{\frac{p}{p-q}} + (\mu\|b\|_\infty)^{\frac{p}{p-q}} < \Lambda_0$, then there exists (u_0^+, v_0^+) in $\mathcal{N}^+_{\lambda,\mu}$ satisfying $E_{\lambda,\mu}(u_0^+, v_0^+) = \inf_{(u,v)\in\mathcal{N}^+_{\lambda,\mu}} E_{\lambda,\mu}(u,v)$.*

Proof. Since $E_{\lambda,\mu}$ is bounded below on $\mathcal{N}_{\lambda,\mu}$ and so is on $\mathcal{N}^+_{\lambda,\mu}$. Then, there exists $\{(u_n^+, v_n^+)\} \subset \mathcal{N}^+_{\lambda,\mu}$ a sequence such that

$$E_{\lambda,\mu}(u_n^+, v_n^+) \to \inf_{(u,v)\in\mathcal{N}^+_{\lambda,\mu}} E_{\lambda,\mu}(u,v) \quad \text{as } n\to\infty.$$

Since $E_{\lambda,\mu}$ is coercive, $\{u_n, v_n\}$ is bounded in E. Then there exists a subsequence, still denoted by (u_n^+, v_n^+) and $(u_0^+, v_0^+) \in E$ such that, as $n\to\infty$,

$$\begin{gathered} u_n^+ \rightharpoonup u_0^+,\ v_n^+ \rightharpoonup v_0^+ \quad \text{weakly in } X_0, \\ u_n^+ \to u_0^+,\ v_n^+ \to v_0^+ \quad \text{strongly in } L^r(\Omega) \text{ for } 1\le r< p_s^*, \\ u_n^+ \to u_0^+,\ v_n^+ \to v_0^+ \quad \text{a.e. in } \Omega. \end{gathered}$$

By Vitali's theorem (see [25, pp. 133]), we claim that

$$\lim_{n\to\infty}\int_\Omega a(x)|u_n^+|^{1-\alpha}dx = \int_\Omega a(x)|u_0^+|^{1-\alpha}dx. \tag{3.3}$$

Indeed, we only need to prove that $\{\int_\Omega a(x)|u_n^+|^{1-\alpha}dx, n\in N\}$ is equi-absolutely-continuous. Note that $\{u_n\}$ is bounded, by the Sobolev embedding theorem, so there exists a constant $C>0$ such that $|u_n|_{p_s^*} \le C < \infty$. Moreover, by Hölder inequality we have

$$\int_\Omega a(x)u^{1-\alpha}dx \le \|a\|_\infty \int_\Omega |u|^{1-\alpha}dx \le \|a\|_\infty |\Omega|^{\frac{p_s^*}{p_s^*+\alpha-1}} |u|^{1-\alpha}_{p_s^*}. \tag{3.4}$$

From (3.4), for every $\varepsilon>0$, setting

$$\delta = \Big(\frac{\varepsilon}{\|a\|_\infty C^{1-\alpha}}\Big)^{\frac{p_s^*}{p_s^*+\gamma-1}},$$

when $A \subset \Omega$ with $\operatorname{meas}(A) < \delta$, we have

$$\int_A a(x)|u_n^+|^{1-\alpha}dx \le \|a\|_\infty \|u\|_{p_s^*}^{1-\alpha}\big(\operatorname{meas} A\big)^{\frac{p_s^*+\alpha-1}{p_s^*}} \le \|a\|_\infty C^{1-\alpha}\delta^{\frac{p_s^*+\alpha-1}{p_s^*}} < \varepsilon.$$

Thus, our claim is true. Similarly,

$$\lim_{n\to\infty}\int_\Omega b(x)|v_n^+|^{1-\beta}dx = \int_\Omega b(x)|v_0^+|^{1-\beta}dx. \tag{3.5}$$

On the other hand, by [2] there exists $l \in L^r(\mathbb{R}^N)$ such that

$$|u_n^+(x)| \le l(x), \quad |v_n^+(x)| \le l(x), \quad \text{as } k \to \infty$$

for $1 \le r < p_s^*$. Therefore by the Dominated convergence Theorem,

$$\int_\Omega \big(\lambda|u_n^+|^q + \mu|v_n^+|^q\big)\,dx \to \int_\Omega \big(\lambda|u_0^+|^q + \mu|v_0^+|^q\big)\,dx.$$

Moreover, by Lemma 2.5, there exists t_0 such that $(t_0u_0^+, t_0v_0^+) \in \mathcal{N}_{\lambda,\mu}^+$. Now, we shall prove that $u_n^+ \to u_0^+$ strongly in X_0, $v_n^+ \to v_0^+$ strongly in X_0. Suppose otherwise, then

$$\|(u_0^+, v_0^+)\|_E \le \liminf_{n\to\infty}\|(u_n^+, v_n^+)\|_E.$$

On the other hand, since $(u_n^+, v_n^+) \in \mathcal{N}_{\lambda,\mu}^+$, one has

$$\begin{aligned}
&\lim_{n\to\infty}\Phi'_{u_n^+,v_n^+}(t_0)\\
&= \lim_{n\to\infty}\Big(kt_0^{p-1}\|(u_n^+,v_n^+)\|^p + lt_0^{p\theta-1}\|(u_n^+,v_n^+)\|^{p\theta}\\
&\quad - t_0^{q-1}\int_\Omega\big(\lambda a(x)|u_n^+|^q + \mu b(x)|v_n^+|^q\big)\,\mathrm{d}x - t_0^{1-\alpha-\beta}\int_\Omega c(x)|u_n^+|^{1-\alpha}|v_n^+|^{1-\beta}\,\mathrm{d}x\Big)\\
&> kt_0^{p-1}\|(u_0^+,v_0^+)\|^p + lt_0^{p\theta-1}\|(u_0^+,v_0^+)\|^{p\theta}\\
&\quad - t_0^{q-1}\int_\Omega\big(\lambda a(x)|u_0^+|^q + \mu b(x)|v_0^+|^q\big)\,\mathrm{d}x\\
&\quad - t_0^{1-\alpha-\beta}\int_\Omega c(x)|u_0^+|^{1-\alpha}|v_0^+|^{1-\beta}\,\mathrm{d}x = \Phi'_{u_0^+,v_0^+}(t_0) = 0.
\end{aligned}$$

So, $\Phi'_{u_n^+,v_n^+}(t_0) > 0$ for n large enough. Moreover, $(u_n^+, v_n^+) \in \mathcal{N}_{\lambda,\mu}^+$, and we can see for all n that $\Phi'_{u_n^+,v_n^+}(t) < 0$ for $t \in (0,t)$ and $\Phi'_{u_n^+,v_n^+}(1) = 0$. Thus we must have $t_0 > 1$. Moreover $\Phi_{u_n^+,v_n^+}(1)$ is decreasing for $t \in (0,t_0)$ and that is

$$E_{\lambda,\mu}(t_0u_0^+, t_0v_0^+) < E_{\lambda,\mu}(u_0^+, v_0^+) = \lim_{n\to\infty}E_{\lambda,\mu}(u_n^+, v_n^+) = \inf_{(u,v)\in\mathcal{N}_{\lambda,\mu}^+}E_{\lambda,\mu}(u,v)$$

which gives a contradiction. Thus, $u_n^+ \to u_0^+$ strongly in X_0, $v_n^+ \to v_0^+$ strongly in X_0 and $E_{\lambda,\mu}(u_0^+, v_0^+) = \inf_{(u,v)\in\mathcal{N}_{\lambda,\mu}^+}E_{\lambda,\mu}(u,v)$. The proof of is complete. □

4. Existence of a minimizer on $\mathcal{N}_{\lambda,\mu}^-$

In this section, we shall show the existence of a solution to problem (1.1) by proving the existence of minimizer of $E_{\lambda,\mu}$ on $\mathcal{N}_{\lambda,\mu}^-$.

Lemma 4.1. *If $0 < (\lambda\|a\|_\infty)^{\frac{p}{p-q}} + (\mu\|b\|_\infty)^{\frac{p}{p-q}} < \Lambda_0$, then for all $(u,v) \in \mathcal{N}_{\lambda,\mu}^+$, $c_{\lambda,\mu}^- > d_0$ for some $d_0 = d_0(\alpha,\beta,p,q,a,b,\lambda,\mu,|\Omega|) > 0$.*

Proof. Let $(u_0^-,v_0^-)\in\mathcal{N}_{\lambda,\mu}^-$, then we have $\Phi''_{u_0^-,v_0^-}(1)<0$ which from (2.1) gives

$$\int_\Omega c(x)|u|^{1-\alpha}|v|^{1-\beta}\,\mathrm{d}x>\frac{k(p-q)}{2-\alpha-\beta-q}\|(u,v)\|^p+\frac{l(p\theta-q)}{2-\alpha-\beta-q}\|(u,v)\|^{p\theta}. \tag{4.1}$$

Therefore using (2.5), we obtain

$$\|(u,v)\|>\frac{1}{\|c\|_\infty}S^{-\frac{2-\alpha-\beta}{p(p+\alpha+\beta-2)}}\Big(\frac{k(p-q)}{2-\alpha-\beta-q}\Big)^{-\frac{1}{p+\alpha+\beta-2}}. \tag{4.2}$$

Hence, using (2.4) and (4.2), one has

$$\begin{aligned}
E_{\lambda,\mu}(u,v)&\ge k\big(\frac{1}{p}-\frac{1}{2-\alpha-\beta}\big)\|(u,v)\|^p-\big(\frac{1}{q}-\frac{1}{2-\alpha-\beta}\big)|\Omega|^{\frac{p_s^*-q}{p_s^*}}\\
&\quad\times S^{-\frac{q}{p}}\Big((\lambda\|a\|_\infty)^{\frac{p}{p-q}}+(\mu\|b\|_\infty)^{\frac{p}{p-q}}\Big)^{\frac{p-q}{p}}\|(u,v)\|^q\\
&=\|(u,v)\|^q\Big[k\big(\frac{1}{p}-\frac{1}{2-\alpha-\beta}\big)\|(u,v)\|^{p-q}-\big(\frac{1}{q}-\frac{1}{2-\alpha-\beta}\big)|\Omega|^{\frac{p_s^*-q}{p_s^*}}\\
&\quad\times S^{-\frac{q}{p}}\Big((\lambda\|a\|_\infty)^{\frac{p}{p-q}}+(\mu\|b\|_\infty)^{\frac{p}{p-q}}\Big)^{\frac{p-q}{p}}\Big]\\
&>\|(u,v)\|^q\Big[k\big(\frac{1}{p}-\frac{1}{2-\alpha-\beta}\big)S^{\frac{(p-q)}{p}}\Big(\frac{p-q}{2-\alpha-\beta-q}\Big)^{\frac{q-p}{p+\alpha+\beta-2}}\\
&\quad-\big(\frac{1}{q}-\frac{1}{2-\alpha-\beta}\big)|\Omega|^{\frac{p_s^*-q}{p_s^*}}S^{-\frac{q}{p}}\Big((\lambda\|a\|_\infty)^{\frac{p}{p-q}}+(\mu\|b\|_\infty)^{\frac{p}{p-q}}\Big)^{\frac{p-q}{p}}\Big].
\end{aligned}$$

Thus, if $0<(\lambda\|a\|_\infty)^{\frac{p}{p-q}}+(\mu\|b\|_\infty)^{\frac{p}{p-q}}<\Lambda_0$, then $E_{\lambda,\mu}(u,v)>d_0$ for all $(u,v)\in\mathcal{N}_{\lambda,\mu}^-$ for some $d_0=d_0(\alpha,\beta,p,q,a,b,\lambda,\mu,|\Omega|)>0$. Therefore $c_{\lambda,\mu}^->d_0$ follows from the definition $c_{\lambda,\mu}^-$. This completes the proof. □

Theorem 4.2. *If $0<(\lambda\|a\|_\infty)^{\frac{p}{p-q}}+(\mu\|b\|_\infty)^{\frac{p}{p-q}}<\Lambda_0$, then there exists (u_0^-,v_0^-) in $\mathcal{N}_{\lambda,\mu}^-$ satisfying $E_{\lambda,\mu}(u_0^-,v_0^-)=\inf_{(u,v)\in\mathcal{N}_{\lambda,\mu}^-}E_{\lambda,\mu}(u,v)$.*

Proof. Since $E_{\lambda,\mu}$ is bounded below on $\mathcal{N}_{\lambda,\mu}$ and so on $\mathcal{N}_{\lambda,\mu}^-$. Then, there exists $\{(u_n^-,v_n^-)\}\subset\mathcal{N}_{\lambda,\mu}^-$, a sequence such that

$$E_{\lambda,\mu}(u_n^-,v_n^-)\to\inf_{(u,v)\in\mathcal{N}_{\lambda,\mu}^-}E_{\lambda,\mu}(u,v)\quad\text{as } n\to\infty.$$

Since $E_{\lambda,\mu}$ is coercive, $\{(u_n,v_n)\}$ is bounded in E. Then there exists a subsequence, still denoted by (u_n^-,v_n^-) and $(u_0^-,v_0^-)\in E$ such that, as $n\to\infty$,

$$\begin{gathered}
u_n^+\rightharpoonup u_0^-,\ v_n^-\rightharpoonup v_0^-\quad\text{weakly in } X_0,\\
u_n^-\to u_0^-,\ v_n^-\to v_0^-\quad\text{strongly in } L^r(\Omega)\text{ for } 1\le r<p_s^*,\\
u_n^-\to u_0^-,\ v_n^-\to v_0^-\quad\text{a.e. in }\Omega.
\end{gathered}$$

Moreover, as in Lemma 3.2, we have

$$\begin{gathered}
\lim_{n\to\infty}\int_\Omega|u_n^-|^{1-\alpha}dx=\int_\Omega|u_0^-|^{1-\alpha}dx,\\
\lim_{n\to\infty}\int_\Omega|v_n^-|^{1-\beta}dx=\int_\Omega|v_0^-|^{1-\beta}dx,\\
\int_\Omega\big(\lambda a(x)|u_n^+|^q+\mu b(x)|v_n^+|^q\big)dx\to\int_\Omega\big(\lambda a(x)|u_0^+|^q+\mu b(x)|v_0^+|^q\big)dx.
\end{gathered}$$

Moreover, by Lemma 2.5, there exists t_1 such that $(t_1u_0^-, t_1v_0^-) \in \mathcal{N}_{\lambda,\mu}^-$. Now, we prove that $u_n^- \to u_0^-$ strongly in X_0, $v_n^- \to v_0^-$ strongly in X_0. Suppose otherwise, then

$$\|(u_0^-, v_0^-)\|_E \le \liminf_{n\to\infty} \|(u_n^-, v_n^-)\|_E.$$

Thus, since $(u_n^-, v_n^-) \in \mathcal{N}_{\lambda,\mu}^-$, $E_{\lambda,\mu}(tu_0^-, tv_0^-) \le E_{\lambda,\mu}(u_0^-, v_0^-)$, for all $t \ge 0$ we have

$$E_{\lambda,\mu}(t_1u_0^-, t_1v_0^-) < \lim_{n\to\infty} E_{\lambda,\mu}(t_1u_n^-, t_1v_n^-) \le \lim_{n\to\infty} E_{\lambda,\mu}(u_n^-, v_n^-) = c_{\lambda,\mu}^-.$$

which gives a contradiction. Thus, $u_n^- \to u_0^-$ strongly in X_0, $v_n^- \to v_0^-$ strongly in X_0 and $E_{\lambda,\mu}(u_0^-, v_0^-) = \inf_{(u,v)\in\mathcal{N}_{\lambda,\mu}^-} E_{\lambda,\mu}(u,v)$. The proof is complete. □

Proof of Theorem 1.2. Let us start by proving the existence of non-negative solutions. First, by Theorems 3.2, 4.2, there exist $(u_0^+, v_0^+) \in \mathcal{N}_{\lambda,\mu}^+$, $(u_0^-, v_0^-) \in \mathcal{N}_{\lambda,\mu}^-$ satisfying

$$\begin{gathered} E_{\lambda,\mu}(u_0^+, v_0^+) = \inf_{(u,v)\in\mathcal{N}_{\lambda,\mu}^+} E_{\lambda,\mu}(u,v), \\ E_{\lambda,\mu}(u_0^-, v_0^-) = \inf_{(u,v)\in\mathcal{N}_{\lambda,\mu}^-} E_{\lambda,\mu}(u,v). \end{gathered}$$

Moreover, since $E_{\lambda,\mu}(u_0^+, v_0^+) = E_{\lambda,\mu}(|u_0^+|, |v_0^+|)$ and $(|u_0^+|, |v_0^+|) \in \mathcal{N}_{\lambda,\mu}^+$. Similarly we have $E_\lambda(u_0^-, v_0^-) = E_{\lambda,\mu}(|u_0^-|, |v_0^-|)$ and $(|u_0^-|, |v_0^-|) \in \mathcal{N}_{\lambda,\mu}^-$, so we may assume $(u_0^\pm, v_0^\pm) \ge 0$. By Lemma 2.2, $(u_0^\pm, v_0^\pm)$ are the nontrivial non-negatives solutions of problem (1.1). Finally, it remain to show that the solutions found in Theorems 3.2, 4.2, are distinct. Since $\mathcal{N}_{\lambda,\mu}^- \cap \mathcal{N}_{\lambda,\mu}^+ = \emptyset$, then $(u_0^\pm, v_0^\pm)$ are distinct. The proof of complete. □

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