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MIXED LOCAL AND NONLOCAL SCHRÖDINGER-POISSON TYPE SYSTEM INVOLVING VARIABLE EXPONENTS

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Abstract. We consider the existence of solutions for a class of Schrödinger-Poisson type equations with mixed local and nonlocal p-Laplacian. More precisely, we obtain two distinct nontrivial solutions for the problem involving variable exponents growth by the variational methods. Moreover, the phenomena of concentration and multiplicity of solutions are also investigated as $\lambda \to \infty$.

1. Introduction

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with smooth boundary and let α, β be two positive parameters. The purpose of this article is to investigate the existence and asymptotic behavior, as $\lambda \to \infty$, of solutions of the Schrödinger-Poisson type system

$$\begin{gathered}\mathcal{L}u + \lambda V(x)|u|^{p-2}u + \phi|u|^{p-2}u = \alpha|u|^{p(x)-2}u + \beta|u|^{q(x)-2}u \quad \text{in } \Omega,\\ -\Delta\phi = u^p \quad \text{in } \Omega,\\ u = \phi = 0 \quad \text{in } \mathbb{R}^N \setminus \Omega,\end{gathered} \tag{1.1}$$

where λ is a positive parameter, and $V(x) \in C(\mathbb{R}^N)$ is a potential function. Here, $\alpha|u|^{p(x)-2}u + \beta|u|^{q(x)-2}u$ is assumed to be a concave-convex nonlinearity with variable functions $p(x), q(x) \in C(\Omega)$. We stress that the operator $\mathcal{L}$ appearing (1.1) represents the superposition of a p-Laplacian and a fractional p-Laplacian, defined as

$$\mathcal{L} := -\Delta_p + (-\Delta)_p^s \quad \text{for some } s \in (0,1),$$

where $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ and $(-\Delta)_p^s$ is the nonlocal operator given by

$$(-\Delta)_p^s\varphi = \text{P.V.} \int_{\mathbb{R}^N} \frac{|\varphi(x) - \varphi(y)|^{p-2}\big(\varphi(x) - \varphi(y)\big)}{|x-y|^{N+ps}} dy.$$

Here P.V. denotes the principle value of the integral. See [15, 25] for further details on the fractional p-Laplacian.

The operator $\mathcal{L}$ models diffusion patterns over a variety of time scales. So it is used in applications such as of optimal search methods, biomathematics, and animal foraging; see for example [16, 17] and their references. The mixed local

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and nonlocal operator $\mathcal{L}$ is remarkably similar to the mixed (s,t)-order operator in mathematical research. For the significance, applications and some results of mixed (s,t)-order operators, see see [27, 33].

Problem (1.1) will play a crucial part while associated with standing wave solutions $\psi(x,t)=u(x)e^{-\imath t}$ to the time-dependent Schrödinger-Poisson system

$$\begin{gathered} -i\frac{\partial\psi}{\partial t}=-\Delta\psi+\phi(x)\psi-f(\psi)\quad\text{in }\Omega,\\ -\Delta\phi=|\psi|^2\quad\text{in }\Omega,\\ \psi=\phi=0\quad\text{on }\partial\Omega. \end{gathered}\tag{1.2}$$

This system is employed in quantum mechanics to describe the interaction of a charged particle with an electrostatic field, where ψ and ϕ represent the wave functions connected to the particle and the electric potentials, respectively. The nonlinearity $f(\psi)$ is usually used to model the interaction between multiple particles. For more information on the physical background we refer the readers to [32]. Under various assumptions, the existence of solution for (1.2) have received a lot of interest in recent years.

Existence, nonexistence, and asymptotic behavior of solutions to Schrödinger-Poisson systems have been studied by Du, Tian, Wang and Zhang [18]. This is done under suitable assumptions on potential well $V(x)$ and the nonlinearity $f(\cdot)$ (and asymptotically 3-linear), using variational methods. Recently, Jeanjean and Le [23] obtained multiple normalized solutions for Sobolev critical Schrödinger-Poisson-Slater equation. For more results see [1, 21].

Let us now review topics associated with the so-called concave-convex nonlinearity. The emergence of substantial literature on concave-convex nonlinear elliptic problems started with the pioneering work by Ambrosetti, Brezis and Cerami [4], for Laplacian problem in a bounded domain,

$$\begin{gathered} -\Delta z=\lambda|z|^{q-2}z+|z|^{h-2}z\quad\text{in }\Omega,\\ z=0\quad\text{on }\partial\Omega, \end{gathered}\tag{1.3}$$

where $1<q<2<h<2^*$. Brändle et al. [9] studied the concave-convex-type elliptic problems driven by a nonlocal integro-differential operator. Subsequently, Ho and Sim [22] discussed the multiplicity of nontrivial solutions to degenerate $p(x)$-Laplacian equations involving concave-convex type nonlinearities with two parameters. For studying variable exponential problems, there are two main reasons: One is that they frequently occur in many different applications, including electrorheological fluids, image processing and elastic mechanics, see [12, 29]. The other reason is from a pure mathematical standpoint, as variable exponents have more complex nonlinearities than problems with constant exponents. For more results on variable exponent problems see [2, 7, 11]. Regarding the unbalanced double-phase problem with variable exponent, we would like to remark that Kim et al. [24] obtained L^∞-bound of solutions via the De Giorgi iteration method and a truncated energy technique. Related to double-phase problem, we refer to [14, 26, 37] and their references.

For the Schrödinger-Poisson system with concave and convex terms

$$\begin{gathered} -\Delta u+V(x)u+\mu\phi u=a(x)|u|^{p-2}u+b(x)|u|^{q-2}u\quad\text{in }\mathbb{R}^3,\\ -\Delta\phi=u^2\quad\text{in }\mathbb{R}^3, \end{gathered}\tag{1.4}$$

Sun, Su, and Zhao [30] showed that it has infinitely many solutions, under the constraint that $\phi(x) \to 0$ as $|x| \to +\infty$. By developing a novel constraint approach, Sun and Wu [31] showed that (1.4) has at least two positive solutions, under the assumption that $V(x)$ satisfying a steep potential well condition. By quantitative deformation lemma, Yang and Ou [36] proved that the Schrödinger-Poisson system with concave-convex nonlinearity admits a nodal solution in a bounded domain.

Inspired by the above facts, our main goal is to consider the Schrödinger-Poisson system for mixed order p-Laplacian with variable exponent growth. More precisely, we show that there exist two distinct solutions of (1.1), and then discuss the concentration and multiplicity of solutions as $\lambda \to \infty$. To the best of our knowledge, this is the first attempt to investigate the existence of solution for mixed order Schrödinger-Poisson system involving variable exponent growth.

To obtain our main results, we use the following assumptions: On the continuous potential function $V : \Omega \to [0, \infty)$, we assume that

- (A1) $Z = \operatorname{int}(V^{-1}(0)) \subset \Omega$ is a nonempty bounded domain;
- (A2) there exists a nonempty open domain $\Omega_0 \subset Z$ such that $V(x) \equiv 0$ for all $x \in \overline{\Omega_0}$.

On the variable exponents $p(x), q(x) \in C(\overline{\Omega})$ we assume that

- (A3) $2p < p(x) < \frac{Np}{N-ps}$ for all $x \in \overline{\Omega}$;
- (A4) $1 < q(x) < p$ for all $x \in \overline{\Omega}$;
- (A5) Let α and β be two positive parameters such that

$$\alpha \leq \frac{p-q^+}{Ap(p^+-q^+)}, \quad \alpha^{p-q^+}\beta^{p^+-p} \leq \Big(\frac{p-q^+}{Ap(p^+-q^+)}\Big)^{p-q^+}\Big(\frac{p^+-p}{Bp(p^+-q^+)}\Big)^{p^+-p},$$

where constants A, B will be specified in Lemma 3.3.

Our first main results reads as follows.

Theorem 1.1. *Assume* (A1)–(A5) *hold. Then for all $\lambda > 0$, Problem* (1.1) *has at least two distinct nontrivial solutions.*

To be precise, Theorem 1.1 shows that (1.1) has a positive energy solution u_λ^1 and a negative energy solution u_λ^2. The basic strategies for proving Theorem 1.1 were used by Alves and Ferreira [2], and are based on the mountain pass theorem (cf. [5]) and Ekeland variational principle (cf.[20]). The process in [2] does not seem to be entirely applicable in our setting because our consideration involves a nonlocal term ϕ and a mixed order operator.

The following result is associated with the concentration behavior of the solutions from Theorem 1.1 as $\lambda \to \infty$.

Theorem 1.2. *Let u_λ^1 and u_λ^2 be two solutions of* (1.1) *from Theorem 1.1. Suppose that* (A1)—(A5) *hold, then there exists $u^1, u^2 \in W_0^{1,p}(\Omega)$ such that $u_\lambda^1 \to u^1$ and $u_\lambda^2 \to u^2$ in $W_0^{1,p}(\Omega)$ as $\lambda \to \infty$. Moreover, $u^1 \neq u^2$ are two nontrivial solutions of the problem*

$$\begin{gathered}
\mathcal{L}u + \phi(x)|u|^{p-2}u = \alpha|u|^{p(x)-2}u + \beta|u|^{q(x)-2}u \quad \text{in } \Omega_0, \\
-\Delta\phi = |u|^p \quad \text{in } \Omega_0, \\
u = \phi = 0 \quad \text{in } \mathbb{R}^N \setminus \Omega_0.
\end{gathered} \tag{1.5}$$

Theorem 1.2 is proved by analyzing the convergence property of u_λ^1 and u_λ^2 as $\lambda \to \infty$. In this process, we need to confirm that $u^1 = u^2 = 0$ in $\mathbb{R}^N \setminus \Omega_0$, which is fulfilled by the idea of Bartsh, Pankov and Wang in [8].

Finally, we are to establish the existence of infinitely many solutions for (1.5).

Theorem 1.3. *Assume that* (A3)–(A5) *hold. Then* (1.5) *admits infinitely many solutions.*

The rest of this article is organized as follows. In Section 2, the variational framework and some preliminaries are recalled. We devote Section 3 to two distinct nontrivial weak solutions for the problem (1.1) by using mountain pass theorem and Ekeland variational principle. In Section 4, the concentration of the weak solutions of the problem (1.1) is considered. Finally, we focus on the existence of infinitely many solutions of the problem (1.5) based on the symmetric mountain pass theorem in Section 5.

2. Preliminaries

In this section we introduce a functional framework related to problem (1.1). Let us begin with reviewing some notation and valuable conclusion concerning the variable exponent Lebesgue spaces that will be used later. Throughout this article, we assume that $p(x) \in C(\overline{\Omega}) : \overline{\Omega} \to (1, \infty)$ and denote

$$p^- := \operatorname{ess\,inf}_{x\in\Omega} p(x), \quad p^+ := \operatorname{ess\,sup}_{x\in\Omega} p(x).$$

The variable exponent Lebesgue space is defined by

$$L^{p(x)}(\Omega) = \big\{u : \Omega \to \mathbb{R} \text{ is a measurable function; } \rho_{p(x)}(u) = \int_\Omega |u(x)|^{p(x)} dx < \infty\big\}$$

with the Luxemburg norm

$$\|u\|_{L^{p(x)}(\Omega)} = \inf\big\{\mu > 0 : \rho_{p(x)}(\mu^{-1}u) = \int_\Omega \big|\frac{u(x)}{\mu}\big|^{p(x)} \le 1\big\}.$$

Here, $p(x)$ is said to be bounded if p^+ is finite. For this case, it is easy to see that

$$\begin{gathered}
\|u\|^{p^-}_{L^{p(x)}(\Omega)} \le \rho_{p(x)}(u) \le \|u\|^{p^+}_{L^{p(x)}(\Omega)} \quad \text{if } \|u\|_{L^{p(x)}(\Omega)} \ge 1, \\
\|u\|^{p^+}_{L^{p(x)}(\Omega)} \le \rho_{p(x)}(u) \le \|u\|^{p^-}_{L^{p(x)}(\Omega)} \quad \text{if } \|u\|_{L^{p(x)}(\Omega)} \le 1.
\end{gathered}$$

Definition 2.1. The dual space of $L^{p(x)}(\Omega)$ is $L^{p'(x)}(\Omega)$, where the conjugate exponent $p(x)'$ is defined as $p(x)' = p(x)/(p(x) - 1)$.

If $1 < p^- \le p^+ < \infty$, then the variable exponent Lebesgue space $L^{p(x)}(\Omega)$ is a reflexive uniformly convex Banach space. Obviously, for $L^{p(x)}(\Omega)$, the Hölder inequality is still valid.

Lemma 2.2 ([28])**.** *For all* $u \in L^{p(x)}(\Omega)$, $v \in L^{p'(x)}(\Omega)$ *with* $p(x) \in (1, \infty)$, *it holds*

$$\int_\Omega |uv| dx \le \Big(\frac{1}{p^-} + \frac{1}{(p')^-}\Big)\|u\|_{L^{p(x)}(\Omega)}\|v\|_{L^{p'(x)}(\Omega)} \le 2\|u\|_{L^{p(x)}(\Omega)}\|v\|_{L^{p'(x)}(\Omega)}.$$

For more information on variable exponent Lebesgue spaces, we refer the readers to [28]. For any $1 < p < \infty$, $W^{1,p}(\Omega)$ is the usual Sobolev space equipped with norm

$$\|u\|_{W^{1,p}(\Omega)} := \|u\|_{L^p(\Omega)} + \|\nabla u\|_{L^p(\Omega)}.$$

The closure of $C_c^\infty(\Omega)$ in $W^{1,p}(\Omega)$ is denoted by $W_0^{1,p}(\Omega)$. In what follows, we review several fundamental results related to the fractional Sobolev space $W^{s,p}(\mathbb{R}^N)$, for more details see [15]. Before we do that, let us recall the so-called Gagliardo seminorm of u

$$[u]_{s,p}=\Big(\iint_{\mathbb{R}^{2N}}\frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}\,dx\,dy\Big)^{1/p}.$$

Definition 2.3. Let u be a measurable function, and let

$$W^{s,p}(\mathbb{R}^N):=\big\{u\in L^p(\mathbb{R}^N):\iint_{\mathbb{R}^{2N}}\frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}\,dx\,dy<\infty\big\}$$

endowed with the norm

$$\|u\|_{W^{s,p}(\mathbb{R}^N)}=\Big(\int_{\mathbb{R}^N}|u(x)|^pdx+\iint_{\mathbb{R}^{2N}}\frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}\,dx\,dy\Big)^{1/p}.$$

The space $W^{s,p}(\Omega)$ is defined similarly by confining to a domain Ω. Obviously, the fractional Sobolev space with zero boundary value

$$W_0^{s,p}(\Omega)=\big\{u\in W^{s,p}(\mathbb{R}^N):u=0 \text{ on } \mathbb{R}^N\setminus\Omega\big\}$$

is a reflexive Banach space. The next result asserts that $W^{1,p}(\Omega)$ is continuously embedded in $W^{s,p}(\Omega)$, see [15, Proposition 2.2].

Lemma 2.4. *For $0<s<1<p<\infty$, there exists a constant $C=C(N,p,s)>0$ such that*

$$\|u\|_{W^{s,p}(\Omega)}\le C\|u\|_{W^{1,p}(\Omega)},\quad \forall u\in W^{1,p}(\Omega).$$

The simultaneous existence of the local operator Δ_p enables us to work in a simpler space. More precisely, see [10, Lemma 2.1].

Lemma 2.5. *There exists a constant $C=C(N,p,s,\Omega)$ such that*

$$[\bar u]_{s,p}^p:=\iint_{\mathbb{R}^{2N}}\frac{|\bar u(x)-\bar u(y)|^p}{|x-y|^{N+ps}}\,dx\,dy\le C\int_\Omega|\nabla u|^pdx\quad \forall u\in W_0^{1,p}(\Omega),$$

where $\bar u$ is the extension as zero of u in all $\mathbb{R}^N$.

For a zero extension with $u=0$ in $\mathbb{R}^N\setminus\Omega$ in (1.1), it does not matter if we write $[\bar u]_{s,p}$ as $[u]_{s,p}$. Moreover, the mixed norm on the space $W_0^{1,p}(\Omega)$ is denoted as

$$\|u\|_{W_0^{1,p}(\Omega)}:=\Big(\int_\Omega|\nabla u|^pdx+\iint_{\mathbb{R}^{2N}}\frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}\,dx\,dy\Big)^{1/p}.$$

For a proof of the following Sobolev embedding we refer the reader to [15].

Lemma 2.6. *The embedding*

$$W_0^{1,p}(\Omega)\hookrightarrow\begin{cases}L^t(\Omega) & \text{for } t\in[1,p^*],\ \text{if } p\in(1,N)\,,\\ L^t(\Omega) & \text{for } t\in[1,\infty],\ \text{if } p=N,\\ L^\infty(\Omega) & \text{if } p>N,\end{cases}$$

are continuous. More precisely, the above embeddings are compact, except for $t=p^*=\frac{Np}{N-p}$, if $p\in(1,N)$.

In regards to the existence of the potential $V(x)$ in Problem (1.1), it is necessary to introduce the function spaces

$$E = \Big\{u \in W_0^{1,p}(\Omega) : [u]_{s,p}^p + \int_\Omega |\nabla u|^p dx + \int_\Omega V(x)|u|^p dx < \infty\Big\}$$

equipped with the inner product

$$\begin{aligned}\langle u, v\rangle_E = &\iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+ps}}\,dx\,dy \\ &+ \int_\Omega |\nabla u|^{p-2} uv\,dx + \int_\Omega V(x)|u|^{p-2}uv\,dx \quad \text{for all } u, v \in E\end{aligned}$$

and the corresponding norm $\|u\|_E^p = \langle u, u\rangle_E$. For $\lambda > 0$, we also need the inner product

$$\begin{aligned}\langle u, v\rangle_\lambda = &\iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+ps}}\,dx\,dy \\ &+ \int_\Omega |\nabla u|^{p-2} uv\,dx + \lambda\int_\Omega V(x)|u|^{p-2}uv\,dx \quad \text{for all } u, v \in E\end{aligned}$$

and the corresponding norm $\|u\|_\lambda^p = \langle u, u\rangle_\lambda$. Clearly, $\|u\|_E \le \|u\|_\lambda$ for all $\lambda \ge 1$. Obviously, $E_\lambda = (E, \|\cdot\|_\lambda)$ is a reflexive Banach space.

Lemma 2.7. *Let $p(x) : \overline{\Omega} \to \mathbb{R}$ be a continuous function satisfying* (A3) *and $p(x) \in (1, Np/(N-ps))$ for any $x \in \overline{\Omega}$. Then, the embedding $E_\lambda \hookrightarrow L^{p(x)}(\Omega)$ is continuous and compact. More precisely, there exist a constant $C_p = C(N, p^+, p, s) > 0$ such that*

$$\begin{aligned}\int_\Omega |u(x)|^{p(x)} dx &\le \max\big\{\|u\|_{L^{p(x)}}^{p^+}, \|u\|_{L^{p(x)}}^{p^-}\big\} \\ &\le \max\Big\{C_p^{p^+}[u]_{s,p}^{p^+}, C_p^{p^-}[u]_{s,p}^{p^-}\Big\} \\ &\le \max\Big\{C_p^{p^+}\|u\|_\lambda^{p^+}, C_p^{p^-}\|u\|_\lambda^{p^-}\Big\}.\end{aligned} \tag{2.1}$$

It is important that System (1.1) can be reduced into one single Schrödinger equation with a nonlocal term, see for instance [3, 6]. For every fixed $u \in W_0^{1,p}(\Omega)$, by applying the so-called Newton potential, we find a function $\phi_u \in H_0^1(\Omega)$ satisfying

$$\begin{gathered}-\Delta\phi = |u|^p \quad \text{in } \Omega, \\ \phi = 0 \quad \text{in } \mathbb{R}^N \setminus \Omega.\end{gathered}$$

Then, we use a standard argument to obtain that ϕ_u satisfies the following properties, see [19].

Lemma 2.8. *For any $u \in W_0^{1,p}(\Omega)$, we have*

(1) *$\phi_u \ge 0$ and $\phi_{tu} = t^p\phi_u$ for any $t \ge 0$;*
(2) *there exists a constant $C > 0$ such that*
$$\|\nabla\phi_u\|_{L^2(\Omega)} = \int_\Omega \phi_u|u|^p dx \le C\|u\|_\lambda^p;$$
(3) *if $u_n \rightharpoonup u$ in $W_0^{1,p}(\Omega)$, then $\phi_{u_n} \rightharpoonup \phi_u$ in $H_0^1(\Omega)$ and*
$$\lim_{n\to\infty}\int_\Omega \phi_{u_n} u_n^{p-2}u_n\phi\,dx = \int_\Omega \phi_u u^{p-2}u\phi\,dx \quad \text{for all } \phi \in W_0^{1,p}(\Omega).$$

Putting $\phi = \phi_u$ into the first equation of (1.1), our problem (1.1) reduces to the single fractional Schrödinger equation

$$\begin{gathered} \mathcal{L}u + \lambda V(x)|u|^{p-2}u + \phi_u|u|^{p-2}u = \alpha|u|^{p(x)-2}u + \beta|u|^{q(x)-2}u \quad \text{in } \Omega, \\ u = 0 \quad \text{in } \mathbb{R}^N \setminus \Omega. \end{gathered} \tag{2.2}$$

We are now in a position to give the definition of weak solution to (2.2).

Definition 2.9. We say that $u \in E_\lambda$ is a weak solution of (2.2), if u satisfies

$$\begin{aligned} &\iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+ps}}\,dx\,dy \\ &+ \int_\Omega |\nabla u|^{p-2}uv\,dx + \int_\Omega \lambda V(x)|u|^{p-2}uv + \phi_u|u|^{p-2}uv\,dx \\ &= \int_\Omega \big(\alpha|u|^{p(x)-2}uv + \beta|u|^{q(x)-2}uv\big)dx \end{aligned}$$

for any $v \in E_\lambda$.

Note that, if u is the solution of Schrödinger equation (2.2), then u is the solution of (1.1). We would like to emphasize that the existence of solutions for (2.2) can be established by a variational method. Clearly, the energy functional $I_\lambda : E_\lambda \to \mathbb{R}$ associated with Problem (2.2) is

$$\begin{aligned} I_\lambda(u) &= \frac{1}{p}\int_\Omega |\nabla u|^p dx + \frac{1}{p}\iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}\,dx\,dy + \frac{\lambda}{p}\int_\Omega V(x)|u|^p dx \\ &\quad + \frac{1}{2p}\int_\Omega \phi_u|u|^p dx - \int_\Omega \Big(\frac{\alpha}{p(x)}|u|^{p(x)} + \frac{\beta}{q(x)}|u|^{q(x)}\Big)dx \\ &= \frac{1}{p}\|u\|_\lambda^p + \frac{1}{2p}\int_\Omega \phi_u|u|^p dx - \int_\Omega \Big(\frac{\alpha}{p(x)}|u|^{p(x)} + \frac{\beta}{q(x)}|u|^{q(x)}\Big)dx. \end{aligned}$$

Then, we employ the argument used in [34] to prove that $I_\lambda(u)$ is well-defined, of class C^1 in E_λ and

$$\langle I'_\lambda(u), v\rangle = \langle u, v\rangle_\lambda + \int_\Omega \phi_u|u|^{p-2}uv\,dx - \int_\Omega \Big(\frac{\alpha}{p(x)}|u|^{p(x)} + \frac{\beta}{q(x)}|u|^{q(x)}\Big)dx$$

for all $u, v \in E_\lambda$. Hence, if $u \in E_\lambda$ is a critical point of the functional I_λ, then it leads to u being a solution of (2.2).

3. Proof of Theorem 1.1

To show the existence of solutions for (2.2), let us first recall the definition of Palais-Smale sequence.

Definition 3.1. A sequence $\{u_n\}_n \subset E_\lambda$ is called a $(PS)_c$ sequence, if $J(u_n) \to c$ and $J'(u_n) \to 0$. We say J satisfies $(PS)_c$ condition if any $(PS)_c$ sequence admits a converging subsequence.

The following general mountain pass theorem (cf. [5]), which allows us to find a $(PS)_c$ sequence.

Theorem 3.2. *Let $\mathbb{X}$ be a real Banach space, and $J \in C^1(\mathbb{X}, \mathbb{R})$ with $J(0) = 0$. Suppose that*

(i) *there exist two constants $\rho, \delta > 0$ such that $J(u) \geq \delta$ for $u \in \mathbb{X}$ with $\|u\|_{\mathbb{X}} = \rho$;*

(ii) *there exists an* $e \in \mathbb{X}$ *satisfying* $\|e\|_{\mathbb{X}} > \rho$ *such that* $J(e) < 0$.

If we define $\Gamma = \{\gamma \in C^1([0,1];\mathbb{X}) : \gamma(0) = 0, \gamma(1) = e\}$. *Then*

$$c = \inf_{\gamma\in\Gamma} \max_{0\le t\le 1} J\big(\gamma(t)\big) \ge \delta,$$

and there exists a $(PS)_c$ *sequence* $\{u_n\}_n \subset \mathbb{X}$.

Before using the mountain pass theorem to prove Theorem 1.1, we verify that I_λ possesses the mountain pass geometry (i) and (ii).

Lemma 3.3. *Assume that* (A1)–(A5) *hold. Then for all* $\lambda > 0$, *there exists two positive constants* δ *and* ρ *such that* $I_\lambda(u) \ge \delta > 0$, *for any* $u \in E_\lambda$ *with* $\|u\|_\lambda = \rho$, *where* δ *is independent of* λ.

Proof. Using (A3)–(A4) and (2.1), for all $u \in E_\lambda$, we conclude that

$$\begin{aligned}
&\int_\Omega \Big(\frac{\alpha}{p(x)}|u|^{p(x)} + \frac{\beta}{q(x)}|u|^{q(x)}\Big)dx \\
&\le \frac{\alpha}{p^-}\int_\Omega |u|^{p(x)}dx + \frac{\beta}{q^-}\int_\Omega |u|^{q(x)}dx \\
&\le \frac{\alpha}{p^-}\max\big\{C_p^{p^+}\|u\|_\lambda^{p^+}, C_p^{p^-}\|u\|_\lambda^{p^-}\big\} + \frac{\beta}{q^-}\max\big\{C_q^{q^+}\|u\|_\lambda^{q^+}, C_q^{q^-}\|u\|_\lambda^{q^-}\big\},
\end{aligned} \tag{3.1}$$

where $C_q, C_p > 0$ are the constants defined in Lemma 2.7. Now, we deduce from Lemma 2.8-(1) and (3.1) that

$$I_\lambda(u) \ge \frac{1}{p}\|u\|_\lambda^p - \frac{\alpha}{p^-}\max\big\{C_p^{p^+}, C_p^{p^-}\big\}\|u\|_\lambda^{p^+} - \frac{\beta}{q^-}\max\big\{C_q^{q^+}, C_q^{q^-}\big\}\|u\|_\lambda^{q^+}$$

for all $u \in E_\lambda$ with $\|u\|_\lambda \ge 1$.

Next, let us introduce two functions $\Phi(t) : [0,\infty) \to \mathbb{R}$ and $\Psi(t) : [0,\infty) \to \mathbb{R}$ as follows:

$$\Phi(t) = \Psi(t)t^{q^+}, \quad \Psi(t) = \frac{1}{p}t^{p-q^+} - A\alpha t^{p^+-q^+} - B\beta;$$

where

$$A := \frac{\max\{C_p^{p^+}, C_p^{p^-}\}}{p^-} > 0, \quad B := \frac{\max\{C_q^{q^+}, C_q^{q^-}\}}{q^-} > 0.$$

According to assumption (A5),

$$B\beta < \Big(\frac{p-q^+}{Ap(p^+-q^+)}\Big)^{\frac{p-q^+}{p^+-p}} \frac{p^+-p}{p(p^+-q^+)}.$$

It is easy to check that $\Psi(t)$ attains its maximum at

$$t = t^* = [\frac{p-q^+}{A\alpha p(p^+-q^+)}]^{\frac{1}{p^+-p}};$$

that is, $\Psi(t^*) = \max_{t\ge 0}\Psi(t) > 0$. Meanwhile, one easily ensures that

$$t^* = \Big[\frac{p-q^+}{A\alpha p(p^+-q^+)}\Big]^{\frac{1}{p^+-p}} \ge 1.$$

provided that

$$\alpha \le \frac{p-q^+}{Ap(p^+-q^+)},$$

which is guaranteed by Condition (A5). Hence, the conclusion follows by letting $\rho = t^* > 0$ and $\delta = \Phi(t^*) > 0$. $\square$

Lemma 3.4. *Suppose that* (A1)–(A5) *hold. Then there exists an* $e \in E_\lambda$ *with* $\|e\|_\lambda > \rho$ *such that* $I_\lambda(e) < 0$ *for all* $\lambda > 0$*, where* $\rho > 0$ *is obtained in Lemma 3.3.*

Proof. Choosing a function $u_0 \in E_\lambda$ such that

$$\|u_0\|_\lambda = 1 \quad \text{and} \quad \int_\Omega |u_0|^{p(x)}dx > 0.$$

Then using Lemma 2.8-(2), we have

$$\begin{aligned} I_\lambda(tu_0) &= \frac{t^p}{p}\|u_0\|_\lambda^p + \frac{t^{2p}}{2p}\|\nabla\phi_{u_0}\|^2_{L^2(\Omega)} - \int_\Omega \Big(\frac{\alpha}{p(x)}|tu_0|^{p(x)} + \frac{\beta}{q(x)}|tu_0|^{q(x)}\Big)dx \\ &\le \frac{t^p}{p}\|u_0\|_\lambda^p + \frac{t^{2p}}{2p}\|u_0\|_\lambda^{2p} - \frac{\alpha}{p(x)}\int_\Omega |tu_0|^{p(x)}dx \\ &\le \frac{t^p}{p} + \frac{t^{2p}}{2p} - \frac{\alpha t^{p^-}}{p^+}\int_\Omega |u_0|^{p(x)}dx. \end{aligned}$$

Thus, considering $p < 2p < p^-$ we see that there exists $t_0 \ge 1$ large enough such that $\|t_0u_0\|_\lambda > \rho$ and $I_\lambda(t_0u_0) < 0$. The proof is completed by letting $e = t_0u_0$. $\square$

With Lemma 3.3, Lemma 3.4, and Theorem 3.2 in hand, for all $\lambda > 0$, the $(PS)_{c_\lambda}$ sequence of the functional $I_\lambda(u)$ at the level

$$c_\lambda := \inf_{\gamma\in\Gamma}\max_{0\le t\le 1} I_\lambda\big(\gamma(t)\big) \ge \delta > 0 \tag{3.2}$$

can be constructed, where the set of paths is defined by

$$\Gamma = \{\gamma \in C^1([0,1]; E_\lambda) : \gamma(0) = 0, \gamma(1) = e\}.$$

Lemma 3.5. *Assume that* (A1)–(A5) *hold. If* $\{u_n\}_n \subset E_\lambda$ *is a* (PS) *sequence, then there exists* $C > 0$ *such that* $\|u_n\|_\lambda \le C$ *for all* $\lambda > 0$.

Proof. Let $\{u_n\}_{n\in\mathbb{N}} \subset E_\lambda$ be a Palais-Smale sequence of the functional I_λ, which implies that

$$c + o(1) = \frac{1}{p}\|u_n\|_\lambda^p + \frac{1}{2p}\int_\Omega \phi_{u_n}|u_n|^p dx - \int_\Omega \frac{\alpha}{p(x)}|u_n|^{p(x)} + \frac{\beta}{q(x)}|u_n|^{q(x)}dx \tag{3.3}$$

and

$$o(1) = \langle I'_\lambda(u_n), u_n\rangle = \|u_n\|_\lambda^p + \int_\Omega \phi_{u_n}|u_n|^p dx - \int_\Omega \alpha|u_n|^{p(x)} + \beta|u_n|^{q(x)}dx. \tag{3.4}$$

Taking into account (A3), (A4), and (2.1), we obtain that

$$\begin{aligned} &c + o(1) \\ &= I_\lambda(u_n) - \frac{1}{p^-}\langle I'_\lambda(u_n), u_n\rangle \\ &= \Big(\frac{1}{p} - \frac{1}{p^-}\Big)\|u_n\|_\lambda^p + \Big(\frac{1}{2p} - \frac{1}{p^-}\Big)\int_\Omega \phi_{u_n}|u_n|^p dx \\ &\quad - \int_\Omega \alpha\Big(\frac{1}{p(x)} - \frac{1}{p^-}\Big)|u_n|^{p(x)} - \int_\Omega \beta\Big(\frac{1}{q(x)} - \frac{1}{p^-}\Big)|u_n|^{q(x)}dx \\ &\ge \Big(\frac{1}{p} - \frac{1}{p^-}\Big)\|u_n\|_\lambda^p - \beta\Big(\frac{1}{q^-} - \frac{1}{p^-}\Big)\int_\Omega |u_n|^{q(x)}dx \\ &\ge \Big(\frac{1}{p} - \frac{1}{p^-}\Big)\|u_n\|_\lambda^p - \beta\Big(\frac{1}{q^-} - \frac{1}{p^-}\Big)\max\big\{C_q^{q^+}\|u_n\|_\lambda^{q^+}, C_q^{q^-}\|u_n\|_\lambda^{q^-}\big\}. \end{aligned} \tag{3.5}$$

Next we argue by contradiction. We assume that $\{u_n\}_n$ is not bounded in E_λ. Then there exists a subsequence still denoted by $\{u_n\}_n$ such that $\|u_n\|_\lambda \to \infty$ as $n \to \infty$. Thus, from (3.5) it holds that

$$\frac{c+o(1)}{\|u_n\|_\lambda^p} \geq \Big(\frac{1}{p}-\frac{1}{p^-}\Big) - \beta\Big(\frac{1}{q^-}-\frac{1}{p^-}\Big)\max\big\{C_q^{q^+}\|u_n\|_\lambda^{q^+-p}, C_q^{q^-}\|u_n\|_\lambda^{q^--p}\big\},$$

which contradicts that $q^- < q^+ < p < p^- < p^+$. This completes the proof. □

Lemma 3.6. *Suppose that* (A1)–(A5) *hold. Then the functional I_λ satisfies the $(PS)_c$ condition in E_λ for all $c \in \mathbb{R}$ and $\lambda > 0$.*

Proof. Let us choose a Palais-Smale sequence $\{u_n\}_n \subset E_\lambda$ of I_λ with $c \in \mathbb{R}$, which up to a subsequence, is bounded via Lemma 3.5. Thus, there exists a function $u \in E_\lambda$ such that

$$\begin{gathered} u_n \rightharpoonup u \quad \text{in } E_\lambda, \\ u_n \to u \quad \text{a.e. in } \mathbb{R}, \\ |u_n|^{p(x)-2}u_n \rightharpoonup |u|^{p(x)-2}u \quad \text{in } L^{p'(x)}(\Omega). \end{gathered} \tag{3.6}$$

Next we prove that $u_n \to u$ in E_λ. In fact, as the first matter of all, we use the Hölder inequality and Lemma 2.8(3) to infer that

$$\begin{aligned} &\int_\Omega (\phi_{u_n}|u_n|^{p-2}u_n - \phi_u|u|^{p-2}u)(u_n-u)dx \\ &\leq \|\phi_{u_n}\|_{L^{2^*}(\Omega)}\big\|\,|u_n|^{p-2}u_n(u_n-u)\big\|_{L^{(2^*-1)/2^*}(\Omega)} \\ &\quad + \|\phi_u\|_{L^{2^*}(\Omega)}\big\|\,|u|^{p-2}u(u_n-u)\big\|_{L^{(2^*-1)/2^*}(\Omega)} \\ &\leq C\|u_n\|_\lambda^p\big\|\,|u_n|^{p-2}u_n(u_n-u)\big\|_{L^{(2^*-1)/2^*}(\Omega)} \\ &\quad + C\|u\|_\lambda^p\big\|\,|u|^{p-2}u(u_n-u)\big\|_{L^{(2^*-1)/2^*}(\Omega)} \\ &\leq C\big(\|u_n\|_\lambda^p\|u_n\|_{L^p(\Omega)}^{p-1} + \|u\|_\lambda^p\|u\|_{L^p(\Omega)}^{p-1}\big)\|u_n-u\|_{L^{2^*p/(2^*-p)}(\Omega)} \to 0, \end{aligned} \tag{3.7}$$

as $n \to \infty$, where we used Lemma 2.6.

Next, by Lemma 2.6, we know that $u_n \to u$ in $L^{p(x)}(\Omega)$ and $L^{q(x)}(\Omega)$, respectively. Thus,

$$\lim_{n\to\infty}\int_\Omega |u_n-u|^{p(x)}dx = 0, \tag{3.8}$$

$$\lim_{n\to\infty}\int_\Omega |u_n-u|^{q(x)}dx = 0. \tag{3.9}$$

Finally, it follows from (3.3)-(3.4) and (3.6)-(3.7) that

$$\begin{aligned} o(1) &= \langle I_\lambda'(u_n) - I_\lambda'(u), u_n-u\rangle \\ &= \langle u_n-u, u_n-u\rangle_\lambda + \int_\Omega (\phi_{u_n}|u_n|^{p-2}u_n - \phi_u|u|^{p-2}u)(u_n-u)dx \\ &\quad - \alpha\int_\Omega \big(|u_n|^{p(x)-2}u_n - |u|^{p(x)-2}u\big)(u_n-u)dx \\ &\quad - \beta\int_\Omega \big(|u_n|^{q(x)-2}u_n - |u|^{q(x)-2}u\big)(u_n-u)dx \\ &= \|u_n-u\|_\lambda - \int_\Omega |u_n-u|^{p(x)}dx - \int_\Omega |u_n-u|^{q(x)}dx, \end{aligned}$$

which together with (3.8)-(3.9) implies

$$\lim_{n\to\infty} \|u_n - u\|_\lambda = 0,$$

which completes the proof. □

Theorem 3.7. *Assume that* (A1)–(A5) *hold. Then* (2.2) *has a nontrivial solution* u_λ^1 *in* E_λ *with* $I_\lambda(u_\lambda^1) > 0$.

Proof. Thanks to Lemma 3.3, Lemma 3.4 and Theorem 3.2, we deduce that for all $\lambda > 0$ there exists a $(PS)_{c_\lambda}$ sequence $\{u_n\}_n$ for I_λ on E_λ. In view of Lemma 3.6, we know that I_λ satisfies $(PS)_{c_\lambda}$ condition, and there exists $u_\lambda^1 \in E_\lambda$ such that $I'_\lambda(u_\lambda^1) = 0$ and $I_\lambda(u_\lambda^1) = c_\lambda$ for all $\lambda > 0$. Thus, u_λ^1 is a solution of (2.2). □

The following proposition plays a fundamental role in giving the second solution for Problem (1.1).

Proposition 3.8 (Ekeland variational principle, [20, Theorem 1.1])**.** *Let* V *be a Banach space and* $F : V \to \mathbb{R}\cup\{+\infty\}$ *be lower semicontiuous, bounded from below. Then, for each* $\varepsilon > 0$*, there exists a point* $\nu \in V$ *with*

$$F(\nu) \le \inf_V F + \varepsilon, \quad F(\nu) \le F(w) + \varepsilon d(\nu, w) \quad \text{for all } w \in V.$$

In the next proof, we set $B_\rho = \{u \in E_\lambda : \|u\|_\lambda < \rho\}$, where $\rho > 0$ is given by Lemma 3.3.

Theorem 3.9. *Suppose that* (A1)–(A5) *hold. Then* (1.1) *admits another nontrivial solution* u_λ^2 *in* E_λ *with* $I_\lambda(u_\lambda^2) < 0$.

Proof. Let us denote $\widetilde{c}_\lambda = \inf_{u\in\overline{B_\rho}} I_\lambda$, and pick $0 < \varepsilon < \inf_{u\in\partial B_\rho} I_\lambda - \widetilde{c}_\lambda$. By Lemma 3.8, we can choose u_ε such that

$$I_\lambda(u_\varepsilon) \le \widetilde{c}_\lambda + \varepsilon, \tag{3.10}$$

$$I_\lambda(u_\varepsilon) \le I_\lambda(u) + \varepsilon\|u_\varepsilon - u\|_\lambda \tag{3.11}$$

for all $u \in \overline{B_\rho}$ and $u \ne u_\varepsilon$. This implies that $u_\varepsilon \in B_\rho$ because $I_\lambda(u_\varepsilon) \le \widetilde{c}_\lambda + \varepsilon < \inf_{u\in\partial B_\rho} I_\lambda$. Let us set

$$u = u_\varepsilon + \tau v, \quad \forall v \in B_1 := \{v \in E_\lambda : \|v\|_\lambda \le 1\},$$

where $\tau > 0$ small enough that $0 < \tau \le \rho - \|u_\varepsilon\|_\lambda$ for fixed n large. Then

$$\|u\|_\lambda = \|u_\varepsilon + \tau v\|_\lambda \le \|u_\varepsilon\|_\lambda + \tau \le \rho,$$

which implies that $u \in \overline{B_\rho}$. Thus, it follows from (3.11) that

$$0 \le \frac{I_\lambda(u_\varepsilon + \tau v) - I_\lambda(u_\varepsilon)}{\tau} + \varepsilon\|v\|_\lambda,$$

Therefore, letting $\tau \to 0^+$, we obtain

$$\langle I'_\lambda(u_\varepsilon), v\rangle + \varepsilon\|v\|_\lambda \ge 0.$$

By choosing $\tau < 0$ such that $|\tau|$ small enough, we use a similar discussion as above to obtain

$$-\langle I'_\lambda(u_\varepsilon), v\rangle + \varepsilon\|v\|_\lambda \ge 0.$$

We immediately conclude that $\big|\langle I'_\lambda(u_\varepsilon), v\rangle\big| \le \varepsilon\|v\|_\lambda$, for any $v \in \overline{B_1}$. Hence we know

$$\big|\langle I'_\lambda(u_\varepsilon), v\rangle\big| \le \varepsilon. \tag{3.12}$$

Using (3.10) and (3.12), we can choose a sequence $\{u_n\} \subset B_\rho$ such that

$$I_\lambda(u_n) \to \widetilde{c}_\lambda, \quad \|I'_\lambda(u_n)\|_{E^*_\lambda} \to 0, \tag{3.13}$$

as $n \to \infty$. Thus, $\{u_n\}$ is a bounded $(PS)_{\widetilde{c}_\lambda}$ sequence in the reflexive Banach space E_λ due to Lemma 3.5. Repeating the process as Lemma 3.6, there exists u^2_λ and a subsequence of $\{u_n\}$ still denoted by $\{u_n\}$ such that $u_n \to u^2_\lambda$ in E_λ.

Next, we claim that $\widetilde{c}_\lambda < 0$. To this end, let us take a nonnegative function $\omega_0 \in B_\rho$ such that

$$\int_\Omega |\omega_0|^{q(x)} dx > 0.$$

Then, for $\tau \in (0,1)$ small enough we infer from Lemma 2.8-(2) that

$$\begin{aligned}
&I_\lambda(\tau\omega_0) \\
&\le \frac{1}{p}\|\tau\omega_0\|^p_\lambda + \frac{1}{2p}\int_\Omega \phi_{\tau\omega_0}|\tau\omega_0|^p dx - \int_\Omega \frac{\alpha}{p(x)}|\tau\omega_0|^{p(x)} + \frac{\beta}{q(x)}|\tau\omega_0|^{q(x)} dx \\
&\le \frac{1}{p}\|\tau\omega_0\|^p_\lambda + \frac{1}{2p}\|\tau\omega_0\|^{2p}_\lambda - \frac{\beta}{q^+}\int_\Omega |\tau\omega_0|^{q(x)} dx < 0,
\end{aligned} \tag{3.14}$$

where we have used $q^+ < p < 2p$. Thus, we conclude $\widetilde{c}_\lambda < 0$. More precisely, it follows from (3.14) that

$$\widetilde{c}_\lambda \le \frac{1}{p}\rho^p + \frac{1}{2p}\rho^{2p} := \kappa < 0, \tag{3.15}$$

where constant κ is independent of λ.

In summary, we obtain a nontrivial solution u^2_λ of (2.2) satisfying

$$I_\lambda(u^2_\lambda) = \widetilde{c}_\lambda \le \kappa < 0 \quad \text{and} \quad \|u^2_\lambda\|_\lambda < \rho,$$

which completes the proof. □

Proof of Theorem 1.1. The result follows immediately by combining Theorems 3.7 and 3.9. □

4. Asymptotic behavior of solutions

In this section we study the concentration of solutions for Problem (1.1), which is stated by Theorem 1.2. Our main idea is motivated by the recent papers [8, 38, 35].

Let $I_\lambda|_{W^{1,p}_0(\Omega_0)}$ be a restriction of I_λ on $W^{1,p}_0(\Omega_0)$. Note that

$$\begin{aligned}
I_\lambda|_{W^{1,p}_0(\Omega_0)} &= \frac{1}{p}[u]^p_{s,p} + \frac{1}{p}\int_\Omega |\nabla u|^p dx + \frac{1}{2p}\int_\Omega \phi_{u_n}|u_n|^p dx \\
&\quad - \int_\Omega \Big(\frac{\alpha}{p(x)}|u|^{p(x)} + \frac{\beta}{q(x)}|u|^{q(x)}\Big) dx,
\end{aligned}$$

By the same method as in the proofs of Lemmas 3.3 and 3.4, we deduce that $I_\lambda|_{W^{1,p}_0(\Omega_0)}$ also satisfies mountain pass geometry. Then the $(PS)_{m(\Omega_0)}$ sequence of $I_\lambda|_{W^{1,p}_0(\Omega_0)}$ at the level

$$m(\Omega_0) = \inf_{\gamma \in \tilde{\Gamma}} \max_{0 \le t \le 1} I_\lambda|_{W^{1,p}_0(\Omega_0)}\big(\gamma(t)\big),$$

can also be constructed, where the set of paths is defined by

$$\widetilde{\Gamma} = \{\gamma \in C^1([0,1]; W^{1,p}_0(\Omega_0)) : \gamma(0) = 0, \gamma(1) = e\}.$$

Clearly, $m(\Omega_0)$ is independent of λ. Since $W_0^{1,p}(\Omega_0) \subset E_\lambda$, one has

$$0 < \delta \le c_\lambda \le m(\Omega_0) \le M_0 < \infty \tag{4.1}$$

for all $\lambda > 0$.

Proof of Theorem 1.2. For the sequence $\{\lambda_n\}$ with $1 \le \lambda_n \to \infty$ as $n \to \infty$, let $u_n^i := u_{\lambda_n}^i$ be the critical points of the energy functional I_{λ_n} obtained as in Theorem 1.1 for $i = 1, 2$, that is to say,

$$\begin{aligned} I'_{\lambda_n}(u_n^1) = 0, \quad I_{\lambda_n}(u_n^1) = c_{\lambda_n}, \\ I'_{\lambda_n}(u_n^2) = 0, \quad I_{\lambda_n}(u_n^2) = \widetilde{c}_{\lambda_n}. \end{aligned}$$

Thus, from (3.2), (3.15), and (4.1)

$$I_{\lambda_n}(u_n^2) \le \kappa < 0 < \delta \le I_{\lambda_n}(u_n^1) \le m(\Omega_0). \tag{4.2}$$

By a similar argument as in Lemma 3.5, it is clear that

$$\begin{aligned} I_{\lambda_n}(u_n^i) &= I_{\lambda_n}(u_n^i) - \frac{1}{p^-}\langle I'_{\lambda_n}(u_n^i), u_n^i\rangle \\ &\ge \Big(\frac{1}{p} - \frac{1}{p^-}\Big)\|u_n^i\|_\lambda^p + \Big(\frac{1}{2p} - \frac{1}{p^-}\Big)\int_\Omega \phi_{u_n^i}|u_n^i|^p dx \\ &\quad - \beta\Big(\frac{1}{q^-} - \frac{1}{p^-}\Big)\int_\Omega |u_n^i|^{q(x)} dx \\ &\ge \Big(\frac{1}{p} - \frac{1}{p^-}\Big)\|u_n^i\|_\lambda^p - \beta\Big(\frac{1}{q^-} - \frac{1}{p^-}\Big)\max\big\{C_q^{q^+}\|u_n^i\|_\lambda^{q^+}, C_q^{q^-}\|u_n^i\|_\lambda^{q^-}\big\}. \end{aligned}$$

Therefore, we can deduce from (4.2) that there exists a constant $C > 0$ independent of λ_n such that

$$\|u_n^i\|_{\lambda_n} \le C,$$

which shows that $\{u_n^i\}_n$ are uniformly bounded. Passing to a subsequence if necessary, we may assume that there exists $u^i \in W_0^{1,p}(\Omega_0)$ satisfying $u_n^i \rightharpoonup u^i$ in $W_0^{1,p}(\Omega_0)$. Thanks to Lemma 3.6, we immediately conclude that $u_n^i \to u^i$ in $L^{p(x)}(\Omega_0)$ and $L^{q(x)}(\Omega_0)$, respectively. It follows from Fatou's lemma that

$$\int_\Omega V(x)|u^i|^p dx \le \liminf_{n\to\infty}\int_\Omega V(x)|u_n^i|^p dx \le \liminf_{n\to\infty}\frac{\|u_n^i\|_{\lambda_n}^p}{\lambda_n} = 0.$$

Thus, $u^i = 0$ a.e. in $\Omega\backslash\Omega_0$, and $u^i \in W_0^{1,p}(\Omega_0)$ because of assumption (A2). Similar to the proof of Theorem 1.1, we obtain that $u^i \in W_0^{1,p}(\Omega_0)$ for $i = 1, 2$ are solutions of Problem (1.5).

Then from (4.2) there exist two positive constants δ, κ independent of λ_n satisfying

$$I(u^2) = \lim_{n\to\infty} I(u_n^2) \le \kappa < 0 < \delta \le \lim_{n\to\infty} I(u_n^1) = I(u^1),$$

which ensures that $u^i \ne 0$ and $u^1 \ne u^2$. The proof is complete. $\square$

5. Proof of Theorem 1.3

We are now in a position to consider the existence of infinitely many solutions of (1.5). The energy functional $I : W_0^{1,p}(\Omega_0) \to \mathbb{R}$ associated with (1.5) is

$$I(u) = \frac{1}{p}\|u\|^p + \frac{1}{2p}\int_{\Omega_0} \phi_u |u|^p dx - \int_{\Omega_0} \frac{\alpha}{p(x)}|u|^{p(x)}dx - \int_{\Omega_0} \frac{\beta}{q(x)}|u|^{q(x)}dx,$$

where $\|u\|^p := [u]_{s,p}^p + \int_{\Omega_0} |\nabla u|^p dx$. Clearly, $I \in C^1(W_0^{1,p}(\Omega_0))$ and the critical points of I are the weak solutions of (1.5).

To achieve the desired results, we need to verify that I satisfies the following symmetric mountain pass theorem [13, Theorem 2.2].

Lemma 5.1. *Let $\mathbb{X}$ be an infinite-dimensional Banach space. Suppose that $J \in C^1(\mathbb{X}, \mathbb{R})$ satisfies the $(PS)_c$ condition and the following conditions:*

1. *$J(0) = 0$ and J is even;*
2. *there exists two constants $\rho, \delta > 0$ satisfying $J(u) \geq \delta$ for all $u \in \mathbb{X}$ with $\|u\|_{\mathbb{X}} = \rho$;*
3. *for all finite dimensional subspaces $Y \subset \mathbb{X}$ there exists $R = R(Y) > 0$ such that $J(u) \leq 0$ for all $u \in \mathbb{X} \setminus B_R(Y)$, where $B_R(Y) = \{u \in Y : \|u\|_{\mathbb{X}} \leq R\}$. Then J poses an unbounded sequence of critical values characterized by a minimax argument.*

We first verify that I satisfies Lemma 5.1(3).

Lemma 5.2. *Assume that* (A3)–(A5) *hold. Then, for any finite dimensional subspace W of $W_0^{1,p}(\Omega_0)$, there exists $R_0 = R_0(W)$ such that $I(u) < 0$ for all $u \in W_0^{1,p}(\Omega_0) \setminus B_{R_0}(W)$, where $B_{R_0}(W) = \{u \in W : \|u\| \leq R_0\}$.*

Proof. Let W be a fixed finite dimensional subspace of $W_0^{1,p}(\Omega)$ and $R = R(W) > 1$, for any $u \in W$ such that $\|u\| > R$. Thus, we have

$$\begin{aligned}
I(u) &= \frac{1}{p}\|u\|^p + \frac{1}{2p}\int_{\Omega_0} \phi_u |u|^p dx - \int_{\Omega_0} \frac{\alpha}{p(x)}|u|^{p(x)}dx - \int_{\Omega_0} \frac{\beta}{q(x)}|u|^{q(x)}dx \\
&\leq \frac{1}{p}\|u\|^p + \frac{1}{2p}\|u\|^{2p} - \frac{1}{p^+}\int_{\Omega_0} |u|^{p(x)}dx \\
&\leq \frac{1}{p}\|u\|^p + \frac{1}{2p}\|u\|^{2p} - \frac{\alpha}{p^+}\min\big\{\|u\|^{p^-}_{L^{p(x)}(\Omega_0)}, \|u\|^{p^+}_{L^{p(x)}(\Omega_0)}\big\}.
\end{aligned}$$

Note that there exists $C_W > 0$ such that $\|u\|_{L^{p(x)}(\Omega_0)} \geq C_W\|u\|$, since all norms are equivalent on finite dimensional Banach space W. Hence, by $p^+ > p^- > 2p$, we obtain

$$I(u) \leq \frac{1}{p}\|u\|^p + \frac{1}{2p}\|u\|^{2p} - \frac{\alpha}{p^+}\|u\|^{p^-} \to -\infty, \quad \text{as } R \to \infty.$$

Therefore, there exists $R_0 > 0$ large enough that $I(u) < 0$ for all $u \in W_0^{s,p}(\Omega_0)$, with $\|u\| = R$ and $R > R_0$. Thus the assertion holds. □

Proof of Theorem 1.2. It is obviously to see that $I(0) = 0$, and I is an even functional. Moreover, the functional I satisfies Lemma 5.1(2) via the proof of Lemma 3.3. Similar to the proof of Lemma 3.6, one can show that I satisfies the $(PS)_c$ condition for any $c \in \mathbb{R}$. As a consequence of Lemma 5.1, there exists an unbounded sequence of solutions for Problem (1.5). □

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