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# EXISTENCE OF POSITIVE PERIODIC SOLUTIONS FOR A NONLINEAR SYSTEM OF SECOND-ORDER ORDINARY DIFFERENTIAL EQUATIONS

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Abstract. This article concerns the positive periodic solutions for a system of second-order nonlinear ordinary differential equations, in which the nonlinear term is sublinear in one equation and superlinear in the other equation. By using the fixed point theorem of cone expansion and compression we obtain the existence of positive periodic solutions.

## 1. Introduction

System of ordinary differential equations appear in fields such as applied mathematics, mathematical physics, mechanical engineering, etc. To find special solutions, for example, radial symmetric solutions of elliptic system, it is natural to consider systems of ordinary differential equations (see [3, 4, 11, 12]). In recent decades, the existence of solutions for ordinary differential equation and related questions have attracted extensive attention (see [1, 2, 5, 7, 8, 13, 14]). Dunninger and Wang [3] considered positive and radial symmetric solutions for a class of elliptic systems. The corresponding problem was reduced to a Dirichlet boundary value problem for a system of ordinary differential equations. In their work, the nonlinear terms of the two equations are either both sublinear or superlinear, which means that the corresponding solution operators have the properties of the cone compression or the cone expansion. Thus, the result on existence of positive solutions can be obtained by constructing a single cone in the product space $C[0,1]\times C[0,1]$ and applying the fixed point theorem of cone compression or expansion. Later, Cheng and Zhong [2] studied the Dirichlet boundary value problem of a system of the second-order ordinary differential equations in which the nonlinear terms have the different growth properties and proved the existence of positive solutions by investigating the properties of the fixed point index of the Cartesian product of two cones in the space $C[0,1]$.

This article is mainly concerned with the periodic behavior of solutions to ordinary differential equations. Such a problem has always been the focus in the study of ordinary differential equation. Guerrero-Flores et al. [9] studied the seasonal SIQRS models with nonlinear infection terms and proved the existence of periodic solutions by using Leray-Schauder degree theory. Kobilzoda and Naimov

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[10] considered a class of systems of nonlinear ordinary differential equations on the plane and obtained the existence of positive periodic solutions by giving suitable estimations and applying the theory of rotation of vector fields. In [6, 16] the authors studied periodic solutions of a system of (generalized) ordinary differential equations by using the bifurcations method. Here we hope to develop the method of fixed point theorem in cone to investigate the periodic solutions of the system of ordinary differential equations. To do so, the first problem we face is to give the Green's function for the linear ordinary differential equation and study its properties. Then, we need to construct the suitable cone by analyzing the nonlinear problem. Finally, we develop the fixed point theorem in a cone to establish the existence of periodic solutions.

We consider the existence of positive periodic solutions for the system of second-order nonlinear ordinary differential equations

$$\begin{gathered} -u''(t)+u(t)=g_1(t,u)+h_1(u,v), \quad 0<t<1, \\ -v''(t)+v(t)=g_2(t,u)+h_2(u,v), \quad 0<t<1, \\ u(0)=u(1), \quad u'(0)=u'(1), \quad v(0)=v(1), \quad v'(0)=v'(1), \end{gathered} \tag{1.1}$$

where $g_i \in C([0,1]\times\mathbb{R}^+,\mathbb{R}^+)$ are 1-periodic in $t$ and $h_i \in C(\mathbb{R}^+\times\mathbb{R}^+,\mathbb{R}^+)$ for $i=1,2$, $\mathbb{R}^+=[0,+\infty)$. In this article we assume the folloing hypothesis on $g_i$ and $h_i$ $(i=1,2)$:

- (H1) $\limsup_{u\to 0^+}\max_{t\in[0,1]}\frac{g_1(t,u)}{u}<1<\liminf_{u\to+\infty}\min_{t\in[0,1]}\frac{g_1(t,u)}{u}$;
- (H2) $\limsup_{v\to +\infty}\max_{t\in[0,1]}\frac{g_2(t,v)}{v}<1<\liminf_{v\to 0^+}\min_{t\in[0,1]}\frac{g_2(t,v)}{v}$;
- (H3) $\lim_{u\to 0^+}\frac{h_1(u,v)}{u}=0$ uniformly for $v\in\mathbb{R}^+$;
- (H4) $\lim_{v\to+\infty}\frac{h_2(u,v)}{v}=0$ uniformly for $u\in\mathbb{R}^+$, and for any fixed constant $M>0$, $\lim_{u\to+\infty}h_2(u,v)=0$ uniformly for $v\in[0,M]$.

From (H1) it follows that $g_1(t,u)$ is superlinear with respect to $u$ at 0 and $+\infty$. Condition (H2) implies that $g_2(t,v)$ is sublinear with respect to $v$ at 0 and $+\infty$. (H3) and (H4) show that $h_1(u,v)$ is superlinear with respect to $u$ at 0, and $h_2(u,v)$ is sublinear with respect to $v$ at $+\infty$.

The main results of this article read as follows.

**Theorem 1.1.** *Assume that $g_i\in C([0,1]\times\mathbb{R}^+,\mathbb{R}^+)$ are 1-periodic in $t$ and satisfy* (H1) *and* (H2)*, and $h_i\in C(\mathbb{R}^+\times\mathbb{R}^+,\mathbb{R}^+)$ satisfy* (H3) *and* (H4) *for $i=1,2$. Then* (1.1) *has at least one positive periodic solution.*

This article is organized as follows. We first give some preliminaries and construct the Green's function for the corresponding homogeneous linear problem in Section 2. Then the proof of Theorem 1.1 is completed in Section 3 and some examples are presented in Section 4 as the application of our main result.

## 2. Preliminaries

In this section, we first construct a cone which can be viewed as the Cartesian product of two cones in $C[0,1]$, and then we shall transform the problem of finding the positive periodic solutions of (1.1) into a fixed-point index problem in this cone.

As we know, $C[0,1]$ is a Banach space with the norm

$$\|u\|=\max_{t\in[0,1]}|u(t)|, \quad \forall u\in C[0,1].$$

The space of non-negative functions belonging to $C[0,1]$ is defined by

$$C^+[0,1] = \{u \in C[0,1] : u(t) \geq 0\}.$$

To find positive periodic solutions of (1.1), we would like to transform the original problem into a fixed point problem of some integral system; thus we first need to construct the Green's function $G(t,s)$ of the corresponding linear problem

$$-u''(t) + u(t) = 0, \quad 0 < t < 1, \tag{2.1}$$

$$u(0) = u(1), \quad u'(0) = u'(1). \tag{2.2}$$

To do so, we consider the Cauchy problems of equation (2.1) with the initial conditions

$$u(0) = 1, \quad u'(0) = 0,$$

and

$$u(0) = 0, \quad u'(0) = 1,$$

respectively. It is obvious that

$$u_1(t) = \cosh t = \frac{e^t + e^{-t}}{2}$$

and

$$u_2(t) = \sinh t = \frac{e^t - e^{-t}}{2}$$

are, respectively, the solutions of the above Cauchy problems.

We denote

$$\kappa = u_1(1) + u_2'(1) - 2 = e + e^{-1} - 2.$$

Then, a not very complicated calculation shows that the Green function is

$$G(t,s) = \frac{u_2(1)}{\kappa} u_1(t) u_1(s) - \frac{u_1'(1)}{\kappa} u_2(t) u_2(s) + r(t,s),$$

with

$$r(t,s) = \begin{cases} \frac{u_2'(1)-1}{\kappa} u_1(t) u_2(s) - \frac{u_1(1)-1}{\kappa} u_1(s) u_2(t), & 0 \leq s \leq t \leq 1, \\ \frac{u_2'(1)-1}{\kappa} u_1(s) u_2(t) - \frac{u_1(1)-1}{\kappa} u_1(t) u_2(s), & 0 \leq t \leq s \leq 1. \end{cases}$$

From the expressions $u_1(t)$ and $u_2(t)$ and noting that

$$u_1(1) = u_2'(1) = \frac{e + e^{-1}}{2}, \quad u_1'(1) = u_2(1) = \frac{e - e^{-1}}{2},$$

it is easy to see that

$$G(t,s) = \begin{cases} \frac{e-1}{2\kappa}(e^{t-s-1} + e^{s-t}), & 0 \leq s \leq t \leq 1, \\ \frac{e-1}{2\kappa}(e^{s-t-1} + e^{t-s}), & 0 \leq t \leq s \leq 1. \end{cases} \tag{2.3}$$

**Lemma 2.1.** *The Green function $G(t,s)$ given by* (2.3) *has the following properties:*

(i) $G(t,s) \geq 0$, $\forall t, s \in (0,1)$;
(ii) $G(t,s) \leq G(s,s)$, $\forall t, s \in [0,1]$;
(iii) $G(t,s) \geq \frac{2\sqrt{e}}{e+1} G(s,s)$, *for all* $t, s \in [0,1]$.

*Proof.* Firstly, from the expression of $G(t,s)$ given by (2.3), it is easy to obtain the property (i).

Because of the symmetry of $G(t,s)$ with respect to $t$ and $s$, it is sufficient for us to consider one of the cases, for example, the case of $0 \le s \le t \le 1$, in which

$$G(t,s) = \frac{e-1}{2\kappa}(e^{t-s-1} + e^{s-t}).$$

For this case, it is obvious that

$$G(s,s) = \frac{e-1}{2\kappa}(e^{-1} + 1),$$

and

$$\frac{G(s,s)}{G(t,s)} = \frac{e^{-1}+1}{e^{t-s-1} + e^{s-t}}.$$

Denote $x = t - s$ and define

$$f(x) = \frac{e^{-1}+1}{e^{x-1} + e^{-x}}.$$

Then, in this case (that is, $0 \le s \le t \le 1$), we have $0 \le x \le 1$ and

$$f(x) = \frac{G(s,s)}{G(t,s)}.$$

A simple calculation yields that

$$\max_{x\in[0,1]} f(x) = \frac{1+e}{2e^{1/2}}, \quad \min_{x\in[0,1]} f(x) = 1,$$

which shows that

$$1 \le f(x) \le \frac{1+e}{2e^{1/2}}$$

holds for $0 \le x \le 1$. Therefore, we have

$$1 \le \frac{G(s,s)}{G(t,s)} \le \frac{1+e}{2e^{1/2}},$$

which implies the conclusions (ii) and (iii). The proof is complete. $\square$

For each $h \in C[0,1]$, we consider the non-homogeneous problem associated with (2.1) and (2.2) having the form

$$-u''(t) + u(t) = h(t), \ 0 < t < 1, \tag{2.4}$$

$$u(0) = u(1), u'(0) = u'(1). \tag{2.5}$$

From the Green function $G(t,s)$ given by (2.3), the periodic solution of (2.4)–(2.5) can be expressed as

$$u(t) = \int_0^1 G(t,s)h(s)ds. \tag{2.6}$$

Thus, the problem of finding positive periodic solutions of (1.1) is transformed into the fixed point problem of the integral system

$$\begin{aligned} u(t) &= \int_0^1 G(t,s)\Big(g_1(s,u(s)) + h_1(u(s),v(s))\Big)ds, \\ v(t) &= \int_0^1 G(t,s)\Big(g_2(s,u(s)) + h_2(u(s),v(s))\Big)ds, \end{aligned}$$

in $C^+[0,1] \times C^+[0,1]$.

To prove the existence of fixed point for the above integral system, we introduce a family of operators as following. For each $\theta \in [0,1]$ and $u, v \in C^+[0,1]$, we define

$$A_v(\theta,u)(t) = \int_0^1 G(t,s)\Big((1-\theta)u^2(s) + \theta\Big(g_1(s,u(s)) + h_1(u(s),v(s))\Big)\Big)ds, \quad (2.7)$$

$$B_u(\theta,v)(t) = \int_0^1 G(t,s)\Big((1-\theta)\sqrt{v(s)} + \theta\Big(g_2(s,u(s)) + h_2(u(s),v(s))\Big)\Big)ds. \quad (2.8)$$

It is easy to check that, for any $\theta \in [0,1]$, the operators $A_v(\theta,\cdot)$ and $B_u(\theta,\cdot)$ : $C^+[0,1] \to C^+[0,1]$.

Now we define the vector operator

$$T_\theta(u,v) = \Big(A_v(\theta,u)(t), B_u(\theta,v)(t)\Big), \quad (2.9)$$

then $T_\theta(\cdot,\cdot) : C^+[0,1] \times C^+[0,1] \to C^+[0,1] \times C^+[0,1]$ for all $\theta \in [0,1]$, and the positive periodic solutions of (1.1) correspond to the fixed points of the vector operator $T_1$ in $C^+[0,1] \times C^+[0,1]$.

We define the cone $K$ in $C^+[0,1]$ by

$$K = \Big\{u \in C^+[0,1] : u(0) = u(1),\ u(t) \geq \frac{2\sqrt{e}}{e+1}\|u\|,\ \forall t \in [\frac{1}{4}, \frac{3}{4}]\Big\},$$

and, for a constant $r > 0$, we define

$$K_r = \{u \in K : \|u\| < r\}, \quad \partial K_r = \{u \in K : \|u\| = r\}. \quad (2.10)$$

**Lemma 2.2.** *For each $\theta \in [0,1]$, the operator $T_\theta : K \times K \to K \times K$ is completely continuous.*

*Proof.* We first prove that for any $\theta \in [0,1]$, the operator $T_\theta$ maps $K \times K$ into $K \times K$. In fact, for any $u, v \in K$, by using the properties of the Green function $G(t,s)$ given in Lemma 2.1, and taking into consideration the definition of operator $A_v(\theta,u)$ given in (2.7), it is easy to see that

$$\begin{aligned} A_v(\theta,u)(t) &= \int_0^1 G(t,s)\Big((1-\theta)u^2(s) + \theta\Big(g_1(s,u(s)) + h_1(u(s),v(s))\Big)\Big)ds \\ &\geq \frac{2\sqrt{e}}{e+1}\int_0^1 G(s,s)\Big((1-\theta)u^2(s) + \theta\Big(g_1(s,u(s)) + h_1(u(s),v(s))\Big)\Big)ds \\ &\geq \frac{2\sqrt{e}}{e+1}\|A_v(\theta,u)\| \end{aligned}$$

holds for all $t \in [1/4, 3/4]$. In a similar way we obtain

$$B_u(\theta,v)(t) \geq \frac{2\sqrt{e}}{e+1}\|B_u(\theta,v)\|$$

for $t \in [1/4, 3/4]$. Thus, we have

$$T_\theta(u,v) \in K \times K, \quad \forall (u,v) \in K \times K.$$

Finally, using the Arzelà-Ascoli theorem, it is not difficult to prove that $T_\theta$ is completely continuous. □

At the end of this section, we make some remarks and introduce two lemmas. Let $X$ be a Banach space and $P \subset X$ be a closed convex cone. Assume that $\Omega \subset X$ is a bounded open set and the operator $A : P \cap \overline{\Omega} \to P$ is completely continuous. If $Au \neq u$ for all $u \in P \cap \partial\Omega$, then the fixed point index $i(A, P \cap \Omega, P)$ can be defined as in [15]. Furthermore, if $i(A, P \cap \Omega, P) \neq 0$, the operator $A$ possesses a fixed point in $P \cap \Omega$.

**Lemma 2.3** ([2, 15])**.** *Assume that $A : P_r \to P_r$ is completely continuous, where $P_r = \{u \in P : \|u\| < r\}$ with $\partial P_r = \{u \in P : \|u\| = r\}$ for some constant $r > 0$.*

- (i) *If $\|Au\| > \|u\|$, for all $u \in \partial P_r$, then $i(A, P_r, P) = 0$;*
- (ii) *If $\|Au\|\| < u\|$, for all $u \in \partial P_r$, then $i(A, P_r, P) = 1$.*

**Lemma 2.4** (Product rule for fixed point index, see [2])**.** *Assume that $P_i \subset X$ are closed convex cone in Banach space $X$ and $A_i : P_i \to P_i$ are completely continuous operators for $i = 1, 2$. If $A_i u_i \neq u_i$ for any $u_i \in \partial P_i$, then*

$$i(A, P_{r_1} \times P_{r_2}, P_1 \times P_2) = i(A_1, P_{r_1}, P_1) \cdot i(A_2, P_{r_2}, P_2),$$

*where $A(u, v) = (A_1(u), A_2(v))$ for $(u, v) \in P_1 \times P_2$, $P_{r_i} = \{u \in P_i : \|u\| < r_i\}$ and $\partial P_{r_i} = \{u \in P_i : \|u\| = r_i\}$ for some constants $r_i > 0$.*

## 3. Proof of main result

In this section, we shall prove the existence of positive periodic solutions of (1.1). To do so, we first give the fixed point index of $T_0$, and then we apply the homotopy invariance to obtain the fixed point index of $T_1$. In this process, the following theorem plays a fundamental role.

**Theorem 3.1.** *There exist constants $0 < r_i < R_i$ for $i = 1, 2$ such that, for any $\theta \in [0, 1]$, we have*

$$T_\theta(u, v) \neq (u, v), \quad \forall (u, v) \in \partial\Big((K_{R_1} \backslash \overline{K}_{r_1}) \times (K_{R_2} \backslash \overline{K}_{r_2})\Big),$$

*where $K_{R_i}$ and $K_{r_i}$ are defined by (2.10).*

*Proof.* The proof is divided into the following four steps.

**Step 1.** Denote

$$r_0 = \Big(\int_0^1 G(s, s) ds\Big)^{-1} = \frac{2(e + e^{-1} - 2)}{e - e^{-1}}. \tag{3.1}$$

Then, by the assumptions (H1) and (H3), there exist $\varepsilon \in (0, 1/2)$ and $0 < r_1 < \min\{r_0, 1 - \varepsilon\}$, such that

$$g_1(t, u) \leq (1 - 2\varepsilon)u, \quad \forall t \in [0, 1],\ 0 \leq u \leq r_1, \tag{3.2}$$

$$h_1(u, v) \leq \varepsilon u, \quad \forall v \geq 0,\ 0 \leq u \leq r_1. \tag{3.3}$$

Thus, for any $\theta \in [0, 1]$, it is not difficult to verify that

$$T_\theta(u, v) \neq (u, v), \quad \forall (u, v) \in \partial K_{r_1} \times K.$$

In fact, if there exist $\theta_0 \in [0, 1]$ and $(u_0, v_0) \in \partial K_{r_1} \times K$ such that

$$T_{\theta_0}(u_0, v_0) = (u_0, v_0).$$

Then, by (2.7) and (2.9), $u_0$ satisfies

$$-u_0''(t) + u_0(t) = (1 - \theta)u_0^2(t) + \theta\Big(g_1(t, u_0(t)) + h_1(u_0(t), v_0(t))\Big), \tag{3.4}$$

$$u_0(0) = u_0(1), u_0'(0) = u_0'(1). \tag{3.5}$$

Noting that $0 < r_1 < 1 - \varepsilon$, by (3.2) and (3.3), we have

$$-u_0''(t) + u_0(t) \le (1-\theta)(1-\varepsilon)u_0(t) + \theta(1-\varepsilon)u_0(t) = (1-\varepsilon)u_0(t),$$

for $0 \le u \le r_1$. Integrating this inequality on $[0,1]$ yields

$$\int_0^1 u_0(t)dt \le (1-\varepsilon)\int_0^1 u_0(t)dt,$$

which implies $1 \le 1-\varepsilon$ because of $u_0 \in C^+[0,1]$. This contradicts $\varepsilon \in (0, \frac{1}{2})$.

**Step 2.** According to assumption (H2), there exist $\varepsilon > 0$ and $0 < \xi < \frac{1}{(1+\varepsilon)^2}$ such that

$$g_2(t,v) \ge (1+\varepsilon)v, \quad \forall t \in [0,1],\ 0 \le v \le \xi.$$

From $0 \le v \le \xi$ and $0 < \xi < \frac{1}{(1+\varepsilon)^2}$, we have $\sqrt{v} \ge (1+\varepsilon)v$. Taking $r_2 \in (0, \min\{r_0, \xi\})$ and noting $h_2(t) \ge 0$, a similar proof as Step 1 shows that

$$T_\theta(u,v) \ne (u,v), \quad \forall (u,v) \in K \times \partial K_{r_2},$$

for any fixed $\theta \in [0,1]$.

**Step 3.** By assumption (H1), there exist $\varepsilon > 0$ and $M_1 > 0$ such that

$$g_1(t,u) \ge (1+\varepsilon)u,\ \forall t \in [0,1],\ u \ge M_1.$$

Furthermore, since $g_1$ is continuous, there exists a constant $C_1 > 0$ such that

$$g_1(t,u) \ge (1+\varepsilon)u - C_1, \quad \forall t \in [0,1],\ u \ge 0, \tag{3.6}$$

$$u^2 \ge (1+\varepsilon)u - (1+\varepsilon)^2 \ge (1+\varepsilon)u - C_1, \quad u \ge 0. \tag{3.7}$$

If there exist $\theta_0 \in [0,1]$ and $(u_0, v_0) \in K \times K$ such that $T_{\theta_0}(u_0,v_0) = (u_0,v_0)$, then (3.4) and (3.5) hold. Noting $h_1(t) \ge 0$, by (3.4), (3.6) and (3.7), we have

$$-u_0''(t)+u_0(t) \ge (1-\theta)\Big((1+\varepsilon)u_0(t)-C_1\Big)+\theta\Big((1+\varepsilon)u_0(t)-C_1\Big) = (1+\varepsilon)u_0(t)-C_1.$$

Integrating this inequality on $[0,1]$ yields

$$\int_0^1 u_0(t)dt \ge (1+\varepsilon)\int_0^1 u_0(t)dt - C_1,$$

which shows that $C_1 \ge \varepsilon \int_0^1 u_0(t)dt$. Furthermore, by the definition of $K$, we have

$$C_1 \ge \varepsilon \int_0^1 u_0(t)dt \ge \varepsilon \int_{1/4}^{3/4} \frac{2\sqrt{e}}{e+1}\|u_0\|dt = \varepsilon\frac{\sqrt{e}}{e+1}\|u_0\|,$$

which shows

$$\|u_0\| \le \frac{C_1(e+1)}{\sqrt{e}\varepsilon} =: \overline{R}_1.$$

Taking $R_1 > \max\{R_0, \overline{R}_1\}$ with

$$R_0 = \Big(\big(\frac{2e^{1/2}}{e+1}\big)^3 \int_{1/4}^{3/4} G(s,s)ds\Big)^{-1} = \frac{(e+1)^3(e+e^{-1}-2)}{2e^{3/2}(e-e^{-1})}, \tag{3.8}$$

for any $\theta \in [0,1]$, we have

$$T_\theta(u,v) \ne (u,v), \quad \forall (u,v) \in \partial K_{R_1} \times K.$$

**Step 4.** By assumptions (H2) and (H4), there exist $0 < \varepsilon \ll 1$ and $M_2 > 0$ such that

$$g_2(t,v) \le (1-2\varepsilon)v \quad \text{and} \quad h_2(u,v) \le \varepsilon v, \quad \forall t \in [0,1],\ v \ge M_2, u \ge 0.$$

Also, we have

$$g_2(t,v) + h_2(u,v) \le (1-\varepsilon)v + C_2, \quad \forall t \in [0,1],\ v \ge 0,\ u \ge 0,$$

where

$$0 < C_2 = \frac{1}{1-\varepsilon} + \max_{t\in[0,1],0\le v\le M_2,u\ge 0} \Big(g_2(t,v) + h_2(u,v)\Big) < +\infty,$$

because of the continuity of $g_2$ and $h_2$ and the assumption (H4).

Obviously, we have

$$\sqrt{v} \le (1-\varepsilon)v + C_2, \quad \forall t \in [0,1].$$

Let $\overline{R}_2 = C_2(e+1)/(\sqrt{e}\varepsilon)$. Taking $R_2 > \max\{R_0, \overline{R}_2\}$, a similar proof as in Step 3 shows that

$$T_\theta(u,v) \ne (u,v), \quad \forall (u,v) \in K \times \partial K_{R_2},$$

for each fixed $\theta \in [0,1]$. Finally, the conclusion of this theorem is derived from the results of steps 1–4, and thus the proof is completed. □

*Proof of Theorem 1.1.* By Lemma 2.1 and the definitions of $A_v(\theta,u)$ and $A_u(\theta,v)$ given in (2.7) and (2.8), for any $u, v \in K$, we have

$$A_v(0,u) = \int_0^1 G(t,s)u^2(s)ds \le \int_0^1 G(s,s)u^2(s)ds,$$
$$B_u(0,v) = \int_0^1 G(t,s)\sqrt{v(s)}ds \le \int_0^1 G(s,s)\sqrt{v(s)}ds.$$

Therefore,

$$\|A_v(0,u)\| \le \int_0^1 G(s,s)ds\|u\|^2,$$
$$\|B_u(0,v)\| \le \int_0^1 G(s,s)ds\|v\|^{1/2}.$$

Moreover, by Lemma 2.1, we have

$$\|A_v(0,u)\| \ge \int_0^1 G(\frac{1}{2},s)u^2(s)ds \ge \frac{8e^{3/2}}{(e+1)^3}\int_{1/4}^{3/4} G(s,s)ds\|u\|^2,$$

and

$$\|B_u(0,v)\| \ge \int_0^1 G(\frac{1}{2},s)\sqrt{v(s)}ds \ge \Big(\frac{2e^{1/2}}{e+1}\Big)^{3/2}\int_{1/4}^{3/4} G(s,s)ds\|v\|^{1/2}.$$

Let

$$\overline{R}_0 = \Big(\int_0^1 G(s,s)ds\Big)^2 = \Big(\frac{e-e^{-1}}{2(e+e^{-1}-2)}\Big)^2,$$
$$\bar{r}_0 = \Big(\Big(\frac{2e^{1/2}}{e+1}\Big)^{3/2}\int_{1/4}^{3/4} G(s,s)ds\Big)^2 = \frac{2e^{3/2}}{(e+1)^3}\overline{R}_0.$$

It is obvious that $\overline{R}_0 > \bar{r}_0$, and thus we have

$$\|A_v(0,u)\| < \|u\|, \quad \forall r \in (0,r_0),\ u \in \partial K_r,$$

$$
\begin{gathered}
\|A_v(0,u)\| > \|u\|, \quad \forall R \in (R_0, +\infty),\ u \in \partial K_R,\\
\|B_u(0,v)\| > \|v\|, \quad \forall \bar{r} \in (0, \bar{r}_0),\ v \in \partial K_{\bar{r}},\\
\|B_u(0,v)\| < \|v\|, \quad \forall \overline{R} \in (\overline{R}_0, +\infty),\ u \in \partial K_{\overline{R}},
\end{gathered}
$$

where $r_0$ and $R_0$ are given by (3.1) and (3.8), respectively.

By Lemma 2.3, we have

$$
\begin{gathered}
i(A_v(0,\cdot), K_r, K) = 1, \quad \forall r \in (0, r_0),\\
i(A_v(0,\cdot), K_R, K) = 0, \quad \forall R \in (R_0, +\infty),\\
i(B_u(0,\cdot), K_{\bar{r}}, K) = 0, \quad \forall \bar{r} \in (0, \bar{r}_0),\\
i(B_u(0,\cdot), K_{\overline{R}}, K) = 1, \quad \forall \overline{R} \in (\overline{R}_0, +\infty).
\end{gathered}
$$

Thus, by Lemma 2.4, we have

$$
\begin{aligned}
& i\Big(T_0, (K_R \backslash \overline{K_r}) \times (K_{\overline{R}} \backslash \overline{K_{\bar{r}}}), K \times K\Big)\\
&= i(A_v(0,\cdot), K_R \backslash \overline{K_r}, K) \cdot i(B_u(0,\cdot), K_{\overline{R}} \backslash \overline{K_{\bar{r}}}, K) = -1.
\end{aligned}
$$

Therefore, by Theorem 3.1 and applying the homotopy invariance, we have

$$
\begin{aligned}
& i\Big(T_\theta, (K_{R_1} \backslash \overline{K}_{r_1}) \times (K_{R_2} \backslash \overline{K}_{r_2}), K \times K\Big)\\
&= i\Big(T_0, (K_{R_1} \backslash \overline{K}_{r_1}) \times (K_{R_2} \backslash \overline{K}_{r_2}), K \times K\Big) = -1,
\end{aligned}
$$

for any fixed $\theta \in [0,1]$, where $R_i$ and $r_i$ $(i = 1,2)$ are given in Theorem 3.1 and satisfy $r_1 \in (0, r_0)$, $R_1 > R_0$, $r_2 \in (0, \bar{r}_0)$ and $R_2 > \overline{R}_0$. Consequently,

$$
i\Big(T_1, (K_{R_1} \backslash \overline{K}_{r_1}) \times (K_{R_2} \backslash \overline{K}_{r_2}), K \times K\Big) = -1,
$$

which implies that problem (1.1) has a positive periodic solution in $K \times K$. The proof is complete. □

## 4. Applications

To demonstrate the usefulness of our main theorem, in this section we consider the the existence of positive periodic solutions for

$$
\begin{gathered}
-u''(t) + u(t) = \xi(t)u^{p+1} + u^{p+1}\frac{|\sin v|}{v}, \quad 0 < t < 1,\\
-v''(t) + v(t) = \eta(t)v^{1-q} + v^{1-q}\frac{|\sin u|}{u}, \quad 0 < t < 1,\\
u(0) = u(1), u'(0) = u'(1), v(0) = v(1), v'(0) = v'(1),
\end{gathered} \tag{4.1}
$$

where $p > 0$, $0 < q < 1$ are constants, and $\xi(t)$, $\eta(t) \in C([0,1];\mathbb{R}^+)$ are positive, 1-periodic, continuous functions.

Set

$$
\begin{gathered}
g_1(t,u) = \xi(t)u^{p+1}, \quad g_2(t,v) = \eta(t)v^{1-q},\\
h_1(u,v) = \begin{cases} u^{p+1}\frac{|\sin v|}{v}, & v > 0,\\ u^{p+1}, & v = 0, \end{cases} \quad h_2(u,v) = \begin{cases} v^{1-q}\frac{|\sin u|}{u}, & u > 0,\\ v^{1-q}, & u = 0. \end{cases}
\end{gathered}
$$

Then a simple computation shows that all the conditions in Theorem 1.1 are satisfied. Thus we have the following result.

**Proposition 4.1.** *Let $p > 0$, $0 < q < 1$. Then, for any positive, 1-periodic, continuous functions $\xi(t)$, $\eta(t) \in C([0,1];\mathbb{R}^+)$, system (4.1) has at least one positive 1-periodic solution.*

More specifically, we present the following example.

**Example 4.2.** Consider the system of ordinary differential equations

$$\begin{gathered}
-u''(t) + u(t) = (2+\sin(\pi t))u^{\frac{5}{2}} + u^2(t)\frac{|\sin v|}{v}, \quad 0 < t < 1, \\
-v''(t) + v(t) = (2+\sin(\pi t))v^{\frac{1}{3}} + v^{\frac{1}{4}}(t)\frac{|\sin u|}{u}, \quad 0 < t < 1, \\
u(0) = u(1), \quad u'(0) = u'(1), \quad v(0) = v(1), \quad v'(0) = v'(1).
\end{gathered} \tag{4.2}$$

By choosing $\xi(t) = \eta(t) = 2 + \sin(\pi t)$ in system (4.1), it is reduced to the above specific example. By Proposition 4.1, we can conclude that system (4.2) has at least one positive 1-periodic solution.

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