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INTERNAL STABILIZATION OF INTERCONNECTED HEAT-WAVE EQUATIONS

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Abstract. This article concerns the internal stabilization problem of 1-D interconnected heat-wave equations, where information exchange and the two actuators occur at the adjacent side of the two equations. By designing an inverse back-stepping transformation, the original system is converted into a dissipative target system. Moreover, we investigate the eigenvalues distribution and the corresponding eigenfunctions of the closed-loop system by an asymptotic analysis method. This shows that the spectrum of the system can be divided into two families: one distributed along the a line parallel to the left side of the imaginary axis and symmetric to the real axis, and the other on the left half real axis. Then we work on the properties of the resolvent operator and we verify that the root subspace is complete. Finally, we prove that the closed-loop system is exponentially stable.

1. Introduction

Parabolic-hyperbolic equations, as one of linear PDE-PDE systems or nonlinear, has been studied for several decades. It attracts many researchers because of the important applications, such as fluid structure interaction models [3, 4, 21, 30], mathematical biology [8], electromagnetic fields [6] and so on. For parabolic-hyperbolic systems, many works have been done: polynomial stability [21, 30], regularity analysis [11], boundary controllability [2], global existence and asymptotic behavior of smooth solutions [31], and optimal linear-quadratic control [1].

Heat-wave coupled system is one of typical parabolic-hyperbolic systems. In [30], a 1-D heat-wave system coupled through an interface without any control input is proved to be polynomially stable by spectral analysis and it is also studied that null-controllability problem of the above system with the controller at one boundary. In [11], it is considered that regularity analysis for an abstract system of coupled heat-wave equations, and in [18], an optimal regularity result in Sobolev spaces is proved for the heat-wave system with mass on the interface. In [12], it is considered that spectrum and stability of a 1-D heat-wave coupled system, where dynamical boundary control is designed at the Neumann boundary of the wave equation. The stabilities of the heat-wave coupled systems in $\mathbb{R}^N$ and $\mathbb{R}^2$ have presented in [21] and [5] respectively. More results of heat-wave coupled systems can also be refer to the references in [5, 11, 12, 18, 21, 30]. All of those works mentioned above

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force the control at the boundary of the coupled heat-wave systems. Compared to those references and the results, in this paper, we put the controls at the internal connection point between the heat and wave equations, and both the heat and wave equations suffer the unstable terms of the system.

For distributed parameter systems, the Riemannian geometrical method introduced in [28] is an efficient tool for the stabilization problem of the variable-coefficient systems. In [16], the Euler-Bernoulli plate with variable coefficients is stabilized by nonlinear internal feedback using geometry method. The backstepping method is another control strategy, which is applied to design controllers to stabilize systems in many aspects, such as to deal with PDEs with space-dependent diffusivity or time-dependent reactivity [24], a wave equation with an internal spatially varying antidamping term [23], coupled ODE-hyperbolic equations [25]. In [13], it is considered that the stabilization problem of a 1-d wave equation where the instability is at its free end and control is on the opposite end. By backstepping transformation, the controller and the observer are then designed. However, for some complex systems, one controller can not achieve desirable effect, which inspires us to add the number of controllers or other paths to improve system performance. In [9], two controllers are designed to stabilize the Orr-Sommerfeld equation, the Squire equation and ODE cascaded system with matched disturbances, and in [27], the pointwise feedback stabilization problem of a 1-D Euler-Bernoulli beam equation with time delay outputs are considered, where two collocated sensors are presented at the arbitrary internal position.

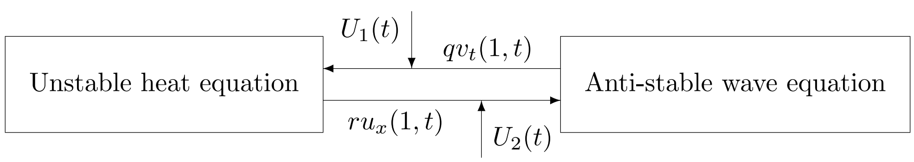

FIGURE 1. An interconnected heat-wave system

In this article, we consider the stabilization of 1-D coupled heat-wave system where it is free at $x = 0$ and $x = 2$ while the information of heat and wave equation are exchanged with each other and two inputs flow into two equations respectively at the point $x = 1$ (see Figure 1).

$$
\begin{gathered}
u_t(x,t) = u_{xx}(x,t) + cu(x,t), \quad 0 < x < 1,\ t > 0, \\
v_{tt}(x,t) = v_{xx}(x,t) + 2dv_t(x,t), \quad 1 < x < 2,\ t > 0, \\
u(0,t) = v(2,t) = 0, \quad t > 0, \\
u(1,t) = qv_t(1,t) + U_1(t), \quad v_x(1,t) = ru_x(1,t) + U_2(t), \quad t > 0, \\
u(x,0) = u_0(x), \quad v(x,0) = v_0(x),\ v_t(x,0) = v_1(x),
\end{gathered}
\tag{1.1}
$$

where $c > 0$, $d > 0$ are constants, q, $r \neq 0$, $u(x,t)$, $w(x,t)$ are the state of the heat and wave equations respectively, $U_1(t)$ and $U_2(t)$ are two input controls forced on the heat and wave respectively, and $u_0(x)$, $v_0(x)$, $v_1(x)$ are the initial datum. In the middle point $x = 1$, there is the weak connection conditions between the heat and wave equations,

$$
u(1,t) = qv_t(1,t), \quad v_x(1,t) = ru_x(1,t),
$$

where the output $qv_t(1,t)$ of the wave is flowing into the heat equation, and the heat flux $ru_x(1,t)$ is feeding into the wave equation. It is known that (1) when c is large enough, the heat equation has the finite unstable eigenvalues; and (2) owing to the term $2dv_t(x,t)$, the wave equation has infinitely many unstable eigenvalues. So we need to design the controls $U_1(t)$ and $U_2(t)$ to make the closed-loop system exponentially stable.

This article is organized as follows. By an invertible backstepping transformation, two boundary feedback controllers are designed to stabilize the original system in Section 2. In Section 3, we show the target system is well-posed by the semigroup approach. After investigating the eigenvalues and the corresponding eigenfunctions, we prove the root subspace is complete in Section 4 and the closed-loop system is exponentially stable by the Riesz basis method in Section 5.

2. Transformations

By defining $w(x,t) = v(2-x,t)$, $0 < x < 1$, $t > 0$, system (1.1) is transformed into the following coupled PDEs in the domain $\{(x,t) : 0 < x < 1, t > 0\}$,

$$\begin{gathered}
u_t(x,t) = u_{xx}(x,t) + cu(x,t), \\
w_{tt}(x,t) = w_{xx}(x,t) + 2dw_t(x,t), \\
u(0,t) = w(0,t) = 0, \\
u(1,t) = qw_t(1,t) + U_1(t),\ w_x(1,t) = -ru_x(1,t) - U_2(t), \\
u(x,0) = u_0(x), \quad w(x,0) = w_0(x), \quad w_t(x,0) = w_1(x).
\end{gathered} \tag{2.1}$$

where constants $c > 0$, $d > 0$, $q \neq 0$, $r \neq 0$. Let $H_L^1(0,1) = \{f \in H^1(0,1) | f(0) = 0\}$ with H^1 norm, and we define the projection

$$\begin{gathered}
\Pi : L^2(0,1) \times H_L^1(0,1) \times L^2(0,1) \to L^2(0,1) \times H_L^1(0,1) \times L^2(0,1) \\
(u(x), w(x), w_t(x)) \mapsto (y(x), z(x), z_t(x)),
\end{gathered}$$

which maps system (2.1) into the target system

$$\begin{gathered}
y_t(x,t) = y_{xx}(x,t), \quad 0 < x < 1,\ t > 0, \\
z_{tt}(x,t) = z_{xx}(x,t) - 2\alpha z_t(x,t) - \alpha^2 z(x,t), \quad 0 < x < 1,\ t > 0, \\
y(0,t) = z(0,t) = 0, \quad t > 0, \\
y(1,t) = pz_t(1,t), \quad z_x(1,t) = -py_x(1,t), \quad t > 0, \\
y(x,0) = y_0(x), \quad z(x,0) = z_0(x), \quad z_t(x,0) = z_1(x), \quad 0 < x < 1,
\end{gathered} \tag{2.2}$$

where $\alpha > 0$, and $p \neq 0$ are two constants. Moreover, the projection satisfies

$$y(x) = u(x) - \int_0^x k_1(x,y)u(y)dy, \tag{2.3}$$

and

$$z(x) = h(x)w(x) - \int_0^x k_2(x,y)w(y)dy - \int_0^x k_3(x,y)w_t(y)dy, \tag{2.4}$$

$$\begin{aligned}
z_t(x) &= h(x)w_t(x) + k_{3y}(x,x)w(x) - k_3(x,x)w_x(x) + k_3(x,0)w_x(0) \\
&\quad - \int_0^x k_{3yy}(x,y)w(y)dy - \int_0^x [k_2(x,y) + 2dk_3(x,y)]w_t(y)dy\,.
\end{aligned} \tag{2.5}$$

The control laws are

$$\begin{aligned}
U_1(t) &= \int_0^1 k_1(1,y)u(y,t)dy - p\int_0^1 k_{3yy}(1,y)w(y,t)dy \\
&\quad - p\int_0^1 [k_2(1,y)+2dk_3(1,y)]w_t(y,t)dy - pk_3(1,1)w_x(1,t) \\
&\quad + pk_3(1,0)w_x(0,t) + pk_{3y}(1,1)w(1,t) + [ph(1)-q]w_t(1,t), \\
U_2(t) &= -r\int_0^1 k_{1x}(1,y)u(y,t)dy - \frac{r}{p}\int_0^1 k_{2x}(1,y)w(y,t)dy \\
&\quad - \frac{r}{p}\int_0^1 k_{3x}(1,y)w_t(y,t)dy - rk_1(1,1)u(1,t) - \frac{r}{p}k_3(1,1)w_t(1,t) \\
&\quad + \frac{r}{p}[h'(1) - k_2(1,1)]w(1,t) + \frac{rh(1)-p}{p}w_x(1,t),
\end{aligned}$$

where $h(x) = \cosh(d+\alpha)x$. The kernel functions $k_1(x,y)$, $k_2(x,y)$, and $k_3(x,y)$ satisfy

$$\begin{gathered}
k_{1xx}(x,y) - k_{1yy}(x,y) = ck_1(x,y), \\
\frac{d}{dx}(k_1(x,x)) = -\frac{c}{2}, \quad k_1(x,0) = 0,
\end{gathered} \tag{2.6}$$

and

$$\begin{gathered}
k_{2xx}(x,y) - k_{2yy}(x,y) - 2ak_{3yy}(x,y) = \alpha^2 k_2(x,y), \\
k_{3xx}(x,y) - k_{3yy}(x,y) = 2ak_2(x,y) + (\alpha^2 + 4ad)k_3(x,y), \\
k_2(x,x) = (a+d)\sinh ax + \big(\frac{a^2}{2} - \frac{\alpha^2}{2} - ad\big)x\cosh ax, \\
k_3(x,x) = -\sinh ax, \quad k_2(x,0) = 0,\ k_3(x,0) = 0,
\end{gathered} \tag{2.7}$$

where $a = d+\alpha$, the domain is the triangle

$$\Omega_1 = \{(x,y) : 0 \le x \le 1,\ 0 \le y \le x\}. \tag{2.8}$$

The computation process is shown in the appendix.

Now we consider the existence of kernel functions (2.6) and (2.7). Using successive approximations method, (2.6) has solution

$$k_1(x,y) = -cy\frac{I_1(\sqrt{c(x^2-y^2)})}{\sqrt{c(x^2-y^2)}}, \quad x \ne y, \tag{2.9}$$

where $I_1(\cdot)$ is the modified Bessel function of order one (see [14, Chapter 4]). The existence of (2.7) is included in the following theorem.

Theorem 2.1. *System* (2.7) *have a unique solution* $(k_2(x,y), k_3(x,y)) \in C^2(\Omega_1)$.

Proof. We introduce the following variable transformation:

$$\xi = x+y, \quad \eta = x-y,$$

and denote

$$k_2(x,y) = G(\xi,\eta), \quad k_3(x,y) = H(\xi,\eta). \tag{2.10}$$

Then (2.7) can be rewritten as

$$G_{\xi\eta}(\xi,\eta) = \frac{\alpha^2}{4}G(\xi,\eta) + \frac{a}{2}H_{\xi\xi}(\xi,\eta) - aH_{\xi\eta}(\xi,\eta) + \frac{a}{2}H_{\eta\eta}(\xi,\eta), \tag{2.11}$$

$$H_{\xi\eta}(\xi,\eta) = \frac{a}{2}G(\xi,\eta) + adH(\xi,\eta), \tag{2.12}$$

$$G(\xi,0)=g(\frac{\xi}{2}),\quad H(\xi,0)=-\sinh\frac{a}{2}\xi, \tag{2.13}$$
$$G(\xi,\xi)=0,\quad H(\xi,\xi)=0, \tag{2.14}$$

where the domain is

$$\Omega_2=\{(\xi,\eta)|\ 0\le\xi+\eta\le 2,\ 0\le\eta\le\xi\}, \tag{2.15}$$

and

$$g(\frac{\xi}{2})=(a+d)\sinh\frac{a}{2}\xi+\Big(\frac{a^2}{4}-\frac{\alpha^2}{4}-\frac{ad}{2}\Big)\xi\cosh\frac{a}{2}\xi. \tag{2.16}$$

Integrating (2.11) and (2.12) with respect to η from 0 to η, and then with respect to ξ from η to ξ, we obtain

$$\begin{aligned}
G(\xi,\eta)&=G(\eta,\eta)+G(\xi,0)-G(\eta,0)+\frac{\alpha^2}{4}\int_\eta^\xi\int_0^\eta G(\tau,s)\,ds\,d\tau\\
&\quad+\frac{a}{2}\int_\eta^\xi\int_0^\eta H_{\xi\xi}(\tau,s)\,ds\,d\tau-a\int_\eta^\xi\int_0^\eta H_{\xi\eta}(\tau,s)\,ds\,d\tau\\
&\quad+\frac{a}{2}\int_\eta^\xi\int_0^\eta H_{\eta\eta}(\tau,s)\,ds\,d\tau,\\
H(\xi,\eta)&=H(\eta,\eta)+H(\xi,0)-H(\eta,0)+\frac{a}{2}\int_\eta^\xi\int_0^\eta G(\tau,s)\,ds\,d\tau\\
&\quad+ad\int_\eta^\xi\int_0^\eta H(\tau,s)\,ds\,d\tau.
\end{aligned}$$

By (2.13) and (2.14), we have

$$\begin{aligned}
G(\xi,\eta)&=g(\frac{\xi}{2})-g(\frac{\eta}{2})+\frac{\alpha^2}{4}\int_\eta^\xi\int_0^\eta G(\tau,s)\,ds\,d\tau-a\int_\eta^\xi\int_0^\eta H_{\xi\eta}(\tau,s)\,ds\,d\tau\\
&\quad+\frac{a}{2}\int_\eta^\xi\int_0^\eta H_{\xi\xi}(\tau,s)\,ds\,d\tau+\frac{a}{2}\int_\eta^\xi\int_0^\eta H_{\eta\eta}(\tau,s)\,ds\,d\tau,
\end{aligned}\tag{2.17}$$

$$\begin{aligned}
H(\xi,\eta)&=\sinh\frac{a}{2}\eta-\sinh\frac{a}{2}\xi+\frac{a}{2}\int_\eta^\xi\int_0^\eta G(\tau,s)\,ds\,d\tau\\
&\quad+ad\int_\eta^\xi\int_0^\eta H(\tau,s)\,ds\,d\tau,
\end{aligned}\tag{2.18}$$

where $g(\cdot)=(a+d)\sinh a\cdot+\cdot\big(\frac{a^2}{2}-\frac{\alpha^2}{2}-ad\big)\cosh a\cdot$. Differentiating $H(\xi,\eta)$ with respect to ξ and η, we have

$$\begin{aligned}
H_\xi(\xi,\eta)&=-\frac{a}{2}\cosh\frac{a}{2}\xi+\frac{a}{2}\int_0^\eta G(\xi,s)ds+ad\int_0^\eta H(\xi,s)ds,\\
H_\eta(\xi,\eta)&=\frac{a}{2}\cosh\frac{a}{2}\eta+\frac{a}{2}\int_\eta^\xi G(\tau,\eta)d\tau-\frac{a}{2}\int_0^\eta G(\eta,s)ds\\
&\quad+ad\int_\eta^\xi H(\tau,\eta)d\tau-ad\int_0^\eta H(\eta,s)ds,\\
H_{\xi\eta}(\xi,\eta)&=\frac{a}{2}G(\xi,\eta)+adH(\xi,\eta),
\end{aligned}$$

$$
\begin{aligned}
H_{\xi\xi}(\xi,\eta) &= -\frac{a^2}{4}\sinh\frac{a}{2}\xi + \frac{a}{2}\int_0^\eta G_\xi(\xi,s)ds + ad\int_0^\eta H_\xi(\xi,s)ds,\\
H_{\eta\eta}(\xi,\eta) &= \frac{a^2}{4}\sinh\frac{a}{2}\eta + \frac{a}{2}\int_\eta^\xi G_\eta(\tau,\eta)d\tau - \frac{a}{2}\int_0^\eta G_\eta(\eta,s)ds\\
&\quad + ad\int_\eta^\xi H_\eta(\tau,\eta)d\tau - ad\int_0^\eta H_\eta(\eta,s)ds.
\end{aligned}
$$

We set up the following recursion for $n = 0, 1, 2, \dots$:

$$
\begin{aligned}
H^{n+1}(\xi,\eta) &= \frac{a}{2}\int_\eta^\xi\int_0^\eta G^n(\tau,s)\,ds\,d\tau + ad\int_\eta^\xi\int_0^\eta H^n(\tau,s)\,ds\,d\tau,\\
H_\xi^{n+1}(\xi,\eta) &= \frac{a}{2}\int_0^\eta G^n(\xi,s)ds + ad\int_0^\eta H^n(\xi,s)ds,\\
H_\eta^{n+1}(\xi,\eta) &= \frac{a}{2}\int_\eta^\xi G^n(\tau,\eta)d\tau - \frac{a}{2}\int_0^\eta G^n(\eta,s)ds + ad\int_\eta^\xi H^n(\tau,\eta)d\tau\\
&\quad - ad\int_0^\eta H^n(\eta,s)ds,\\
H_{\xi\eta}^{n+1}(\xi,\eta) &= \frac{a}{2}G^n(\xi,\eta) + adH^n(\xi,\eta),\\
H_{\xi\xi}^{n+1}(\xi,\eta) &= \frac{a}{2}\int_0^\eta G_\xi^n(\xi,s)ds + ad\int_0^\eta H_\xi^n(\xi,s)ds,\\
H_{\eta\eta}^{n+1}(\xi,\eta) &= \frac{a}{2}\int_\eta^\xi G_\eta^n(\tau,\eta)d\tau - \frac{a}{2}\int_0^\eta G_\eta^n(\eta,s)ds + ad\int_\eta^\xi H_\eta^n(\tau,\eta)d\tau\\
&\quad - ad\int_0^\eta H_\eta^n(\eta,s)ds,\\
G^{n+1}(\xi,\eta) &= \frac{\alpha^2}{4}\int_\eta^\xi\int_0^\eta G^n(\tau,s)\,ds\,d\tau + \frac{a}{2}\int_\eta^\xi\int_0^\eta H_{\xi\xi}^n(\tau,s)\,ds\,d\tau\\
&\quad - a\int_\eta^\xi\int_0^\eta H_{\xi\eta}^n(\tau,s)\,ds\,d\tau + \frac{a}{2}\int_\eta^\xi\int_0^\eta H_{\eta\eta}^n(\tau,s)\,ds\,d\tau,
\end{aligned}
$$

with initial values

$$
\begin{gathered}
G^0(\xi,\eta) = g(\frac{\xi}{2}) - g(\frac{\eta}{2}),\quad H^0(\xi,\eta) = \sinh\frac{a}{2}\eta - \sinh\frac{a}{2}\xi,\\
H_\xi^0(\xi,\eta) = -\frac{a}{2}\cosh\frac{a}{2}\xi,\quad H_\eta^0(\xi,\eta) = \frac{a}{2}\cosh\frac{a}{2}\eta,\quad H_{\xi\eta}^0(\xi,\eta) = 0,\\
H_{\xi\xi}^0(\xi,\eta) = -\frac{a^2}{4}\sinh\frac{a}{2}\xi,\quad H_{\eta\eta}^0(\xi,\eta) = \frac{a^2}{4}\sinh\frac{a}{2}\eta.
\end{gathered}
$$

We denote

$$
M = \max\Big\{\sup_{x\in\Omega_2}|g'(x)|, \sup_{x\in\Omega_2}|a\cosh\frac{a}{2}x|, \sup_{x\in\Omega_2}|\frac{a^2}{4}\sinh\frac{a}{2}x|\Big\}, \tag{2.19}
$$

and we have

$$
\begin{gathered}
|G^0(\xi,\eta)| = |g(\frac{\xi}{2}) - g(\frac{\eta}{2})| \le \frac{|\xi-\eta|}{2}\sup_{x\in\Omega_2}|g'(x)| \le \sup_{x\in\Omega_2}|g'(x)| \le M,\\
|H^0(\xi,\eta)| = |\sinh\frac{a}{2}\eta - \sinh\frac{a}{2}\xi| \le 2\sup_{x\in\Omega_2}|\frac{a}{2}\cosh\frac{a}{2}x| \le M,
\end{gathered}
$$

$$
\begin{gathered}
|H_\xi^0(\xi,\eta)| = |\frac{a}{2}\cosh\frac{a}{2}\xi| \le \sup_{x\in\Omega_2}|\frac{a}{2}\cosh\frac{a}{2}x| \le M,\\
|H_\eta^0(\xi,\eta)| = |\frac{a}{2}\cosh\frac{a}{2}\eta| \le \sup_{x\in\Omega_2}|\frac{a}{2}\cosh\frac{a}{2}x| \le M,\\
|H_{\xi\xi}^0(\xi,\eta)| = |\frac{a^2}{4}\sinh\frac{a}{2}\xi| \le \sup_{x\in\Omega_2}|\frac{a^2}{4}\sinh\frac{a}{2}x| \le M,\\
|H_{\eta\eta}^0(\xi,\eta)| = |\frac{a^2}{4}\sinh\frac{a}{2}\eta| \le \sup_{x\in\Omega_2}|\frac{a^2}{4}\sinh\frac{a}{2}x| \le M,\ |H_{\xi\eta}^0(\xi,\eta)| = 0 \le M.
\end{gathered}
$$

Assume that the following expressions hold for some $n\in\mathbb{N}$:

$$
\begin{gathered}
|G^n(\xi,\eta)| \le MK^n\frac{(\xi+\eta)^n}{n!},\quad |G_\xi^n(\xi,\eta)| \le MK^n\frac{(\xi+\eta)^n}{n!},\\
|G_\eta^n(\xi,\eta)| \le MK^n\frac{(\xi+\eta)^n}{n!},\quad |H^n(\xi,\eta)| \le MK^n\frac{(\xi+\eta)^n}{n!},\\
|H_\xi^n(\xi,\eta)| \le MK^n\frac{(\xi+\eta)^n}{n!},\quad |H_\eta^n(\xi,\eta)| \le MK^n\frac{(\xi+\eta)^n}{n!},\\
|H_{\xi\eta}^n(\xi,\eta)| \le MK^n\frac{(\xi+\eta)^{n-1}}{(n-1)!},\quad |H_{\xi\xi}^n(\xi,\eta)| \le MK^n\frac{(\xi+\eta)^{n-1}}{(n-1)!},\\
|H_{\eta\eta}^n(\xi,\eta)| \le MK^n\frac{(\xi+\eta)^{n-1}}{(n-1)!},
\end{gathered}
\tag{2.20}
$$

where M is given by (2.19) and K is a positive constant. Then we obtain

$$
\begin{aligned}
|H^{n+1}(\xi,\eta)| &\le \frac{|a|}{2}\int_\eta^\xi\int_0^\eta |G^n(\tau,s)|\,ds\,d\tau + |a|d\int_\eta^\xi\int_0^\eta |H^n(\tau,s)|\,ds\,d\tau\\
&\le \Big(\frac{|a|}{2}+|a|d\Big)\frac{MK^n}{n!}\int_\eta^\xi\int_0^\eta(\tau+s)^n\,ds\,d\tau\\
&\le \Big(\frac{|a|}{2}+|a|d\Big)\frac{MK^n}{(n+1)!}|\xi-\eta|(\xi+\eta)^{n+1}\\
&\le \Big(|a|+2|a|d\Big)\frac{MK^n}{(n+1)!}(\xi+\eta)^{n+1},\\
|H_\xi^{n+1}(\xi,\eta)| &\le \frac{|a|}{2}\int_0^\eta |G^n(\xi,s)|ds + |a|d\int_0^\eta |H^n(\xi,s)|ds\\
&\le \Big(\frac{|a|}{2}+|a|d\Big)\frac{MK^n}{n!}\int_0^\eta(\xi+s)^n ds\\
&\le \Big(\frac{|a|}{2}+|a|d\Big)\frac{MK^n}{(n+1)!}(\xi+\eta)^{n+1},\\
|H_\eta^{n+1}(\xi,\eta)| &\le \frac{|a|}{2}\int_\eta^\xi |G^n(\tau,\eta)|d\tau + \frac{|a|}{2}\int_0^\eta |G^n(\eta,s)|ds\\
&\quad + |a|d\int_\eta^\xi |H^n(\tau,\eta)|d\tau + |a|d\int_0^\eta |H^n(\eta,s)|ds\\
&\le \Big(\frac{|a|}{2}+|a|d\Big)\frac{MK^n}{n!}\int_0^\xi(s+\eta)^n ds\\
&\le \Big(\frac{|a|}{2}+|a|d\Big)\frac{MK^n}{(n+1)!}(\xi+\eta)^{n+1},
\end{aligned}
$$

$$\begin{aligned}
|H^{n+1}_{\xi\eta}(\xi,\eta)| &\le \frac{|a|}{2}|G^n(\xi,\eta)| + |a|d|H^n(\xi,\eta)| \\
&\le (\frac{|a|}{2} + |a|d)\frac{MK^n}{n!}(\xi+\eta)^n, \\
|H^{n+1}_{\xi\xi}(\xi,\eta)| &\le \frac{|a|}{2}\int_0^\eta |G^n_\xi(\xi,s)|ds + |a|d\int_0^\eta |H^n_\xi(\xi,s)|ds \\
&\le \Big(\frac{|a|}{2} + |a|d\Big)\frac{MK^n}{(n+1)!}(\xi+\eta)^{n+1} \\
&\le \Big(\frac{|a|}{2} + |a|d\Big)\frac{MK^n}{n!}(\xi+\eta)^n, \\
|H^{n+1}_{\eta\eta}(\xi,\eta)| &\le \frac{|a|}{2}\int_0^\xi |G^n_\eta(\tau,\eta)|d\tau + |a|d\int_0^\xi |H^n_\eta(\tau,\eta)|d\tau \\
&\le \Big(\frac{|a|}{2} + |a|d\Big)\frac{MK^n}{n!}(\xi+\eta)^n,
\end{aligned}$$

and

$$\begin{aligned}
&|G^{n+1}(\xi,\eta)| \\
&\le \frac{\alpha^2}{4}\int_\eta^\xi\int_0^\eta |G^n(\tau,s)|\,ds\,d\tau + \frac{|a|}{2}\int_\eta^\xi\int_0^\eta |H^n_{\xi\xi}(\tau,s)|\,ds\,d\tau \\
&\quad + |a|\int_\eta^\xi\int_0^\eta |H^n_{\xi\eta}(\tau,s)|\,ds\,d\tau + \frac{|a|}{2}\int_\eta^\xi\int_0^\eta |H^n_{\eta\eta}(\tau,s)|\,ds\,d\tau \\
&\le \frac{\alpha^2}{4}\frac{MK^n}{n!}\int_\eta^\xi\int_0^\eta (\tau+s)^n\,ds\,d\tau + \frac{2|a|MK^n}{(n-1)!}\int_\eta^\xi\int_0^\eta (\tau+s)^{n-1}\,ds\,d\tau \\
&= \frac{\alpha^2}{4}\frac{MK^n}{(n+1)!}\int_\eta^\xi [(\tau+\eta)^{n+1} - \tau^{n+1}]d\tau + \frac{2|a|MK^n}{n!}\int_\eta^\xi [(\tau+\eta)^n - \tau^n]d\tau \\
&\le \frac{\alpha^2}{4}\frac{MK^n}{(n+1)!}\int_\eta^\xi (\tau+\eta)^{n+1}d\tau + \frac{2|a|MK^n}{n!}\int_\eta^\xi (\tau+\eta)^n d\tau \\
&\le \frac{\alpha^2}{4}\frac{MK^n}{(n+1)!}|\xi-\eta|(\xi+\eta)^{n+1} + \frac{2|a|MK^n}{(n+1)!}(\xi+\eta)^{n+1} \\
&\le \Big(2|a| + \frac{\alpha^2}{2}\Big)\frac{MK^n}{(n+1)!}(\xi+\eta)^{n+1}.
\end{aligned}$$

Let

$$K = \max\big\{|a| + 2|a|d,\ 2|a| + \frac{\alpha^2}{2}\big\}.$$

Then (2.20) is true for $n+1$. Hence, the two series

$$\sum_{n=0}^\infty G^n(\xi,\eta), \quad \sum_{n=0}^\infty H^n(\xi,\eta),$$

are convergent absolutely in Ω_2 given by (2.15), and then $G(\xi,\eta)$, $H(\xi,\eta)$ exist, which are given by

$$G(\xi,\eta) = \sum_{n=0}^\infty G^n(\xi,\eta), \quad H(\xi,\eta) = \sum_{n=0}^\infty H^n(\xi,\eta). \tag{2.21}$$

Since $G(\xi,\eta)$, $H(\xi,\eta)$ satisfy (2.17) and (2.18), we know that $G(\xi,\eta)$, $H(\xi,\eta)$ are of C^2 in Ω_2. This together with (2.10) yields the desired result. $\square$

Now, we prove that the projection Π^{-1} exists, where

$$\begin{gathered}\Pi^{-1}: L^2(0,1)\times H_L^1(0,1)\times L^2(0,1)\to L^2(0,1)\times H_L^1(0,1)\times L^2(0,1)\\ (y(x),z(x),z_t(x))\mapsto (u(x),w(x),w_t(x)).\end{gathered}$$

Actually, we let $a=d+\alpha$,

$$u(x,t)=y(x,t)+\int_0^x s_1(x,y)y(y,t)dy, \tag{2.22}$$

$$w(x,t)=l(x)z(x,t)+\int_0^x s_2(x,y)z(y,t)dy+\int_0^x s_3(x,y)z_t(y,t)dy, \tag{2.23}$$

where

$$\begin{gathered} s_{1xx}(x,y)-s_{1yy}(x,y)=cs_1(x,y),\\ \frac{d}{dx}(s_1(x,x))=-\frac{c}{2},\ s_1(x,0)=0,\\ s_{2xx}(x,y)-s_{2yy}(x,y)+2as_{3yy}(x,y)=-\alpha^2 s_2(x,y)+2\alpha^2 a s_3(x,y),\\ s_{3xx}(x,y)-s_{3yy}(x,y)=(4a\alpha-\alpha^2)s_3(x,y)-2as_2(x,y),\\ s_2(x,x)=-\frac{a^2+\alpha^2+d\alpha}{a}\sinh ax-\frac{d^2}{2}x\cosh ax,\\ s_3(x,x)=-\sinh ax,\ s_2(x,0)=0,\ s_3(x,0)=0, \end{gathered}\tag{2.24}$$

which are obtained from Appendix 6. In a similar way as the proof of Theorem 2.1, we obtain $s_1(x,y)$, $s_2(x,y)$ and $s_3(x,y)$ all exist, which implies that projection Π is invertible.

3. Well-posedness of system

We consider system (2.2) in the Hilbert space

$$\mathcal{H}=L^2(0,1)\times H_L^1(0,1)\times L^2(0,1),\ H_L^1(0,1)=\{f\in H^1(0,1)|f(0)=0\},$$

with the inner product

$$\begin{aligned}\langle (g_1,\phi_1,\theta_1),(g_2,\phi_2,\theta_2)\rangle &=\int_0^1 g_1(x)\overline{g_2(x)}dx+\alpha^2\int_0^1\phi_1(x)\overline{\phi_2(x)}dx\\ &\quad+\int_0^1\big[\phi_1'(x)\overline{\phi_2'(x)}+\theta_1(x)\overline{\theta_2(x)}\big]dx.\end{aligned}$$

We define a linear operator $\mathcal{A}: D(\mathcal{A})\subset\mathcal{H}\to\mathcal{H}$ as follows

$$\begin{gathered}\mathcal{A}(g,\phi,\theta)=(g'',\theta,\phi''-2\alpha\theta-\alpha^2\phi),\quad \forall (g,\phi,\theta)\in D(\mathcal{A}),\\ D(\mathcal{A})=\big\{(g,\phi,\theta)\in H^2(0,1)\times(H^2(0,1)\cap H_L^1(0,1))\times H_L^1(0,1),\\ g(0)=0,\ g(1)=p\theta(1),\ \phi'(1)=-pg'(1)\big\},\end{gathered}\tag{3.1}$$

and system (2.2) can be rewritten as

$$\frac{d}{dt}\left(y(t),z(t),z_t(t)\right)=\mathcal{A}\left(y(t),z(t),z_t(t)\right),\ (y(0),z(0),z_t(0))=(y_0,z_0,z_1). \tag{3.2}$$

First, we have the following result.

Lemma 3.1. *The operator $\mathcal{A}$ defined by (3.1) is dissipative and generates a C_0-semigroup $e^{\mathcal{A}t}$ of contractions in $\mathcal{H}$. Hence, system (3.2) is well-posed in the sense that for initial datum $(y(\cdot,0), z(\cdot,0), z_t(\cdot,0)) \in \mathcal{H}$, system (3.2) admits a unique solution $(y(\cdot,t), z(\cdot,t), z_t(\cdot,t)) \in C(0,\infty;\mathcal{H})$*

$$(y(\cdot,t), z(\cdot,t), z_t(\cdot,t)) = e^{\mathcal{A}t}(y(\cdot,0), z(\cdot,0), z_t(\cdot,0)). \tag{3.3}$$

Proof. Let $(r,m,n) \in \mathcal{H}$. Solving $\mathcal{A}(g,\phi,\theta) = (r,m,n)$, we have

$$\begin{gathered} g''(x) = r(x), \quad \theta(x) = m(x), \\ \phi''(x) - 2\alpha\theta(x) - \alpha^2\phi(x) = n(x), \\ g(0) = 0, \quad \phi(0) = 0, \quad \theta(0) = 0, \quad g(1) = p\theta(1), \quad \phi'(1) = -pg'(1), \end{gathered} \tag{3.4}$$

which yields

$$\begin{gathered} g(x) = pm(1)x - x\int_0^1\int_0^\tau r(s)\,ds\,d\tau + \int_0^x\int_0^\tau r(s)\,ds\,d\tau, \\ m(0) = 0, \quad \theta(x) = m(x), \\ \phi(x) = C\sinh\alpha x + \int_0^x \alpha^{-1}\sinh\alpha(x-s)[2\alpha m(s) + n(s)]ds, \\ \begin{aligned} C = &\int_0^1\int_0^\tau \frac{pr(s)}{\alpha\cosh\alpha}\,ds\,d\tau - \frac{p^2 m(1)}{\alpha\cosh\alpha} \\ &- \int_0^1 \frac{pr(s) + [2\alpha m(s) + n(s)]\cosh\alpha(1-s)}{\alpha\cosh\alpha}ds. \end{aligned} \end{gathered} \tag{3.5}$$

Hence, (3.4) has a unique solution, which implies that $\mathcal{A}^{-1}$ exists and is compact. Moreover, the spectrum of $\mathcal{A}$, $\sigma(\mathcal{A})$, consists of isolated eigenvalues of finite algebraic multiplicity only.

$$\begin{aligned} &\langle \mathcal{A}(g,\phi,\theta), (g,\phi,\theta)\rangle \\ &= \langle (g'', \theta, \phi'' - 2\alpha\theta - \alpha^2\phi), (g,\phi,\theta)\rangle \\ &= \int_0^1 g''(x)\overline{g(x)}dx + \alpha^2\int_0^1 \theta(x)\overline{\phi(x)}dx \\ &\quad + \int_0^1 \big[\theta'(x)\overline{\phi'(x)} + [\phi''(x) - 2\alpha\theta(x) - \alpha^2\phi(x)]\overline{\theta(x)}\big]dx \\ &= \overline{g(x)}g'(x)\big|_0^1 - \int_0^1 |g'(x)|^2dx + \int_0^1 \theta'(x)\overline{\phi'(x)}dx + \int_0^1 \phi''(x)\overline{\theta(x)}dx \\ &\quad - 2\alpha\int_0^1 |\theta(x)|^2dx + \alpha^2\int_0^1 [\theta(x)\overline{\phi(x)} - \phi(x)\overline{\theta(x)}]dx \\ &= \overline{g(x)}g'(x)\big|_0^1 + \phi'(x)\overline{\theta(x)}\big|_0^1 - \int_0^1 |g'(x)|^2dx - 2\alpha\int_0^1 |\theta(x)|^2dx \\ &\quad + \int_0^1 [\theta'(x)\overline{\phi'(x)} - \overline{\theta'(x)}\phi'(x)]dx + \alpha^2\int_0^1 [\theta(x)\overline{\phi(x)} - \phi(x)\overline{\theta(x)}]dx. \end{aligned}$$

Since $\alpha > 0$, we obtain

$$\operatorname{Re}\langle \mathcal{A}(g,\phi,\theta), (g,\phi,\theta)\rangle = -\int_0^1 [g'(x)]^2dx - 2\alpha\int_0^1 \theta^2(x)dx \le 0. \tag{3.6}$$

Hence, $\mathcal{A}$ is dissipative and generates a C_0-semigroup $e^{\mathcal{A}t}$ of contractions in $\mathcal{H}$ by Lumer-Philips Theorem ([20]). Moreover, system (3.2) is well-posed satisfying (3.3). □

Next, we consider the eigenvalues problem of operator $\mathcal{A}$. Let $\mathcal{A}(g,\phi,\theta) = \lambda(g,\phi,\theta)$, $(g,\phi,\theta) \in D(\mathcal{A})$. We have $\theta(x) = \lambda\phi(x)$ and

$$\begin{gathered} g''(x) = \lambda g(x), \quad \phi''(x) - 2\alpha\lambda\phi(x) - \alpha^2\phi(x) = \lambda^2\phi(x), \\ g(0) = 0,\ \phi(0) = 0, \quad g(1) = p\lambda\phi(1),\ \phi'(1) = -pg'(1), \end{gathered} \tag{3.7}$$

which has general solution $(g(x), \phi(x))$,

$$g(x) = c_1 e^{\sqrt{\lambda}x} + c_2 e^{-\sqrt{\lambda}x}, \quad \phi(x) = c_3 e^{(\lambda+\alpha)x} + c_4 e^{-(\lambda+\alpha)x}, \tag{3.8}$$

where c_1, c_2, c_3 and c_4 are constants to be fixed. According to boundary conditions of the last line in (3.7), we obtain

$$\begin{gathered} c_1 + c_2 = 0, \quad c_3 + c_4 = 0, \\ e^{\sqrt{\lambda}}c_1 + e^{-\sqrt{\lambda}}c_2 - p\lambda e^{\lambda+\alpha}c_3 - p\lambda e^{-(\lambda+\alpha)}c_4 = 0, \\ p\sqrt{\lambda}e^{\sqrt{\lambda}}c_1 - p\sqrt{\lambda}e^{-\sqrt{\lambda}}c_2 + (\lambda+\alpha)e^{\lambda+\alpha}c_3 - (\lambda+\alpha)e^{-(\lambda+\alpha)}c_4 = 0. \end{gathered} \tag{3.9}$$

Therefore, (3.9) has a nontrivial solution (c_1, c_2, c_3, c_4) if and only if

$$\begin{vmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ e^{\sqrt{\lambda}} & e^{-\sqrt{\lambda}} & -p\lambda e^{\lambda+\alpha} & -p\lambda e^{-(\lambda+\alpha)} \\ p\sqrt{\lambda}e^{\sqrt{\lambda}} & -p\sqrt{\lambda}e^{-\sqrt{\lambda}} & (\lambda+\alpha)e^{\lambda+\alpha} & -(\lambda+\alpha)e^{-(\lambda+\alpha)} \end{vmatrix} = 0,$$

which yields

$$\begin{aligned} &(\lambda+\alpha)(e^{\sqrt{\lambda}} - e^{-\sqrt{\lambda}})(e^{\lambda+\alpha} + e^{-(\lambda+\alpha)}) \\ &+ p^2\lambda^{\frac{3}{2}}(e^{\sqrt{\lambda}} + e^{-\sqrt{\lambda}})(e^{\lambda+\alpha} - e^{-(\lambda+\alpha)}) = 0. \end{aligned} \tag{3.10}$$

Moreover, by some linear algebra calculations, operator $\mathcal{A}$ has two branches of eigenfunctions $(g_1, \phi_1, \lambda\phi_1)$ and $(g_2, \phi_2, \lambda\phi_2)$, where

$$\begin{aligned} g_1(x) &= \begin{vmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ e^{\sqrt{\lambda}x} & e^{-\sqrt{\lambda}x} & 0 & 0 \\ p\sqrt{\lambda}e^{\sqrt{\lambda}} & -p\sqrt{\lambda}e^{-\sqrt{\lambda}} & (\lambda+\alpha)e^{\lambda+\alpha} & -(\lambda+\alpha)e^{-(\lambda+\alpha)} \end{vmatrix} \\ &= -4(\lambda+\alpha)\cosh(\lambda+\alpha)\sinh\sqrt{\lambda}x, \end{aligned} \tag{3.11}$$

$$\begin{aligned} \phi_1(x) &= \begin{vmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & e^{(\lambda+\alpha)x} & e^{-(\lambda+\alpha)x} \\ p\sqrt{\lambda}e^{\sqrt{\lambda}} & -p\sqrt{\lambda}e^{-\sqrt{\lambda}} & (\lambda+\alpha)e^{\lambda+\alpha} & -(\lambda+\alpha)e^{-(\lambda+\alpha)} \end{vmatrix} \\ &= 4p\sqrt{\lambda}\cosh\sqrt{\lambda}\sinh(\lambda+\alpha)x, \end{aligned} \tag{3.12}$$

$$\begin{aligned} g_2(x) &= \begin{vmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ e^{\sqrt{\lambda}} & e^{-\sqrt{\lambda}} & -p\lambda e^{\lambda+\alpha} & -p\lambda e^{-(\lambda+\alpha)} \\ e^{\sqrt{\lambda}x} & e^{-\sqrt{\lambda}x} & 0 & 0 \end{vmatrix} \\ &= -4p\lambda\sinh(\lambda+\alpha)\sinh\sqrt{\lambda}x, \end{aligned} \tag{3.13}$$

$$\phi_2(x) = \begin{vmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ e^{\sqrt{\lambda}} & e^{-\sqrt{\lambda}} & -p\lambda e^{\lambda+\alpha} & -p\lambda e^{-(\lambda+\alpha)} \\ 0 & 0 & e^{(\lambda+\alpha)x} & e^{-(\lambda+\alpha)x} \end{vmatrix} = -4\sinh\sqrt{\lambda}\sinh(\lambda+\alpha)x. \tag{3.14}$$

Hence, the eigenvalues and its corresponding eigenfunctions of $\mathcal{A}$ are obtained in the following Theorem.

Theorem 3.2. *Let $\mathcal{A}$ be defined by* (3.1) *and*

$$\begin{aligned} \Delta(\lambda) &= (\lambda+\alpha)(e^{\sqrt{\lambda}} - e^{-\sqrt{\lambda}})(e^{\lambda+\alpha} + e^{-(\lambda+\alpha)}) \\ &\quad + p^2\lambda^{\frac{3}{2}}(e^{\sqrt{\lambda}} + e^{-\sqrt{\lambda}})(e^{\lambda+\alpha} - e^{-(\lambda+\alpha)}). \end{aligned} \tag{3.15}$$

(1) *We have*

$$\sigma(\mathcal{A}) = \sigma_p(\mathcal{A}) = \{\lambda \in \mathbb{C} | \ \Delta(\lambda) = 0\}. \tag{3.16}$$

Moreover, the real part of elements in $\sigma(\mathcal{A})$ are negative, which means that there is no eigenvalues distributed on the imaginary axis.

(2) *The eigenvalues $\{\lambda_{1n}, \overline{\lambda_{1n}}, \lambda_{2n}\}$ have the following asymptotic expansions*

$$\begin{aligned} \lambda_{1n} &= -\alpha - \frac{k_1}{p^2\sqrt{\alpha^2+n^2\pi^2}} + \Big(n\pi + \frac{k_2}{p^2\sqrt{\alpha^2+n^2\pi^2}}\Big)i + \mathcal{O}(n^{-1}), \\ \lambda_{2n} &= -\frac{(2n+1)^2}{4}\pi^2 + 2p^{-2} + \mathcal{O}(n^{-1}), \end{aligned} \tag{3.17}$$

where $n \in \mathbb{N}$ is sufficiently large, $\overline{\lambda_{1n}}$ and λ_{1n} are conjugate and

$$k_1 = \sqrt{\frac{-\alpha + \sqrt{\alpha^2+n^2\pi^2}}{2}}, \quad k_2 = \sqrt{\frac{\alpha + \sqrt{\alpha^2+n^2\pi^2}}{2}}. \tag{3.18}$$

(3) *The corresponding eigenfunctions $\{\Phi_{1n}(x), \overline{\Phi_{1n}(x)}, \Phi_{2n}(x)\}$, with*

$$\begin{aligned} \Phi_{1n}(x) &= (g_{1n}(x), \phi_{1n}(x), \lambda_{1n}\phi_{1n}(x))^T, \\ \overline{\Phi_{1n}(x)} &= \Big(\overline{g_{1n}(x)}, \overline{\phi_{1n}(x)}, \overline{\lambda_{1n}}\overline{\phi_{1n}(x)}\Big)^T, \\ \Phi_{2n}(x) &= (g_{2n}(x), \phi_{2n}(x), \lambda_{2n}\phi_{2n}(x))^T, \end{aligned} \tag{3.19}$$

satisfy the asymptotic expressions

$$\begin{pmatrix} g_{1n}(x) \\ \phi_{1n}'(x) \\ \lambda_{1n}\phi_{1n}(x) \end{pmatrix} = \begin{pmatrix} \mathcal{O}(e^{-\sqrt{n}}) \\ e^{(-l_1+l_2i)x} + e^{(l_1-l_2i)x} + \mathcal{O}(n^{-1}) \\ e^{(-l_1+l_2i)x} - e^{(l_1-l_2i)x} + \mathcal{O}(n^{-1}) \end{pmatrix}, \tag{3.20}$$

$$\begin{pmatrix} g_{2n}(x) \\ \phi_{2n}'(x) \\ \lambda_{2n}\phi_{2n}(x) \end{pmatrix} = \begin{pmatrix} e^{\frac{2n+1}{2}\pi ix} - e^{-\frac{2n+1}{2}\pi ix} \\ \beta_n(x) \\ -\beta_n(x) \end{pmatrix} + \mathcal{O}(n^{-1}), \tag{3.21}$$

with

$$\begin{aligned} l_1 &= \frac{k_1}{p^2\sqrt{\alpha^2+n^2\pi^2}}, \quad l_2 = n\pi + \frac{k_2}{p^2\sqrt{\alpha^2+n^2\pi^2}}, \\ \beta_n(x) &= (-1)^{n+1}2ip^{-1}e^{(\alpha+\widetilde{\lambda}_{2n})(1-x)}, \quad \widetilde{\lambda}_{2n} = -\frac{(2n+1)^2}{4}\pi^2 + 2p^{-2}. \end{aligned} \tag{3.22}$$

Proof. (1) From Lemma 3.1, we know $\mathrm{Re}(\lambda) \le 0$. Now we show the real part of eigenvalues is strictly less than zero, that is, there is no eigenvalues located on imaginary axis. Let $\lambda = i\mu^2$ with real number $\mu \neq 0$ and $\Phi(x) = (g(x), \phi(x), \theta(x)) \in D(\mathcal{A})$ be its associated eigenfunction. Then it then follows from (3.6) that

$$g'(x) = 0, \quad \theta(x) = 0.$$

According to $\theta(x) = i\mu^2\phi(x)$ and (3.7), we have $g(x) = 0$, $\phi(x) = 0$ and $\Phi(x) = 0$. The proof of part (1) is complete.

(2) It follows from (1) that $\mathrm{Re}(\lambda) \le 0$ and $\arg\lambda \in (\frac{\pi}{2}, \frac{3\pi}{2})$. Let $\lambda = \rho^2$, and then we have $\arg\rho \in (\frac{\pi}{4}, \frac{3\pi}{4})$ and (3.10) becomes

$$(e^{\rho} + e^{-\rho})\big[e^{\rho^2+\alpha} - e^{-(\rho^2+\alpha)}\big] + \frac{\rho^2+\alpha}{p^2\rho^3}(e^{\rho} - e^{-\rho})\big[e^{\rho^2+\alpha} + e^{-(\rho^2+\alpha)}\big] = 0. \tag{3.23}$$

To solve (3.23) with respect to ρ, we divide $(\pi/4, 3\pi/4)$ into three sectors S_1, S_2, S_3, where

$$S_1 = \big\{\rho \in \mathbb{C} : \arg\rho \in (\frac{\pi}{4}, \frac{5\pi}{16})\big\}, \quad S_2 = \{\rho \in \mathbb{C} : \arg\rho \in (\frac{11\pi}{16}, \frac{3\pi}{4})\},$$
$$S_3 = \{\rho \in \mathbb{C} \mid \arg\rho \in [\frac{5\pi}{16}, \frac{11\pi}{16}]\}.$$

(i) When $\arg\rho \in S_1$, we have $\arg\rho^2 \in (\frac{\pi}{2}, \frac{5\pi}{8})$ and

$$\mathrm{Re}(-\rho) = -|\rho|\cos\arg\rho < |\rho|\cos\frac{5\pi}{16} < 0,$$

which leads to $e^{\rho} \to \infty$, as $|\rho| \to \infty$,

$$e^{\rho^2+\alpha} - e^{-(\rho^2+\alpha)} + \mathcal{O}(\rho^{-1}) = 0, \tag{3.24}$$

$$p^{-2}\rho^{-3}e^{-\rho}\Delta(\rho^2) = e^{\rho^2+\alpha} - e^{-(\rho^2+\alpha)} + \mathcal{O}(\rho^{-1}), \tag{3.25}$$

where $\Delta(\rho^2)$ is given by (3.15). It can be verified that $\{\widetilde{\rho}_n^2 = -\alpha \pm n\pi i, n \in \mathbb{N}\}$ is the solution of $e^{\rho^2+\alpha} - e^{-(\rho^2+\alpha)} = 0$. By Rouche's theorem, when $n \in \mathbb{N}$ is sufficiently large, (3.24) has a solution

$$\lambda_{1n} = \rho_n^2 = -\alpha + n\pi i + \mathcal{O}(n^{-\frac{1}{2}}).$$

By Taylor expansion and substituting $\rho_n = \sqrt{-\alpha + n\pi i} + \mathcal{O}(n^{-1})$ into (3.23), we have a more accurate result,

$$\mathcal{O}(n^{-\frac{1}{2}}) = -\frac{\rho_n^2+\alpha}{p^2\rho_n^3}\cdot\frac{1-e^{-2\rho_n}}{1+e^{-2\rho_n}} + \mathcal{O}(\rho_n^{-2}) = -\frac{1}{p^2\sqrt{-\alpha+n\pi i}} + \mathcal{O}(n^{-1}),$$
$$\lambda_{1n} = -\alpha - \frac{k_1}{p^2\sqrt{\alpha^2+n^2\pi^2}} + \Big(n\pi + \frac{k_2}{p^2\sqrt{\alpha^2+n^2\pi^2}}\Big)i + \mathcal{O}(n^{-1}),$$

where k_1, k_2 given by (3.18) are obtained from $x^2 + \alpha x - \frac{1}{4}n^2\pi^2 = 0$.

(ii) When $\arg\rho \in S_2$, we have $\arg\rho^2 \in (\frac{11\pi}{8}, \frac{3\pi}{2})$ and

$$\mathrm{Re}(\rho) = |\rho|\cos\arg\rho < |\rho|\cos\frac{11\pi}{16} < 0.$$

Hence, $e^{-\rho} \to \infty$, as $|\rho| \to \infty$. Like in case i), we have

$$\overline{\lambda_{1n}} = -\alpha - \frac{k_1}{p^2\sqrt{\alpha^2+n^2\pi^2}} - \Big(n\pi + \frac{k_2}{p^2\sqrt{\alpha^2+n^2\pi^2}}\Big)i + \mathcal{O}(n^{-1}), \tag{3.26}$$

and (3.25) holds.

(iii) When $\arg\rho \in S_3$, we have $\arg\rho^2 \in [\frac{5\pi}{8}, \frac{11\pi}{8}]$ and

$$\mathrm{Re}(\rho^2) = |\rho^2|\cos\arg\rho^2 \le |\rho^2|\cos\frac{5\pi}{8} < 0,$$

Hence, $e^{-\rho^2} \to \infty$, as $|\rho| \to \infty$ and

$$-p^{-2}\rho^{-3}e^{\rho^2+\alpha}\Delta(\rho^2) = e^{\rho} + e^{-\rho} + \mathcal{O}(\rho^{-1}). \tag{3.27}$$

Equation (3.23) can be rewritten as

$$e^{\rho} + e^{-\rho} + \mathcal{O}(\rho^{-1}) = 0. \tag{3.28}$$

By applying Rouche's theorem again, we have

$$\rho_n = \widetilde{\rho} + \mathcal{O}(n^{-1}), \quad \widetilde{\rho} = \frac{2n+1}{2}\pi i. \tag{3.29}$$

Furthermore, $\rho_n^2 = \widetilde{\rho}^2 + \mathcal{O}(1)$. Substituting (3.29) into (3.23), we have

$$\begin{gathered}
\mathcal{O}(n^{-1}) = -\frac{\rho_n^2+\alpha}{p^2\rho_n^3}\frac{e^{2(\rho_n^2+\alpha)}+1}{e^{2(\rho_n^2+\alpha)}-1} + \mathcal{O}(\rho_n^{-2}) = p^{-2}\widetilde{\rho}^{-1} + \mathcal{O}(\widetilde{\rho}^{-2}),\\
\lambda_{2n} = \rho_n^2 = \widetilde{\rho}^2 + 2p^{-2} + \mathcal{O}(\widetilde{\rho}^{-1}) = -\frac{(2n+1)^2}{4}\pi^2 + 2p^{-2} + \mathcal{O}(n^{-1}),
\end{gathered}$$

for sufficiently large integer n.

Collecting the above three cases above, the eigenvalues of $\mathcal{A}$ are given by (3.17).

(3) Now we are in a position to solve the corresponding eigenfunctions. Let

$$\begin{gathered}
\Phi_{1n}(x) = \begin{pmatrix} g_{1n}(x)\\ \phi_{1n}(x)\\ \lambda_{1n}\phi_{1n}(x)\end{pmatrix} = \frac{1/\cosh\sqrt{\lambda_{1n}}}{2p\lambda_{1n}^{\frac{3}{2}}}\begin{pmatrix} g_1(x,\lambda_{1n})\\ \phi_1(x,\lambda_{1n})\\ \lambda_1\phi_1(x,\lambda_{1n})\end{pmatrix},\\
\Phi_{2n}(x) = \begin{pmatrix} g_{2n}(x)\\ \phi_{2n}(x)\\ \lambda_{2n}\phi_{2n}(x)\end{pmatrix} = \frac{1/\sinh(\lambda_{2n}+\alpha)}{-2p\lambda_{2n}}\begin{pmatrix} g_2(x,\lambda_{2n})\\ \phi_2(x,\lambda_{2n})\\ \lambda_2\phi_2(x,\lambda_{2n})\end{pmatrix},
\end{gathered}$$

where $g_1(x)$, $\phi_1(x)$, $g_2(x)$, $\phi_2(x)$ are given by (3.11), (3.12), (3.13) and (3.14). Then

$$\begin{aligned}
&(g_{1n}(x), \phi_{1n}'(x), \lambda_{1n}\phi_{1n}(x))^T\\
&= \frac{1/\cosh\sqrt{\lambda_{1n}}}{2p\lambda_{1n}^{\frac{3}{2}}}\begin{pmatrix} -4(\lambda_{1n}+\alpha)\cosh(\lambda_{1n}+\alpha)\sinh\sqrt{\lambda_{1n}}x\\ 4p\sqrt{\lambda_{1n}}(\lambda_{1n}+\alpha)\cosh\sqrt{\lambda_{1n}}\cosh(\lambda_{1n}+\alpha)x\\ 4p\lambda_{1n}^{\frac{3}{2}}\cosh\sqrt{\lambda_{1n}}\sinh(\lambda_{1n}+\alpha)x\end{pmatrix}\\
&= \begin{pmatrix} \frac{-2(\lambda_{1n}+\alpha)\cosh(\lambda_{1n}+\alpha)\sinh\sqrt{\lambda_{1n}}x}{p\lambda_{1n}^{\frac{3}{2}}\cosh\sqrt{\lambda_{1n}}}\\ \frac{2(\lambda_{1n}+\alpha)\cosh(\lambda_{1n}+\alpha)x}{\lambda_{1n}}\\ 2\sinh(\lambda_{1n}+\alpha)x.\end{pmatrix},
\end{aligned} \tag{3.30}$$

and

$$
\begin{aligned}
&(g_{2n}(x), \phi'_{2n}(x), \lambda_{2n}\phi_{2n}(x))^T \\
&= \frac{1/\sinh(\lambda_{2n}+\alpha)}{-2p\lambda_{2n}} \begin{pmatrix} -4p\lambda_{2n}\sinh(\lambda_{2n}+\alpha)\sinh\sqrt{\lambda_{2n}}x \\ -4(\lambda_{2n}+\alpha)\sinh\sqrt{\lambda_{2n}}\cosh(\lambda_{2n}+\alpha)x \\ -4\lambda_{2n}\sinh\sqrt{\lambda_{2n}}\sinh(\lambda_{2n}+\alpha)x \end{pmatrix} \\
&= \begin{pmatrix} 2\sinh\sqrt{\lambda_{2n}}x \\ \frac{2(\lambda_{2n}+\alpha)\sinh\sqrt{\lambda_{2n}}\cosh(\lambda_{2n}+\alpha)x}{p\lambda_{2n}\sinh(\lambda_{2n}+\alpha)} \\ \frac{2\sinh\sqrt{\lambda_{2n}}\sinh(\lambda_{2n}+\alpha)x}{p\sinh(\lambda_{2n}+\alpha)} \end{pmatrix}.
\end{aligned} \tag{3.31}
$$

For (3.30), we have the following estimates for (3.17):

$$
\begin{gathered}
\frac{-2(\lambda_{1n}+\alpha)\cosh(\lambda_{1n}+\alpha)\sinh\sqrt{\lambda_{1n}}x}{p\lambda_{1n}^{\frac{3}{2}}\cosh\sqrt{\lambda_{1n}}} = \frac{-2(\lambda_{1n}+\alpha)(1-e^{-2\sqrt{\lambda_{1n}}x})}{p\lambda_{1n}^{\frac{3}{2}}e^{\sqrt{\lambda_{1n}}(1-x)}(1+e^{-2\sqrt{\lambda_{1n}}})} = \mathcal{O}(e^{-\sqrt{n}}), \\
\frac{2(\lambda_{1n}+\alpha)\cosh(\lambda_{1n}+\alpha)x}{\lambda_{1n}} = e^{(-l_1+l_2 i)x} + e^{(l_1-l_2 i)x} + \mathcal{O}(n^{-1}), \\
2\sinh(\lambda_{1n}+\alpha)x = e^{(-l_1+l_2 i)x} - e^{(l_1-l_2 i)x} + \mathcal{O}(n^{-1}),
\end{gathered}
$$

where l_1, l_2 are given by (3.22). Using (3.17) and (3.29), we estimate (3.31), where the first term satisfies

$$
2\sinh\sqrt{\lambda_{2n}}x = e^{\frac{2n+1}{2}\pi i x} - e^{-\frac{2n+1}{2}\pi i x} + \mathcal{O}(n^{-1}),
$$

the second term satisfies

$$
\begin{aligned}
&\frac{2(\lambda_{2n}+\alpha)\sinh\sqrt{\lambda_{2n}}\cosh(\lambda_{2n}+\alpha)x}{p\lambda_{2n}\sinh(\lambda_{2n}+\alpha)} \\
&= \frac{2\sinh(\frac{2n+1}{2}\pi i)\cosh(\alpha-\frac{(2n+1)^2}{4}\pi^2+2p^{-2})x}{p\sinh(\alpha-\frac{(2n+1)^2}{4}\pi^2+2p^{-2})} + \mathcal{O}(n^{-1}) \\
&= \frac{e^{2[\alpha-\frac{(2n+1)^2}{4}\pi^2+2p^{-2}]x}+1}{e^{2[\alpha-\frac{(2n+1)^2}{4}\pi^2+2p^{-2}]}-1}(-1)^n 2ip^{-1}e^{[\alpha-\frac{(2n+1)^2}{4}\pi^2+2p^{-2}](1-x)} + \mathcal{O}(n^{-1}) \\
&= (-1)^{n+1}2ip^{-1}e^{[\alpha-\frac{(2n+1)^2}{4}\pi^2+2p^{-2}](1-x)} + \mathcal{O}(n^{-1}),
\end{aligned}
$$

and the third term satisfies

$$
\begin{aligned}
&\frac{2\sinh\sqrt{\lambda_{2n}}\sinh(\lambda_{2n}+\alpha)x}{p\sinh(\lambda_{2n}+\alpha)} \\
&= \frac{2\sinh(\frac{2n+1}{2}\pi i)\sinh(\alpha-\frac{(2n+1)^2}{4}\pi^2+2p^{-2})x}{p\sinh(\alpha-\frac{(2n+1)^2}{4}\pi^2+2p^{-2})} + \mathcal{O}(n^{-1}) \\
&= \frac{e^{2[\alpha-\frac{(2n+1)^2}{4}\pi^2+2p^{-2}]x}-1}{e^{2[\alpha-\frac{(2n+1)^2}{4}\pi^2+2p^{-2}]}-1}(-1)^n 2ip^{-1}e^{[\alpha-\frac{(2n+1)^2}{4}\pi^2+2p^{-2}](1-x)} + \mathcal{O}(n^{-1}) \\
&= (-1)^{n}2ip^{-1}e^{[\alpha-\frac{(2n+1)^2}{4}\pi^2+2p^{-2}](1-x)} + \mathcal{O}(n^{-1}).
\end{aligned}
$$

Hence, the eigenfunctions of operator $\mathcal{A}$ satisfy (3.20) and (3.21). □

Moreover, the eigenvalues of $\mathcal{A}$ satisfy the following.

Theorem 3.3. *Let $\mathcal{A}$ be defined by* (3.1)*. Then all $\lambda = \rho^2 \in \sigma(\mathcal{A})$ with sufficiently large moduli are algebraically simple.*

Proof. One can check that the multiplicity of each $\lambda \in \sigma(\mathcal{A})$ with sufficiently large modulus, as a pole of $R(\lambda, \mathcal{A})$, is less than or equal to the multiplicity of λ as a zero of the entire function $\Delta(\rho^2)$ with respect to ρ. On the other hand, it can be verified that λ is geometrically simple. Moreover, all zeros of $\Delta(\rho^2) = 0$ in (3.23) with large moduli are simple, then the result follows from the general formula $m_a \le p \cdot m_g$ (see [17, p. 148]), where p denotes the order of the pole of the resolvent operator and m_a, m_g denote the algebraic and geometric multiplicities respectively. □

Now, we investigate the eigenvalues and eigenfunctions of $\mathcal{A}^*$, the adjoint of operator $\mathcal{A}$, which is defined as

$$\begin{gathered} \mathcal{A}^*(g, \phi, \theta) = (g'', -\theta, -\phi'' - 2\alpha\theta + \alpha^2\phi), \quad \forall (g, \phi, \theta) \in D(\mathcal{A}^*) \\ D(\mathcal{A}^*) = \big\{(g, \phi, \theta) \in H^2(0,1) \times (H^2(0,1) \cap H_L^1(0,1)) \times H_L^1(0,1), \\ g(0) = 0,\ g(1) = p\theta(1),\ \phi'(1) = pg'(1)\big\}. \end{gathered} \tag{3.32}$$

Let $\mathcal{A}^*(g, \phi, \theta) = \lambda^*(g, \phi, \theta)$, $(g, \phi, \theta) \in D(\mathcal{A}^*)$ and we have $\theta(x) = -\lambda^*\phi(x)$ and

$$\begin{gathered} g''(x) = \lambda^* g(x), \\ -\phi''(x) + 2\alpha\lambda^*\phi(x) + \alpha^2\phi(x) = -(\lambda^*)^2\phi(x), \\ g(0) = 0,\ \phi(0) = 0,\ g(1) = -p\lambda^*\phi(1),\ \phi'(1) = pg'(1). \end{gathered} \tag{3.33}$$

A straightforward computation shows that λ^* satisfies (3.10) and it follows that $\sigma(\mathcal{A}^*) = \sigma(\mathcal{A}) = \{\lambda_{1n}, \overline{\lambda_{1n}}, \lambda_{2n}\}$ given by (3.17). Moreover, $\mathcal{A}^*$ has the eigenfunctions

$$\begin{gathered} \Psi_{1n}(x) = (g_{1n}(x), \phi_{1n}(x), -\lambda_{1n}\phi_{1n}(x))^T, \\ \overline{\Psi_{1n}(x)} = \Big(\overline{g_{1n}(x)}, \overline{\phi_{1n}(x)}, -\overline{\lambda_{1n}}\overline{\phi_{1n}(x)}\Big)^T, \\ \Psi_{2n}(x) = \big(g_{2n}(x), \phi_{2n}(x), -\lambda_{2n}\phi_{2n}(x)\big)^T, \end{gathered} \tag{3.34}$$

satisfying (3.20) and (3.21).

4. Completeness of root subspaces

In this section, we show the root subspace of system (3.2) is complete. First, we give the following lemma.

Lemma 4.1 ([10])**.** *Let $D(\lambda) = 1 + \sum_{i=1}^n Q_i(\lambda)e^{\alpha_i\lambda}$, where Q_i are polynomials of λ, α_i, $i = 1, 2, \dots, n$ are some complex numbers and $n \in \mathbb{N}^+$. Then for all λ outside those circles of radius $\epsilon > 0$ that centered at the roots of $D(\cdot)$, we have $|D(\lambda)| \ge C(\epsilon) > 0$ for some constant $C(\epsilon)$ that depends only on ϵ.*

Theorem 4.2. *Let $\mathcal{A}$ be defined by* (3.1) *and for $\lambda \in \rho(\mathcal{A})$, let $R(\lambda, \mathcal{A}) = (\lambda I - \mathcal{A})^{-1}$ be the resolvent operator of $\mathcal{A}$. Then there exists a constant $M > 0$ independent of λ such that*

$$R(\lambda, \mathcal{A}) \le M(1 + |\lambda|), \tag{4.1}$$

for all $\lambda = \rho^2$ with $\rho \in \mathbb{C}$ lying outside all circles of radius $\epsilon > 0$ that are centered at the zeros of $\Delta(\rho^2)$.

Proof. For any $(g_1, \phi_1, \theta_1) \in \mathcal{H}$, if

$$(\lambda I - \mathcal{A})(g, \phi, \theta) = (g_1, \phi_1, \theta_1), \tag{4.2}$$

and $R(\lambda,\mathcal{A})=(\lambda I-\mathcal{A})^{-1}$ is the resolvent operator of $\mathcal{A}$, then

$$(g,\phi,\theta)=R(\lambda,\mathcal{A})(g_1,\phi_1,\theta_1). \tag{4.3}$$

Let $\lambda\neq 0$ and $\lambda=\rho^2\in\rho(\mathcal{A})$, the resolvent set of $\mathcal{A}$, which leads to $\Delta(\lambda)\neq 0$. Solving (4.2), we obtain $\theta(x)=\lambda\phi(x)-\phi_1(x)$ and

$$\begin{gathered} g''(x)-\lambda g(x)=-g_1(x),\\ \phi''(x)-(\lambda+\alpha)^2\phi(x)=-\lambda\phi_1(x)-\theta_1(x)-2\alpha\phi_1(x),\\ g(0)=0,\quad \phi(0)=0,\quad g(1)-p\lambda\phi(1)=-p\phi_1(1),\quad \phi'(1)+pg'(1)=0. \end{gathered} \tag{4.4}$$

Now, we set the following functions, for $0\le x\le 1$,

$$\begin{gathered} Q_1(x,\xi)=\frac{1}{4}\operatorname{sign}(x-\xi)\rho^{-1}\left[e^{\rho(x-\xi)}-e^{-\rho(x-\xi)}\right],\\ Q_2(x,\xi)=\frac{1}{4}\operatorname{sign}(x-\xi)(\rho^2+\alpha)^{-1}\big[e^{(\rho^2+\alpha)(x-\xi)}-e^{-(\rho^2+\alpha)(x-\xi)}\big],\\ F_0(x)=-\int_0^1 Q_1(x,\xi)g_1(\xi)d\xi,\\ H_0(x)=-\int_0^1 Q_2(x,\xi)\big[(\rho^2+2\alpha)\phi_1(\xi)+\theta_1(\xi)\big]d\xi. \end{gathered} \tag{4.5}$$

According to the method of Green function [26, 10], for any $(g_1,\phi_1,\theta_1)\in\mathcal{H}$, the solution (g,ϕ,θ) of (4.3) has the following expressions

$$g(x)=\frac{F(x,\rho)}{\Delta(\rho^2)},\quad \phi(x)=\frac{H(x,\rho)}{\Delta(\rho^2)},\quad \theta(x)=\rho^2\phi(x)-\phi_1(x), \tag{4.6}$$

where

$$F(x,\rho)=\begin{vmatrix} e^{\rho x} & e^{-\rho x} & 0 & 0 & F_0(x)\\ 1 & 1 & 0 & 0 & U_1\\ 0 & 0 & 1 & 1 & U_2\\ e^{\rho} & e^{-\rho} & -p\rho^2 e^{\rho^2+\alpha} & -p\rho^2 e^{-(\rho^2+\alpha)} & U_3+p\phi_1(1)\\ p\rho e^{\rho} & -p\rho e^{-\rho} & (\rho^2+\alpha)e^{\rho^2+\alpha} & -(\rho^2+\alpha)e^{-(\rho^2+\alpha)} & U_4 \end{vmatrix},$$

$$H(x,\rho)=\begin{vmatrix} 0 & 0 & e^{(\rho^2+\alpha)x} & e^{-(\rho^2+\alpha)x} & H_0(x)\\ 1 & 1 & 0 & 0 & U_1\\ 0 & 0 & 1 & 1 & U_2\\ e^{\rho} & e^{-\rho} & -p\rho^2 e^{\rho^2+\alpha} & -p\rho^2 e^{-(\rho^2+\alpha)} & U_3+p\phi_1(1)\\ p\rho e^{\rho} & -p\rho e^{-\rho} & (\rho^2+\alpha)e^{\rho^2+\alpha} & -(\rho^2+\alpha)e^{-(\rho^2+\alpha)} & U_4 \end{vmatrix},$$

with

$$\begin{gathered} U_1=\frac{1}{4\rho}\int_0^1\big(e^{-\rho\xi}-e^{\rho\xi}\big)g_1(\xi)d\xi,\\ U_2=\frac{1}{4(\rho^2+\alpha)}\int_0^1\left[e^{-(\rho^2+\alpha)\xi}-e^{(\rho^2+\alpha)\xi}\right]\big[(\rho^2+2\alpha)\phi_1(\xi)+\theta_1(\xi)\big]d\xi,\\ U_3=\frac{p\rho^2}{4(\rho^2+\alpha)}\int_0^1\left[e^{(\rho^2+\alpha)(1-\xi)}-e^{-(\rho^2+\alpha)(1-\xi)}\right]\big[(\rho^2+2\alpha)\phi_1(\xi)+\theta_1(\xi)\big]d\xi\\ -\frac{1}{4\rho}\int_0^1\big[e^{\rho(1-\xi)}-e^{-\rho(1-\xi)}\big]g_1(\xi)d\xi, \end{gathered}$$

$$
\begin{aligned}
U_4 = &-\frac{1}{4}\int_0^1 \big[e^{(\rho^2+\alpha)(1-\xi)} + e^{-(\rho^2+\alpha)(1-\xi)}\big]\big[(\rho^2+2\alpha)\phi_1(\xi)+\theta_1(\xi)\big]d\xi \\
&-\frac{p}{4}\int_0^1 \big[e^{\rho(1-\xi)} + e^{-\rho(1-\xi)}\big]g_1(\xi)d\xi.
\end{aligned}
$$

Considering $\rho \in \bar{S} = \{\rho \in \mathbb{C} \mid \arg\rho \in (\pi/4, \pi/2]\}$, we have

$$
\operatorname{Re}(-\rho) \le 0 \quad \text{or} \quad \operatorname{Re}(\rho^2) \le 0, \tag{4.7}
$$

and

$$
\Delta(\rho^2) = \rho(\rho^2+\alpha)e^{\rho-\rho^2-\alpha}\Delta_1(\rho), \tag{4.8}
$$

where $\Delta(\rho^2)$ is given by (3.15) with $\lambda = \rho^2$, and

$$
\Delta_1(\rho) = \frac{p^2\rho^2}{\rho^2+\alpha}(1+e^{-2\rho})\big[e^{2(\rho^2+\alpha)}-1\big] + \rho^{-1}(1-e^{-2\rho})\big[e^{2(\rho^2+\alpha)}+1\big]. \tag{4.9}
$$

More accurately, $\Delta(\rho^2)$ are expressed by (3.25), (3.27) in S_1, S_2, S_3, respectively. Applying the following transformation for determinants $F(x,\rho)$, $H(x,\rho)$, that is, multiplying

$$
\begin{gathered}
\text{the first column by } \frac{1}{4\rho}\int_0^1 e^{-\rho\xi}g_1(\xi)d\xi, \\
\text{the second column by} \frac{1}{4\rho}\int_0^1 e^{\rho\xi}g_1(\xi)d\xi, \\
\text{the third column by } -\frac{1}{4(\rho^2+\alpha)}\int_0^1 e^{-(\rho^2+\alpha)\xi}\big[(\rho^2+2\alpha)\phi_1(\xi)+\theta_1(\xi)\big]d\xi, \\
\text{the fourth column by } -\frac{1}{4(\rho^2+\alpha)}\int_0^1 e^{(\rho^2+\alpha)\xi}\big[(\rho^2+2\alpha)\phi_1(\xi)+\theta_1(\xi)\big]d\xi,
\end{gathered}
$$

and adding this expression to the last column, we have

$$
\begin{gathered}
F(x,\rho) = (\rho^2+\alpha)e^{\rho-\rho^2-\alpha}F_1(x,\rho), \\
H(x,\rho) = \rho e^{\rho-\rho^2-\alpha}H_1(x,\rho), \\
H_x(x,\rho) = \rho(\rho^2+\alpha)e^{\rho-\rho^2-\alpha}H_{1x}(x,\rho),
\end{gathered} \tag{4.10}
$$

where

$$
F_1(x,\rho) = \begin{vmatrix}
e^{-\rho(1-x)} & e^{-\rho x} & 0 & 0 & \rho^{-1}\widetilde{F}_0(x) \\
e^{-\rho} & 1 & 0 & 0 & \rho^{-1}\widetilde{U}_1 \\
0 & 0 & 1 & e^{\rho^2+\alpha} & \frac{\widetilde{U}_2}{\rho^2+\alpha} \\
1 & e^{-\rho} & -p\rho^2 e^{\rho^2+\alpha} & -p\rho^2 & \widetilde{U}_3 + p\phi_1(1) \\
\frac{p\rho}{\rho^2+\alpha} & \frac{-p\rho}{\rho^2+\alpha}e^{-\rho} & e^{\rho^2+\alpha} & -1 & \frac{\widetilde{U}_4}{\rho^2+\alpha},
\end{vmatrix},
$$

$$
H_1(x,\rho) = \begin{vmatrix}
0 & 0 & e^{(\rho^2+\alpha)x} & e^{(\rho^2+\alpha)(1-x)} & \frac{\widetilde{H}_0(x)}{\rho^2+\alpha} \\
e^{-\rho} & 1 & 0 & 0 & \rho^{-1}\widetilde{U}_1 \\
0 & 0 & 1 & e^{\rho^2+\alpha} & \frac{\widetilde{U}_2}{\rho^2+\alpha} \\
\rho^{-1} & \rho^{-1}e^{-\rho} & -p\rho e^{\rho^2+\alpha} & -p\rho & \frac{\widetilde{U}_3+p\phi_1(1)}{\rho} \\
p\rho & -p\rho e^{-\rho} & (\rho^2+\alpha)e^{\rho^2+\alpha} & -(\rho^2+\alpha) & \widetilde{U}_4
\end{vmatrix},
$$

$$H_{1x}(x,\rho)=\begin{vmatrix} 0 & 0 & e^{(\rho^2+\alpha)x} & -e^{(\rho^2+\alpha)(1-x)} & \frac{\widetilde{H}_0'(x)}{\rho^2+\alpha} \\ e^{-\rho} & 1 & 0 & 0 & \rho^{-1}\widetilde{U}_1 \\ 0 & 0 & 1 & e^{\rho^2+\alpha} & \frac{\widetilde{U}_2}{\rho^2+\alpha} \\ \rho^{-1} & \rho^{-1}e^{-\rho} & -p\rho e^{\rho^2+\alpha} & -p\rho & \frac{\widetilde{U}_3+p\phi_1(1)}{\varrho} \\ p\rho & -p\rho e^{-\rho} & (\rho^2+\alpha)e^{\rho^2+\alpha} & -(\rho^2+\alpha) & \widetilde{U}_4 \end{vmatrix},$$

and

$$\begin{aligned}
\widetilde{F}_0(x)&=\frac{1}{2}\int_0^x e^{-\rho(x-\xi)}g_1(\xi)d\xi+\frac{1}{2}\int_x^1 e^{-\rho(\xi-x)}g_1(\xi)d\xi,\\
\widetilde{H}_0(x)&=-\frac{1}{2}\int_0^x e^{(\rho^2+\alpha)(x-\xi)}[(\rho^2+2\alpha)\phi_1(\xi)+\theta_1(\xi)]d\xi\\
&\quad-\frac{1}{2}\int_x^1 e^{(\rho^2+\alpha)(\xi-x)}[(\rho^2+2\alpha)\phi_1(\xi)+\theta_1(\xi)]d\xi,\\
\widetilde{H}_0'(x)&=\frac{1}{2}\int_x^1 e^{(\rho^2+\alpha)(\xi-x)}[(\rho^2+2\alpha)\phi_1(\xi)+\theta_1(\xi)]d\xi\\
&\quad-\frac{1}{2}\int_0^x e^{(\rho^2+\alpha)(x-\xi)}[(\rho^2+2\alpha)\phi_1(\xi)+\theta_1(\xi)]d\xi,\\
\widetilde{U}_1&=\frac{1}{2}\int_0^1 e^{-\rho\xi}g_1(\xi)d\xi,\\
\tilde{U}_2&=-\frac{1}{2}\int_0^1 e^{(\rho^2+\alpha)\xi}[(\rho^2+2\alpha)\phi_1(\xi)+\theta_1(\xi)]d\xi,\\
\widetilde{U}_3&=\frac{1}{2\rho}\int_0^1 e^{-\rho(1-\xi)}g_1(\xi)d\xi\\
&\quad+\frac{p\rho^2}{2(\rho^2+\alpha)}\int_0^1 e^{(\rho^2+\alpha)(1-\xi)}[(\rho^2+2\alpha)\phi_1(\xi)+\theta_1(\xi)]d\xi,\\
\widetilde{U}_4&=-\frac{p}{2}\int_0^1 e^{-\rho(1-\xi)}g_1(\xi)d\xi\\
&\quad-\frac{1}{2}\int_0^1 e^{(\rho^2+\alpha)(1-\xi)}[(\rho^2+2\alpha)\phi_1(\xi)+\theta_1(\xi)]d\xi.
\end{aligned}$$

According to (4.7), $F_1(x,\rho)$, $H_1(x,\rho)$, $H_{1x}(x,\rho)$ are bounded after a direct calculation of matrix determinant. From (4.8) and (4.10), (4.6) becomes

$$\begin{gathered}
g(x)=\frac{F_1(x,\rho)}{\rho\Delta_1(\rho)},\quad \phi(x)=\frac{H_1(x,\rho)}{(\rho^2+\alpha)\Delta_1(\rho)},\quad \phi'(x)=\frac{H_{1x}(x,\rho)}{\Delta_1(\rho)},\\
\theta(x)=\rho^2\phi(x)-\phi_1(x)=\frac{\rho^2H_1(x,\rho)}{(\rho^2+\alpha)\Delta_1(\rho)}-\phi_1(x).
\end{gathered}$$

It follows from Lemma 4.1 that there exists a constant $M_1>0$, such that

$$\begin{aligned}
|g(x)|&\le\frac{M_1}{|\rho|}\Big\{\int_0^1[|g_1(\xi)|+(\rho^2+2\alpha)|\phi_1(\xi)|+|\theta_1(\xi)|]d\xi+|p\phi_1(1)|\Big\},\\
|\phi'(x)|&\le M_1\Big\{\int_0^1[|g_1(\xi)|+(\rho^2+2\alpha)|\phi_1(\xi)|+|\theta_1(\xi)|]d\xi+|p\phi_1(1)|\Big\},
\end{aligned}$$

$$|\theta(x)| \le M_1\Big\{\int_0^1 [|g_1(\xi)| + (\rho^2+2\alpha)|\phi_1(\xi)| + |\theta_1(\xi)|]d\xi + |p\phi_1(1)|\Big\} + |\phi_1(x)|.$$

It is known that $|f(x)| \le \|f'\|_{L^1} \le \|f'\|_{L^2}$ and $\|f\|_{L^1} \le \|f\|_{L^2}$ hold for any function $f(x) \in H_L^1(0,1)$. Hence,

$$\begin{aligned}
|\lambda|^{-1}|g(x)| &\le \frac{M_1}{|\rho|}\Big[|\lambda|^{-1}\|g_1(\xi)\|_{L^2} + (1+2\alpha|\lambda|^{-1})\|\phi_1(\xi)\|_{L^2} \\
&\quad + |\lambda|^{-1}\|\theta_1(\xi)\|_{L^2} + |\lambda|^{-1}\|p\phi_1'\|_{L^2}\Big], \\
|\lambda|^{-1}|\phi'(x)| &\le M_1\Big[|\lambda|^{-1}\|g_1(\xi)\|_{L^2} + (1+2\alpha|\lambda|^{-1})\|\phi_1(\xi)\|_{L^2} \\
&\quad + |\lambda|^{-1}\|\theta_1(\xi)\|_{L^2} + |\lambda|^{-1}\|p\phi_1'\|_{L^2}\Big], \\
|\lambda|^{-1}|\theta(x)| &\le M_1\Big[|\lambda|^{-1}\|g_1(\xi)\|_{L^2} + (1+2\alpha|\lambda|^{-1})\|\phi_1(\xi)\|_{L^2} \\
&\quad + |\lambda|^{-1}\|\theta_1(\xi)\|_{L^2} + |\lambda|^{-1}\|p\phi_1'\|_{L^2}\Big] + |\lambda|^{-1}\|\phi_1'\|_{L^2},
\end{aligned}$$

which leads to

$$\begin{gathered}
\|(g,\phi,\theta)\| \le M_2|\lambda|\|(g_1,\phi_1,\theta_1)\|, \quad |\lambda| > K > 1, \\
\|(g,\phi,\theta)\| \le M_3\|(g_1,\phi_1,\theta_1)\|, \quad M_3 > M_2, \quad |\lambda| \le K,
\end{gathered}$$

where M_2, M_3, K are positive constants independent of λ. Therefore,

$$\|(g,\phi,\theta)\| \le M(1+|\lambda|)\|(g_1,\phi_1,\theta_1)\|,$$

for some constant $M > 0$ independent of λ and all $\lambda = \rho^2$ with $\rho \in \bar{S}$ lying outside all circles of radius $\epsilon > 0$ that are centered at the zeros of $\Delta(\rho^2)$. Finally, we have similar results for $\rho \in \widehat{S} = \{\rho \in \mathbb{C} : \arg\rho \in [\pi/2, 3\pi/4]\}$ ([19, pp. 56-60]) which yields (4.1). The proof is complete. □

Now, we show the root subspace of system (3.2) is complete.

Theorem 4.3. *Let $\mathcal{A}$ be defined by* (3.1)*. Then both the root subspace of $\mathcal{A}$ and $\mathcal{A}^*$ are complete in $\mathcal{H}$, that is, $Sp(\mathcal{A}^*) = Sp(\mathcal{A}) = \mathcal{H}$.*

Proof. We only show the completeness for the root subspace of $\mathcal{A}$ since the proof for $\mathcal{A}^*$ is almost the same. From [7, Lemma 5 p.2355] the following orthogonal decomposition holds:

$$\mathcal{H} = \sigma_\infty(\mathcal{A}^*) \oplus Sp(\mathcal{A}),$$

where $\sigma_\infty(\mathcal{A}^*)$ consists of those $Y \in \mathcal{H}$ so that $R(\lambda, \mathcal{A}^*)Y$ is an analytic function of λ anywhere in the whole complex plane. Hence, $Sp(\mathcal{A}) = \mathcal{H}$ if and only if $\sigma_\infty(\mathcal{A}^*) = \{0\}$. Let $Y \in \sigma_\infty(\mathcal{A}^*)$. According to $\|R(\lambda, \mathcal{A}^*)\| = \|R(\overline{\lambda}, \mathcal{A})\|$ and Theorem 4.2, we obtain

$$\|R(\lambda, \mathcal{A}^*)Y\| \le M(1+|\lambda|)Y, \quad \forall \lambda \in \mathbb{C},$$

for some constant $M > 0$ by the maximum modulus principle. From [15, Theorem 1 p. 3], $R(\lambda, \mathcal{A}^*)Y$ is a polynomial with degree of λ at most 1, that is, $R(\lambda, \mathcal{A}^*)Y = Y_0 + \lambda Y_1$ for some $Y_0, Y_1 \in \mathcal{H}$. Thus $Y = (\lambda - \mathcal{A}^*)(Y_0 + \lambda Y_1)$. Since $\mathcal{A}^*$ is a closed operator, $Y_1 \in D(\mathcal{A}^*)$ and so does Y_0. Therefore,

$$\lambda^2 Y_1 + \lambda(Y_0 - \mathcal{A}^*Y_1) - \mathcal{A}^*Y_0 = Y, \quad \forall \lambda \in \mathbb{C}.$$

Comparing the coefficient of λ^2, λ and λ_0 in both sides of the above equation, we obtain $Y_1 = Y_0 = Y = 0$. The proof is complete. □

5. Riesz basis property and exponential stability

In this section, we show the Riesz basis generation and the exponential stability of system (3.2). For this purpose, we recall the following lemmas:

Lemma 5.1. *Let $\{e_i\}_{i=1}^{\infty}$ and $\{e_i^*\}_{i=1}^{\infty}$ be two approximately normalized and biorthogonal sequences. Then $\{e_i\}_{i=1}^{\infty}$ and $\{e_i^*\}_{i=1}^{\infty}$ are Riesz basis for a Hilbert space $\mathcal{H}$ if and only if* [29, p. 27]

(a) both $\{e_i\}_{i=1}^{\infty}$ and $\{e_i^*\}_{i=1}^{\infty}$ are complete in $\mathcal{H}$;

(b) both $\{e_i\}_{i=1}^{\infty}$ and $\{e_i^*\}_{i=1}^{\infty}$ are Bessel sequences in $\mathcal{H}$, that is, for any $f \in \mathcal{H}$, two sequences $\{\langle f, e_i\rangle\}_{i=1}^{\infty}$ and $\{\langle f, e_i^*\rangle\}_{i=1}^{\infty}$ belong to ℓ^2.

Lemma 5.2 ([22, Lemma 3.2])**.** *Let $\{\mu_n\}$ be a sequence which has asymptotic expressions*

$$\mu_n = \alpha(n + \mathrm{i}\beta \ln n) + \mathcal{O}(1), \quad \alpha \neq 0,\ n = 1, 2, 3, \dots,$$

where β is a real number. If μ_n satisfies $\sup_n \mathrm{Re}(\mu_n) < \infty$, the sequence $\{e^{\mu_n x}\}_{n=1}^{\infty}$ is a Bessel sequence in $L^2(0, 1)$.

Lemma 5.3. *The sequences*

$$\{e^{(-l_1+l_2 i)x}\}_{n=0}^{\infty}, \quad \{e^{(l_1-l_2 i)x}\}_{n=0}^{\infty}, \quad \{e^{\frac{2n+1}{2}\pi i x}\}_{n=0}^{\infty},$$
$$\{e^{-\frac{2n+1}{2}\pi i x}\}_{n=0}^{\infty}, \quad \{e^{(\alpha-\frac{(2n+1)^2}{4}\pi^2+2p^{-2})(1-x)}\}_{n=0}^{\infty},$$

are Bessel sequences in $L^2(0, 1)$, where l_1, l_2 are given by (3.22)*.*

Proof. (i) Let $\alpha_1 = \pi i$, $\beta_1 = 0$, $\alpha_2 = -\pi i$, $\beta_2 = 0$, we have

$$\{e^{(-l_1+l_2 i)x}\}_{n=0}^{\infty}, \quad \{e^{(l_1-l_2 i)x}\}_{n=0}^{\infty},$$

are Bessel sequences from Lemma 5.2.

(ii) Let $\alpha_3 = \pi i$, $\beta_3 = 0$, $\alpha_4 = -\pi i$, $\beta_4 = 0$, we have

$$\{e^{\frac{2n+1}{2}\pi i x}\}_{n=0}^{\infty}, \quad \{e^{-\frac{2n+1}{2}\pi i x}\}_{n=0}^{\infty},$$

are two Bessel sequences. With $\alpha_5 = -\pi^2$, $\beta_5 = 0$, we obtain

$$\{e^{(\alpha-\frac{(2n+1)^2}{4}\pi^2+2p^{-2})(1-x)}\}_{n=0}^{\infty},$$

is also a Bessel sequence. The proof is complete. □

Now we can establish the Riesz basis property of system (3.2).

Theorem 5.4. *Let $\mathcal{A}$ be defined by* (3.1)*. Then the generalized eigenfunctions of $\mathcal{A}$ form a Riesz basis for $\mathcal{H}$.*

Proof. Let $\sigma(\mathcal{A}) = \{\lambda_{1n}, \overline{\lambda_{1n}}, \lambda_{2n}\}_{n=1}^{\infty}$ be the eigenvalues of $\mathcal{A}$. By Theorem 3.2 and Theorem 3.3, we have that each eigenvalue of $\mathcal{A}$ with sufficient large modulus is simple, and hence there exists an integer $N > 0$ such that all $\lambda_{1n}, \overline{\lambda_{1n}}, \lambda_{2n}$ with $n \geq N$, are algebraically simple. If the algebraic multiplicities of λ_{1n}, λ_{2n} for $n \leq N$ are m_{1n}, m_{2n} respectively, we have the generalized eigenfunctions $\Phi_{1n,1}, \Phi_{2n,1}$ satisfy

$$(\mathcal{A} - \lambda_{1n})^{m_{1n}}\Phi_{1n,1} = 0, \quad (\mathcal{A} - \lambda_{1n})^{m_{1n}-1}\Phi_{1n,1} \neq 0,$$
$$(\mathcal{A} - \lambda_{2n})^{m_{2n}}\Phi_{2n,1} = 0, \quad (\mathcal{A} - \lambda_{2n})^{m_{2n}-1}\Phi_{2n,1} \neq 0,$$

and

$$\Phi_{1n,j} = (\mathcal{A} - \lambda_{1n})^{j-1}\Phi_{1n,1}, \quad j = 2, 3, \dots, m_{1n},$$
$$\Phi_{2n,p} = (\mathcal{A} - \lambda_{2n})^{p-1}\Phi_{2n,1}, \quad p = 2, 3, \dots, m_{2n},$$

where $\Phi_{1n,m_{1n}}$ and $\Phi_{2n,m_{2n}}$ are the eigenfunctions of $\mathcal{A}$ with respect to λ_{1n} and λ_{2n} respectively. Assume Φ_{1n} and Φ_{2n} are the normalized eigenfunctions of $\mathcal{A}$ corresponding to λ_{1n} and λ_{2n} with $n \geq N$ respectively. Then $\mathcal{A}$ and $\mathcal{A}^*$ have linearly independent bi-orthogonal generalized eigenfunctions

$$\big\{\{\Phi_{1n,j}, \overline{\Phi_{1n,j}}\}_{j=1}^{m_{1n}}, \{\Phi_{2n,p}\}_{p=1}^{m_{2n}}\big\}_{n<N} \cup \big\{\Phi_{1n}, \overline{\Phi_{1n}}, \Phi_{2n}\big\}_{n\geq N}, \tag{5.1}$$
$$\big\{\{\Phi^*_{1n,j}, \overline{\Phi^*_{1n,j}}\}_{j=1}^{m_{1n}}, \{\Phi^*_{2n,p}\}_{p=1}^{m_{2n}}\big\}_{n<N} \cup \big\{\Phi^*_{1n}, \overline{\Phi^*_{1n}}, \Phi^*_{2n}\big\}_{n\geq N}, \tag{5.2}$$

and it follows from Theorem 4.3 that the sequences (5.1) and (5.2) are complete in $\mathcal{H}$.

Now we show the Riesz basis property of the system. First, we show that

$$\{\Phi_{1n}, \ \overline{\Phi_{1n}}, \ \Phi_{2n}\}_{n\geq N}, \quad \{\Psi_{1n}, \ \overline{\Psi_{1n}}, \ \Psi_{2n}\}_{n\geq N},$$

are Bessel sequences in $\mathcal{H}$. It can be verified that $\{\Phi^*_{1n}, \overline{\Phi^*_{1n}}, \Phi^*_{2n}\}_{n\geq N}$ is a Bessel sequence if and only if $\{\Psi_{1n}, \overline{\Psi_{1n}}, \Psi_{2n}\}_{n\geq N}$ is a Bessel sequence by normalizing the latter (see [26, Theorem 5.4]). From (3.19) and (3.34),

$$\Phi_{1n} = (g_{1n}, \phi_{1n}, \lambda_{1n}\phi_{1n})^T, \quad \Phi_{2n} = (g_{2n}, \phi_{2n}, \lambda_{2n}\phi_{2n})^T,$$
$$\Psi_{1n} = (g_{1n}, \phi_{1n}, -\lambda_{1n}\phi_{1n})^T, \quad \Psi_{2n} = (g_{2n}, \phi_{2n}, -\lambda_{2n}\phi_{2n})^T,$$

and it then follows from (3.20), (3.21) and Lemma 5.3 that for $n \geq N$,

$$\{g_{1n}\}, \quad \{\phi'_{1n}\}, \quad \{\pm\lambda_{1n}\phi_{1n}\}, \quad \{g_{2n}\}, \quad \{\phi'_{2n}\}, \quad \{\pm\lambda_{2n}\phi_{2n}\},$$

are Bessel sequences in $L^2(0,1)$. Therefore, $\{\Phi_{1n}, \overline{\Phi_{1n}}, \Phi_{2n}\}_{n\geq N}$ and $\{\Psi_{1n}, \overline{\Psi_{1n}}, \Psi_{2n}\}_{n\geq N}$ are both Bessel sequences in $\mathcal{H}$ and thus $\{\Phi^*_{1n}, \overline{\Phi^*_{1n}}, \Phi^*_{2n}\}_{n\geq N}$ is also a Bessel sequence in $\mathcal{H}$. This together with two finitely many sequences

$$\big\{\{\Phi_{1n,j}, \overline{\Phi_{1n,j}}\}_{j=1}^{m_{1n}}, \{\Phi_{2n,p}\}_{p=1}^{m_{2n}}\big\}_{n<N}, \quad \Big\{\{\Phi^*_{1n,j}, \overline{\Phi^*_{1n,j}}\}_{j=1}^{m_{1n}}, \{\Phi^*_{2n,p}\}_{p=1}^{m_{2n}}\Big\}_{n<N},$$

leads to the sequences (5.1) and (5.2) are Bessel sequence in $\mathcal{H}$. Hence, the generalized eigenfunctions of $\mathcal{A}$ form a Riesz basis for $\mathcal{H}$ from Lemma 5.1. The proof is complete. □

Theorem 5.5. *Let $\mathcal{A}$ be defined by* (3.1). *Then the spectrum-determined growth condition $\omega(\mathcal{A}) = s(\mathcal{A})$ holds true. Moreover, system* (3.2) *is exponentially stable, that is, there exist two positive constants L and ω such that*

$$\|e^{\mathcal{A}t}\| \leq Le^{-\omega t}.$$

Proof. From Theorem 5.4, the spectrum-determined growth condition holds true, that is, $\omega(\mathcal{A}) = s(\mathcal{A})$. By (3.17), we have $\mathrm{Re}(\lambda) < 0$ for each $\lambda \in \sigma(\mathcal{A})$. Hence, semigroup $e^{\mathcal{A}t}$ is exponentially stable. The proof is complete. □

6. Appendix

In this appendix, we present the derivations of (2.6), (2.7), and (2.24). From (2.2), (2.3), (2.4), and (2.5), we have

$$\begin{aligned}
&y_t(x,t) - y_{xx}(x,t)\\
&= \big[c + 2\frac{d}{dx}(k_1(x,x))\big]u(x,t) + k_1(x,0)u_x(0,t)\\
&\quad + \int_0^x [k_{1xx}(x,y) - k_{1yy}(x,y) - ck_1(x,y)]u(y,t)dy = 0,
\end{aligned}$$

$$\begin{aligned}
&z_{tt}(x,t) - z_{xx}(x,t) + 2\alpha z_t(x,t) + \alpha^2 z(x,t)\\
&= \big[-h''(x) + \alpha^2 h(x) + 2\frac{d}{dx}(k_2(x,x)) + 2(d+\alpha)k_{3y}(x,x)\big]w(x,t)\\
&\quad - 2[(d+\alpha)k_3(x,x) + h'(x)]w_x(x,t)\\
&\quad + 2\big[(d+\alpha)h(x) + \frac{d}{dx}(k_3(x,x))\big]w_t(x,t)\\
&\quad + \int_0^x [k_{2xx} - k_{2yy} - \alpha^2 k_2 - 2(d_1+\alpha)k_{3yy}](x,y)w(y,t)dy\\
&\quad + \int_0^x [k_{3xx} - k_{3yy} - 2(\alpha+d)k_2 - (\alpha^2 + 4d(d+\alpha))k_3](x,y)w_t(y,t)dy\\
&\quad + [k_2(x,0) + 2(d+\alpha)k_3(x,0)]w_x(0,t) + k_3(x,0)w_{xt}(0,t) = 0.
\end{aligned}$$

Hence,

$$k_{1xx}(x,y) - k_{1yy}(x,y) = ck_1(x,y), \quad \frac{d}{dx}(k_1(x,x)) = -\frac{c}{2}, \quad k_1(x,0) = 0, \tag{6.1}$$

$$k_{2xx}(x,y) - k_{2yy}(x,y) = \alpha^2 k_2(x,y) + 2ak_{3yy}(x,y), \quad k_2(x,0) = 0, \tag{6.2}$$

$$2\frac{d}{dx}(k_2(x,x)) = h''(x) - \alpha^2 h(x) - 2ak_{3y}(x,x), \tag{6.3}$$

$$k_{3xx}(x,y) - k_{3yy}(x,y) = 2ak_2(x,y) + (\alpha^2 + 4ad)k_3(x,y), \quad k_3(x,0) = 0, \tag{6.4}$$

$$\frac{d}{dx}(k_3(x,x)) = -ah(x), \quad ak_3(x,x) = -h'(x), \tag{6.5}$$

where $0 \le x \le 1, 0 \le y \le x$ and $a = d + \alpha$. From (6.5), we have

$$h(x)h'(x) = k_3(x,x)\frac{d}{dx}k_3(x,x). \tag{6.6}$$

Integrating (6.6), we have $h^2(x) = k_3^2(x,x) + 1$, where $h(0) = 1$. From (6.5), we obtain $\frac{h'(x)}{\sqrt{h^2(x)-1}} = \pm a$, and

$$h(x) = \cosh ax, \quad k_3(x,x) = \frac{-h'(x)}{a} = -\sinh ax, \tag{6.7}$$

which yields

$$k_2(x,x) = \frac{a^2 - \alpha^2}{2a}\sinh ax - a\int_0^x k_{3y}(\tau,\tau)d\tau, \tag{6.8}$$

by integrating (6.3) from 0 to x. A direct computation gives

$$\begin{aligned} k_{3xx}(x,x)-k_{3yy}(x,x) &= \frac{d}{dx}(k_{3x}(x,x)-k_{3y}(x,x)) \\ &= -ah'(x)-2\frac{d}{dx}k_{3y}(x,x). \end{aligned} \tag{6.9}$$

Let $f(x) = k_{3y}(x,x)$. By $\frac{d}{dx}k_3(x,x) = k_{3x}(x,x) + k_{3y}(x,x)$, $k_3(x,0) = 0$, we combine (6.5) with (6.7), (6.8), (6.9) to obtain that $f(x)$ satisfies

$$f'(x) - a^2\int_0^x f(\tau)d\tau = (2ad - a^2 + \alpha^2)\sinh ax, \quad f(0) = -a,$$

which is equivalent to solving the ODE

$$f''(x) - a^2 f(x) = (2a^2 d - a^3 + a\alpha^2)\cosh ax, \quad f(0) = -a, \quad f'(0) = 0.$$

Then we obtain $f(x) = \big(ad - \frac{a^2}{2} + \frac{\alpha^2}{2}\big)x\sinh ax - a\cosh ax$. It follows from (6.8) that

$$g(x) := k_2(x,x) = (a+d)\sinh ax + \Big(\frac{a^2}{2} - \frac{\alpha^2}{2} - ad\Big)x\cosh ax. \tag{6.10}$$

Therefore, the kernel functions $k_2(x,y)$ and $k_3(x,y)$ satisfy

$$\begin{gathered} k_{2xx}(x,y) - k_{2yy}(x,y) - 2ak_{3yy}(x,y) = \alpha^2 k_2(x,y), \\ k_{3xx}(x,y) - k_{3yy}(x,y) = 2ak_2(x,y) + (\alpha^2 + 4ad)k_3(x,y), \\ k_2(x,x) = g(x),\ k_3(x,x) = -\sinh ax,\ k_2(x,0) = 0,\ k_3(x,0) = 0, \end{gathered}$$

where $a = d + \alpha$, $g(x)$ is given by (6.10) and the domain is

$$\Omega_1 = \{(x,y) : 0 \le x \le 1,\ 0 \le y \le x\}.$$

Similarly, from (2.1), (2.22) and (2.23), we have

$$\begin{aligned} 0 &= u_t(x,t) - u_{xx}(x,t) - cu(x,t) \\ &= \int_0^x [s_{1yy}(x,y) - s_{1xx}(x,y) - cs_1(x,y)]y(y,t)dy \\ &\quad - \big[c + 2\frac{d}{dx}(s_1(x,x))\big]y(x,t) - s_1(x,0)y_x(0,t), \end{aligned}$$

$$\begin{aligned} 0 &= w_{tt}(x,t) - w_{xx}(x,t) - 2dw_t(x,t) \\ &= \Big[2(\alpha+d)s_{3y}(x,x) - \alpha^2 l(x) - 2\frac{d}{dx}(s_2(x,x)) - l''(x)\Big]z(x,t) \\ &\quad - 2[(\alpha+d)s_3(x,x) + l'(x)]z_x(x,t) + [2(\alpha+d)s_3(x,0) - s_2(x,0)]z_x(0,t) \\ &\quad - 2\big[\frac{d}{dx}(s_3(x,x)) + (\alpha+d)l(x)\big]z_t(x,t) - s_3(x,0)z_{xt}(0,t) \\ &\quad + \int_0^x [s_{2yy}(x,y) - s_{2xx}(x,y) - 2(\alpha+d)s_{3yy}(x,y) - \alpha^2 s_2(x,y) \\ &\quad + 2\alpha^2(d+\alpha)s_3(x,y)]z(y,t)dy \\ &\quad + \int_0^x [s_{3yy}(x,y) - s_{3xx}(x,y) - 2(\alpha+d)s_2(x,y) \\ &\quad + (4a\alpha - \alpha^2)s_3(x,y)]z_t(y,t)dy, \end{aligned}$$

$l(x) = \cosh ax$, and $s_2(x,x) = -\frac{a^2+\alpha^2+d\alpha}{a}\sinh ax - \frac{d^2}{2}x\cosh ax$, which gives (2.24).

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