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SEMIGROUP THEORY AND ASYMPTOTIC PROFILES OF SOLUTIONS FOR A HIGHER-ORDER FISHER-KPP PROBLEM IN $\mathbb{R}^N$

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Abstract. We study a reaction-diffusion problem formulated with a higher-order operator, a non-linear advection, and a Fisher-KPP reaction term depending on the spatial variable. The higher-order operator induces solutions to oscillate in the proximity of an equilibrium condition. Given this oscillatory character, solutions are studied in a set of bounded domains. We introduce a new extension operator, that allows us to study the solutions in the open domain $\mathbb{R}^N$, but departing from a sequence of bounded domains. The analysis about regularity of solutions is built based on semigroup theory. In this approach, the solutions are interpreted as an abstract evolution given by a bounded continuous operator. Afterward, asymptotic profiles of solutions are studied based on a Hamilton-Jacobi equation that is obtained with a single point exponential scaling. Finally, a numerical assessment, with the function bvp4c in Matlab, is introduced to discuss on the validity of the hypothesis.

1. Description of the problem and main results

Reaction-diffusion models have been a source of research under different scopes. From a physical perspective, an intuition of diffusion is introduced by the concept of Random Walk (see [28] and references listed there) that permits to model complex scenarios related with spatially distributed motion including heterogeneous media. Other physical approach to diffusion has been pursuit based on the Landau-Ginzburg free energy concept [12, 13]. The free energy permits to consider a generalization beyond the classical Fick law. In particular, the authors in [12] derived a mathematical expression for the free energy in a heterogeneous media leading to the formation of spatial patterns of solutions. It is particularly relevant to briefly discuss the mathematical arguments used by the authors in [12]: Firstly, they considered that the free energy shall be dependant of the gradient of a concentration, v, i.e. $\frac{1}{2}k(\nabla v)^2$. Considering this general form and making use of the chemical potential, the authors obtained a parabolic partial differential equation with an order four spatial operator, generally of the form $v_t = -v_{xxxx}$.

Let us return now to the origin: Nonlinear reaction-diffusion models were formally and systematically introduced by Kolmogorov, Petrovskii and Piskunov in [24] and by Fisher in [18] to study the behaviour of flames in combustion theory

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and the interaction of genes respectively. The approaches followed by the authors were supported by the classical Fick law that ends in a gaussian order two operator. The authors employed a paramount technique, that opened new areas for exploring analytical expressions for the solutions. These were named as Travelling Waves and were particularly interesting to predict the dynamics of diffusion acting in a wave front. The Fisher-KPP model is ubiquitous and is given in different applied scopes (refer to [1, 2, 3] as representative examples). Recently, the Fisher-KPP models have been studied with different kind of parabolic operators: higher order operators [20], fractional operators [8] and with a p-Laplacian Porous Medium Equation [4].

In addition to the mention ideas, the higher order operators can be regarded as perturbation terms that supplement the regular order two diffusion operators (see [32, 14, 30, 6] for some extensions of the Fisher-Kolmogorov equation).

Some other notable analyses can be mentioned in relation with heterogeneous diffusion and applications. In [31], the authors studied a biological dynamics with advection, that precluded a non-linear diffusion. Further, the authors in [22] studied the haptotaxis cancer invasion given by a degenerate diffusivity and by spectral stability methods.

The proposed equation, under discussion in this analysis, is formed of a higher-order operator (that make solutions to oscillate and proceed in a non-homogeneous evolution), a non-linear advection and a Fisher-KPP term:

$$\begin{gathered} v_t = -\Delta^2 v + c \cdot \nabla v^p + v(g(x) - v), \\ v_0(x), g(x) \in W_0^{m,p}(\mathbb{R}^N) \cap C_0(\mathbb{R}^N), \quad N > 1, \end{gathered} \tag{1.1}$$

where $c \in \mathbb{R}^N$ is the advection vector, $m = 4$ for our purposes and typically $p = 2$.

Given the oscillatory character of solutions, induced by the higher order operator (see [20]), the analysis presented in this paper is provided in a set of bounded domains. Based on the proposed study, we think that the oscillating character of solutions can be assessed, in a more comprehensive view, for a free domain as $\mathbb{R}^N$. This idea is in fact the main motivation of the presented analysis and requires to include some evidences to support the extension from the sequence of bounded domains to the whole space $\mathbb{R}^N$. The analyses of regularity, existence and uniqueness of the solutions are given by making use of the semigroup theory. The operator $-\Delta^2$ is briefly shown to be an infinitesimal generator of a strongly continuous semigroup. Afterward, an exponential scaling, along with some hypothesis under an asymptotic approach, are introduced to explore forms of analytical profiles of solutions. Eventually, a numerical assessment, for the equation (1.1), is provided in order to compare it with the asymptotic solutions and state about their validity.

2. Preliminary results

To start we consider the set $\Gamma_r = \{x \in B(0,r) \subset \mathbb{R}^N\}$ with C^a-boundary ($a \geq 1$).

Definition 2.1. The extension operator is defined as

$$E : W^{m,p}(\Gamma_r) \to W^{m,p}(\mathbb{R}^N), \tag{2.1}$$

where $(Ev)(x) = v(x)$ a.e. in Γ_r.

Proposition 2.2. *The extension operator E satisfies*

$$\|Ev\|_{W^{m,p}(\mathbb{R}^N)} \leq C \|v\|_{W_0^{m,p}(\Gamma_r)}, \tag{2.2}$$

where C is a constant obtained as a relation of Borel measures in partitions of unity.

Proof. With no loss of generality, first we assume $r \to \infty$ and $p = 2$. Then $v \in H^m(\Gamma_{r\to\infty})$ if

$$(1 + |\xi|^2)^{m/2}\hat{v}(\xi) \in L^2(\Gamma_{r\to\infty}), \tag{2.3}$$

where $\hat{v}$ refers to the Fourier transformation. In addition, for $k < m - \frac{N}{2}$ and a suitable constant C_1, it is easily checked that

$$\|(1 + |\xi|^2)^{\frac{k}{2}}\hat{v}(\xi)\|_{L^2(\Gamma_{r\to\infty})} \le C_1\|Ev\|_{W^{m,p}(\Gamma_{r\to\infty})}. \tag{2.4}$$

Then, it follows to state that each $\partial^\beta v$ is bounded and continuous (in the sense of Holder) for $|\beta| < m - \frac{N}{2}$. Based on the denseness property for $W^{m,p}$, given any $v \in W^{m,p}(\overline{\Gamma}_r)$, define a sequence

$$v_r = \{|\max v(\partial\Gamma_r)|,\ r = 1, 2, 3, dots\}. \tag{2.5}$$

Based on the ordered property, it is possible to find regions of $\partial\Gamma_r$ where the following holds:

$$|\max v(\mathbb{R}^N)| = \lim_{r\to\infty} |\max v(\partial\Gamma_r)| < \infty. \tag{2.6}$$

As $|\max v(\mathbb{R}^N)|$ is finite and Γ_r is bounded with smooth C^a-boundary ($a \ge 1$), it is possible to approximate any function in $W^{m,p}(\overline{\Gamma}_r)$ by functions in $C^\infty(\mathbb{R}^N)$ restricted to $\overline{\Gamma}_r$. In addition, there exists a partition of unity for each r represented as $\{\rho_j^r\}_{j\in J}$, with J being a set of indexes. Then, if $v_r \in W^{m,p}(\overline{\Gamma}_r)$, then $\rho_j^r v_r \in W^{m,p}(\overline{\Gamma}_r)$. Further, it holds that $spt(\rho_j^r v_r) \subset \Gamma_r$ and is compact.

Consider now the standard mollifier ϕ (See [17]). Given a small $\varepsilon_j > 0$, we define function

$$h_j^r := (\rho_j^r v_r) * \phi_{\varepsilon_j} \in C_0^\infty(\overline{\Gamma}_r). \tag{2.7}$$

In the limit for $r \to \infty$, a unity partition is given as $\{\rho_j\} = \lim_{r\to\infty}\{\rho_j^r\}$, such that the following function is defined:

$$h_j := (\rho_j v) * \phi_{\varepsilon_j} \in C_0^\infty(\mathbb{R}^N). \tag{2.8}$$

Given the bound of each $\partial^\beta v$, $|\beta| < m - \frac{N}{2}$, as previously shown, considering a Borel measure μ in each $\{\rho_j^r\}$ and by a spatial translation to make h_j below h_j^r, the following holds

$$(\mu(\rho_j)v) * \phi_{\varepsilon_j} \le (\mu(\rho_j^r)v_r) * \phi_{\varepsilon_j}, \tag{2.9}$$

uniformly in $\overline{\Gamma}_r$ and $\Gamma_{r\to\infty}$.

Considering the involved norms in $W_0^{m,p}$ and the denseness properties of $W_0^{m,p}$ (see [23]) in Γ_r with $\cup_{r=1}^\infty \Gamma_r = \mathbb{R}^N$,

$$\|Ev\|_{W^{m,p}(\mathbb{R}^N)} \le \|v\|_{W^{m,p}(\mathbb{R}^N)} \le \frac{\mu(\rho_j^r)}{\mu(\rho_j)}\|v\|_{W_0^{m,p}(\Gamma_r)}, \tag{2.10}$$

as intended. □

Given the basic equation $v_t = -\Delta^2 v$, the function $S : \mathbb{R}^+ \to \mathbb{R}^+$ with $S(t) = e^{-\Delta^2 t}$ defines a family $\{S(t)\}_{t\in\mathbb{R}^+}$ of bounded linear operators in L^2. The boundedness of each mapping follows from the Plancherel´s theorem and after application of the standard norm in L^2. The family $\{S(t)\}_{t\in\mathbb{R}^+}$ satisfies the basic condition of the semigroup theory: $S(0) = I$, $S(t + s) = S(t)S(s)$, $S(-t) = (S(t))^{-1}$.

Consequently, the operator $-\Delta^2$ can be regarded as the infinitesimal generator of the semigroup family $\{S(t)\}_{t\in\mathbb{R}^+}$. Indeed, it complies with:

$$-\Delta^2 = \lim_{t\to 0^+} \frac{S(t)-I}{t}, \tag{2.11}$$

where $I = \lim_{t\to 0^+} S(t) = I$. In addition, the following proposition holds.

Proposition 2.3. *For the bounded linear operator $-\Delta^2$ in L^2, the family $\{S(t)\}_{t\in\mathbb{R}^+}$ where*

$$e^{-\Delta^2 t} := \sum_{j=1}^{\infty} \frac{(-\Delta^2)^j\, t^j}{j!}, \tag{2.12}$$

with $(-\Delta^2)^0 = I$, forms a uniformly continuous semigroup in L^2 where $-\Delta^2$ is the infinitesimal generator.

Proof. The proof follows from standard ideas of the semigroup theory (see [29]). Nonetheless, some basic principles are introduced, and in particular for the operator $-\Delta^2$,

$$\|S(t)\|_2 = \|e^{-\Delta^2 t}\|_2 \le \sum_{j=1}^{\infty} \frac{(\|-\Delta^2\|_2\, t)^j}{j!} = e^{\|-\Delta^2\|_2\, t}, \quad t>0. \tag{2.13}$$

Consequently, the family $\{S(t)\}_{t\in\mathbb{R}^+}$ is well defined and satisfies $S(0)=I$, being I, the identity map in L^2. To show the uniform continuity,

$$\|\frac{S(t)-I}{t} - (-\Delta^2)\|_2 \le \frac{1}{t}\sum_{j=2}^{\infty} \frac{(\|-\Delta^2\|_2\, t)^j}{j!} = \frac{1}{t}(e^{\|-\Delta^2\|_2 t} - I - t\|-\Delta^2\|_2), \tag{2.14}$$

that tends to zero whenever $t\to 0^+$. Therefore, $-\Delta^2$ is the infinitesimal generator of the uniformly continuous semigroup family $\{S(t)\}_{t\in\mathbb{R}^+}$ with domain $D(-\Delta^2) = L^2$. □

The next objective is to show that the family $\{S(t)\}_{t\in\mathbb{R}^+}$ is strongly continuous (also referred as C^0-semigroup).

Proposition 2.4. *Assume that $S(t)$ is a C^0-semigroup. Then, there exist two constants, m and w, such that for $t\ge 0$,*

$$\|S(t)\|_2 \le m e^{wt}. \tag{2.15}$$

Proof. Previously, we showed that $S(t)$ is uniformly continuous, then for $m\ge 1$ and $\tau>0$ with $0\le t\le \tau$:

$$\|S(t)\|_2 \le m. \tag{2.16}$$

Assume that this last inequality does not hold, this is, there exists a sequence $\{t_n\}$ converging to zero such that $\|S(t_n)\|_2 \ge n$ as $n\to\infty$. Based on the semigroup property $S(t_n)v \to v$ as $n\to\infty$, we state that $\{S(t_n)v\}$ is bounded $\forall v\in L^2$. Considering the Banach-Steinhause theorem, we conclude on the boundedness of $\{S(t_n)\}$, which is a contradiction to the initial assumption. Then, the initial inequality (2.15) holds as $e^{wt}\ge 1$, $\forall t\ge 0$. □

Based on the above Proposition, the strongly continuous condition for $S(t)$ follows easily: For any $\tau>0$, it holds that:

$$\|S(t+\tau)v - S(t)v\|_2 \le \|S(t)\|_2\, \|S(\tau)v - v\|_2 \le m\, e^{wt}\|S(\tau)v - v\|_2. \tag{2.17}$$

Then, for $\tau\to 0^+$, it follows that $\|S(t+\tau)v - S(t)v\|_2 \to 0$.

Definition 2.5. The following norm is defined to support the analysis of existence and uniqueness of solutions:

$$\|v\|_{\Upsilon}^2 = \int_{\Gamma_r} \Upsilon(\xi) \sum_{k=0}^{4} |D^k v(\xi)|^2 d\xi, \quad \xi \in \Gamma_r \subset \mathbb{R}^N \tag{2.18}$$

where $D^k = \frac{\partial^{|k|}}{\partial \xi_1^{k_1} \partial \xi_2^{k_2} \dots \partial \xi_N^{k_N}}$, with $|k| = \sum_{i=1}^N k_i$, and $k = (k_1, k_2, \dots, k_N)$ belongs to $(\mathbb{N} \cup \{0\})^{\mathbb{N}}$. Further, $v \in H^4_{\Upsilon}(\Gamma_r) \subset L^2_{\Upsilon}(\Gamma_r) \subset L^2(\Gamma_r)$ and the weight Υ is given as per the following expression (see [20] together with [27]):

$$\Upsilon(\xi) = \exp\Big(a_0|\xi|^{4/3} - \frac{1}{|\xi|^q} \frac{1}{t^\gamma} \int_0^t (\|c \cdot \nabla v(\xi, s)\|_2^p + 1) ds\Big), \tag{2.19}$$

such that $|\xi| = \sum_{i=1}^N |\xi_i|$, $a_0 > 0$ is sufficiently small and $\gamma > p + 1$.

2.1. **Inequalities and relations in functional spaces.** Let $L = (-\Delta^2 + pv^{p-1}c \cdot \nabla)$ be the spatial operator and consider the basic equation

$$v_t = Lv. \tag{2.20}$$

Although the initial data was requested to belong to $W_0^{m,p}(\mathbb{R}^N) \cap C_0(\mathbb{R}^N)$, further conditions will be specified on v_0 in this section to support the embedding analysis to come.

Lemma 2.6. *Given $v_0 \in L^2(\mathbb{R}^N)$, we have*

$$\|v\|_{L^2} \le \|v_0\|_{L^2}. \tag{2.21}$$

For $r \in \mathbb{R}^+$, and considering $v_0 \in H^r(\mathbb{R}^N) \cap L^2(\mathbb{R}^N)$, now we have

$$\|v\|_{H^r} \le \|v_0\|_{H^r}, \tag{2.22}$$

$$\|v\|_{H^r}^2 \le e^{\frac{r^2}{8t}} \|v_0\|_{L^2}^2. \tag{2.23}$$

In addition,

$$\|v\|_{\Upsilon} \le \kappa \|v\|_{H^r} \le \kappa \|v_0\|_{H^r}, \quad \kappa^2 = 5 \sup_{|\xi| \in \mathbb{R}} \{|v|^2, |D^1 v|^2, |D^2 v|^2, |D^3 v|^2, |D^4 v|^2\}.$$

Proof. A fundamental solution to the basic equation in (2.20) is given by

$$v(x,t) = e^{tL} v_0(x), \tag{2.24}$$

and considering the Fourier transformation in the domain (ξ),

$$\hat{v}(\xi, t) = e^{t(-|\xi|^4 + p\hat{v}^{p-1} c \cdot \xi i)} \hat{v}_0(\xi). \tag{2.25}$$

Considering Plancherel‘s theorem or the isometric Fourier condition in L^2:

$$\begin{aligned} \|v\|_{L^2}^2 &= \int_{-\infty}^{\infty} |e^{t2(-|\xi|^4 + p\hat{v}^{p-1} c \cdot \xi i)}||\hat{v}_0(\xi)|^2 d\xi \\ &= \int_{-\infty}^{\infty} e^{-2|\xi|^4 t} |\hat{v}_0(\xi)|^2 d\xi \\ &\le \sup_{|\xi| \in \mathbb{R}} (e^{-2|\xi|^4 t}) \int_{-\infty}^{\infty} |\hat{v}_0(\xi)|^2 d\xi = \|v_0\|_{L^2}^2. \end{aligned} \tag{2.26}$$

Then $\|v\|_{L^2} \le \|v_0\|_{L^2}$.

Now, consider a mollifying norm for $r \in \mathbb{R}^+$ and $0 \le t \le \tau < \infty$ that complies with the A_p-condition (see [21]) for $p = 1$,

$$\|v\|_{H^r}^2 = \int_{-\infty}^{\infty} e^{r|\xi|^2} \, |\hat{v}(\xi,t)|^2 d\xi. \tag{2.27}$$

Then

$$\begin{aligned} \|v\|_{H^r}^2 &= \int_{-\infty}^{\infty} e^{r|\xi|^2} |\hat{v}(\xi,t)|^2 d\xi \\ &= \int_{-\infty}^{\infty} e^{r|\xi|^2} |e^{t2(-|\xi|^4 + p\hat{v}^{p-1} c\cdot \xi i)}||\hat{v}_0(\xi)|^2 d\xi \\ &\le \sup_{|\xi|\in\mathbb{R}} (e^{-2|\xi|^4 t}) \int_{-\infty}^{\infty} e^{r|\xi|^2} |\hat{v_0}(\xi)|^2 d\xi = \|v_0\|_{H^r}^2. \end{aligned} \tag{2.28}$$

Now we consider $v_0 \in L^2(\mathbb{R}^N)$. Then

$$\|v\|_{H^r}^2 = \int_{-\infty}^{\infty} e^{r|\xi|^2} |\hat{v}(\xi,t)|^2 d\xi \le \sup_{|\xi|\in\mathbb{R}} (e^{r|\xi|^2} \, e^{-2|\xi|^4 t}) \int_{-\infty}^{\infty} |\hat{v}_0(\xi)|^2 d\xi. \tag{2.29}$$

Making standard operations and rearranging terms,

$$\|v\|_{H^r}^2 \le e^{\frac{r^2}{8t}} \|v_0\|_{L^2}^2, \tag{2.30}$$

as initially stated. Eventually,

$$\begin{aligned} \|v\|_{\Upsilon}^2 &= \int_{\Gamma_r} \Upsilon(\xi) \sum_{k=0}^{4} |D^k v(\xi)|^2 d\xi \\ &\le \int_{\Gamma_r} e^{r|\xi|^2} \sum_{k=0}^{4} |D^k v(\xi)|^2 d\xi \\ &\le \kappa^2 \int_{\Gamma_r} e^{r|\xi|^2} |v(\xi)|^2 d\xi \\ &\le \kappa^2 \|v\|_{H^r}^2, \end{aligned} \tag{2.31}$$

where $\kappa^2 = 5 \sup_{|\xi|\in\mathbb{R}} \{|v|^2, |D^1 v|^2, |D^2 v|^2, |D^3 v|^2, |D^4 v|^2\}$.

The scaling variable κ is defined in accordance with the results about continuous inclusions in Sobolev spaces ([23], p. 79). Indeed, assume that the function v is regularly differentiable up to the third order. The fourth order derivative in κ can be considered as a controlling variable. If such fourth order derivative is regular, then the mollifying norm (2.27) bounds the norm $\|\cdot\|_\Gamma$ □

As previously shown, $-\Delta^2$ is the infinitesimal generator of a strongly continuous semigroup. As a consequence, the following representation holds based on the Duhamel‘s principle,

$$v(t) = e^{-\Delta^2 t} v_0 + \int_0^t \left[c\cdot \nabla (e^{-\Delta^2 (t-s)p} v^p(s)) + e^{-\Delta^2 (t-s)} v(s)(g(x) - v(s)) \right] ds. \tag{2.32}$$

Consider now the basic problem $v_t = -\Delta^2 v$ with $v(x,0) = \delta(x)$, a solution is given by the Fourier transformation

$$\hat{v}(t) = e^{-|\xi|^4 t} \hat{v}_0(\xi). \tag{2.33}$$

As a consequence, the kernel can be obtained as

$$\begin{aligned} K(x,t) = F^{-1}(e^{-|\xi|^4 t}) &= \frac{1}{(2\pi)^{N/2}} \int_{\mathbb{R}^N} e^{-|\xi|^4 t - i\xi\cdot x} d\xi \\ &= \frac{1}{(2\pi)^{N/2}} \int_{\mathbb{R}^N} e^{-|\xi|^4 t} cos(\xi \cdot x) d\xi. \end{aligned} \tag{2.34}$$

Note that the left-hand integral is bounded for ξ in $\mathbb{R}^N$. Hence, the abstract evolution in (2.19) can be rewritten in terms of such existing kernel. But firstly, the following operator in $H^4_\Upsilon(\Gamma_r)$ is defined,

$$T_{v_0,t} : H^4_\Upsilon(\Gamma_r) \to H^4_\Upsilon(\Gamma_r), \tag{2.35}$$

which is given as

$$\begin{aligned} T_{v_0,t}(v) = K(x,t) * v_0(x) + \int_0^t \big[& c \cdot \nabla K(x,t-s) * v^p(s) \\ & + K(x,t-s) * v(x,s)(g(x) - v(x,s))\big] \, ds, \end{aligned} \tag{2.36}$$

where ' $*$ ' refers to the spatial convolution and t is a single parameter. Note that in the previous expression, the following assessment has been implicitly conducted in the advection term

$$\begin{aligned} K(x,t) * c \cdot \nabla v^p(x,s) &= \int_{-\infty}^{\infty} K(x-\beta,t) c \cdot \nabla v^p(\beta,s) \, d\beta \\ &= -\int_{-\infty}^{\infty} v^p(\beta,s) c \cdot \nabla K(x-\beta,t) \, d\beta \\ &= -\int_{-\infty}^{\infty} v^p(\beta,s) c \cdot \nabla_{(x-\beta)} K(x-\beta,t) \frac{\partial(x-\beta)}{\partial \beta} d\beta \\ &= \int_{-\infty}^{\infty} v^p(\beta,s) c \cdot \nabla_{(x-\beta)} K(x-\beta,t) \\ &= c \cdot \nabla K(x,t) * v^p(x,t). \end{aligned}$$

The next step is to show the boundedness properties of the single parametric operator $T_{v_0,t}$. Then, the following lemma holds.

Lemma 2.7 (Operator bound)**.** *The single parametric operator $T_{v_0,t}$ is bounded in $H^4_\Upsilon(\Gamma_r)$ with the norm (2.18). In addition, the extended operator given by $ET_{v_0,t}$ (see Definition 2.1) is bounded in $H^4(\mathbb{R}^N)$.*

Proof. Firstly, we shoe inequality

$$k_0 \|v_0\|_\Upsilon \le \|v\|_\Upsilon. \tag{2.37}$$

Indeed,

$$
\begin{aligned}
\|v\|_{\Upsilon}^2 &= \int_{\Gamma_r} \Upsilon(\xi) \sum_{k=0}^{4} |D^k \hat{v}(\xi)|^2 d\xi \\
&= \int_{\Gamma_r} \Upsilon(\xi) \sum_{k=0}^{4} |D^k \big[e^{t(-|\xi|^4 + p\hat{v}^{p-1} c\cdot\xi i)} \hat{v}_0\big]|^2 d\xi \\
&\geq \int_{\Gamma_r} \Upsilon(\xi) \sum_{k=0}^{4} \big|D^k \big[e^{t(-|\xi|^4 + p\hat{v}^{p-1} c\cdot\xi i)}\big]\big|^2 \sum_{k=0}^{4} \big|D^k \hat{v}_0\big|^2 d\xi \\
&\geq k_0^2 \int_{\Gamma_r} \Upsilon(\xi) \sum_{k=0}^{4} |D^k \hat{v}_0|^2 d\xi \ = k_0^2 \|v_0\|_{\Gamma}^2,
\end{aligned}
\tag{2.38}
$$

such that

$$
k_0^2 = \inf_{\xi \in B_r} \{ \sum_{k=0}^{4} \big|D^k \big[e^{t(-|\xi|^4 + p\hat{w}^{p-1} c\cdot\xi i)}\big]\big|^2 \} > 0, \tag{2.39}
$$

in $B_r = \{\xi : |\xi| < r\}$, for any $r > 0$.

Now, we return to the operator $T_{v_0,t}$.

$$
\begin{aligned}
&\|T_{v_0,t}(v)\|_{\Upsilon} \\
&\leq \|T_{v_0,t}\|_{\Upsilon} \, \|v\|_{\Upsilon} \leq \|K\|_{\Upsilon} \|v_0\|_{\Upsilon} \\
&\quad + \int_0^t \big[\|c \cdot \nabla K\|_{\Upsilon} \|v^p\|_{\Upsilon} + \|K\|_{\Upsilon} \, \|v\|_{\Upsilon} \|g(x) - v\|_{\Upsilon}\big] ds \\
&\leq \big[\|K\|_{\Upsilon} \frac{1}{k_0 \, t} + \int_0^t \big[\|c \cdot \nabla K\|_{\Upsilon} \, \|v_0^{p-1}\|_{H^r} + \|K\|_{\Upsilon} |\|g(x)\|_{\Upsilon} - k_0 \|v_0\|_{\Upsilon}|\big] ds\big] t \|v\|_{\Gamma}.
\end{aligned}
$$

Note that inequalities (2.28) and (2.31) have been employed for the term $\|v^{p-1}\|_{\Upsilon}$, indeed,

$$
\|v^{p-1}\|_{\Upsilon} \leq \|v^{p-1}\|_{H^r} \leq \|v_0^{p-1}\|_{H^r}. \tag{2.40}
$$

Then

$$
\begin{aligned}
\|T_{v_0,t}\|_{\Upsilon} \leq \big[\|K\|_{\Upsilon} \frac{1}{k_0 \, t} &+ \int_0^t \big[\|c \cdot \nabla K\|_{\Upsilon} \|v_0^{p-1}\|_{H^r} \\
&+ \|K\|_{\Upsilon} |\|g(x)\|_{\Upsilon} - k_0 \|v_0\|_{\Upsilon}|\big] ds\big] t < \infty,
\end{aligned}
\tag{2.41}
$$

for $0 < t < \infty$. Further, note that the right-hand side is locally bounded for any single parameter t value. The operator action can be extended to $\mathbb{R}^N$ by a simple application of the extension operator given in Proposition 2.2 and the relation $\|T_{v_0,t}\|_{H_0^4(\Gamma_r)} \leq \|T_{v_0,t}\|_{\Upsilon}$, a.e. in Γ_r. Then

$$
\|ET_{v_0,t}\|_{H^4(\mathbb{R}^N)} \leq C \|T_{v_0,t}\|_{H_0^4(\Gamma_r)} \leq C \|T_{v_0,t}\|_{\Upsilon}.
$$

□

Now, we obtain existence results for any solution to (1.1) in $H_0^4(\Gamma_r)$. To this end, the following theorem holds.

Theorem 2.8. *Assume that locally $v(t) = T_{v_0,t}(v(t))$ for $0 < t \leq \tau$. Then there exists a weak solutions to (1.1) and it satisfies $v_1 \in L^2(0, \tau; H_0^4(\Gamma_r))$.*

Proof. Given the sequence subset Γ_r and the space $H_0^4(\Gamma_r)$, it is well known that there exist eigenfunctions of a reproducing kernel that form an orthonormal basis (refer to [9] for polynomial basis as a representative example). Consequently, assume $\{\varphi_j(x)\}$, $j = 1, 2, \dots$ is an orthogonal basis of $H_0^4(\Gamma_r)$. Then, the following sequence of solutions $\{v_n\}$ is defined for problem (1.1),

$$v_n(x,t) = \sum_{j=1}^{n} g_{n,j}(t)\varphi_j(x), \quad n = 1, 2, \dots \tag{2.42}$$

Note that each element of the sequence v_n satisfies an evolution ODE

$$(v_{n,t}, \varphi_j) - (\nabla(\nabla \cdot \nabla v_n), \nabla\varphi_j) + (cv_n^p, \nabla\varphi_j) - (v_n(g(x) - v_n), \varphi_j) = 0, \tag{2.43}$$

where $(\cdot, \cdot)$ denotes the inner product in H^4.

The initial condition for each element of the sequence is expressed as

$$v_n(x,0) = \sum_{j=1}^{n} g_{n,j}(0)\varphi_j(x), \quad n = 1, 2, \dots \tag{2.44}$$

where $g_{n,j}(0) = (v_n(x,0), \varphi_j(x))$ are constants.

Considering standard results of ODE theory, in particular the Peano‘s theorem, there exist solutions for

$$g_{n,j}(t) \in C^1([0,\tau]; H_0^4(\Gamma_r)), \tag{2.45}$$

so that, $v_n \in C^1([0,\tau]; H_0^4(\Gamma_r))$.

The intention now is to obtain estimates for each element $v_n(t)$. Firstly, we consider equation (2.43) and multiply by $g_{n,j}(t)$. After the sum from $j = 1$ to $j = n$:

$$\frac{1}{2}\frac{d}{dt}\|v_n\|_2^2 + \|\nabla \cdot \nabla v_n\|_2^2 + \|cv_n^p\|_2\|\nabla v_n\|_2 \le \|v_n\|_2^2\|g(x)\|_\infty. \tag{2.46}$$

Given the Sobolev space $W^{m,p}(\Gamma_r)$, we define $\alpha = \text{int}\{m - \frac{N}{p}\}$, then the following Holder-continuous inclusion holds (refer to [23, page 79]),

$$W^{m,p}(\Gamma_r) \hookrightarrow C^\alpha(\Gamma_r). \tag{2.47}$$

In the present analysis, any solution shall be generally differentiable up to order fourth, hence $m = 4$, and typically $p = 2$, then $\alpha = int\{4 - \frac{N}{2}\}$. Given the continuity embedding in (2.47) and the compact support for each function $\varphi_j(x) \in H_0^4(\Gamma_r)$, it is possible to conclude that the terms $\|\nabla \cdot \nabla v_n\|_2^2$ and $\|cv_n^p\|_2\|\nabla v_n\|_2$ are bounded in Γ_r. In addition and since $g(x) \in W_0^{m,p}(\mathbb{R}^N) \cap C_0(\mathbb{R}^N)$, it is possible to define $M = \|g(x)\|_\infty < \infty$ in Γ_r. As a consequence, inequality (2.46) is reduced to

$$\frac{1}{2}\frac{d}{dt}\|v_n\|_2^2 \le M\|v_n\|_2^2, \tag{2.48}$$

such that the following estimate is obtained for each element of the sequence $\|v_n\|_2^2 \le e^{2Mt}$, $0 < t \le \tau$.

Now, integrating (2.48) on $(0,\tau]$,

$$\frac{1}{2}\|v_n\|_2^2 - \frac{1}{2}\|v_n(0)\|_2^2 \le M\int_0^\tau \|v_n\|_2^2 \le \frac{1}{2}e^{2M\tau}, \tag{2.49}$$

which yields a global bound of each element of the sequence for $0 < t \le \tau$.

Considering the obtained bounds for $0 < t \le \tau$, there exists a function $v \in L^2(0,\tau; H_0^4(\Gamma_r))$ together with a sub-sequence $\{v_n\}$, $n = 1, 2, \dots$ such that given any $t \in (0,\tau]$ and for $n \to \infty$, it holds that

$$v_n \to v, \tag{2.50}$$

in a weak sense in $L^2(0,\tau;H_0^4(\Gamma_r))$.

Now, consider the global bound in (2.49) for $0<t\le\tau$, the weak converge of $\{v_n\}$ as described and the Aubin-Lions-Dubinskii compactness theorem (see [11]): Then, we state that $v_n\to v$ strongly in $C(0,\tau;L^2(\Gamma_r))$.

We recall that the Aubin-Lions-Dubinskii theorem requires $\frac{\partial v_n}{\partial t}$ to be bounded in a Banach space. This can be found easily from inequality (2.48) in $0<t\le\tau$.

Once the convergence of the sequence $\{v_n\}$ has been shown, the following formulation holds for the solution v,

$$(v_t,\varphi_j)-(\nabla(\nabla\cdot\nabla v),\nabla\varphi_j)+(c\,v^p,\nabla\varphi_j)-(v(g(x)-v),\varphi_j)=0,\quad \forall j. \tag{2.51}$$

For any arbitrary function Ψ obtained as a linear combination of $\{\varphi_j\}$, it holds that

$$(v_t,\Psi)-(\nabla(\nabla\cdot\nabla v),\nabla\Psi)+(cv^p,\nabla\Psi)-(v(g(x)-v),\Psi)=0. \tag{2.52}$$

This last expression, along with the condition $v_0(x)\in W_0^{m,p}(\mathbb{R}^N)\cap C_0(\mathbb{R}^N)$ $(m=4,\ p=2)$ and the convergence in these dense spaces for any initial data distribution, permit to conclude that the problem (1.1) admits weak solutions in $0<t<\tau$ where $\tau\le\infty$. □

Note that in Lemma 2.7, we showed the global bound of solutions by the bounding properties of the single parametric operator $T_{v_0,t}$. In addition, in Theorem 2.8, the solutions have been shown to exist locally in time based on arguments related with convergence. Such convergence can be extended to any $\tau\gg 1$ while keeping the global bound and the local convergence shown. As a consequence, the existence of solutions is shown as a global basis for any $0\le t\le\tau<\infty$. Further, the results can be made applicable to $\mathbb{R}^N$ upon application of the extended operator in Definition 2.1 together with $\|ET_{v_0,t}\|_{H^4(\mathbb{R}^N)}\le C\|T_{v_0,t}\|_{H_0^4(\Gamma_r)}$.

Based on the observations made, we conclude on the existence of global solutions $v(x,t)$ for $(x,t)\in\mathbb{R}^N\times(0,\tau]$.

2.2. **Uniqueness of the solution.** The uniqueness analysis is based on a fix point argument. The map $T_{v_0,t}$ in (2.35) shall comply with $v(x,t)=T_{v_0,t}(v(x,t))$, $(x,t)\in\Gamma_r\times(0,\tau]$. For this, consider that there exist two solutions $v_1(x,t)$ and $v_2(x,t)$ satisfying (2.36) and with the same initial data $v_0(x)$, such that making the difference and for any $t\in(0,\tau]$,

$$\begin{aligned}
&\|T_{v_0,t}(v_1)-T_{v_0,t}(v_2)\|_\Upsilon\\
&\le\int_0^t\|c\cdot\nabla K(x,t-s)*(v_1^p-v_2^p)\\
&\quad+K(x,t-s)*[v_1(g-v_1)-v_2(g-v_2)]\|_\Upsilon ds\\
&=\int_0^t\|\int_t^s\{c\cdot\nabla K(x,t-s-r)(v_1^p-v_2^p)\\
&\quad+K(x,t-s-r)[v_1(g-v_1)-v_2(g-v_2]\}dr\|_\Upsilon ds\\
&\le\int_0^t\int_t^s\{\|c\cdot\nabla K(x,t-s-r)(v_1^p-v_2^p)\|_\Upsilon\\
&\quad+\|K(x,t-s-r)[v_1(g-v_1)-v_2(g-v_2)]\|_\Upsilon\}dr\,ds\\
&=\int_0^t\int_t^s\{\|c\cdot\nabla K(x,t-s-r)\|_\Upsilon\|v_1^p-v_2^p\|_\Upsilon\\
&\quad+\|K(x,t-s-r)\|_\Upsilon\|v_1(g-v_1)-v_2(g-v_2)\|_\Upsilon\}dr\,ds
\end{aligned}$$

$$\le M_1 \int_0^t \int_t^s \{\|v_1^p - v_2^p\|_\Upsilon + \|v_1(g-v_1) - v_2(g-v_2)\|_\Upsilon\} dr\, ds, \tag{2.53}$$

Note that K and ∇K are bounded as per the expression (2.34), hence

$$M_1 = \sup\{\|K(x,t-s-r)\|_\Upsilon, \|c\cdot\nabla K(x,t-s-r)\|_\Upsilon; \quad \forall t \in (0,\tau],\ x \in \Gamma_r\}, \tag{2.54}$$

and for any $s > 0$ and $r > 0$.

To assess the integrals involved in (2.53), consider the function

$$A(\varepsilon, s) = \begin{cases} \frac{v_1(\varepsilon,s)^p - v_2(\varepsilon,s)^p}{v_1(\varepsilon,s) - v_2(\varepsilon,s)} & \text{for } v_1 \not\equiv v_2 \\ p v_1^{p-1} & \text{otherwise.} \end{cases} \tag{2.55}$$

For a fixed value in ε and with $s = \tau$, the last expression is bounded and satisfies $0 \le A(\epsilon, s) \le C_0(p, \|v_0\|_\infty, \tau)$. Then

$$\|v_1^p - v_2^p\|_\Upsilon \le C_0^* \|v_1 - v_2\|_\Upsilon, \tag{2.56}$$

where $C_0^* = \|C_0\|_\Gamma$.

The remaining term involving the reaction-absorption in (2.53) is assessed based on the definition of a norm given in (2.18).

$$\begin{aligned}
&\|[v_1(g-v_1) - v_2(g-v_2)]\|_\Upsilon^2 \\
&= \int_{\Gamma_r} \Upsilon(\xi) \sum_{k=0}^{4} |D^k[v_1(g-v_1) - v_2(g-v_2)]|^2 d\xi \\
&= \int_{\Gamma_r} \Upsilon(\xi) \Big\{ |v_1(g-v_1) - v_2(g-v_2)|^2 \\
&\quad + \sum_{k=1}^{4} |D^k[v_1(g-v_1) - v_2(g-v_2)]|^2 \Big\} d\xi \\
&= \int_{\Gamma_r} \Upsilon(\xi) \Big\{ |(v_1 - v_2)(g - (v_1 - v_2))|^2 \\
&\quad + \sum_{k=1}^{4} \sum_{i=1}^{k} \Big| \binom{k}{i} (v_1 - v_2)^{(i)} (a - (v_1 - v_2)^{(k-i)}) \Big|^2 \Big\} d\xi \\
&\le 25B^2 \int_{\Gamma_r} \Upsilon(\xi) \Big\{ |(v_1 - v_2)|^2 + \sum_{k=1}^{4} \sum_{i=1}^{k} \Big| \binom{k}{i} (v_1 - v_2)^{(i)} \Big|^2 \Big\} d\xi \\
&= 25B^2 \int_{\Gamma_r} \Upsilon(\xi) \sum_{k=0}^{4} |D^k[v_1 - v_2]|^2 d\xi \\
&= 25B^2 \|v_1 - v_2\|_\Upsilon^2,
\end{aligned} \tag{2.57}$$

where $B^2 = \sup\{|g - (v_1 - v_2)|^2, |g - (v_1 - v_2)]^{k-i}|^2\}$. Eventually, it holds

$$\begin{aligned}
\|T_{v_0,t}(v_1) - T_{v_0,t}(v_2)\|_\Upsilon &\le M_1(5B + C_0^*) \int_0^t \int_t^s \|v_1 - v_2\|_\Upsilon ds dr \\
&= M_1(5B + C_0^*) t(t-s) \|v_1 - v_2\|_\Upsilon.
\end{aligned} \tag{2.58}$$

Given a local time interval with $0 < t < s \le \tau$, the uniqueness is shown for $v_1 \swarrow v_2$ leading to a contractive mapping $T_{v_0,t}$, that complies with the convergence condition $T_{v_0,t}(v_1) \swarrow v_1$ in the space of functions H_Υ^4.

The uniqueness result can be made applicable to solutions in $\mathbb{R}^N$. For this, consider the extended operator in Definition 2.1 with the bound $\|ET_{v_0,t}\|_{H^4(\mathbb{R}^N)} \leq C\|T_{v_0,t}\|_{H_0^4(\Gamma_r)}$.

3. Profiles of solutions

Profiles of solutions are obtained under the non-linear point transformation:

$$v = e^w. \tag{3.1}$$

For particular discussions in relation to the proposed exponential scaling, the reader can consult [7] together with the formal introduction in [26, 25, 5].

Generally, the function w shall be understood as a complex mapping, and this can be foreseen given the oscillatory nature of the solutions profiles (see [20]): $w : X \times [0, \tau] \to \mathbb{C}$.

Now, following some previous ideas discussed in [10], the function w satisfies the Hamilton-Jacobi type of equation

$$\begin{gathered} w_t = H_4(w, \nabla w) + P_4\Big(w, \frac{\partial^{|k|} w}{\partial x_1^{k_1} \partial x_2^{k_2} \dots \partial x_N^{k_N}}\Big), \\ |k| = \sum_{i=1}^N k_i, \quad k = (k_1, k_2, \dots, k_N) \in (\mathbb{N} \cup \{0\})^{\mathbb{N}}. \end{gathered} \tag{3.2}$$

where

$$H_4(w) = -(\nabla w)^2 (\nabla w)^2 + cp\nabla w e^{(p-1)w} + g(x) - e^w. \tag{3.3}$$

Furthermore,

$$\begin{aligned} P_4(w) = &-\Delta^2 w - \Delta(\nabla w \cdot \nabla w) - 2\nabla w \cdot \nabla \Delta w - 2(\nabla w \cdot \nabla w)\Delta w \\ &- 2\nabla w \cdot \nabla(\nabla w \cdot \nabla w) - (\Delta w)^2. \end{aligned} \tag{3.4}$$

The operator P_4 is of algebraic order 3 while the Hamilton-Jacobi operator H_4 is of algebraic order 4. This can be easily shown by considering a smooth function $\Sigma \in C_0^\infty$, such that

$$|P_4(\lambda\Sigma)| = O(\lambda^3) \ll H_4(\lambda\Sigma) = O(\lambda^4), \tag{3.5}$$

for $\lambda \gg 1$. Balancing the leading terms, the equation (1.1) can be rewritten as

$$w_t = -(\nabla w)^2 \, (\nabla w)^2 + c\, p\, \nabla w\, e^{(p-1)w} + g(x) - e^w. \tag{3.6}$$

Now, the idea is to explore standing wave solutions to (3.6). Assume that such solutions are given in the form of separated variables (refer to [10] for further details),

$$w(x,t) = (\sigma + t)^{-1/3} \Theta(x), \tag{3.7}$$

where $\sigma < t \leq \tau$.

Upon substitution in (3.6) and in the asymptotic approach with $\sigma \gg 1$,

$$-\frac{1}{3}\Theta = (\nabla\Theta)^4 + c\, p(\sigma + t)\nabla\Theta + (g(x) - 1)(\sigma + t)^{4/3}. \tag{3.8}$$

Recall that our intention is to introduce an asymptotic solution. Consequently, in the last expression, we have considered that

$$e^w = e^{(\sigma+t)^{-1/3}\Theta(x)} \to 1, \quad e^{(p-1)w} \to 1, \tag{3.9}$$

with $t \to \infty$. We consider now the leading terms in (3.8):

$$-\frac{1}{3}\Theta = cp(\sigma + t)\nabla\Theta + g^*(\cdot, t), \tag{3.10}$$

where σ, t are parameters and $g^*(\cdot, t) = |(g(\cdot)-1)(\sigma+t)^{4/3}|$. Note that the function g^* is expressed as a function of t to stress the asymptotic evaluation of g for $t \to \infty$ and the point ' $\cdot$ ' indicates that the function is assessed in each x. Hence

$$\Theta(x) = 3\Big(e^{-\frac{x}{3|c|p\,(\sigma+t)}} - g^*(\cdot, t)\Big), \tag{3.11}$$

where $|c| = \sum_{i=1}^N |c_i|$.

Now, in the asymptotic approximation $t \to \tau \gg 1$, there exists a moving wave front as

$$|x| = 3|c|p\ln(g^*(\cdot, t))t. \tag{3.12}$$

Note that the time exponent in the function g^* and the Log function recall the results in [20], where a Log-Front shift is analyzed for another higher order problem. This is a confirmation of the ubiquity of the Log shift in the moving front.

Now, making the balance of the nabla distribution $O(\nabla\Theta) < O(\sigma+t)$ and making the asymptotic condition with $|x| \gg 1$ and $\sigma \gg 1$,

$$-\frac{1}{3}\Theta = (\nabla\theta)^4 + g^*(\cdot, t), \tag{3.13}$$

for which a general solution is

$$\Theta(x) = 3\Big(\frac{1}{4}J(i)|x|\Big)^{4/3} - 3g^*(\cdot, t), \tag{3.14}$$

being $J(i) = (-1)^{\frac{1}{4}}$ and i the imaginary unit. Based on all exposed, a profile of asymptotic solution to w is

$$w(x, t) = 3t^{-1/3}\Big(\Big(\frac{1}{4}J(i)|x|\Big)^{4/3} - g^*(\cdot, t)\Big). \tag{3.15}$$

In return to the original scaling (3.1), a profile of asymptotic solution to (1.1), for $t \to \tau \gg 1$, is

$$v(x, t) = e^{-3g^*(\cdot,t)t^{-1/3}} \exp\Big(3t^{-1/3}\Big(\frac{1}{4}J(i)|x|\Big)^{4/3}\Big). \tag{3.16}$$

Note on the existence of oscillations in the obtained profile. This is affirmative given the complex number $J(i)$. The oscillation behaviour of solutions in higher-order operators has been widely discussed in [20] and will be further discussed in the numerical approach introduced in the next section.

4. Numerical validation of the asymptotic approach

Once an analytical asymptotic solution has been shown to hold in the expression (3.16), and under certain hypothesis, it is the intention to compare the asymptotic approach with a numerical solution of (1.1). The numerical assessment is based on the function bvp4c in Matlab. The initial condition and the reaction function shall satisfy $v_0(x), g(x) \in H_0^4(\mathbb{R}^3) \cap C_0(\mathbb{R}^3)$. For the sake of simplicity, and with no loss of generality, both functions have been considered as the same: $v_0(x) = g(x) = 1 - c_s\,|x|^4$ where c_s is a suitable constant to be determined based on the numerical trials and has the intention of providing fully readable representations. Note that the bvp4c function set is based on a Runge-Kutta implicit algorithm

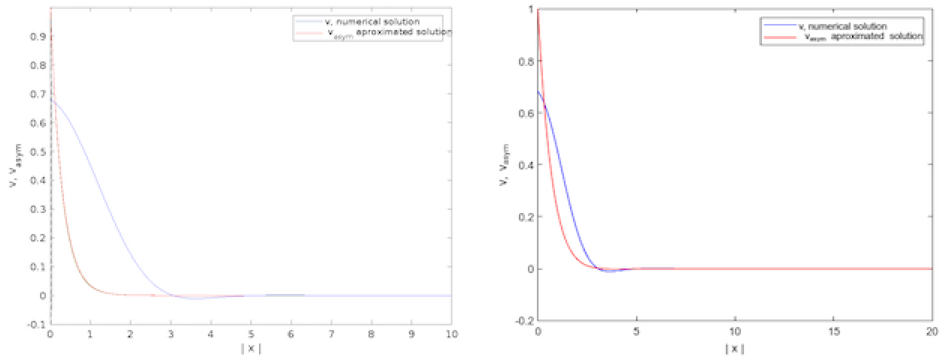

FIGURE 1. Solutions for $t = 1$ (left) and $t = 10$ (right). In both cases, for $|x| > x_m = 5.352$, the global distance between the numerical solutions and the asymptotic profile (3.16) is $\leq 10^{-3}$.

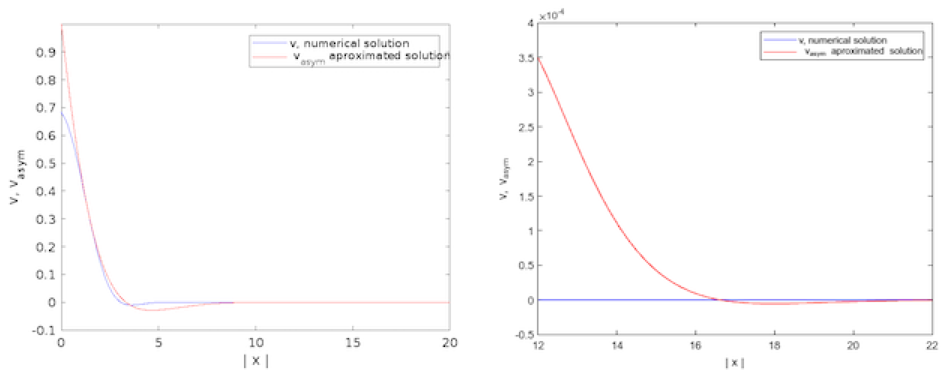

FIGURE 2. Solutions for $t = 100$ as a global picture (left) and a zoom (right). For $|x| > x_m = 11.781$, the global distance between the numerical solutions and the asymptotic profile (3.16) is $\leq 10^{-3}$. Note that in this case the oscillatory behaviour of solutions is made apparent.

with interpolant extensions [16]. The pseudo-boundary condition, required by the collocation methods in the function bvp4c, has been considered as $v(|x| \gg 1) = 0$.

The domain of integration is considered sufficiently large so as to minimize any influence of the collocation method at the boundaries. In this case, the integration domain is given by $|x| \in (-500, 500)$. The number of nodes to execute the Runge-Kutta is 100000 with absolute error of 10^{-5}.

Given the difficulty to make a full representation in several dimensions, the results are given in different time levels and for a single spatial variable of the form $|x| = \sum_{i=1}^{3} |x_i|$. As it can be observed, the asymptotic profile in (3.16) fits with the numerical solution of (1.1) for $|x| \gg 1$ as it was required to obtained the transport equation (3.13). To properly characterize the adequacy of the solutions, it is considered that both solutions are sufficiently close whenever the global distance

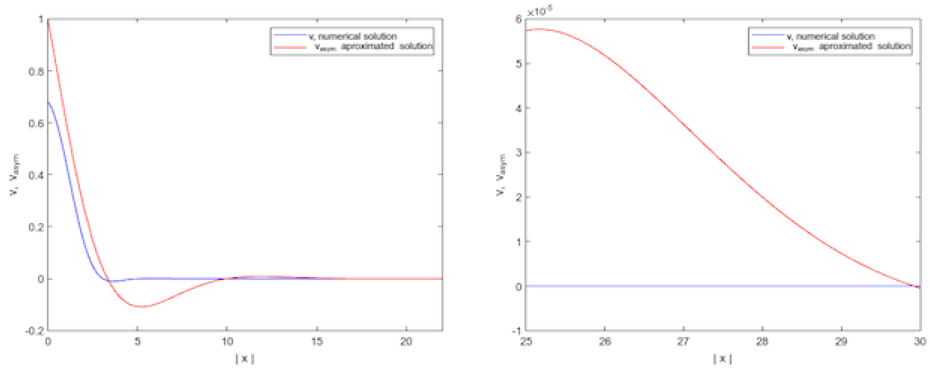

FIGURE 3. Solutions for $t = 500$ globally (left) and with a zoom (right). For $|x| > x_m = 22.612$, the global distance between the numerical solutions and the asymptotic profile (3.16) is $\le 10^{-3}$. Note that the oscillatory behaviour increases when increasing the time levels.

(or error) between the numerical solution and the asymptotic profile is $|v_{num} - v_{asym}| \le 10^{-3}$ for $|x| > x_m$ (see each figure footprint for specific values of x_m).

5. Conclusions

The Fisher-KPP problem in $\mathbb{R}^N$, $(N > 1)$ with a nonlinear advection and a higher order diffusion was studied with analytical and numerical approaches. The operator $-\Delta^2$ was shown to be the infinitesimal generator of a strongly continuous semigroup by application of standard techniques. Consequently, the analyses about regularity and uniqueness of the solutions were supported by the semigroup theory. Afterward, the asymptotic profiles of solutions were obtained with a single point exponential scaling, that led to a Hamilton-Jacobi equation. Eventually, a numerical assessment, with the function bvp4c in Matlab, permitted to validate the hypothesis made during the asymptotic analysis.

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