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DYNAMICS OF A PARTIALLY DEGENERATE REACTION-DIFFUSION CHOLERA MODEL WITH HORIZONTAL TRANSMISSION AND PHAGE-BACTERIA INTERACTION

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Abstract. We propose a cholera model with coupled reaction-diffusion equations and ordinary differential equations for discussing the effects of spatial heterogeneity, horizontal transmission, environmental viruses and phages on the spread of *vibrio cholerae*. We establish the well-posedness of this model which includes the existence of unique global positive solution, asymptotic smoothness of semiflow, and existence of a global attractor. The basic reproduction number $\mathcal{R}_0$ is obtained to describe the persistence and extinction of the disease. That is, the disease-free steady state is globally asymptotically stable for $\mathcal{R}_0 \leq 1$, while it is unstable for $\mathcal{R}_0 > 1$. And, the disease is persistence and the model has the phage-free and phage-present endemic steady states in this case. Further, the global asymptotic stability of phage-free and phage-present endemic steady states are discussed for spatially homogeneous model. Finally, some numerical examples are displayed in order to illustrate the main theoretical results and our opening questions.

1. Introduction

Cholera is an acute intestinal infectious disease which is caused by the bacteria *vibrio cholerae*; it has been around for hundreds of years. Currently, cholera is transmitted in two main ways: one is environmental transmission (environment to human), that is, a person becomes infected by ingesting water or food contaminated with *vibrio cholerae*; the other is horizontal transmission (human to human), such as close contact with people infected with cholera, or contact with the excrement of cholera patients, etc., may be followed by infection. When people get infected with cholera, it causes symptoms like vomiting, muscle cramps, severe copious watery diarrhea, and so on, if not treated promptly, the infection may lead to death after 1 or 2 days. Although modern technology, medicine and public health conditions have improved dramatically compared to the past, but cholera is still not been eliminated and remains endemic in Asia, India, Africa, and Latin America. This is a major threat to the public health of low-income groups in developing countries in particular. In recent years, several areas in large-scale outbreak of cholera: nearly 100,000 cases are reported in Zimbabwe from 2008 to 2009 [6, 19, 31]; 545,000 cases

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is reported in Haiti from 2010 to 2012 [2, 37, 47]; 1,115,378 cases and 2310 deaths are reported in Yemen from 2017 to 2018 [5].

Recently, many research works have studied the transmission mechanism of cholera and put forward effective prevention and control measures. In particular, from the aspect of mathematical dynamics modeling, Codeço [9] incorporated cholera bacteria in aquatic reservoirs into SIR epidemic model, proposed a SIR-B model to simulate the transmission of cholera, and proved the stability of disease-free and endemic equilibria. Considering the movement of humans, Capone et al. [4] employed a reaction-diffusion model based on the model in [9], and studied the impact of population movements on the existence and stability of disease-free and endemic equilibrium; here they neglected the human-to-human transmission of *vibrio cholerae*. Wang et al. [49] conducted a diffusive cholera model combining horizontal and environmental transmission and investigated the effect of diffusive spatial spread on the transmission of this disease spread; They revealed that incorporating spatial diffusion does not produce a Turing instability in some extent. Chen et al. [7] purposed a reaction-diffusive cholera model with nonlinear incidence rate, and obtained the global stability of the disease-free and endemic equilibria. In addition, Zhou et al. [56] proposed a reaction-diffusive model with waterborne pathogen and general incidence rate, and investigated the extinction and persistence of disease which are described by the basic reproduction number. Note that the above mentioned models are discussed in a homogeneous space. However, differences in spatial location, water availability and sanitation have a important impact on the transmission of diseases, so it is necessary to consider reaction-diffusion models with spatial heterogeneity [48, 51, 52, 54].

Since the frequent outbreak of cholera brought the local public health a heavy burden, how to prevent and intervene cholera is particularly important. Common intervention methods of cholera include rehydration therapy, antibiotics, vaccination and water treatment [3, 13, 28, 29, 30, 38, 46]. In terms of reducing *vibrio cholerae* in the environment, Misra et al. [30] introduced a delay SIRS-B compartment model to simulate the transmission of water-born disease, and discussed the effects of disinfectants on the control of diseases. Note that, phages, the natural organisms presents in the aquatic reservoirs, are viruses that live on *vibrio cholerae*, which injects its genetic material into bacterial cells and replicates within the host cell. When phages reproduce to a certain number, it can cause the bacterial cells lysis and releases additional phage into the environment, and interaction of phages and bacteria can be described by the predator-prey relationship. Now, phages had been proved in modulating cholera epidemics to play a crucial role, so the introduction of phages in the environment is helpful for combating cholera [11, 12]. Besides, Malik et al. [24] pointed out that using the basic laboratory equipment preparation phages can be quickly and easily. Alternatively, one can use the freeze-dried, spray drying, emulsification and micro capsule preparation phages, these methods can keep the stability of the years. These also imply that introducing phages may be an effective strategy for *vibrio cholerae* control.

To discuss the role of phages in cholera prevention and control, Jensen et al. [18] further extended the model in [9] by including a phage compartment P, and revealed that incorporating phages can effectively decrease the concentration of bacteria, and then it can reduce infection in humans. And then, Misra et al. [26] investigated a reaction-diffusion cholera model with phages control, investigated the global dynamics of this model through constructing an appropriate Lyapunov function. Other studies on the dynamical models of cholera with phages can be found in [20, 27] and the references therein, name just a few.

Based on the above discussion and the interrelationship between *vibrio cholerae*, phages and hosts, we propose a novel dynamical model with mutually coupled reaction-diffusion equations and ordinary differential equations. Here, we consider only the spatial dispersal behavior of the host because of the relatively large movement of the host and the relatively small movement of *vibrio cholerae* and phages in the environment. This results in the solution semiflow of the proposed model is not compact since these equation of *vibrio cholerae* and phages have no diffusion terms. The rest of this article is organized as follows. In Section 2, we present this model and prove the well-posedness. The basic reproduction number and the global asymptotic stability of the disease-free steady state are obtained in Sections 3 and 4, respectively. In Sections 5 and 6, we focus on the existence and stability of phage-free and phage-present endemic steady states for spatially heterogeneous or homogeneous cases, respectively. Some numerical simulations are performed to support theoretic results and conjectures in Section 7. A brief conclusions are found in Section 8.

2. Model formulation and well-posedness

Let Ω be a bounded domain with smooth boundary $\partial\Omega$ in $\mathbb{R}^n$, and $\frac{\partial}{\partial\nu}$ is the normal derivative along the outward ν to $\partial\Omega$. Based on the idea of compartment modeling, the host population at domain Ω is divided into three classes: susceptible, infected and recovered, and their quantities or density are denoted by $S(x,t)$, $I(x,t)$ and $R(x,t)$ for location x and time t, respectively. Further, $V(x,t)$ and $P(x,t)$ correspond to the concentration of *vibrio cholerae* and phages. From the interrelationships between *vibrio cholerae*, phages and hosts, the cholera model with horizontal and environmental transmission reads

$$
\begin{gathered}
\frac{\partial S(x,t)}{\partial t} = D_S\Delta S + \Lambda(x) - \alpha(x)SI - \frac{\beta(x)SV}{k(x)+V} - d(x)S, \quad x\in\Omega,\ t>0,\\
\frac{\partial I(x,t)}{\partial t} = D_I\Delta I + \alpha(x)SI + \frac{\beta(x)SV}{k(x)+V} - (\gamma(x)+d(x))I, \quad x\in\Omega,\ t>0,\\
\frac{\partial V(x,t)}{\partial t} = \eta(x)I + \mu(x)V - \mu_0(x)V - \frac{\xi(x)VP}{m(x)+V}, \quad x\in\Omega,\ t>0,\\
\frac{\partial P(x,t)}{\partial t} = \frac{\theta(x)\xi(x)VP}{m(x)+V} - \delta(x)P, \quad x\in\Omega,\ t>0,\\
S(x,0)=S_0(x),\ I(x,0)=I_0(x),\ V(x,0)=V_0(x), P(x,0)=P_0(x), \quad x\in\Omega,
\end{gathered}
\tag{2.1}
$$

with the Neumann boundary condition

$$
\frac{\partial S(x,t)}{\partial\nu} = \frac{\partial I(x,t)}{\partial\nu} = 0, \quad x\in\partial\Omega,\ t>0,
$$

and the equation of recovered class

$$
\begin{gathered}
\frac{\partial R(x,t)}{\partial t} = D_R\Delta R + \gamma(x)I - d(x)R, \quad x\in\Omega,\ t>0,\\
\frac{\partial R(x,t)}{\partial\nu} = 0, \quad x\in\partial\Omega,\ t>0.
\end{gathered}
$$

Here, the horizontal transmission described by the mass action $\alpha(x)SI$ and environmental transmission described by the saturation incidence $\frac{\beta(x)SV}{k(x)+V}$. The meaning of other model parameters is as follows: $D_S > 0$, $D_I > 0$ and $D_R > 0$ are diffusion coefficients of susceptible, infected and recovered hosts, respectively; $\Lambda(x)$ is the recruitment rate of susceptible hosts; $d(x)$, $\mu_0(x)$ are the natural death rates of susceptible, infected and recovered hosts, bacteria, respectively; $\delta(x)$ is the loss rate of phages; $\gamma(x)$ is the removal rate of infected hosts expect for natural death; $\alpha(x)$ is the horizontal transmission rate from infected hosts to susceptible hosts; $\beta(x)$ is the environmental transmission rate from bacteria to susceptible hosts; $k(x)$ is the half-saturation coefficient which depicts the concentration of bacteria that leads 50% chance of contracting *vibrio cholerae*; $\eta(x)$ is the shedding rate of bacteria from infected hosts; $\mu(x)$ is the self-growth rate of bacteria; $\xi(x)$ is the adsorption rate of phages; $\theta(x)$ is burst size which represents the concentration of phages produced per *vibrio cholerae*; $m(x)$ is the half-saturation coefficient.

Since the recovered $R(x,t)$ is decoupled from the other state variables, we only need to study the dynamical behaviors of model (2.1) for the full system. In addition, if spatial heterogeneity is ignored, then (2.1) degenerates to a homogeneous form

$$
\begin{gathered}
\frac{\partial S(x,t)}{\partial t} = D_S\Delta S + \Lambda - \alpha SI - \frac{\beta SV}{k+V} - dS, \quad x\in\Omega,\ t>0,\\
\frac{\partial I(x,t)}{\partial t} = D_I\Delta I + \alpha SI + \frac{\beta SV}{k+V} - (\gamma+d)I, \quad x\in\Omega,\ t>0,\\
\frac{\partial V(x,t)}{\partial t} = \eta I + \mu V - \mu_0 V - \frac{\xi VP}{m+V}, \quad x\in\Omega,\ t>0,\\
\frac{\partial P(x,t)}{\partial t} = \frac{\theta\xi VP}{m+V} - \delta P, \quad x\in\Omega,\ t>0,\\
S(x,0) = S_0(x), \quad I(x,0) = I_0(x), \quad x\in\Omega,\\
V(x,0) = V_0(x), \quad P(x,0) = P_0(x), \quad x\in\Omega.
\end{gathered}
\tag{2.2}
$$

Throughout this article, we make the following assumptions:

(H1) Functions $\Lambda(x)$, $\alpha(x)$, $\beta(x)$, $d(x)$, $\gamma(x)$, $\delta(x)$, $\eta(x)$, $\mu(x)$, $\mu_0(x)$, $\xi(x)$, $\theta(x)$, $\delta(x)$, $k(x)$, and $m(x)$ are continuous, uniformly bounded and strictly positive on domain $\bar\Omega$.

(H2) $\min_{x\in\bar\Omega}(\theta(x)\xi(x)-\delta(x)) > 0$ and $\bar\mu - \underline{\mu_0} < 0$, where $\bar\mu := \max_{x\in\bar\Omega}\mu(x)$ and $\underline{\mu_0} := \min_{x\in\bar\Omega}\mu_0(x)$.

Now, we consider well-posedness of (2.1). To do this, denote $\mathbb{X} := C(\bar\Omega,\mathbb{R}^4)$ be the Banach space with the supremum norm $\|\cdot\|_{\mathbb{X}}$, and $\mathbb{X}^+ := C(\bar\Omega,\mathbb{R}^4_+)$ indicates its positive condition. Let Γ be the Green function associated with the operator Δ, obeying the Neumann boundary condition. Denote $T_1(t)$, $T_2(t)$: $C(\bar\Omega,\mathbb{R}) \to C(\bar\Omega,\mathbb{R})$ by

$$
\begin{gathered}
(T_1(t)\phi)(\cdot) = e^{-d(\cdot)t}\int_\Omega \Gamma(D_S t,\cdot,y)\phi(y)\mathrm{d}y, \quad \phi\in C(\bar\Omega,\mathbb{R}),\ t>0,\\
(T_2(t)\phi)(\cdot) = e^{-(\gamma(\cdot)+d(\cdot))t}\int_\Omega \Gamma(D_I t,\cdot,y)\phi(y)\mathrm{d}y, \quad \phi\in C(\bar\Omega,\mathbb{R}),\ t>0,
\end{gathered}
$$

are the $\mathcal{C}_0$-semigroups of $D_S - d(\cdot)$ and $D_I - (\gamma(\cdot)+d(\cdot))$ with Neumann boundary condition, respectively. By [42, Corollary 7.2.3], it follows that $T_i(t) : C(\bar\Omega,\mathbb{R}) \to$

$C(\bar{\Omega},\mathbb{R})$, $t>0$ $(i=1,2)$ are compact and strong positive. Further, let

$$(T_3(t)\phi)(\cdot)=e^{-(\mu_0(\cdot)-\mu(\cdot))t}\phi(\cdot),\quad (T_4(t)\phi)(\cdot)=e^{-\delta(\cdot)t}\phi(\cdot),$$

for $\phi\in C(\bar{\Omega},\mathbb{R})$ and $t>0$. Thus, $\mathcal{T}(t):=(T_1(t),T_2(t),T_3(t),T_4(t)):\mathbb{X}\to\mathbb{X}$, $t\geq 0$ forms a $\mathcal{C}_0$-semigroup [36].

For $\phi=(\phi_1,\cdots,\phi_4)\in\mathbb{X}^+$ and $x\in\bar{\Omega}$, we define $\mathcal{F}:=(F_1,\cdots,F_4):\mathbb{X}^+\to\mathbb{X}$, where

$$F_1(\phi)(\cdot)=\Lambda(\cdot)-\alpha(\cdot)\phi_1\phi_2-\frac{\beta(\cdot)\phi_1\phi_3}{k(\cdot)+\phi_3},\quad F_2(\phi)(\cdot)=\alpha(\cdot)\phi_1\phi_2+\frac{\beta(\cdot)\phi_1\phi_3}{k(\cdot)+\phi_3},$$
$$F_3(\phi)(\cdot)=\eta(\cdot)\phi_2-\frac{\xi(\cdot)\phi_3\phi_4}{m(\cdot)+\phi_3},\quad F_4(\phi)(\cdot)=\frac{\theta(\cdot)\xi(\cdot)\phi_3\phi_4}{m(\cdot)+\phi_3}.$$

Hence (2.1) can be rewritten as

$$u(t)=\mathcal{T}(t)\phi+\int_0^t\mathcal{T}(t-s)\mathcal{F}(u(s))\mathrm{d}s,\ u(t)=(S,I,V,P).$$

Theorem 2.1. *For any initial value function $\phi=(\phi_1,\phi_2,\phi_3,\phi_4)\in\mathbb{X}^+$, model* (2.1) *admits a unique mild solution $u(\cdot,t;\phi):=(u_1(\cdot,t),u_2(\cdot,t),u_3(\cdot,t),u_4(\cdot,t))$ on $[0,\tau_{\max})$ with $u(\cdot,0;\phi)=\phi$, where $\tau_{\max}\leq\infty$. Furthermore, $u(\cdot,t;\phi)\in\mathbb{X}^+$ for $t\in[0,\tau_{\max})$ and $u(\cdot,t;\phi)$ is also a classical solution of* (2.1).

Proof. Note that $\mathcal{T}(t)$ corresponds to the linear part of (2.1), then, for any $\phi\in\mathbb{X}^+$ and $h\geq 0$, one can obtain

$$\phi+h\mathcal{F}(\phi)=\begin{pmatrix}\phi_1+h\big[\Lambda(\cdot)-\alpha(\cdot)\phi_1\phi_2-\frac{\beta(\cdot)\phi_1\phi_3}{k(\cdot)+\phi_3}\big]\\ \phi_2+h\big[\alpha(\cdot)\phi_1\phi_2+\beta(\cdot)\frac{\phi_1\phi_3}{k(\cdot)+\phi_3}\big]\\ \phi_3+h\big[\eta(\cdot)\phi_2-\frac{\xi(\cdot)\phi_3\phi_4}{m(\cdot)+\phi_3}\big]\\ \phi_4+h\frac{\theta(\cdot)\xi(\cdot)\phi_3\phi_4}{m(\cdot)+\phi_3}\end{pmatrix}\geq\begin{pmatrix}\phi_1\big[1-h\big(\bar{\alpha}\phi_2+\bar{\beta}\big)\big]\\ \phi_2\\ \phi_3\big[1-h\frac{\bar{\xi}\phi_4}{\underline{m}+\phi_3}\big]\\ \phi_4\end{pmatrix}.$$

Thus, the limit $\lim_{h\to 0^+}\frac{1}{h}\operatorname{dist}(\phi+h\mathcal{F}(\phi),\mathbb{X}^+)=0$ holds for any $\phi\in\mathbb{X}^+$. According to [42, Theorem 7.3.1], model (2.1) has a unique positive solution on $[0,\tau_{\max})$, where $0<\tau_{\max}\leq\infty$. This proof is complete. $\square$

Now we consider an auxiliary system, for $t>0$,

$$\begin{gathered}\frac{\partial W(x,t)}{\partial t}=D_W\Delta W+\Lambda(x)-d(x)W,\ W(x,0)=W_0(x)\not\equiv 0,\ x\in\Omega,\\ \frac{\partial W(x,t)}{\partial\nu}=0,\ x\in\partial\Omega,\end{gathered}\tag{2.3}$$

where, $\Lambda(x)>0$ and $d(x)>0$ for $x\in\bar{\Omega}$. Directly from [21, Lemma 1], we have the following statement.

Lemma 2.2. *System* (2.3) *has a globally asymptotically stable positive steady state $\bar{U}(x)$ in $C(\bar{\Omega},\mathbb{R}_+)$.*

Using Theorem 2.1 and Lemma 2.2, the global existence for (2.1) is given by the following theorem.

Theorem 2.3. *For any $\phi\in\mathbb{X}^+$, model* (2.1) *has a unique solution $u(\cdot,t;\phi)$ on the interval $[0,\infty)$ satisfying $u(\cdot,0;\phi)=\phi$. The semiflow generated by model* (2.1) *is denoted by $\Phi(t):\mathbb{X}^+\to\mathbb{X}^+$ and satisfies*

$$\Phi(t)\phi=u(\cdot,t;\phi)=(S(\cdot,t;\phi),I(\cdot,t;\phi),V(\cdot,t;\phi),P(\cdot,t;\phi)),\quad x\in\bar{\Omega},\ t\geq 0.$$

Furthermore, $\Phi(t)$ is point dissipative.

Proof. On the uniform boundedness of $S(\cdot,t)$, this yields from the first equation of (2.1) that

$$\frac{\partial S}{\partial t} = D_S\Delta S(\cdot,t) + \Lambda(\cdot) - d(\cdot)S(\cdot,t), \quad x \in \Omega,\ t > 0,$$

with $\frac{\partial S(\cdot,t)}{\partial \nu} = 0$, $x \in \partial\Omega$, $t > 0$. By Lemma 2.2 and the comparison principle, one directly obtain

$$\limsup_{t\to\infty} S(\cdot,t) \le \bar{U}(\cdot), \quad \text{uniformly for } x \in \bar{\Omega}. \tag{2.4}$$

Hence,

$$\limsup_{t\to\infty} \|S(\cdot,t)\| \le \|\bar{U}(\cdot)\| := \mathbb{M}_0. \tag{2.5}$$

From inequality (2.5), $S(\cdot,t)$ is ultimately bounded.

Adding the equations of S and I in (2.1), and integrating over Ω, we have

$$\frac{\partial}{\partial t}\int_\Omega (S(\cdot,t) + I(\cdot,t))\mathrm{d}x \le \bar{\Lambda}|\Omega| - \underline{d}\int_\Omega (S(\cdot,t) + I(\cdot,t))\mathrm{d}x.$$

where $|\Omega|$ is the measurement of Ω. Thus, $\limsup_{t\to\infty}(\|S(\cdot,t)\|_1 + \|I(\cdot,t)\|_1) \le \frac{\bar{\Lambda}|\Omega|}{\underline{d}} := M_{11}$. Similarly, we multiply P with $\frac{1}{\theta(\cdot)}$, then add V and $\frac{P}{\theta(\cdot)}$ equation and integrate over Ω,

$$\frac{\partial}{\partial t}\int_\Omega \Big(V(\cdot,t) + \frac{P(\cdot,t)}{\theta(\cdot)}\Big)\mathrm{d}x \le \bar{\eta}M_{11} - K\int_\Omega \Big(V(\cdot,t) + \frac{P(\cdot,t)}{\theta(\cdot)}\Big)\mathrm{d}x,$$

where $K = \min\left\{\underline{\delta}, \underline{\mu_0} - \bar{\mu}\right\} > 0$. Thus,

$$\limsup_{t\to\infty}\Big(\|(V(\cdot,t)\|_1 + \|\frac{P(\cdot,t)}{\theta(\cdot)}\|_1\Big) \le \frac{\bar{\eta}M_{11}}{K} := M_{12},$$

From

$$\frac{1}{\bar{\bar{\theta}}}\int_\Omega P(\cdot,t)\mathrm{d}x \le \int_\Omega \frac{P(\cdot,t)}{\theta(\cdot)}\mathrm{d}x,$$

one can obtain $\limsup_{t\to\infty} \|(P(\cdot,t)\|_1 \le \bar{\theta}M_{12} := M_{13}$. Summarizing, there is a positive constant $\mathbb{M}_1$ such that

$$\limsup_{t\to\infty}(\|S(\cdot,t)\|_1 + \|I(\cdot,t)\|_1 + \|(V(\cdot,t)\|_1 + \|(P(\cdot,t)\|_1) \le \mathbb{M}_1. \tag{2.6}$$

Next, we turn to the uniform boundedness of $I(\cdot,t), V(\cdot,t)$ and $P(\cdot,t)$. It claims that $I(\cdot,t)$ satisfies the L^{2^k} bounded estimate, i.e., for $k \ge 0$, there is a positive constant $\mathbb{M}_{2^k}$ such that

$$\limsup_{t\to\infty} \|I(\cdot,t)\|_{2^k} \le \mathbb{M}_{2^k}. \tag{2.7}$$

The proof of inequality (2.7) follows by mathematical induction. Obviously, from (2.6) it follows that $k = 0$ holds. Assume that (2.7) holds for $k-1$, i.e., there exists a constant $\mathbb{M}_{2^{k-1}} > 0$ such that

$$\limsup_{t\to\infty} \|I(\cdot,t)\|_{2^{k-1}} \le \mathbb{M}_{2^{k-1}}. \tag{2.8}$$

We multiply I equation with I^{2k-1} and integrate it over Ω,

$$\begin{aligned}\frac{1}{2^k}\frac{\partial}{\partial t}\int_\Omega I^{2^k}\mathrm{d}x \le D_I\int_\Omega I^{2^k-1}\Delta I\mathrm{d}x + \int_\Omega \alpha(\cdot)SI^{2^k}\mathrm{d}x \\ + \int_\Omega \beta(\cdot)SI^{2^k-1}\mathrm{d}x - \int_\Omega(\gamma(\cdot)+d(\cdot))I^{2^k}\mathrm{d}x.\end{aligned} \tag{2.9}$$

Recall that

$$\begin{aligned}D_I\int_\Omega I^{2^k-1}\Delta I\mathrm{d}x &= -D_I\int_\Omega \nabla I\cdot\nabla I^{2^k-1}\mathrm{d}x \\ &= -(2^k-1)D_I\int_\Omega(\nabla I\cdot\nabla I)I^{2^k-2}\mathrm{d}x \\ &= -\frac{2^k-1}{2^{2k-2}}D_I\int_\Omega |\nabla I^{2^{k-1}}|^2\mathrm{d}x.\end{aligned}$$

Hence, (2.9) becomes

$$\begin{aligned}\frac{1}{2^k}\frac{\partial}{\partial t}\int_\Omega I^{2^k}\mathrm{d}x \le -Q_k\int_\Omega |\nabla I^{2^{k-1}}|^2\mathrm{d}x + \int_\Omega \alpha(\cdot)SI^{2^k}\mathrm{d}x \\ + \int_\Omega \beta(\cdot)SI^{2^k-1}\mathrm{d}x - \int_\Omega(\gamma(\cdot)+d(\cdot))I^{2^k}\mathrm{d}x,\end{aligned} \tag{2.10}$$

where $Q_k = \frac{2^k-1}{2^{2k-2}}D_I$. By (2.5), there exists $t_0 > 0$ such that for $t \ge t_0$,

$$\int_\Omega \alpha(\cdot)SI^{2^k}\mathrm{d}x \le \bar{\alpha}(\mathbb{M}_0+1)\int_\Omega I^{2^k}\mathrm{d}x \quad\text{and}\quad \int_\Omega \beta(\cdot)SI^{2^k-1}\mathrm{d}x \le \bar{\beta}\int_\Omega SI^{2^k-1}\mathrm{d}x.$$

Making the use of Young's inequality: $ab \le \epsilon a^p + C_\epsilon b^q$, where p, $q > 1$, $\frac{1}{p}+\frac{1}{q} = 1$, a, b, $\varepsilon > 0$, and $C_\epsilon = \epsilon^{-\frac{q}{p}}$. To estimate $\int_\Omega SI^{2^k-1}\mathrm{d}x$, we set $\epsilon_0 = (\mathbb{M}_0+1)^{-2^k}$, $p = 2^k$ and $q = \frac{2^k}{2^k-1}$, then

$$\begin{aligned}\int_\Omega SI^{2^k-1}\mathrm{d}x &\le \epsilon_0\int_\Omega S^{2^k}\mathrm{d}x + C_{\epsilon_0}\int_\Omega I^{2^k}\mathrm{d}x \\ &\le \epsilon_0(\mathbb{M}_0+1)^{2^k}|\Omega| + C_{\epsilon_0}\int_\Omega I^{2^k}\mathrm{d}x \le |\Omega| + C_{\epsilon_0}\int_\Omega I^{2^k}\mathrm{d}x,\ t \ge t_0,\end{aligned}$$

where $C_{\epsilon_0} = \epsilon_0^{-(2^k-1)^{-1}}$. Thus, it follows from (2.10) that

$$\frac{1}{2^k}\frac{\partial}{\partial t}\int_\Omega I^{2^k}\mathrm{d}x \le -Q_k\int_\Omega |\nabla I^{2^{k-1}}|^2\mathrm{d}x + B_k\int_\Omega I^{2^k}\mathrm{d}x + N_k, \quad t \ge t_0, \tag{2.11}$$

where $B_k = \bar{\alpha}(\mathbb{M}_0+1) + \bar{\beta}C_{\epsilon_0}$ and $N_k = \bar{\beta}|\Omega|$.

By utilizing the statement: for any positive constant ϵ, there is $A_\epsilon > 0$ such that

$$\|\zeta\|_2^2 \le \epsilon\|\nabla\zeta\|_2^2 + A_\epsilon\|\zeta\|_1^2,\ \zeta \in W^{1,2}(\Omega),$$

we set $\zeta = I^{2^{k-1}}$ and $\epsilon_1 = \frac{Q_k}{2B_k}$, to obtain

$$-Q_k\int_\Omega |\nabla I^{2^{k-1}}|^2\mathrm{d}x \le -2B_k\int_\Omega I^{2^k}\mathrm{d}x + 2B_kA_{\epsilon_1}\Big(\int_\Omega I^{2^{k-1}}\mathrm{d}x\Big)^2.$$

Thus, (2.11) yields

$$\frac{1}{2^k}\frac{\partial}{\partial t}\int_\Omega I^{2^k}\mathrm{d}x \le -B_k\int_\Omega I^{2^k}\mathrm{d}x + 2B_kA_{\epsilon_1}\Big(\int_\Omega I^{2^{k-1}}\mathrm{d}x\Big)^2 + N_k, \quad t \ge t_0.$$

According to (2.8), we have $\limsup_{t\to\infty}\int_\Omega I^{2^{k-1}}\mathrm{d}x \le \mathbb{M}_{2^{k-1}}^{2^{k-1}}$. Hence,

$$\limsup_{t\to\infty}\|I(\cdot,t)\|_{2^k} \le \mathbb{M}_{2^k} := \sqrt[2^k]{\frac{2B_kA_{\epsilon_1}\mathbb{M}_{2^{k-1}}^{2^k}+N_k}{B_k}}.$$

Applying the continuous embedding $L^q(\Omega)\subset L^p(\Omega)$, $1\le p\le q$, one can obtain $\limsup_{t\to\infty}\|I(\cdot,t)\|_p \le \mathbb{M}_p$, for $\mathbb{M}_p>0$. Similar to the [54, Lemma 2.4], we can obtain $\limsup_{t\to\infty}\|I(\cdot,t)\|\le\mathbb{M}_\infty$. Same way like talked about $\limsup_{t\to\infty}\|V(\cdot,t)\|_1$ and $\limsup_{t\to\infty}\|P(\cdot,t)\|_1$, we have

$$\limsup_{t\to\infty}\|V(\cdot,t)\|\le\mathbb{M}'_\infty=\frac{\bar\eta\mathbb{M}_\infty}{K}\quad\text{and}\quad\limsup_{t\to\infty}\|P(\cdot,t)\|\le\mathbb{M}''_\infty=\bar\theta\mathbb{M}'_\infty.$$

Therefore, the solutions of (2.1) is ultimately bounded. That is, the solution semiflow $\Phi(t)\phi=u(\cdot,t;\phi)=(S(\cdot,t;\phi),I(\cdot,t;\phi),V(\cdot,t;\phi),P(\cdot,t;\phi))$, $x\in\bar\Omega$, $t\ge0$ generated by (2.1) is point dissipative. This proof is complete. □

For $(S,I,V,P)\in\mathbb{X}^+$, let

$$\mathcal{D}=\big\{(S,I,V,P): S(x,t)\le\mathbb{M}_0, I(x,t)\le\mathbb{M}_\infty, V(x,t)\le\mathbb{M}'_\infty, P(x,t)\le\mathbb{M}''_\infty\big\},$$

then $\Phi(t)\phi\in\mathcal{D}$, $t\ge t^*$, $\phi\in\mathbb{X}^+$, for some $t^*\ge0$. Moreover, similar arguments as in [8, Theorem 2.1], we know that $\mathcal{D}$ is positively invariant for the semiflow $\Phi(t)$ and for any bounded set $\mathbb{B}\subset\mathbb{X}^+$, there exists a $t^{**}\ge0$ such that $\Phi(t)\phi\in\mathcal{D}$, $t\ge t^{**}$, $\phi\in\mathcal{D}$.

Because the V and P in equation (2.1) without the diffusion terms, the solution semiflow $\Phi(t)$ is not compact. To solve this problem, one shall prove that $\Phi(t)$ is asymptotically smooth by introducing the Kuratowski measure κ of noncompactness, which is determined as

$$\kappa(\mathbb{B}):=\inf\{r:\mathbb{B}\text{ has a finite cover of diameter less than }r\},$$

for any bounded $\mathbb{B}\subset\mathbb{X}^+$. It is not hard to see that $\mathbb{B}$ is precompact if and only if $\kappa(\mathbb{B})=0$.

For the sake of convenience, we let $\mathbf{x}:=(S,I)$, $\mathbf{y}:=(V,P)$ and define

$$\mathbf{g}(x,t,\mathbf{x},\mathbf{y}):=(g_1(x,t,I,V,P),g_2(x,t,I,V,P)),$$

where $g_1(x,t,I,V,P)=\eta(x)I+\mu(x)V-\mu_0(x)V-\frac{\xi(x)VP}{m(x)+V}$ and $g_2(x,t,I,V,P)=\frac{\theta(x)\xi(x)VP}{m(x)+V}$. Then the Jacobian of $\mathbf{g}(x,t,\mathbf{x},\mathbf{y})$ with respect to $\mathbf{y}$ is calculated as follows

$$\frac{\partial\mathbf{g}(x,t,\mathbf{x},\mathbf{y})}{\partial\mathbf{y}}=\begin{pmatrix}\mu(x)-\mu_0(x)-\frac{\xi(x)m(x)P}{(m(x)+V)^2} & -\frac{\xi(x)V}{m(x)+V}\\ \frac{\theta(x)\xi(x)m(x)P}{(m(x)+V)^2} & \frac{\theta(x)\xi(x)V}{m(x)+V}-\delta(x)\end{pmatrix}.$$

The following lemma is to claim $\Phi(t)$ satisfies κ-contraction condition.

Lemma 2.4. *If there exists a $r>0$ such that*

$$w^T\frac{\partial\mathbf{g}(x,t,\mathbf{x},\mathbf{y})}{\partial\mathbf{y}}w\le -rw^Tw,\quad\forall w\in\mathbb{R}^2,\ x\in\bar\Omega,\ (V,P)\in\mathcal{D},\tag{2.12}$$

then $\Phi(t)$ is κ-contracting, i.e., $\lim_{t\to\infty}\kappa(\Phi(t)\mathbb{B})=0$ for any bounded set $\mathbb{B}\subset\mathbb{X}^+$.

Proof. Similar to the proof of [17, Lemma 4.1], we can prove that $\Phi(t)$ is asymptotically compact on $\mathbb{B}$ in the sense that for any sequences $\phi_n \in \mathbb{B}$ and $t_n \to \infty$, there exists subsequences ϕ_{n_k} and $t_{n_k} \to \infty$ such that $\Phi(t_{n_k})\phi_{n_k}$ converges in $\mathbb{X}$ as $k \to \infty$.

Next, we consider the omega limit set of $\mathbb{B}$ for solution semiflow $\Phi(t)$ on $\mathbb{X}^+$, which is defined as

$$|\omega(\mathbb{B}) = \big\{\phi \in \mathbb{X}^+ : \lim_{k\to\infty} \Phi(t_{n_k})\phi_{n_k} = \phi \text{ for some sequences } \phi_{n_k} \in \mathbb{B}\big\}.$$

By [39, Lemma 23.1(2)], we know that $\omega(\mathbb{B})$ is nonempty, compact, invariant set in $\mathbb{X}^+$, and $\omega(\mathbb{B})$ attracts $\mathbb{B}$. In view of [23, Lemma 2.1(b)], we have

$$\kappa(\Phi(t)\mathbb{B}) \le \kappa(\omega(\mathbb{B})) + \operatorname{dist}(\Phi(t)\mathbb{B}, \omega(\mathbb{B})) = \operatorname{dist}(\Phi(t)\mathbb{B}, \omega(\mathbb{B})) \to 0 \quad \text{as } t \to \infty,$$

where $\operatorname{dist}(\Phi(t)\mathbb{B}, \omega(\mathbb{B}))$ is described as the distance from $\Phi(t)\mathbb{B}$ to $\omega(\mathbb{B})$. Thus, $\Phi(t)$ is κ-contracting. The proof is complete. $\square$

Remark 2.5. A sufficient condition for (2.12) to hold is

$$\frac{1}{4}\frac{\bar{\theta}^2\bar{\xi}^2{\mathbb{M}''_\infty}^2 + \bar{\xi}^2{\mathbb{M}'_\infty}^2}{\underline{m}^2(\underline{\mu_0} - \bar{\mu})} + \frac{\bar{\theta}\bar{\xi}\mathbb{M}'_\infty}{\underline{m}} - \bar{\delta} < 0.$$

Theorem 2.6. *If* (2.12) *holds, then the solution semiflow* $\Phi(t)$ *of* (2.1) *possesses a global attractor* $\mathcal{A}$ *on* $\mathbb{X}^+$.

Proof. From Theorem 2.1 and Theorem 2.3, it follows that the solution of (2.1) is the global and unique. Further, $\Phi(t)$ is point dissipative and κ-contracting on $\mathbb{X}^+$ by Theorem 2.3 and Lemma 2.4. Hence, from [14, Theorem 2.4.6], $\Phi(t)$ exists and attracts any bounded set in $\mathbb{X}^+$. This proof is complete. $\square$

In particular, on the well-posedness of (2.2), the following corollary is obvious from Theorems 2.3 and 2.6.

Corollary 2.7. *For model* (2.2)*, the following results hold.*

(i) *For any function* $\phi \in \mathbb{X}^+$*, model* (2.2) *admits a unique, nonnegative solution* $\widetilde{u}(\cdot, t; \phi)$ *on* $[0, \infty)$ *satisfying* $\widetilde{u}(\cdot, 0; \phi) = \phi$.

(ii) *The solution semiflow* $\widetilde{\Phi}(t) : \mathbb{X}^+ \to \mathbb{X}^+$ *generated by* (2.2) *is denoted by* $\widetilde{\Phi}(t)\phi = \widetilde{u}(\cdot, t; \phi) = (S(\cdot, t; \phi), I(\cdot, t; \phi), V(\cdot, t; \phi), P(\cdot, t; \phi))$, $x \in \bar{\Omega}, t \ge 0$. *Furthermore,* $\widetilde{\Phi}(t)$ *is point dissipative.*

(iii) *If* (2.12) *holds, then the solution semiflow* $\widetilde{\Phi}(t)$ *of* (2.2) *exists a global attractor* $\widetilde{\mathcal{A}}$ *on* $\mathbb{X}^+$.

3. Basic reproduction number

For model (2.1) there exists a unique disease-free steady state $\mathcal{E}_0 = (\bar{U}(x), 0, 0, 0)$. Now, we calculate the basic reproduction number $\mathcal{R}_0$ by the results in [53, Section 3], which describes the average number of the secondary infections while introducing a single infection in a completely susceptible host population.

Linearizing (2.1) at the steady state $\mathcal{E}_0$, one can obtain obtain a linear system, for $t > 0$,

$$\begin{gathered}
\frac{\partial S(x,t)}{\partial t} = D_S\Delta S + \Lambda(x) - \alpha(x)\bar{U}(x)I - \beta(x)\frac{\bar{U}(x)}{k(x)}V - d(x)S, \quad x \in \Omega,\\
\frac{\partial I(x,t)}{\partial t} = D_I\Delta I + \alpha(x)\bar{U}(x)I + \beta(x)\frac{\bar{U}(x)}{k(x)}V - (\gamma(x) + d(x))I, \quad x \in \Omega,\\
\frac{\partial V(x,t)}{\partial t} = \eta(x)I + \mu(x)V - \mu_0(x)V, \quad x \in \Omega,\\
\frac{\partial P(x,t)}{\partial t} = -\delta(x)P, \quad x \in \Omega,\\
\frac{\partial S(x,t)}{\partial \nu} = \frac{\partial I(x,t)}{\partial \nu} = 0, \quad x \in \partial\Omega.
\end{gathered} \tag{3.1}$$

Note that $S(x,t)$ and $P(x,t)$ are decoupled from the equations of $I(x,t)$ and $V(x,t)$, thus we only discuss the subsystem of (3.1), for $t > 0$,

$$\begin{gathered}
\frac{\partial I(x,t)}{\partial t} = D_I\Delta I + \alpha(x)\bar{U}(x)I + \beta(x)\frac{\bar{U}(x)}{k(x)}V - (\gamma(x) + d(x))I, \quad x \in \Omega,\\
\frac{\partial V(x,t)}{\partial t} = \eta(x)I + \mu(x)V - \mu_0(x)V, \quad x \in \Omega,\\
\frac{\partial I(x,t)}{\partial \nu} = 0, \quad x \in \partial\Omega.
\end{gathered} \tag{3.2}$$

Substituting $(I(x,t), V(x,t)) := e^{\lambda t}(\phi_2(x), \phi_3(x))$ into (3.2), we can obtain the eigenvalue problem

$$\begin{gathered}
\lambda\phi_2(x) = D_I\Delta\phi_2(x) + (\alpha(x)\bar{U}(x) - \gamma(x) - d(x))\phi_2(x) + \beta(x)\frac{\bar{U}(x)}{k(x)}\phi_3(x),\\
\lambda\phi_3(x) = \eta(x)\phi_2(x) + \mu(x)\phi_3(x) - \mu_0(x)\phi_3(x),
\end{gathered} \tag{3.3}$$

for $x \in \Omega$ and $\frac{\partial \phi_2(x)}{\partial \nu} = 0$, $x \in \partial\Omega$. Set

$$\begin{aligned}
\mathcal{B} &= \begin{pmatrix} D_I\Delta + \alpha(x)\bar{U}(x) - (\gamma(x) + d(x)) & \beta(x)\frac{\bar{U}(x)}{k(x)} \\ \eta(x) & \mu(x) - \mu_0(x) \end{pmatrix}\\
&= \begin{pmatrix} D_I\Delta - (\gamma(x) + d(x)) & 0 \\ \eta(x) & \mu(x) - \mu_0(x) \end{pmatrix} + \begin{pmatrix} \alpha(x)\bar{U}(x) & \beta(x)\frac{\bar{U}(x)}{k(x)} \\ 0 & 0 \end{pmatrix}\\
&:= B + F,
\end{aligned}$$

and denote $\Pi(t)$ by the solution semigroup of (3.2). Since system (3.2) is cooperative, $\Pi(t)$ is a positive $\mathcal{C}_0$-semigroup generated by $\mathcal{B}$. Let $\widetilde{\Pi}(t)$ is the semigroup generated by B, then $\mathcal{B}$ and B are closed and resolvent positive operators from the Theorem 3.12 in [45]. To derive the basic reproduction number of (2.1), suppose that the bacteria is invaded at time $t = 0$, and the initial distribution of infected hosts described by $\phi(x) = (\phi_2(x), \phi_3(x))^T$. Therefore, as times evolves, the distribution of new infected hosts is $F(x)\widetilde{\Pi}(t)\phi(x)$ at time t. Hence, the total distribution of new infected hosts is $\int_0^\infty F(x)\widetilde{\Pi}(t)\phi(x)\mathrm{d}t$. Define the operator $\mathcal{L}$ as

$$\mathcal{L}(\phi)(x) := \int_0^\infty F(x)\widetilde{\Pi}(t)\phi(x)\mathrm{d}t = F(x)\int_0^\infty \widetilde{\Pi}(t)\phi(x)\mathrm{d}t.$$

Then $\mathcal{L}$ is a continuous and positive operator of (2.1), which maps the initial infected hosts distribution $\phi(x)$ to the total new infected hosts distribution during the average illness period.

Based on [53], the spectral radius of $\mathcal{L}$ as the basic reproduction number of (2.1), that is,

$$\mathcal{R}_0 := r(\mathcal{L}). \tag{3.4}$$

Furthermore, by [53, Theorem 3.1(i)], the following lemma is valid.

Lemma 3.1. *$\mathcal{R}_0 - 1$ has the same sign as $s(\mathcal{B})$, where $s(\mathcal{B}) = \sup\{Re\lambda, \lambda \in \sigma(\mathcal{B})\}$ is the spectral bound of $\mathcal{B}$.*

For the sake of convenience in discussing, we verify that $\mathcal{R}_0$ has the relationship with the important indicators $\widetilde{\lambda}_0$ and τ^0.

Lemma 3.2. *Let $\mathcal{R}_0$ be defined by (3.4). Then*

(i) *$\mathcal{R}_0 = \frac{1}{\widetilde{\lambda}_0}$, where $\widetilde{\lambda}_0$ is the principal eigenvalue of*

$$D_I\Delta\varphi - (\gamma(x) + d(x))\varphi + \big[\alpha(x)\bar{U}(x) + \frac{\beta(x)\bar{U}(x)\eta(x)}{k(x)(\mu_0(x) - \mu(x))}\big]\widetilde{\lambda}\varphi = 0, \quad x \in \Omega,$$

with $\frac{\partial\varphi}{\partial\nu} = 0$, $x \in \partial\Omega$.

(ii) *$\mathcal{R}_0 - 1$ and $s(\mathcal{B})$ has the same sign with τ^0, where τ^0 is the principal eigenvalue of*

$$D_I\Delta\psi + \big[\alpha(x)\bar{U}(x) + \frac{\beta(x)\bar{U}(x)\eta(x)}{k(x)(\mu_0(x) - \mu(x))} - \gamma(x) - d(x)\big]\psi = \tau\psi, \quad x \in \Omega, \tag{3.5}$$

with $\frac{\partial\psi}{\partial\nu} = 0$, $x \in \partial\Omega$.

Proof. Let us first focus on conclusion (i). To use the result in [53, Theorem 3.3], we define

$$F := \begin{pmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{pmatrix}, \quad V := \begin{pmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{pmatrix},$$

where, $F_{21} = F_{22} = V_{12} = 0$, $F_{11} = \alpha(x)\bar{U}(x)$, $F_{12} = \beta(x)\frac{\bar{U}(x)}{k(x)}$, $V_{11} = \gamma(x) + d(x)$, $V_{21} = -\eta(x)$, and $V_{22} := \mu_0(x) - \mu(x)$. It then yields that $\mathcal{R}_0 = r(-BF^{-1}) = r(-B_1^{-1}F_2)$ due to $F_{21} = 0$, $F_{22} = 0$. Here,

$$B_1 = D_I\Delta - \big(V_{11} - V_{12}V_{22}^{-1}V_{21}\big) = D_I\Delta - (\gamma(x) + d(x))$$

and

$$F_2 = F_{11} - F_{12}V_{22}^{-1}V_{21} = \alpha(x)\bar{U}(x) + \frac{\beta(x)\bar{U}(x)\eta(x)}{k(x)(\mu_0(x) - \mu(x))}.$$

Then, for any $\varphi \in C(\bar{\Omega}, \mathbb{R}^2)$, one can obtain

$$-B_1^{-1}F_2\varphi = -[D_I\Delta - (\gamma(x) + d(x))]^{-1}\big[\alpha(x)\bar{U}(x) + \frac{\beta(x)\bar{U}(x)\eta(x)}{k(x)(\mu_0(x) - \mu(x))}\big]\varphi.$$

Therefore, $\mathcal{R}_0$ satisfies

$$-[D_I\Delta - (\gamma(x) + d(x))]^{-1}\big[\alpha(x)\bar{U}(x) + \frac{\beta(x)\bar{U}(x)\eta(x)}{k(x)(\mu_0(x) - \mu(x))}\big]\varphi = \mathcal{R}_0\varphi,$$

that is, for $\varphi \in C(\bar{\Omega}, \mathbb{R}^2)$,

$$D_I\Delta\varphi - (\gamma(x) + d(x))\varphi + \big[\alpha(x)\bar{U}(x) + \frac{\beta(x)\bar{U}(x)\eta(x)}{k(x)(\mu_0(x) - \mu(x))}\big]\frac{1}{\mathcal{R}_0}\varphi = 0. \tag{3.6}$$

The proof of assertion (i) is complete.

Next, we focus on condition (ii). The eigenvalue problem (3.5) has a principal eigenvalue τ^0 with positive eigenfunction $\psi^*(x)$ on $\bar{\Omega}$, namely, for $x \in \Omega$,

$$D_I \Delta\psi^* - (\gamma(x) + d(x))\psi^* + \big[\alpha(x)\bar{U}(x) + \frac{\beta(x)\bar{U}(x)\eta(x)}{k(x)(\mu_0(x) - \mu(x))}\big]\psi^* = \tau^0\psi^*, \quad (3.7)$$

with $\frac{\partial\psi^*}{\partial\nu} = 0$, $x \in \partial\Omega$. Multiplying (3.7) by φ and (3.6) by ψ^*, integrate by parts on Ω and then subtract the equation, we obtain

$$\big(1 - \frac{1}{\mathcal{R}_0}\big) \int_\Omega \big[\alpha(x)\bar{U}(x) + \frac{\beta(x)\bar{U}(x)\eta(x)}{k(x)(\mu_0(x) - \mu(x))}\big]\varphi\psi^* \mathrm{d}x = \tau^0 \int_\Omega \varphi\psi^* \mathrm{d}x.$$

Since $\int_\Omega \Big[\alpha(x)\bar{U}(x) + \frac{\beta(x)\bar{U}(x)\eta(x)}{k(x)(\mu_0(x)-\mu(x))}\Big]\varphi\psi^*\mathrm{d}x$ and $\int_\Omega \varphi\psi^*\mathrm{d}x$ are positive, it follows that $(1 - \frac{1}{\mathcal{R}_0})$ and τ^0 have the same sign. With Lemma 3.1, the proof of assertion (ii) is completed. $\square$

From conclusion (i) in Lemma 3.2, $\mathcal{R}_0$ has the variational formula

$$\mathcal{R}_0 = \frac{1}{\widetilde{\lambda}_0} = \sup_{\varphi \in H^1(\Omega), \varphi \neq 0} \Big\{ \frac{\int_\Omega [\alpha(x)\bar{U}(x) + \frac{\beta(x)\bar{U}(x)\eta(x)}{k(x)(\mu_0(x)-\mu(x))}]\varphi^2 \mathrm{d}x}{\int_\Omega [D_I|\nabla\varphi|^2 + (\gamma(x) + d(x))\varphi^2]\mathrm{d}x} \Big\}.$$

Remark 3.3. For model (2.2), it is easily to check that there is the disease-free steady state $\widetilde{\mathcal{E}}_0 = (\bar{U}, 0, 0, 0)$, where $\bar{U} = \frac{\Lambda}{d}$, and the basic reproduction number is

$$\widetilde{\mathcal{R}}_0 = \frac{\alpha\frac{\Lambda}{d} + \frac{\beta\frac{\Lambda}{d}\eta}{k(\mu_0 - \mu)}}{\gamma + d} = \frac{\alpha\Lambda}{d(\gamma + d)} + \frac{\beta\Lambda\eta}{kd(\mu_0 - \mu)(\gamma + d)}.$$

Note that because ofthe second equation in (3.2) without diffusion term, $\Pi(t)$ is not compact. The following results shows the existence of the principal eigenvalue for (3.3).

Lemma 3.4. *If $\mathcal{R}_0 \geq 1$, then the principal eigenvalue of* (3.3) *is $s(\mathcal{B})$ which associates with a strongly positive eigenfunction.*

Proof. From (3.2), one can obtain that

$$I(x, t; \phi) = T_2(t)\phi_2 + \int_0^t T_2(t - s)\big[\alpha(x)\bar{U}(x)I + \beta(x)\frac{\bar{U}(x)}{k(x)}V\big]\mathrm{d}s,$$
$$V(x, t; \phi) = T_3(t)\phi_3 + \int_0^t T_3(t - s)\eta(x)I(x, s; \phi)\mathrm{d}s,$$

for any $\phi = (\phi_2, \phi_3) \in C(\bar{\Omega}, \mathbb{R}^2_+)$. We decompose $\Pi(t)$ as $\Pi(t) = \Pi_2(t) + \Pi_3(t)$, where

$$\Pi_2(t)\phi = (0, T_3(t)\phi_3) \quad \text{and} \quad \Pi_3(t)\phi = \Big(I(x, t; \phi), \int_0^t T_3(t - s)\eta(x)I(x, s; \phi)\mathrm{d}s\Big).$$

By [54, Lemma 2.5 in], $\Pi_3(t)$ is compact. Further,

$$\|\Pi_2(t)\| = \sup_{\phi \in C(\bar{\Omega}, \mathbb{R}^2_+), \phi \neq 0} \frac{\|T_3(t)\phi_3\|}{\|\phi\|} \leq e^{-Kt} \sup_{\phi \in C(\bar{\Omega}, \mathbb{R}^2_+), \phi \neq 0} \frac{\|\phi_3\|}{\|\phi\|} \leq e^{-Kt}.$$

Hence, for any bound $\mathbb{S} \in C(\bar{\Omega}, \mathbb{R}^2_+)$,

$$\kappa(\Pi(t)\mathbb{S}) \leq \kappa(\Pi_1(t)\mathbb{S}) + \kappa(\Pi(t)_2\mathbb{S}) \leq 0 + \|\Pi_2(t)\|\kappa(\mathbb{S}) \leq e^{-Kt}\kappa(\mathbb{S}).$$

Consequently, $\Pi(t)$ is κ-contracting on $C(\bar{\Omega}, \mathbb{R}_+^2)$. Further, the essential growth bound $\omega_{ess}(\Pi(t)) \leq -K$ and the essential spectral radius $r_e(\Pi(t)) \leq e^{-Kt} < 1$ for $t > 0$. Obviously, $\omega(\Pi(t))$, the exponential growth bound ($\omega(\Pi(t)) := \lim_{t\to\infty} ln\|\Pi(t)\|/t$ such that $\|\Pi(t)\| \leq Me^{\omega(\Pi(t))t}$, for some $M > 0$) satisfies $\omega(\Pi(t)) = \max\{s(\mathcal{B}), \omega_{ess}(\Pi(t))\}$. With the help of Lemma 3.1, $\mathcal{R}_0 \geq 1$ implies that $s(\mathcal{B}) \geq 0$, then the spectral radius of $\Pi(t)$, $r(\Pi(t)) = e^{s(\mathcal{B})t} \geq 1,\ t > 0$. Hence, $r_e(\Pi(t)) < r(\Pi(t))$. Using the Krein-Rutman Theorem (see, [34, Lemma 2.2]), we complete the proof. □

Lemma 3.5. *Suppose that $\mu(x) \equiv \mu$ and $\mu_0(x) \equiv \mu_0$, then the principal eigenvalue of* (3.3) *is $s(\mathcal{B})$.*

Proof. Let

$$L_\lambda\varphi = D_I\Delta\varphi + \alpha(x)\bar{U}(x)\varphi - (\gamma(x) + d(x))\varphi + \frac{\beta(x)\bar{U}(x)\eta(x)}{k(x)[\lambda + (\mu_0 - \mu)]}\varphi,$$

$\lambda > -(\mu_0 - \mu)$,

$$C_1 := \min_{x\in\bar{\Omega}}\big\{\alpha(x)\bar{U}(x)\big\} > 0, \quad \text{and} \quad C_2 := \min_{x\in\bar{\Omega}}\big\{\frac{\beta(x)\bar{U}(x)\eta(x)}{k(x)}\big\} > 0.$$

Note that the eigenvalue problem

$$\begin{gathered}\hat{\omega}\psi = D_I\Delta\psi - (\gamma(x) + d(x))\psi, \quad x \in \Omega;\\ \frac{\partial\psi}{\partial\nu} = 0, \quad x \in \partial\Omega,\end{gathered}$$

has a principal eigenvalue $\hat{\omega}^0$ associated with the positive eigenvector φ^0. We denote by λ^* the larger root of the algebraic equation $\lambda^2 + (\mu_0 - \mu - C_1 - \hat{\omega}^0)\lambda - [C_2 + (C_1 + \hat{\omega}^0)(\mu_0 - \mu)] = 0$, then

$$\lambda^* = \frac{1}{2}\Big\{[\hat{\omega}^0 + C_1 - (\mu_0 - \mu)] + \sqrt{[\hat{\omega}^0 + C_1 + (\mu_0 - \mu)]^2 + 4C_2}\Big\} > -(\mu_0 - \mu).$$

Hence,

$$\begin{aligned}L_{\lambda^*}\varphi^0 &= D_I\Delta\varphi^0 + \alpha(x)\bar{U}(x)\varphi^0 - (\gamma(x) + d(x))\varphi^0 + \frac{\beta(x)\bar{U}(x)\eta(x)}{k(x)\,[\lambda^* + (\mu_0 - \mu)]}\varphi^0\\ &\geq \Big(\hat{\omega}^0 + C_1 + \frac{C_2}{\lambda^* + (\mu_0 - \mu)}\Big)\varphi^0 = \lambda^*\varphi^0.\end{aligned}$$

By [53, Theorem 2.3(i)], the proof is complete. □

Remark 3.6. Lemma 3.5 is the special case of Lemma 3.4, and $s(\mathcal{B})$ is the principal eigenvalue of (3.3) without any limitation. But if μ and μ_0 are both related on x, whether the Lemma 3.5 still holds or not is unknown for $\mathcal{R}_0 < 1$.

4. Extinction of disease

Theorem 4.1. *If $\mathcal{R}_0 < 1$, then $\mathcal{E}_0$ of model* (2.1) *is globally asymptotically stable.*

Proof. By [53, Theorem 3.1], it follows that $\mathcal{E}_0$ is locally asymptotically stable. Thus, we need verify only the global attractivity of $\mathcal{E}_0$. For $\varepsilon_0 > 0$, it yields from (2.4) that there is $t_1 > 0$ such that $0 \leq S(\cdot, t) \leq \bar{U}(\cdot) + \varepsilon_0$, $x \in \bar{\Omega}$, $t > t_1$. With the help of the comparison principle in [25], we have

$$(I(\cdot, t), V(\cdot, t)) \leq (\hat{I}(\cdot, t), \hat{V}(\cdot, t)), \quad x \in \bar{\Omega},\ t > t_1,$$

where $(\hat{I}(\cdot,t),\hat{V}(\cdot,t))$ is the solution of the problem

$$\begin{aligned}\frac{\partial \hat{I}(x,t)}{\partial t} &= D_I\Delta\hat{I} + \alpha(x)(\bar{U}(x)+\varepsilon_0)\hat{I} + \beta(x)\frac{\bar{U}(x)+\varepsilon_0}{k(x)}\hat{V} - (\gamma(x)+d(x))\hat{I},\\ \frac{\partial \hat{V}(x,t)}{\partial t} &= \eta(x)\hat{I} + \mu(x)\hat{V} - \mu_0(x)\hat{V},\end{aligned} \tag{4.1}$$

for $x\in\Omega$, $t>t_1$ and $\frac{\partial \hat{I}(x,t)}{\partial\nu}=0$, $x\in\partial\Omega$, $t>t_1$. We set $\Pi_{\varepsilon_0}(t)$ be the solution semigroup generated by system (4.1) with generator $\mathcal{B}_{\varepsilon_0}$. Similar to Lemma 2.4, one can obtain $\omega_{ess}(\Pi_{\varepsilon_0}(t))\le -K$. Recall that $\omega(\Pi_{\varepsilon_0}(t))=\max\{s(\mathcal{B}_{\varepsilon_0}),\omega_{ess}(\Pi_{\varepsilon_0}(t))\}$, it is known that $\omega(\Pi_{\varepsilon_0}(t))$ has the same sign with $s(\mathcal{B}_{\varepsilon_0})$. From Lemma 3.2(ii), $s(\mathcal{B}_{\varepsilon_0})$ has the same sign as the principal eigenvalue $\tau^0_{\varepsilon_0}$, where $\tau^0_{\varepsilon_0}$ satisfies

$$D_I\Delta\psi - (\gamma(x)+d(x))\psi + \Big[\alpha(x)(\bar{U}(x)+\varepsilon_0) + \frac{\beta(x)(\bar{U}(x)+\varepsilon_0)\eta(x)}{k(x)(\mu_0(x)-\mu(x))}\Big]\psi = \tau\psi,$$

for $x\in\Omega$, with $\frac{\partial\psi}{\partial\nu}=0$, $x\in\partial\Omega$. Again by Lemma 3.2(ii), the assumption that $\mathcal{R}_0<1$ and the continuous dependence of $\tau^0_{\varepsilon_0}$ on ε_0, we can choose $\varepsilon_0>0$ such that $\tau^0_{\varepsilon_0}<0$. Therefore, $\omega(\Pi_{\varepsilon_0}(t))<0$. By the definition of $\omega(\Pi_{\varepsilon_0}(t))$, it follows that $\|\Pi_{\varepsilon_0}(t)\|\le M_{\varepsilon_0}e^{\omega(\Pi_{\varepsilon_0}(t))t}$, for some $M_{\varepsilon_0}>0$. Thus, $(I(\cdot,t),V(\cdot,t))\le(\hat{I}(\cdot,t),\hat{V}(\cdot,t))\to(0,0)$ as $t\to\infty$ uniformly for $x\in\bar{\Omega}$. Moreover, from Lemma 2.2, we can obtain $S(x,t)\to\bar{U}(x), P(x,t)\to 0$ uniformly for $x\in\bar{\Omega}$. This proof is complete. □

Now, we verify that $\mathcal{E}_0$ is globally asymptotically stable when $\mathcal{R}_0=1$. This is similar method to the one used in [54, Theorem 3.12]. Let (Y,d) be a completed metric space, semifolow $F(t):Y\to Y$, $t\ge 0$, be a strongly continuous. Let $y_0\in Y$ be a stable steady state of $F(t)$, and $E\subset Y$ is compact and invariant, i.e., $F(t)E=E$ for all $t\ge 0$. The following lemmas will be used later, similar arguments can be found in [10, 52].

Lemma 4.2. *If $\lim_{t\to\infty}F(t)y=y_0$ for all $y\in E$, then $E=\{y_0\}$.*

Lemma 4.3. (i) *If $S_0(\cdot)>0$, then $S(\cdot,t;\phi)>0$ for all $x\in\bar{\Omega}$, $t>0$;*
(ii) *If $I_0(\cdot)\ne 0$ or $V_0(\cdot)\ne 0$, then $I(\cdot,t;\phi)>0$ and $V(\cdot,t;\phi)>0$ for all $x\in\bar{\Omega}$, $t>0$.*

Proof. (i) It follows form the first equation of (2.1) that

$$\frac{\partial S}{\partial t}\ge D_S\Delta S - \Big(\alpha(x)I + \beta(x)\frac{V}{k(x)+V} + d(x)\Big)S.$$

Thus, we have $S(\cdot,t)\ge S^\sharp(\cdot,t)$, where $S^\sharp(\cdot,t)$ satisfies

$$\begin{gathered}\frac{\partial S^\sharp}{\partial t} = D_S\Delta S^\sharp - \Big(\alpha(x)I + \beta(x)\frac{V}{k(x)+V} + d(x)\Big)S^\sharp,\quad x\in\Omega,\ t>0,\\ \frac{\partial S^\sharp}{\partial\nu}=0,\quad x\in\partial\Omega,\ t>0,\\ S^\sharp(\cdot,0)=S^\sharp_0(\cdot)=S_0(\cdot)>0,\quad x\in\bar{\Omega}.\end{gathered}$$

From the maximum principle, we have $S^\sharp(\cdot,t)>0$ for $x\in\bar{\Omega}$ and $t>0$. Thus, $S(\cdot,t)\ge S^\sharp(\cdot,t)>0$ for $x\in\bar{\Omega}$ and $t>0$. That is, assertion (i) is valid.

(ii) If $I_0(\cdot) \neq 0$, in the view of Theorem 2.1 and the second equation of (2.1), we have $\frac{\partial I}{\partial t} \geq D_I \Delta I - (\gamma(\cdot) + d(\cdot))I$ for $x \in \bar{\Omega}$ and $t > 0$. Let I is the upper solution of the system

$$\begin{gathered}
\frac{\partial I^\sharp}{\partial t} = D_I \Delta I^\sharp - (\gamma(\cdot) + d(\cdot))I^\sharp, \quad x \in \Omega,\ t > 0, \\
\frac{\partial I^\sharp}{\partial \nu} = 0, \quad x \in \partial\Omega,\ t > 0, \\
I^\sharp(\cdot, 0) = I_0^\sharp(\cdot) = I_0(\cdot), \quad x \in \bar{\Omega}
\end{gathered}$$

then, using the strong maximum principle and $I_0(\cdot) \neq 0$, we have $I^\sharp(\cdot, t) > 0$ for $x \in \bar{\Omega}$ and $t > 0$. Therefore, $I(\cdot, t) \geq I^\sharp(\cdot, t) > 0$ for $x \in \bar{\Omega}$ and $t > 0$, from the standard comparison principle.

From the third equation of (2.1) and Theorem 2.1, we have

$$\begin{aligned}
V(\cdot, t) &= V_0(\cdot) e^{\int_0^t \left(\mu(\cdot) - \mu_0(\cdot) - \frac{\xi(\cdot)P(\cdot,s)}{m(\cdot)+V(\cdot,s)}\right) \mathrm{d}s} \\
&\quad + \int_0^t e^{\int_s^t \left(\mu(\cdot) - \mu_0(\cdot) - \frac{\xi(\cdot)P(\cdot,\tau)}{m(\cdot)+V(\cdot,\tau)}\right) \mathrm{d}\tau} \eta(\cdot) I(\cdot, s) \mathrm{d}s,
\end{aligned}$$

for fixed $x \in \bar{\Omega}$. By the comparison principle, we have $V(\cdot, t) > 0$ for $x \in \bar{\Omega}$ and $t > 0$.

If $V_0(\cdot) \neq 0$, it is obvious that $V(\cdot, t) > 0$. From Theorem 2.1 and the second equation of (2.1), it well known that I is strictly positive. The proof of assertion (ii) is complete. □

Theorem 4.4. *If $\mathcal{R}_0 = 1$, then $\mathcal{E}_0$ of model* (2.1) *is globally asymptotically stable.*

Proof. We verify the local stability of $\mathcal{E}_0$ first. Suppose that there is $\rho > 0$ such that $\|\phi - \mathcal{E}_0\| \leq \rho$ for any $\phi \in \mathbb{X}^+$. We denote

$$\Theta(\cdot, t) := \frac{S(\cdot, t)}{\bar{U}(\cdot)} - 1 \quad \text{and} \quad b(t) = \max_{x \in \bar{\Omega}} \{\Theta(\cdot, t), 0\}.$$

The first equation of (2.1) yields

$$\frac{\partial \Theta}{\partial t} - D_S \Delta \Theta - 2D_S \frac{\nabla \bar{U}(\cdot) \cdot \nabla \Theta}{\bar{U}(\cdot)} + \frac{\Lambda(\cdot)}{\bar{U}(\cdot)} \Theta = -\frac{\alpha(\cdot) S I + \beta(\cdot) \frac{SV}{k(\cdot)+V}}{\bar{U}(\cdot)}. \tag{4.2}$$

Let $\hat{T}(t)$ be the positive semigroup generated by $D_S \Delta + 2D_S \frac{\nabla \bar{U}(\cdot) \cdot \nabla}{\bar{U}(\cdot)} - \frac{\Lambda(\cdot)}{\bar{U}(\cdot)}$, satisfying $\|\hat{T}(t)\| \leq \hat{M} e^{-rt}$ for some $\hat{M},\ r > 0$. Solving (4.2), we obtain

$$\Theta(\cdot, t) = \hat{T}(t)\Theta_0 - \int_0^t \hat{T}(t-s) \frac{\alpha(\cdot) S(\cdot, s) I(\cdot, s) + \beta(\cdot) \frac{S(\cdot,s)V(\cdot,s)}{k(\cdot)+V(\cdot,s)}}{\bar{U}(\cdot)} \mathrm{d}s,$$

where $\Theta_0 = \frac{S_0(\cdot)}{\bar{U}(\cdot)} - 1$. Therefore,

$$\begin{aligned}
b(t) &= \max_{x \in \bar{\Omega}} \Big\{ \hat{T}(t)\Theta_0 - \int_0^t \hat{T}(t-s) \frac{\alpha(\cdot) S(\cdot, s) I(\cdot, s) + \beta(\cdot) \frac{S(\cdot,s)V(\cdot,s)}{k(\cdot)+V(\cdot,s)}}{\bar{U}(\cdot)} \mathrm{d}s, 0 \Big\} \\
&\leq \max_{x \in \bar{\Omega}} \big\{ \hat{T}(t)\Theta_0, 0 \big\} \leq \|\hat{T}(t)\Theta_0\| \\
&\leq \hat{M} e^{-rt} \Big\| \frac{S_0(\cdot)}{\bar{U}(\cdot)} - 1 \Big\| \leq \frac{\rho \hat{M} e^{-rt}}{\bar{U}_{\min}},
\end{aligned} \tag{4.3}$$

where $\bar{U}_{\min} = \min_{x\in\bar{\Omega}}\{\bar{U}(\cdot)\}$.

By a zero trick, $(I(\cdot,t),V(\cdot,t))$ satisfies

$$\begin{gathered}
\frac{\partial I}{\partial t} = D_I\Delta I + \alpha(\cdot)\bar{U}(\cdot)I + \beta(\cdot)\frac{\bar{U}(\cdot)}{k(\cdot)}V - (\gamma(\cdot)+d(\cdot))I \\
+ \alpha(\cdot)\bar{U}(\cdot)[\frac{S(\cdot)}{\bar{U}(\cdot)} - 1]I + \beta(\cdot)[\frac{S}{k(\cdot)+V} - \frac{\bar{U}(\cdot)}{k(\cdot)}]V, \quad x\in\Omega,\ t>0, \\
\frac{\partial V}{\partial t} = \eta(\cdot)I + \mu(\cdot)V - \mu_0(\cdot)V - \frac{\xi(\cdot)VP}{m(\cdot)+V}, \quad x\in\Omega,\ t>0, \\
\frac{\partial I}{\partial \nu} = 0, \quad x\in\partial\Omega,\ t>0.
\end{gathered}$$

Thus,

$$\begin{aligned}
\begin{pmatrix} I(\cdot,t) \\ V(\cdot,t)\end{pmatrix} &= \Pi(t)\begin{pmatrix} I_0(\cdot) \\ V_0(\cdot)\end{pmatrix} + \int_0^t \Pi(t-s) \qquad (4.4)\\
&\quad\times \begin{pmatrix} \alpha(\cdot)\bar{U}(\cdot)[\frac{S(\cdot,s)}{\bar{U}(\cdot)} - 1]I(\cdot,s) + \beta(\cdot)[\frac{S(\cdot,s)}{k(\cdot)+V(\cdot,s)} - \frac{\bar{U}(\cdot)}{k(\cdot)}]V(\cdot,s) \\ -\frac{\xi(\cdot)V(\cdot,s)P(\cdot,s)}{m(\cdot)+V(\cdot,s)} \end{pmatrix} \mathrm{d}s \\
&\le \Pi(t)\begin{pmatrix} I_0(\cdot) \\ V_0(\cdot)\end{pmatrix} + \int_0^t \Pi(t-s) \qquad (4.5)\\
&\quad\times \begin{pmatrix} \alpha(\cdot)\bar{U}(\cdot)[\frac{S(\cdot,s)}{\bar{U}(\cdot)} - 1]I(\cdot,s) + \beta(\cdot)[\frac{S(\cdot,s)}{\bar{U}(\cdot)} - 1]\frac{\bar{U}(\cdot)V(\cdot,s)}{k(\cdot)+V(\cdot,s)} \\ 0 \end{pmatrix} \mathrm{d}s.
\end{aligned}$$

Under the conditions of Lemma 3.1 and $\mathcal{R}_0 = 1$, we have $\omega(\Pi(t)) = 0$. Then $\|\Pi(t)\| \le M$ for $t\ge 0$. By (4.3) and (4.5), it follows that

$$\begin{aligned}
&\max\{\|I(\cdot,t)\|, \|V(\cdot,t)\|\} \\
&\le M\max\{\|I_0(\cdot)\|, \|V_0(\cdot)\|\} + M\mathbb{M}_0 \\
&\quad\times \Big[\bar{\alpha}\int_0^t b(s)\|I(\cdot,s)\|\mathrm{d}s + \bar{\beta}\int_0^t b(s)\|\frac{V(\cdot,s)}{k(\cdot)+V(\cdot,s)}\|\mathrm{d}s\Big] \qquad (4.6)\\
&\le M\rho + \mathbb{L}_1\rho\int_0^t e^{-rs}\|I(\cdot,s)\|\mathrm{d}s + \mathbb{L}_2\rho,
\end{aligned}$$

where $\mathbb{L}_1 = M\bar{\alpha}\mathbb{M}_0\hat{M}/\bar{U}_{\min}$, $\mathbb{L}_2 = M\bar{\beta}\mathbb{M}_0\hat{M}/(r\bar{U}_{\min})$. With the help of Gronwall's inequality, we obtain

$$\|I(\cdot,t)\| \le (M+\mathbb{L}_2)\rho e^{\int_0^t \mathbb{L}_1\rho e^{-rs}\mathrm{d}s} \le (M+\mathbb{L}_2)\rho e^{\mathbb{L}_1\rho/r}. \qquad (4.7)$$

From (4.6) and (4.7), we know that

$$\begin{aligned}
\|V(\cdot,t)\| &\le M\rho + \mathbb{L}_2\rho + \mathbb{L}_1\rho(M+\mathbb{L}_2)\rho e^{\mathbb{L}_1\rho/r}\int_0^t e^{-rs}\mathrm{d}s \\
&\le (M+\mathbb{L}_2)\rho\Big(1 + \frac{\mathbb{L}_1\rho e^{\mathbb{L}_1\rho/r}}{r}\Big).
\end{aligned} \qquad (4.8)$$

This, the first equation of (2.1), (4.7), and (4.8) imply that

$$\begin{aligned}
\frac{\partial S}{\partial t} &\ge D_S\Delta S + \Lambda(\cdot) - d(\cdot)S - \alpha(\cdot)(M+\mathbb{L}_2)\rho e^{\mathbb{L}_1\rho/r}S \\
&\quad - \frac{\beta(\cdot)}{k(\cdot)}(M+\mathbb{L}_2)\rho\Big(1 + \frac{\mathbb{L}_1\rho e^{\mathbb{L}_1\rho/r}}{r}\Big)S.
\end{aligned}$$

Now we consider the system

$$\begin{aligned}\frac{\partial \hat{S}}{\partial t} &= D_S\Delta\hat{S} + \Lambda(\cdot) - d(\cdot)\hat{S} - \alpha(\cdot)(M+\mathbb{L}_2)\rho e^{\mathbb{L}_1\rho/r}\hat{S}\\ &\quad - \frac{\beta(\cdot)}{k(\cdot)}(M+\mathbb{L}_2)\rho\Big(1+\frac{\mathbb{L}_1\rho e^{\mathbb{L}_1\rho/r}}{r}\Big)\hat{S}, \quad x\in\Omega,\ t>0,\\ \frac{\partial \hat{S}}{\partial \nu} &= 0, \quad x\in\partial\Omega,\ t>0,\\ \hat{S}(\cdot,0) &= S_0(\cdot), \quad x\in\Omega,\ t>0.\end{aligned}\tag{4.9}$$

From the comparison principle, we have $S(\cdot,t)\geq \hat{S}(\cdot,t)$ for all $x\in\bar{\Omega}$ and $t>0$. Denote by $\hat{S}_\rho(\cdot)$ the positive steady state of (4.9). Let $\tilde{S}(\cdot,t)=\hat{S}(\cdot,t)-\hat{S}_\rho(\cdot)$, then $\tilde{S}(\cdot,t)$ satisfies

$$\begin{aligned}\frac{\partial \tilde{S}}{\partial t} &= D_S\Delta\tilde{S} - \big[d(\cdot) + \alpha(\cdot)(M+\mathbb{L}_2)\rho e^{\mathbb{L}_1\rho/r}\big]\tilde{S}\\ &\quad + \frac{\beta(\cdot)}{k(\cdot)}(M+\mathbb{L}_2)\rho\Big(1+\frac{\mathbb{L}_1\rho e^{\mathbb{L}_1\rho/r}}{r}\Big)\tilde{S}, \quad x\in\Omega,\ t>0,\\ \frac{\partial \tilde{S}}{\partial \nu} &= 0, \quad x\in\partial\Omega,\ t>0,\\ \tilde{S}(\cdot,0) &= \tilde{S}_0(\cdot) = S_0(\cdot)-\hat{S}_\rho(\cdot), \quad x\in\Omega.\end{aligned}\tag{4.10}$$

Notice that $T_1(t)$ is the positive semigroup generated by $D_S\Delta - d(\cdot)$, satisfying $\|T_1(t)\|\leq \tilde{M}e^{-\underline{d}t}$ for some $\tilde{M}>0$. Solving (4.10), it follows that

$$\begin{aligned}\tilde{S}(\cdot,t) &= T_1(t)\tilde{S}_0 - \int_0^t T_1(t-s)\\ &\quad\times\Big[\alpha(\cdot)(M+\mathbb{L}_2)\rho e^{\mathbb{L}_1\rho/r} + \frac{\beta(\cdot)}{k(\cdot)}(M+\mathbb{L}_2)\rho\Big(1+\frac{\mathbb{L}_1\rho e^{\mathbb{L}_1\rho/r}}{r}\Big)\Big]\tilde{S}(\cdot,s)\mathrm{d}s.\end{aligned}$$

Hence,

$$\begin{aligned}\|\tilde{S}(\cdot,t)\| &\leq \tilde{M}e^{-\underline{d}t}\|\tilde{S}_0\| + \int_0^t \tilde{M}e^{-\underline{d}(t-s)}\\ &\quad\times\Big[\bar{\alpha}(M+\mathbb{L}_2)\rho e^{\mathbb{L}_1\rho/r} + \frac{\bar{\beta}}{\underline{k}}(M+\mathbb{L}_2)\rho\Big(1+\frac{\mathbb{L}_1\rho e^{\mathbb{L}_1\rho/r}}{r}\Big)\Big]\|\tilde{S}(\cdot,s)\|\mathrm{d}s.\end{aligned}$$

Again applying Gronwall's inequality, we have $\|\hat{S}(\cdot,t)-\hat{S}_\rho(\cdot)\| = \|\tilde{S}(\cdot,t)\| \leq \tilde{M}\|S_0(\cdot)-\hat{S}_\rho(\cdot)\|e^{\hat{K}t-\underline{d}t}$, where

$$\hat{K} = \tilde{M}\Big[\bar{\alpha}(M+\mathbb{L}_2)\rho e^{\mathbb{L}_1\rho/r} + \frac{\bar{\beta}}{\underline{k}}(M+\mathbb{L}_2)\rho\Big(1+\frac{\mathbb{L}_1\rho e^{\mathbb{L}_1\rho/r}}{r}\Big)\Big].$$

Choosing $\rho>0$ small enough such that $\hat{K}<\underline{d}/2$, it follows that $\|\hat{S}(\cdot,t)-\hat{S}_\rho(\cdot)\|\leq \tilde{M}\|S_0(\cdot)-\hat{S}_\rho(\cdot)\|e^{-\underline{d}t/2}$. By a zero trick, we can obtain

$$\begin{aligned}S(\cdot,t)-\bar{U}(\cdot) &\geq \hat{S}(\cdot,t)-\bar{U}(\cdot) = \hat{S}(\cdot,t)-\hat{S}_\rho(\cdot)+\hat{S}_\rho(\cdot)-\bar{U}(\cdot)\\ &\geq -\tilde{M}\|S_0(\cdot)-\hat{S}_\rho(\cdot)\|e^{-\underline{d}t/2} + \hat{S}_\rho(\cdot)-\bar{U}(\cdot)\\ &\geq -\tilde{M}(\|S_0(\cdot)-\bar{U}(\cdot)\| + \|\hat{S}_\rho(\cdot)-\bar{U}(\cdot)\|) - \|\hat{S}_\rho(\cdot)-\bar{U}(\cdot)\|\\ &\geq -\tilde{M}\rho - (\tilde{M}+1)\|\hat{S}_\rho(\cdot)-\bar{U}(\cdot)\|.\end{aligned}\tag{4.11}$$

Noticing that $b(t) \le \frac{\rho \hat{M} e^{-rt}}{\bar{U}_{\min}}$, one has

$$S(\cdot,t) - \bar{U}(\cdot) = \bar{U}(\cdot)\Big(\frac{S(\cdot,t)}{\bar{U}(\cdot)} - 1\Big) \le \mathbb{M}_0 b(t) \le \frac{\rho \hat{M}\mathbb{M}_0}{\bar{U}_{\min}}. \tag{4.12}$$

Combined this, (4.11), and (4.12), it follows that

$$\|S(\cdot,t) - \bar{U}(\cdot)\| \le \max\big\{\tilde{M}\rho + (\tilde{M}+1)\|\hat{S}_\rho(\cdot) - \bar{U}(\cdot)\|, \frac{\rho\hat{M}\mathbb{M}_0}{\bar{U}_{\min}}\big\}. \tag{4.13}$$

The forth equation in (2.1) yields

$$\frac{\partial P}{\partial t} \le \frac{\bar{\theta}\bar{\xi}}{\underline{m}}(M + \mathbb{L}_2)\rho\Big(1 + \frac{\mathbb{L}_1 \rho e^{\mathbb{L}_1\rho/r}}{r}\Big)P - \delta(\cdot)P. \tag{4.14}$$

Let $\hat{L} = \frac{\bar{\theta}\bar{\xi}}{\underline{m}}(M+\mathbb{L}_2)\rho(1 + \frac{\mathbb{L}_1\rho e^{\mathbb{L}_1\rho/r}}{r})$. Solving (4.14), yields $P(\cdot,t) \le T_4(t)P_0 + \int_0^t T_4(t-s)\hat{L}P(\cdot,s)\mathrm{d}s$, where $T_4(t)$ is defined in Section 2 and $\|T_4(t)\| \le e^{-\underline{\delta}t}$. Hence, we have

$$\|P(\cdot,t)\| \le \rho e^{-\underline{\delta}t} + \int_0^t e^{-\underline{\delta}(t-s)}\hat{L}\|P(\cdot,s)\|\mathrm{d}s.$$

Again applying Gronwall's inequality, we obtain

$$\|P(\cdot,t)\| \le \rho e^{-\underline{\delta}t} e^{\int_0^t \hat{L}e^{-\underline{\delta}(t-s)}\mathrm{d}s} \le \rho e^{\hat{L}t - \underline{\delta}t}.$$

Choosing $\rho > 0$ small enough such that $\hat{L} < \underline{\delta}/2$, then

$$\|P(\cdot,t)\| \le \rho e^{-\underline{\delta}t/2} \le \rho. \tag{4.15}$$

Thus, by (4.7), (4.8), (4.13), (4.15), and $\lim_{\rho\to 0}\hat{S}_\rho(\cdot) = \bar{U}(\cdot)$, one can select ρ small enough such that

$$\|S(\cdot,t) - \bar{U}(\cdot)\| \le \varepsilon, \quad \|I(\cdot,t)\| \le \varepsilon, \quad \|V(\cdot,t)\| \le \varepsilon, \quad \|P(\cdot,t)\| \le \varepsilon,\ t > 0.$$

This verifies the local stability of $\mathcal{E}_0$.

Next, we turn to the global attractivity of $\mathcal{E}_0$. From Theorem 2.6, $\Phi(t)$ possesses a global attractor $\mathcal{A}$ on $\mathbb{X}^+$. From Lemma 3.4, the eigenvalue problem (3.3) has a positive eigenfunction (φ_2, φ_3) associated with $s(\mathcal{B}) = 0$. We define $\partial\mathbb{X}_1 = \{(\breve{S}, \breve{I}, \breve{V}, \breve{P}) \in \mathbb{X} : \breve{I} = \breve{V} = 0\}$. Now, we claim that for any $\phi = (S_0, I_0, V_0, P_0) \in \mathcal{A}$, the omega limit set $\omega(\phi) \subset \partial\mathbb{X}_1$. From (2.4), it follows that $S_0 \le \bar{U}(\cdot)$. If $I_0 = V_0 = 0$, the conclusion directly gets because the fact that $\partial\mathbb{X}_1$ is invariant for $\Phi(t)$. Thus, suppose that either $I_0 \ne 0$ or $V_0 \ne 0$. This follows from Lemma 4.3 that $S(\cdot,t),\ I(\cdot,t),\ V(\cdot,t),\ P(\cdot,t) > 0$ for $x \in \Omega$ and $t > 0$. Therefore, $S(\cdot,t)$ satisfies (2.3) and $S(\cdot,t) < \bar{U}(\cdot)$ for $x \in \Omega$ and $t > 0$.

According to [10, 52, 54], we denote $c(t;\phi) := \inf\{\tilde{c} \in \mathbb{R} : I(\cdot,t) \le \tilde{c}\varphi_2,\ V(\cdot,t) \le \tilde{c}\varphi_3\}$, then $c(t;\phi) > 0$ for all $t > 0$ and $c(t;\phi) > 0$ is strictly decreasing function. In fact, we fix $\tilde{t} > 0$ and $\bar{I}(\cdot,t) = c(\tilde{t};\phi)\varphi_2$, $\bar{V}(\cdot,t) = c(\tilde{t};\phi)\varphi_3$ for $t \ge \tilde{t}$. Combined with $S(\cdot,t) < \bar{U}(\cdot)$, it follows that, for $t > \tilde{t}$,

$$\begin{gathered}
\frac{\partial \bar{I}}{\partial t} > D_I\Delta\bar{I} - (\gamma(\cdot) + d(\cdot))\bar{I} + S\Big(\alpha(\cdot)\bar{I} + \beta(\cdot)\frac{\bar{V}}{k(\cdot) + \bar{V}}\Big), \quad x \in \Omega, \\
\frac{\partial \bar{V}}{\partial t} > \eta(\cdot)\bar{I} + \mu(\cdot)\bar{V} - \mu_0(\cdot)\bar{V}, \quad x \in \Omega, \\
\frac{\partial \bar{I}}{\partial \nu} = 0,\ x \in \partial\Omega; \quad \bar{I}(\cdot,\tilde{t}) \ge I(\cdot,\tilde{t}), \quad \bar{V}(\cdot,\tilde{t}) \ge V(\cdot,\tilde{t}), \quad x \in \Omega.
\end{gathered} \tag{4.16}$$

By the comparison principle, we obtain $(\bar{I}(\cdot,t),\bar{V}(\cdot,t)) \geq (I(\cdot,t),V(\cdot,t))$ for all $x\in\bar{\Omega}$, $t\geq\tilde{t}$. Further from (4.16), one can obtain that $c(\tilde{t};\phi)\varphi_2=\bar{I}(\cdot,t)>I(\cdot,t)$ and $c(\tilde{t};\phi)\varphi_3=\bar{V}(\cdot,t)>V(\cdot,t)$, for all $x\in\bar{\Omega}$ and $t\geq\tilde{t}$. Therefore, $c(t;\phi)$ is strictly decreasing function from the fact that $\tilde{t}>0$ is arbitrary.

Let $c_*=\lim_{t\to\infty}c(t;\phi)$, we confirm $c_*=0$. Actually set $N=(S,I,V,P)\in\omega(\phi)$, there is $\{t_k\}$ with $t_k\to\infty$ such that $\Phi(t_k)\phi\to N$. Since $\lim_{t_k\to\infty}\Phi(t+t_k)\phi=\Phi(t)\lim_{t_k\to\infty}\Phi(t_k)\phi=\Phi(t)N$, it directly implies $c(t;N)=c_*$. If $I\neq0$ or $V\neq0$, we can repeat the previous procedures to prove that $c(t;N)$ is strictly decreasing function on t. This contradicts that $c(t;N)=c_*$. Therefore, $I=0$ and $V=0$.

Lastly, we need to prove that $\mathcal{A}=\{\mathcal{E}_0\}$. When $V\to0$, from the forth equation in (2.1), it also known that $P\to0$ whether $P_0=0$ or not. This together with above arguments, implies that $\{\mathcal{E}_0\}$ is globally attractive in $\partial\mathbb{X}_1$. In addition, $\{\mathcal{E}_0\}$ is the only compact invariant subset in $\partial\mathbb{X}_1$. Further, for any $\phi\in\mathcal{A},\omega(\phi)\subset\partial\mathbb{X}_1$, we obtain $\omega(\phi)=\{\mathcal{E}_0\}$. Because $\mathcal{A}$ is compact invariant in $\mathbb{X}^+$ by Theorem 2.6, this together with the stability of $\{\mathcal{E}_0\}$ and Lemma 4.2 imply that $\mathcal{A}=\{\mathcal{E}_0\}$. Local stability and global attractivity of $\mathcal{E}_0$ indicate that $\mathcal{E}_0$ is globally asymptotically stability. This proof is complete. □

For the spatially homogeneous case, we have the following result.

Theorem 4.5. *If $\widetilde{\mathcal{R}}_0\leq1$, then $\widetilde{\mathcal{E}}_0=(\frac{\Lambda}{d},0,0,0)$ of model* (2.2) *is globally asymptotically stable.*

Proof. It is worth noting that function $g(x)=x-1-\ln x\geq0$ for $x>0$, and $g(x)=0$ if and only if $x=1$. We choose a Lyapunov function as follows,

$$\mathcal{L}_1(t)=\int_\Omega\bar{U}g\Big(\frac{S}{\bar{U}}\Big)\mathrm{d}x+\int_\Omega I\mathrm{d}x+k_1\int_\Omega V\mathrm{d}x+k_2\int_\Omega P\mathrm{d}x,$$

where the constants $k_1>0$ and $k_2>0$ will be determined below, $\bar{U}=\Lambda/d$. By calculating the derivative of $\mathcal{L}_1(t)$, we have

$$\begin{aligned}
\frac{\mathrm{d}\mathcal{L}_1(t)}{\mathrm{d}t}&=\int_\Omega\Big(1-\frac{\bar{U}}{S}\Big)\Big(D_S\Delta S+\Lambda-\alpha SI-\beta\frac{SV}{k+V}-dS\Big)\mathrm{d}x\\
&\quad+\int_\Omega\big[D_I\Delta I+\alpha SI+\beta\frac{SV}{k+V}-(\gamma+d)I\big]\mathrm{d}x\\
&\quad+k_1\int_\Omega\Big(\eta I+\mu V-\mu_0V-\frac{\xi VP}{m+V}\Big)\mathrm{d}x+k_2\int_\Omega\Big(\frac{\theta\xi VP}{m+V}-\delta P\Big)\mathrm{d}x\\
&=-D_S\int_\Omega\Big[\bar{U}\frac{|\nabla S|^2}{S^2}+\Big(1-\frac{\bar{U}}{S}\Big)\Lambda-(\gamma+d)I+\alpha\bar{U}I+\frac{\beta\bar{U}V}{k+V}+k_1\eta I\\
&\quad+\Big(1-\frac{S}{\bar{U}}\Big)d\bar{U}+k_1\mu V-k_1\mu_0V-\frac{k_1\xi VP}{m+V}+\frac{k_2\theta\xi VP}{m+V}-k_2\delta P\Big]\mathrm{d}x
\end{aligned}$$

From $\widetilde{\mathcal{R}}_0\leq1$, we have $\frac{\alpha\bar{U}}{\gamma+d}\leq1$. Hence, we choose the positive constants $k_1=(\gamma+d-\alpha\bar{U})/\eta$ and $k_2=k_1/\theta$. Using that $\Lambda=d\bar{U}$, we have

$$\begin{aligned}
\frac{\mathrm{d}\mathcal{L}_1(t)}{\mathrm{d}t}&=-D_S\int_\Omega\bar{U}\frac{|\nabla S|^2}{S^2}\mathrm{d}x-\int_\Omega d\bar{U}\Big[g\left(\frac{\bar{U}}{S}\right)+g\Big(\frac{S}{\bar{U}}\Big)\Big]\mathrm{d}x\\
&\quad+\int_\Omega\Big\{\Big[\beta\frac{\bar{U}}{k+V}-\frac{\gamma+d-\alpha\bar{U}}{\eta}(\mu_0-\mu)\Big]V-\frac{\gamma+d-\alpha\bar{U}}{\theta\eta}\delta P\Big\}\mathrm{d}x
\end{aligned}$$

$$
\begin{aligned}
&\le -D_S\int_\Omega \bar U\frac{|\nabla S|^2}{S^2}\mathrm{d}x-\int_\Omega d\bar U\Big[g\Big(\frac{\bar U}{S}\Big)+g\Big(\frac{S}{\bar U}\Big)\Big]\mathrm{d}x\\
&\quad+\int_\Omega\Big[(\widetilde{\mathcal{R}}_0-1)\frac{(\mu_0-\mu)(\gamma+d)V}{\eta}-\frac{\gamma+d-\alpha\bar U}{\theta\eta}\delta P\Big]\mathrm{d}x.
\end{aligned}
$$

Therefore, by the assumption $\widetilde{\mathcal{R}}_0\le 1$, we have $\frac{\mathrm{d}\mathcal{L}_1(t)}{\mathrm{d}t}\le 0$. Obviously, $\frac{\mathrm{d}\mathcal{L}_1(t)}{\mathrm{d}t}=0$ if and only if $S=\bar U,\ V=0$ and $P=0$. Then, we have $\frac{\partial V}{\partial t}=0$ which gives $I=0$ according to (2.2). So the largest invariant subset of $\{(S,I,V,P)|\frac{\mathrm{d}\mathcal{L}_1(t)}{\mathrm{d}t}=0\}$ is the singleton $\{\widetilde{\mathcal{E}}_0\}$. Consequently, from invariable principle [15], this concludes that $\widetilde{\mathcal{E}}_0$ is globally asymptotically stable for $\widetilde{\mathcal{R}}_0\le 1$. The proof is complete. □

5. Analysis of the phage-free endemic steady state

We discuss, in this section, the dynamics when the phages do not effect in reservoir. If $\mathcal{R}_0>1$, model (2.1) admits a phage-free endemic steady state $\mathcal{E}_1=(S_1(\cdot),I_1(\cdot),V_1(\cdot),0)$ and its elements satisfy

$$
\begin{gathered}
D_S\Delta S_1(\cdot)+\Lambda(\cdot)-\alpha(\cdot)S_1(\cdot)I_1(\cdot)-\beta(\cdot)\frac{S_1(\cdot)V_1(\cdot)}{k(\cdot)+V_1(\cdot)}-d(\cdot)S_1(\cdot)=0,\quad x\in\Omega,\\
D_I\Delta I_1(\cdot)+\alpha(\cdot)S_1(\cdot)I_1(\cdot)+\beta(\cdot)\frac{S_1(\cdot)V_1(\cdot)}{k(\cdot)+V_1(\cdot)}-(\gamma(\cdot)+d(\cdot))I_1(\cdot)=0,\quad x\in\Omega,\\
\eta(\cdot)I_1(\cdot)+\mu(\cdot)V_1(\cdot)-\mu_0(\cdot)V_1(\cdot)=0,\quad x\in\Omega,\\
\frac{\partial S_1(\cdot)}{\partial\nu}=\frac{\partial I_1(\cdot)}{\partial\nu}=0,\quad x\in\partial\Omega.
\end{gathered}
$$

Let $P=0$ in (2.1), we have the subsystem

$$
\begin{gathered}
\frac{\partial S(x,t)}{\partial t}=D_S\Delta S+\Lambda(x)-\alpha(x)SI-\beta(x)\frac{SV}{k(x)+V}-d(x)S,\quad x\in\Omega,\ t>0,\\
\frac{\partial I(x,t)}{\partial t}=D_I\Delta I+\alpha(x)SI+\beta(x)\frac{SV}{k(x)+V}-(\gamma(x)+d(x))I,\quad x\in\Omega,\ t>0,\\
\frac{\partial V(x,t)}{\partial t}=\eta(x)I+\mu(x)V-\mu_0(x)V,\quad x\in\Omega,\ t>0,
\end{gathered}
\tag{5.1}
$$

with $S(x,0)=S_0(x)$, $I(x,0)=I_0(x)$, $V(x,0)=V_0(x)$, $x\in\Omega$, and $\frac{\partial S(x,t)}{\partial\nu}=\frac{\partial I(x,t)}{\partial\nu}=0$, $x\in\partial\Omega$, $t>0$. Note that model (5.1) has a disease-free steady state $\bar{\mathcal{E}}_0=(\bar U(x),0,0)$. According to the discussions on Section 2 (see Theorems 2.1-2.6), we have the following corollary for model (5.1).

Corollary 5.1. *Let $\mathbb{W}:=C(\bar\Omega,\mathbb{R}^3)$ be the Banach space and its positive cone is denoted by $\mathbb{W}^+$.*

- (i) *For any initial value function $\psi(x)=(\psi_1(x),\psi_2(x),\psi_3(x))=(S_0(x),I_0(x),V_0(x))\in\mathbb{W}^+$, model (5.1) admits a unique, nonnegative solution $\bar u(\cdot,t;\psi)$ on $[0,\infty)$ with $\bar u(\cdot,0;\phi)=\psi$.*
- (ii) *The semiflow $\bar\Phi(t):\mathbb{W}^+\to\mathbb{W}^+$ generated by (5.1) is defined by $\bar\Phi(t)\psi=\bar u(\cdot,t;\psi)=(S(\cdot,t;\psi),I(\cdot,t;\psi),\ V(\cdot,t;\psi))$, $x\in\bar\Omega$, $t\ge 0$. And, $\bar\Phi(t)$ is point dissipative.*
- (iii) *The semiflow $\bar\Phi(t)$ of (5.1) possesses a global attractor $\bar{\mathcal{A}}$ on $\mathbb{W}^+$.*

We define the set $\mathbb{W}_0 := \{\psi(x) = (\psi_1, \psi_2, \psi_3) \in \mathbb{W}^+ : \psi_1 > 0,\ \psi_2 \neq 0,\ \psi_3 \neq 0\}$ and $\partial\mathbb{W}_0 := \mathbb{W}^+ \backslash \mathbb{W}_0 = \{\psi(x) \in \mathbb{W}^+ : \psi_2\psi_3 = 0\}$. Similar to Lemma 4.3, it can obtain that $\mathbb{W}_0$ is the positive invariant set for solution semiflow $\bar{\Phi}(t)$ of (5.1). We define $M_\partial := \{\psi \in \partial\mathbb{W}_0 : \bar{\Phi}(t)\psi \in \partial\mathbb{W}_0,\ \forall t \geq 0\}$, and $\omega(\psi)$ be the omega limit set of the orbit $\mathcal{G}^+ := \{\bar{\Phi}(t)\psi : t \geq 0\}$.

The following lemma is about the uniform weak repulsion of the disease-free steady state $\bar{\mathcal{E}}_0$, which is necessary to verify the persistence of (5.1).

Lemma 5.2. *If $\mathcal{R}_0 > 1$, then there exists $\varepsilon_1 > 0$ such that the solution semiflow $\bar{\Phi}(t)$ of (5.1) satisfies $\limsup_{t\to\infty} \|\bar{\Phi}(t)\psi - \bar{\mathcal{E}}_0\|_{\mathbb{W}^+} \geq \varepsilon_1$ for any $\psi \in \mathbb{W}_0$.*

Proof. Because $\mathcal{R}_0 > 1$, the principal eigenvalue $s(\mathcal{B})$ of (3.3) satisfies $s(\mathcal{B}) > 0$. By continuous dependence, one can choose a positive constant ε_* such that $\bar{U}(\cdot) - \varepsilon_* > 0$ and $s(\mathcal{B}_{\varepsilon_*}) > 0$, where $s(\mathcal{B}_{\varepsilon_*})$ is the principal eigenvalue of equation

$$\begin{gathered}
\lambda\phi_2(\cdot) = D_I\Delta\phi_2(\cdot) + \alpha(\cdot)(\bar{U}(\cdot) - \varepsilon_*)\phi_2(\cdot) \\
+ \beta(\cdot)(\bar{U}(\cdot) - \varepsilon_*)\Big(\frac{1}{k(\cdot)} - \varepsilon_*\Big)\phi_3(\cdot) - (\gamma(\cdot) + d(\cdot))\phi_2(\cdot), \quad x \in \Omega, \\
\lambda\phi_3(\cdot) = \eta(\cdot)\phi_2(\cdot) + \mu(\cdot)\phi_3(\cdot) - \mu_0(\cdot)\phi_3(\cdot), \quad x \in \Omega, \\
\frac{\partial\phi_2(\cdot)}{\partial\nu} = 0, \quad x \in \partial\Omega.
\end{gathered}$$

For this ε_*, there is a positive constant σ_* such that

$$\frac{1}{k(\cdot) + V(\cdot, t)} > \frac{1}{k(\cdot)} - \varepsilon_*, \quad \text{for } V(\cdot, t) < \sigma_*.$$

Let $\varepsilon_1 = \min\{\varepsilon_*, \sigma_*\}$. Suppose, by contradiction, there exists $\psi_0 \in \mathbb{W}_0$ such that $\limsup_{t\to\infty} \|\bar{\Phi}(t)\psi_0 - \bar{\mathcal{E}}_0\|_{\mathbb{W}^+} < \varepsilon_1$. Then there is $t_1 > 0$ such that $\bar{U}(\cdot) - \varepsilon_1 < S(\cdot, t; \psi_0)$, $I(\cdot, t; \psi_0) < \varepsilon_1$, and $V(\cdot, t; \psi_0) < \varepsilon_1$ for $x \in \bar{\Omega}$ and $t \geq t_1$. Here, $I(\cdot, t; \psi_0)$ and $V(\cdot, t; \psi_0)$ satisfy

$$\begin{gathered}
\frac{\partial I}{\partial t} \geq D_I\Delta I + \alpha(\cdot)(\bar{U}(\cdot) - \varepsilon_1)I + \beta(\cdot)(\bar{U}(\cdot) - \varepsilon_*)\Big(\frac{1}{k(\cdot)} - \varepsilon_*\Big)V - (\gamma(\cdot) + d(\cdot))I, \\
\frac{\partial V}{\partial t} = \eta(\cdot)I + \mu(\cdot)V - \mu_0(\cdot)V,
\end{gathered}$$

for $x \in \Omega$, $t \geq t_1$ and $\frac{\partial I}{\partial \nu} = 0$, $x \in \partial\Omega$, $t \geq t_1$. By the above discussion on $\mathbb{W}_0$, we know that $I(\cdot, t) > 0$ and $V(\cdot, t) > 0$ for $x \in \bar{\Omega}$ and $t > 0$. Denote by $\phi^{\varepsilon_1} = (\phi_2^{\varepsilon_1}(\cdot), \phi_3^{\varepsilon_1}(\cdot))$ the strongly positive eigenfunction associated with $s(\mathcal{B}_{\varepsilon_1})$. Choose $\chi > 0$ such that $(I(\cdot, t_1; \psi_0), V(\cdot, t_1; \psi_0)) \geq \chi(\phi_2^{\varepsilon_1}(\cdot), \phi_3^{\varepsilon_1}(\cdot))$, $x \in \Omega$. Since $(\check{I}(\cdot, t_1; \psi_0), \check{V}(\cdot, t_1; \psi_0)) = \chi e^{s(\mathcal{B}_{\varepsilon_1})(t-t_1)}(\phi_2^{\varepsilon_1}(\cdot), \phi_3^{\varepsilon_1}(\cdot))$ is the solution of the system

$$\begin{gathered}
\frac{\partial \check{I}}{\partial t} = D_I\Delta \check{I} + \alpha(\cdot)(\bar{U}(\cdot) - \varepsilon_1)\check{I} + \beta(\cdot)(\bar{U}(\cdot) - \varepsilon_*)\Big(\frac{1}{k(\cdot)} - \varepsilon_*\Big)\check{V} - (\gamma(\cdot) + d(\cdot))\check{I}, \\
\frac{\partial \check{V}}{\partial t} = \eta(\cdot)\check{I} + \mu(\cdot)\check{V} - \mu_0(\cdot)\check{V},
\end{gathered}$$

for $x \in \Omega$, $t \geq t_1$ and $\frac{\partial \check{I}}{\partial \nu} = 0$, $x \in \partial\Omega$, $t \geq t_1$. We can obtain $(I(\cdot, t; \psi_0), V(\cdot, t; \psi_0)) \geq \chi e^{s(\mathcal{B}_{\varepsilon_1})(t-t_1)}(\phi_2^{\varepsilon_1}(\cdot), \phi_3^{\varepsilon_1}(\cdot))$, $x \in \Omega$, $t \geq t_1$ from the comparison principle. From $s(\mathcal{B}_{\varepsilon_1}) > 0$, it follows that $\lim_{t\to\infty} I(\cdot, t; \psi_0) = \infty$ and $\lim_{t\to\infty} V(\cdot, t; \psi_0) = \infty$. This is a contradiction with the boundedness of $(I(\cdot, t), V(\cdot, t))$ by Corollary 5.1. This proof is complete. $\square$

Theorem 5.3. *If $\mathcal{R}_0 > 1$, then there exists $\varrho > 0$ such that for any $\psi = (\psi_1, \psi_2, \psi_3) \in \mathbb{W}^+$ with $\psi_2 \neq 0$ and $\psi_3 \neq 0$, the solution $\bar{u}(\cdot, t; \psi) = (S(\cdot, t; \psi), I(\cdot, t; \psi), V(\cdot, t; \psi))$ of* (5.1) *satisfies $\liminf_{t\to\infty} \bar{u}(\cdot, t; \psi) \geq (\varrho, \varrho, \varrho)$, uniformly for all $x \in \bar{\Omega}$. Moreover, model* (5.1) *has at least one positive steady state.*

Proof. We verify, firstly, that $\omega(\psi) = \{\bar{\mathcal{E}}_0\}$, $\psi \in M_\partial$. For $\psi \in M_\partial$, we have $\bar{\Phi}(t)\psi \in M_\partial$, $t \geq 0$. Thus $I(\cdot, t) \equiv 0$ or $V(\cdot, t) \equiv 0$ for $t \geq 0$. In the case that $I(\cdot, t) \equiv 0$, we can obtain that $V(\cdot, t) \equiv 0$ from the second equation of (5.1). Therefore, (5.1) degenerates to the system (2.3), and this yields that $\lim_{t\to\infty} S(\cdot, t) = \bar{U}(\cdot)$ from Lemma 2.2. That is, $\omega(\psi) = \{\bar{\mathcal{E}}_0\}$. For the other case $V(\cdot, t) \equiv 0$, we have $I(\cdot, t) \equiv 0$ from the third equation of (5.1). Similarly, we also have $\lim_{t\to\infty} S(\cdot, t) = \bar{U}(\cdot)$. This also shows $\omega(\psi) = \{\bar{\mathcal{E}}_0\}$.

We define a function $p : \mathbb{W}^+ \to [0, \infty)$ by

$$p(\psi) = \min\big\{\min_{x\in\bar{\Omega}} \psi_2(x), \min_{x\in\bar{\Omega}} \psi_3(x)\big\}, \quad \psi \in \mathbb{W}^+.$$

Obviously, $p^{-1}(0, \infty) \subseteq \mathbb{W}_0$ and p has the property that if $p(\psi) > 0$ or $\psi \in \mathbb{W}_0$ with $p(\psi) = 0$, then $p(\bar{\Phi}(t)\psi) > 0$, $t > 0$. Thus p is a generalized distance function for semiflow $\bar{\Phi}(t) : \mathbb{W}^+ \to \mathbb{W}^+$ (see, [44]).

For any $\psi \in M_\partial$, then $\omega(\psi) = \{\bar{\mathcal{E}}_0\}$ from the above discussion. Namely, for any forward orbit $\bar{\Phi}(t)$ in M_∂ converges to $\{\bar{\mathcal{E}}_0\}$ as $t \to +\infty$. Hence, no subset of $\{\bar{\mathcal{E}}_0\}$ forms a cycle in $\partial\mathbb{W}_0$. Further, Lemma 5.2 implies that $\{\bar{\mathcal{E}}_0\}$ is a isolated invariant set in $\mathbb{W}^+$ and $W^s(\{\bar{\mathcal{E}}_0\}) \bigcap \mathbb{W}_0 = \emptyset$, where $W^s(\{\bar{\mathcal{E}}_0\})$ is the stable subset of $\{\bar{\mathcal{E}}_0\}$. By [44, Theorem 3], we know that there is a $\varrho_1 > 0$ such that $\min_{\psi\in\omega(\phi)} p(\psi) > \varrho_1$, for $\phi \in \mathbb{W}_0$. Hence, $\liminf_{t\to\infty} I(\cdot, t; \phi) > \varrho_1$ and $\liminf_{t\to\infty} V(\cdot, t; \phi) > \varrho_1$, for $\phi \in \mathbb{W}_0$. On the other hand, from Corollary 5.1, there exists $t_* > 0$ such that $I(\cdot, t) \leq \mathbb{M}_\infty$, for $x \in \Omega$ and $t \geq t_*$. Then $S(\cdot, t)$ satisfies

$$\frac{\partial S}{\partial t} \geq D_S \Delta S + \Lambda(\cdot) - (\alpha(\cdot)\mathbb{M}_\infty + \beta(\cdot) + d(\cdot))S, \quad x \in \Omega,\ t > t_*,$$

with $\frac{\partial S}{\partial \nu} = 0$, $x \in \partial\Omega$, $t > t_*$. Combining this with the standard comparison principle and Lemma 2.2, we have $\liminf_{t\to\infty} S(\cdot, t; \phi) > \varrho_2$, uniformly for all $x \in \bar{\Omega}$. Let $\varrho = \min\{\varrho_1, \varrho_2\}$. The uniform persistence stated is valid.

Finally, by [23, Theorem 3.7 and Remark 3.10], $\bar{\Phi}(t) : \mathbb{W}_0 \to \mathbb{W}_0$ has a global attractor. It then follows from [23, Theorem 4.7] that $\bar{\Phi}(t)$ has a steady state $\hat{u}(\cdot) \in \mathbb{W}_0$. Further, similar to Lemma 4.3, model (5.1) admits a endemic steady state. This completes the proof. □

As a directly consequence of Theorem 5.3, from [55, Theorem 1.3.6], we have the following corollary.

Corollary 5.4. *If $\mathcal{R}_0 > 1$, model* (2.1) *admits at least one phage-free endemic steady state $\mathcal{E}_1 = (S_1(x), I_1(x), V_1(x), 0)$.*

Remark 5.5. Although we obtain that the existence of endemic steady state without phages $\mathcal{E}_1 = (S_1(x), I_1(x), V_1(x), 0)$ for (2.1), it is unknown about its uniqueness and local/global stability. Fortunately, if the heterogeneous space degenerates to the homogeneous space, i.e., model (2.1) degenerates to (2.2), then we can obtain the uniqueness and stability of $\mathcal{E}_1$.

For model (2.2), if $\widetilde{\mathcal{R}}_0 > 1$, then there is a phage-free endemic steady state $\widetilde{\mathcal{E}}_1 = (\widetilde{S}_1, \widetilde{I}_1, \widetilde{V}_1, 0)$, where

$$\widetilde{S}_1 = \frac{\Lambda - (\gamma + d)\widetilde{I}_1}{d}, \quad \widetilde{V}_1 = \frac{\eta \widetilde{I}_1}{\mu_0 - \mu}, \tag{5.2}$$

and $\widetilde{I}_1$ is the positive root of $f(I) = AI^2 + BI + C$. Here,

$$A = -\frac{\alpha\eta(\gamma + d)}{d(\mu_0 - \mu)}, \quad B = \frac{\alpha}{d}\big[\frac{\Lambda\eta}{\mu_0 - \mu} - k(\gamma + d)\big] - \frac{\eta(\gamma + d)}{\mu_0 - \mu}\big(\frac{\beta}{d} + 1\big),$$
$$C = \alpha k\frac{\Lambda}{d} + \beta\frac{\Lambda\eta}{d(\mu_0 - \mu)} - k(\gamma + d) = k(\gamma + d)(\widetilde{\mathcal{R}}_0 - 1).$$

Clearly, we have $f(0) = C > 0$ for $\widetilde{\mathcal{R}}_0 > 1$. From $A < 0$, equation $f(I) = 0$ exists two real roots, one is positive and the other is negative. Furthermore, if $\widetilde{\mathcal{R}}_0 \le 1$, then $B < 0$ and $f(0) = C \le 0$, which implies that $\frac{\mathrm{d}f(I)}{\mathrm{d}I} < 0$ for all $I > 0$. This implies that $f(I)$ has no positive roots if $\widetilde{\mathcal{R}}_0 \le 1$. Thus, (2.2) has a unique phage-free endemic steady state $\widetilde{\mathcal{E}}_1 = (\widetilde{S}_1, \widetilde{I}_1, \widetilde{V}_1, 0)$ for $\widetilde{\mathcal{R}}_0 > 1$. Further, we have the following result on the global stability of the phage-free endemic steady state $\widetilde{\mathcal{E}}_1$.

Theorem 5.6. *If $\widetilde{\mathcal{R}}_0 > 1$ and $\widetilde{V}_1 \le \frac{\delta m}{\theta\xi}$, then $\widetilde{\mathcal{E}}_1$ of model* (2.2) *is globally asymptotically stable.*

Proof. Choose that a Lyapunov function

$$\mathcal{L}_2(t) = \int_\Omega \widetilde{S}_1 g\Big(\frac{S}{\widetilde{S}_1}\Big)\mathrm{d}x + \int_\Omega \widetilde{I}_1 g\Big(\frac{I}{\widetilde{I}_1}\Big)\mathrm{d}x + c_1\int_\Omega \widetilde{V}_1 g\Big(\frac{V}{\widetilde{V}_1}\Big)\mathrm{d}x + c_2\int_\Omega P\mathrm{d}x,$$

where, $g(x) = x - 1 - \ln x$ $(x > 0)$, and constants $c_1 > 0$, $c_2 > 0$ will be determined below. By calculating the derivative of $\mathcal{L}_2(t)$, we have

$$\begin{aligned}
\frac{\mathrm{d}\mathcal{L}_2(t)}{\mathrm{d}t} &= \int_\Omega \Big(1 - \frac{\widetilde{S}_1}{S}\Big)\Big(D_S\Delta S + \Lambda - \alpha SI - \beta\frac{SV}{k+V} - dS\Big)\mathrm{d}x \\
&\quad + \int_\Omega \Big(1 - \frac{\widetilde{I}_1}{I}\Big)\Big(D_I\Delta I + \alpha SI + \beta\frac{SV}{k+V} - (\gamma + d)I\Big)\mathrm{d}x \\
&\quad + c_1\int_\Omega \Big(1 - \frac{\widetilde{V}_1}{V}\Big)\Big(\eta I + \mu V - \mu_0 V - \frac{\xi VP}{m+V}\Big)\mathrm{d}x \\
&\quad + c_2\int_\Omega \Big(\frac{\theta\xi VP}{m+V} - \delta P\Big)\mathrm{d}x.
\end{aligned}$$

Since $(\widetilde{S}_1, \widetilde{I}_1, \widetilde{V}_1, 0)$ is the steady state of (2.2), we further obtain

$$\begin{aligned}
\frac{\mathrm{d}\mathcal{L}_2(t)}{\mathrm{d}t} &= -D_S\int_\Omega \widetilde{S}_1\frac{|\nabla S|^2}{S^2}\mathrm{d}x - D_I\int_\Omega \widetilde{I}_1\frac{|\nabla I|^2}{I^2}\mathrm{d}x - \int_\Omega dS\frac{(S - \widetilde{S}_1)^2}{S^2}\mathrm{d}x \\
&\quad + \int_\Omega \alpha\widetilde{S}_1\widetilde{I}_1\Big(2 - \frac{\widetilde{S}_1}{S} - \frac{S}{\widetilde{S}_1}\Big)\mathrm{d}x + \int_\Omega c_2\Big(\frac{\theta\xi VP}{m+V} - \delta P\Big)\mathrm{d}x \\
&\quad + \int_\Omega \beta\frac{\widetilde{S}_1\widetilde{V}_1}{k+\widetilde{V}_1}\Big[2 - \frac{\widetilde{S}_1}{S} - \frac{I}{\widetilde{I}_1} + \frac{V/(k+V)}{\widetilde{V}_1/(k+\widetilde{V}_1)} - \frac{S\widetilde{I}_1 V/(k+V)}{\widetilde{S}_1 I\widetilde{V}_1/(k+\widetilde{V}_1)}\Big]\mathrm{d}x \\
&\quad + \int_\Omega \Big[c_1\eta\widetilde{I}_1\Big(1 - \frac{V}{\widetilde{V}_1} + \frac{I}{\widetilde{I}_1} - \frac{I\widetilde{V}_1}{\widetilde{I}_1 V}\Big) - c_1\Big(1 - \frac{\widetilde{V}_1}{V}\Big)\frac{\xi VP}{m+V}\Big]\mathrm{d}x.
\end{aligned}$$

Notice that $1-x\le -\ln x$ for all $x>0$. Then we have

$$
\begin{aligned}
&2-\frac{\widetilde{S}_1}{S}-\frac{I}{\widetilde{I}_1}+\frac{V/(k+V)}{\widetilde{V}_1/(k+\widetilde{V}_1)}-\frac{S\widetilde{I}_1V/(k+V)}{\widetilde{S}_1I\widetilde{V}_1/(k+\widetilde{V}_1)}\\
&=\Big[2-\frac{\widetilde{S}_1}{S}-\frac{I}{\widetilde{I}_1}-\frac{S\widetilde{I}_1V/(k+V)}{\widetilde{S}_1I\widetilde{V}_1/(k+\widetilde{V}_1)}+1-\frac{V\widetilde{V}_1/(k+\widetilde{V}_1)}{\widetilde{V}_1V/(k+V)}+\frac{V}{\widetilde{V}_1}\Big]\\
&\quad-\frac{k(V-\widetilde{V}_1)^2}{\widetilde{V}_1(k+\widetilde{V}_1)(k+V)}\\
&\le 3-\frac{\widetilde{S}_1}{S}-\frac{I}{\widetilde{I}_1}-\frac{S\widetilde{I}_1V/(k+V)}{\widetilde{S}_1I\widetilde{V}_1/(k+\widetilde{V}_1)}-\frac{V\widetilde{V}_1/(k+\widetilde{V}_1)}{\widetilde{V}_1V/(k+V)}+\frac{V}{\widetilde{V}_1}\\
&=\Big(\frac{V}{\widetilde{V}_1}-\frac{I}{\widetilde{I}_1}\Big)+\Big(1-\frac{\widetilde{S}_1}{S}\Big)+\Big(1-\frac{S\widetilde{I}_1V/(k+V)}{\widetilde{S}_1I\widetilde{V}_1/(k+\widetilde{V}_1)}\Big)+\Big(1-\frac{V\widetilde{V}_1/(k+\widetilde{V}_1)}{\widetilde{V}_1V/(k+V)}\Big)\\
&\le\Big(\frac{V}{\widetilde{V}_1}-\frac{I}{\widetilde{I}_1}\Big)-\ln\frac{\widetilde{S}_1}{S}-\ln\frac{S\widetilde{I}_1V/(k+V)}{\widetilde{S}_1I\widetilde{V}_1/(k+\widetilde{V}_1)}-\ln\frac{V\widetilde{V}_1/(k+\widetilde{V}_1)}{\widetilde{V}_1V/(k+V)}\\
&=\Big(\frac{V}{\widetilde{V}_1}-\ln\frac{V}{\widetilde{V}_1}\Big)-\Big(\frac{I}{\widetilde{I}_1}-\ln\frac{I}{\widetilde{I}_1}\Big),
\end{aligned}
\tag{5.3}
$$

and

$$
1-\frac{V}{\widetilde{V}_1}+\frac{I}{\widetilde{I}_1}-\frac{I\widetilde{V}_1}{\widetilde{I}_1V}\le\Big(\frac{I}{\widetilde{I}_1}-\ln\frac{I}{\widetilde{I}_1}\Big)-\Big(\frac{V}{\widetilde{V}_1}-\ln\frac{V}{\widetilde{V}_1}\Big).
$$

By (5.3) and the above inequality,

$$
\begin{aligned}
\frac{\mathrm{d}\mathcal{L}_2(t)}{\mathrm{d}t}&\le -D_S\int_\Omega\widetilde{S}_1\frac{|\nabla S|^2}{S^2}\mathrm{d}x-D_I\int_\Omega\widetilde{I}_1\frac{|\nabla I|^2}{I^2}\mathrm{d}x-\int_\Omega dS\frac{(S-\widetilde{S}_1)^2}{S^2}\mathrm{d}x\\
&\quad+\int_\Omega\Big[\alpha\widetilde{S}_1\widetilde{I}_1\Big(2-\frac{\widetilde{S}_1}{S}-\frac{S}{\widetilde{S}_1}\Big)\Big]\mathrm{d}x+\int_\Omega c_2\Big(\frac{\theta\xi VP}{m+V}-\delta P\Big)\mathrm{d}x\\
&\quad+\int_\Omega\beta\frac{\widetilde{S}_1\widetilde{V}_1}{k+\widetilde{V}_1}\Big[\Big(\frac{V}{\widetilde{V}_1}-\ln\frac{V}{\widetilde{V}_1}\Big)-\Big(\frac{I}{\widetilde{I}_1}-\ln\frac{I}{\widetilde{I}_1}\Big)\Big]\mathrm{d}x\\
&\quad+\int_\Omega c_1\Big\{\eta\widetilde{I}_1\Big[\Big(\frac{I}{\widetilde{I}_1}-\ln\frac{I}{\widetilde{I}_1}\Big)-\Big(\frac{V}{\widetilde{V}_1}-\ln\frac{V}{\widetilde{V}_1}\Big)\Big]-\frac{V-\widetilde{V}_1}{V}\frac{\xi VP}{m+V}\Big\}\mathrm{d}x.
\end{aligned}
$$

Choosing $c_1=\beta\widetilde{S}_1\widetilde{V}_1/(\eta\widetilde{I}_1(k+\widetilde{V}_1))$ and $c_2=c_1/\theta$, we have

$$
\frac{\mathrm{d}\mathcal{L}_2(t)}{\mathrm{d}t}\le\int_\Omega\Big[\alpha\widetilde{S}_1\widetilde{I}_1\Big(2-\frac{\widetilde{S}_1}{S}-\frac{S}{\widetilde{S}_1}\Big)-dS\Big(1-\frac{\widetilde{S}_1}{S}\Big)^2+c_1\Big(\frac{\xi\widetilde{V}_1}{m}-\frac{\delta}{\theta}\Big)P\Big]\mathrm{d}x.
$$

Therefore, by $\widetilde{V}_1\le\delta m/\theta\xi$, we have $\mathrm{d}\mathcal{L}_2(t)/\mathrm{d}t\le 0$. Obviously, $\mathrm{d}\mathcal{L}_2(t)/\mathrm{d}t=0$ if and only if $S=\widetilde{S}_1$, $I=\widetilde{I}_1$, $V=\widetilde{V}_1$, and $P=0$. So the largest invariant subset of $\{(S,I,V,P)|\frac{\mathrm{d}\mathcal{L}_2(t)}{\mathrm{d}t}=0\}$ is the singleton $\{\widetilde{\mathcal{E}}_1\}$. Consequently, from the invariable principle [15], we conclude that $\widetilde{\mathcal{E}}_1$ is globally asymptotically stable. □

Remark 5.7. Because of the limitations of the study methodology, we did not obtain global stability of the phage-free endemic steady state $\mathcal{E}_1=(S_1(\cdot),I_1(\cdot),V_1(\cdot),0)$ of (2.1); however, we pose an interesting open question.

Conjecture 5.8. *If $\mathcal{R}_0 > 1$ and $\max\{V_1(x)\} \le \min\{\frac{\delta(x)m(x)}{\theta(x)\xi(x)}\}$ for all $x \in \bar{\Omega}$, then $\mathcal{E}_1 = (S_1(\cdot), I_1(\cdot), V_1(\cdot), 0)$ is globally asymptotically stable.*

6. Phage-present endemic steady state

The existence and stability of the phage-present endemic steady state of model (2.1) are difficult to obtain because of the spatial heterogeneity and the saturation incidence (the transmission rate of *vibrio cholerae* from environment to host), so we only discuss the existence and stability of the phage-present endemic steady state for space homogeneous form, i.e., model (2.2).

Suppose that for (2.2) there exists the phage-present endemic steady state $\widetilde{\mathcal{E}}^* = (\widetilde{S}^*, \widetilde{I}^*, \widetilde{V}^*, \widetilde{P}^*)$, then

$$
\begin{aligned}
&\Lambda - \alpha\widetilde{S}^*\widetilde{I}^* - \beta\frac{\widetilde{S}^*\widetilde{V}^*}{k+\widetilde{V}^*} - d\widetilde{S}^* = 0,\\
&\alpha\widetilde{S}^*\widetilde{I}^* + \beta\frac{\widetilde{S}^*\widetilde{V}^*}{k+\widetilde{V}^*} - (\gamma+d)\widetilde{I}^* = 0,\\
&\eta\widetilde{I}^* + \mu\widetilde{V}^* - \mu_0\widetilde{V}^* - \frac{\xi\widetilde{V}^*\widetilde{P}^*}{m+\widetilde{V}^*} = 0,\\
&\frac{\theta\xi\widetilde{V}^*\widetilde{P}^*}{m+\widetilde{V}^*} - \delta\widetilde{P}^* = 0.
\end{aligned}
\tag{6.1}
$$

Simple calculations yield

$$
\widetilde{S}^* = \frac{\Lambda(k+\widetilde{V}^*)}{(\widetilde{I}^*+d)(k+\widetilde{V}^*)+\beta\widetilde{V}^*}, \quad \widetilde{V}^* = \frac{\delta m}{\theta\xi-\delta}, \quad \widetilde{P}^* = \frac{\theta\eta}{\delta}\widetilde{I}^* - \frac{m\theta(\mu_0-\mu)}{\theta\xi-\delta}
$$

and $\widetilde{I}^*$ is the positive root of $h(I) = \bar{A}I^2 + \bar{B}I + \bar{C}$, where

$$
\bar{A} = -\frac{\alpha(\gamma+d)}{d}, \quad \bar{B} = \frac{\alpha\Lambda}{d} - \frac{\beta\delta m(\gamma+d)}{d[k\theta\xi+\delta(m-k)]} - (\gamma+d),
$$
$$
\bar{C} = \beta\frac{\frac{\Lambda}{d}\cdot\frac{\delta m}{\theta\xi-\delta}}{k+\frac{\delta m}{\theta\xi-\delta}} = \frac{\beta\Lambda\delta m}{d[k\theta\xi+\delta(m-k)]} > 0.
$$

Since $\bar{A} < 0$ and $\bar{C} > 0$, equation $h(I) = 0$ has two real roots, one is positive and one is negative. We define the phage invasion reproduction number as

$$
\widetilde{\mathcal{R}}_0^p = \frac{\eta(\theta\xi-\delta)}{m\delta(\mu_0-\mu)}\widetilde{I}^*.
$$

When $\widetilde{\mathcal{R}}_0^p > 1$, it ensure that $\widetilde{P}^* > 0$. Then (2.2) has a unique phage-present endemic steady state $\widetilde{\mathcal{E}}^* = (\widetilde{S}^*, \widetilde{I}^*, \widetilde{V}^*, \widetilde{P}^*)$. To establish the global asymptotic stability of $\mathcal{E}_1$ of (2.2), we need to do the following preparatory work.

We define the set $\mathbb{X}_0 = \{\psi(x) = (\psi_1, \psi_2, \psi_3, \psi_4) \in \mathbb{X}^+ : \psi_1 > 0,\ \psi_2 \ne 0,\ \psi_3 \ne 0,\ \psi_4 \ne 0\}$ and $\partial\mathbb{X}_0 = \mathbb{X}^+ \backslash \mathbb{X}_0 = \{\psi(x) \in \mathbb{X}^+ : \psi_2\psi_3\psi_4 = 0\}$. Similar to Lemma 4.3, it follows that $\mathbb{X}_0$ is the positive invariant set for solution semiflow $\widetilde{\Phi}(t)$ of (2.2). We set $\widetilde{M_\partial} := \{\psi \in \partial\mathbb{X}_0 : \widetilde{\Phi}(t)\psi \in \partial\mathbb{X}_0,\ \forall t \ge 0\}$, and $\widetilde{\omega}(\psi)$ be the omega limit set of the orbit $\widetilde{\mathcal{G}}^+ := \{\widetilde{\Phi}(t)\psi : t \ge 0\}$.

Lemma 6.1. *If $\widetilde{\mathcal{R}}_0 > 1$, then there exists $\varepsilon_2 > 0$ such that the solution semiflow $\widetilde{\Phi}(t)$ of (2.2) satisfies $\limsup_{t\to\infty} \|\widetilde{\Phi}(t)\psi - \widetilde{\mathcal{E}}_0\|_{\mathbb{X}^+} \ge \varepsilon_2$ for any $\psi \in \mathbb{X}_0$.*

The proof of the above lemma is similar to that of Lemma 5.2, we omit it here.

Lemma 6.2. *If $\widetilde{\mathcal{R}}_0 > 1$, and $\widetilde{V}_1 > \widetilde{V}^*$, then there exists $\varepsilon_3 > 0$ such that for any $\psi \in \mathbb{X}_0$, the solution semiflow $\widetilde{\Phi}(t)$ of (2.2) satisfies $\limsup_{t\to\infty} \|\widetilde{\Phi}(t)\psi - \widetilde{\mathcal{E}}_1\|_{\mathbb{X}^+} \geq \varepsilon_3$, where $\widetilde{V}_1$ is given by (5.2).*

Proof. From $\widetilde{V}_1 > \widetilde{V}^*$, we can choose a small $\varepsilon_3 > 0$ such that

$$\frac{\theta\xi(\widetilde{V}_1 - \varepsilon_3)}{m + \widetilde{V}_1} - \delta > \frac{\theta\xi\widetilde{V}^*}{m + \widetilde{V}^*} - \delta = 0. \tag{6.2}$$

Suppose, by contradiction, there is $\psi_0 \in \mathbb{X}_0$ such that $\limsup_{t\to\infty} \|\widetilde{\Phi}(t)\psi_0 - \widetilde{\mathcal{E}}_1\|_{\mathbb{X}^+} < \varepsilon_3$. This inequality in the sense that there exists $t_3 > 0$ such that $\widetilde{V}_1 - \varepsilon_3 < V(x, t; \psi_0)$. Thus, from the forth equation of (2.2), we obtain

$$\frac{\partial P}{\partial t} \geq \frac{\theta\xi(\widetilde{V}_1 - \varepsilon_3)}{m + \widetilde{V}_1 - \varepsilon_3} P - \delta P > \frac{\theta\xi(\widetilde{V}_1 - \varepsilon_3)}{m + \widetilde{V}_1} P - \delta P, \quad x \in \Omega,\ t > t_3.$$

By the above discussion on $\mathbb{X}_0$, we have $P(x, t) > 0$ for $x \in \bar{\Omega}$ and $t > 0$. Therefore, there is a positive constant b such that $P(x, t_3; \psi_0) \geq bP_0(x)$. Utilizing the standard comparison principle, we obtain

$$P(x, t) \geq bP_0(x)e^{\left[\frac{\theta\xi(\widetilde{V}_1 - \varepsilon_3)}{m + \widetilde{V}_1} - \delta\right](t - t_3)}, \quad x \in \Omega,\ t > t_3.$$

From (6.2) we have $\lim_{t\to\infty} P(x, t) = \infty$. This is a contradiction with the boundedness of $P(x, t)$ by Corollary 2.7. The proof is complete. $\square$

Next, we turn to the uniform persistence of model (2.2).

Theorem 6.3. *If $\widetilde{\mathcal{R}}_0 > 1$, and $\widetilde{V}_1 > \widetilde{V}^*$, then there is a $\tilde{\varrho} > 0$ such that for the initial value $\psi = (\psi_1, \psi_2, \psi_3, \psi_4) \in \mathbb{X}^+$ with $\psi_2 \neq 0$, $\psi_3 \neq 0$ and $\psi_4 \neq 0$, solution $\widetilde{u}(\cdot, t; \psi) = (S(\cdot, t; \psi), I(\cdot, t; \psi), V(\cdot, t; \psi),\ P(\cdot, t; \psi))$ of (2.2) satisfies $\liminf_{t\to\infty} \widetilde{u}(\cdot, t; \psi) \geq (\tilde{\varrho}, \tilde{\varrho}, \tilde{\varrho}, \tilde{\varrho})$, uniformly for all $x \in \bar{\Omega}$.*

Proof. firstly,, we verify that $\widetilde{\omega}(\psi) = \{\widetilde{\mathcal{E}}_0\} \cup \{\widetilde{\mathcal{E}}_1\}$, $\psi \in \widetilde{M}_\partial$. In fact, if $\psi \in \widetilde{M}_\partial$, then $\widetilde{\Phi}(t)\psi \in \widetilde{M}_\partial$, $t \geq 0$. Thus $I(\cdot, t) \equiv 0$ or $V(\cdot, t) \equiv 0$ or $P(\cdot, t) \equiv 0$ for $t \geq 0$. For the case, $I(\cdot, t) \equiv 0$, it follows from the second equation of (2.2) that $\beta(\cdot)\frac{S(\cdot,t;\psi)V(\cdot,t;\psi)}{k(\cdot)+V(\cdot,t;\psi)} = 0$; that is, $V(\cdot, t) \equiv 0$. Then, we have $\lim_{t\to\infty} S(\cdot, t) = \bar{U}(\cdot)$ according to Lemma 2.2. From the forth equation of (2.2), one can obtain that $\lim_{t\to\infty} P(\cdot, t) = 0$. This shows $\widetilde{\omega}(\psi) = \{\widetilde{\mathcal{E}}_0\}$. For case $V(\cdot, t) \equiv 0$, then from the third equation of (2.2), one can obtain $I(\cdot, t) \equiv 0$. Similarly, we also have $\lim_{t\to\infty} S(\cdot, t) = \bar{U}(\cdot)$ and $\lim_{t\to\infty} P(\cdot, t) = 0$. This also shows $\widetilde{\omega}(\psi) = \{\widetilde{\mathcal{E}}_0\}$. In the case that $P(\cdot, t) \equiv 0$, then (2.2) becomes the phage-free subsystem

$$\begin{gathered}
\frac{\partial S(x,t)}{\partial t} = D_S\Delta S + \Lambda - \alpha SI - \beta\frac{SV}{k(x) + V} - dS, \quad x \in \Omega,\ t > 0, \\
\frac{\partial I(x,t)}{\partial t} = D_I\Delta I + \alpha SI + \beta\frac{SV}{k(x) + V} - (\gamma + d)I, \quad x \in \Omega,\ t > 0, \\
\frac{\partial V(x,t)}{\partial t} = \eta I + \mu V - \mu_0 V, \quad x \in \Omega,\ t > 0, \\
\frac{\partial S(x,t)}{\partial \nu} = \frac{\partial I(x,t)}{\partial \nu} = 0, \quad x \in \partial\Omega,\ t > 0.
\end{gathered} \tag{6.3}$$

When $\widetilde{\mathcal{R}}_0 > 1$, model (6.3) has a positive steady state $\widetilde{\mathcal{E}}_1^\natural = (\widetilde{S}_1, \widetilde{I}_1, \widetilde{V}_1)$. We choose that a Lyapunov function

$$\mathcal{L}_3(t) = \int_\Omega \widetilde{S}_1 g\Big(\frac{S}{\widetilde{S}_1}\Big)\mathrm{d}x + \int_\Omega \widetilde{I}_1 g\Big(\frac{I}{\widetilde{I}_1}\Big)\mathrm{d}x + \frac{\beta \widetilde{S}_1 \widetilde{V}_1}{\eta \widetilde{I}_1 (k+\widetilde{V}_1)} \int_\Omega \widetilde{V}_1 g\Big(\frac{V}{\widetilde{V}_1}\Big)\mathrm{d}x.$$

Using an argument similar to the one in the proof of Theorem 5.6, we obtain

$$\frac{\mathrm{d}\mathcal{L}_3(t)}{\mathrm{d}t} \le \int_\Omega \alpha \widetilde{S}_1 \widetilde{I}_1 \Big(2 - \frac{\widetilde{S}_1}{S} - \frac{S}{\widetilde{S}_1}\Big)\mathrm{d}x - \int_\Omega dS\Big(1 - \frac{\widetilde{S}_1}{S}\Big)^2 \mathrm{d}x.$$

Therefore, by the conditions of the theorem, we have $\frac{\mathrm{d}\mathcal{L}_3(t)}{\mathrm{d}t} \le 0$. Obviously, $\frac{\mathrm{d}\mathcal{L}_3(t)}{\mathrm{d}t} = 0$ if and only if $S = \widetilde{S}_1$, $I = \widetilde{I}_1$, $V = \widetilde{V}_1$. So the largest invariant subset of $\{(S,I,V) | \frac{\mathrm{d}\mathcal{L}_3(t)}{\mathrm{d}t} = 0\}$ is the singleton $\{\widetilde{\mathcal{E}}_1^\natural\}$. Then, from invariable principle in [15], we conclude that $\widetilde{\mathcal{E}}_1^\natural$ is globally asymptotically stable. Thus, we have $\lim_{t\to\infty}(S(x,t), I(x,t), V(x,t)) = (\widetilde{S}_1, \widetilde{I}_1, \widetilde{V}_1)$. This also shows $\widetilde{\omega}(\psi) = \{\widetilde{\mathcal{E}}_1\}$. From the above discussion, we know that $\widetilde{\omega}(\psi) = \{\widetilde{\mathcal{E}}_0\} \cup \{\widetilde{\mathcal{E}}_1\}$, for any $\psi \in \widetilde{M_\partial}$.

We define a function $\tilde{p} : \mathbb{X}^+ \to [0,\infty)$ by

$$\tilde{p}(\psi) = \min\big\{ \min_{x\in\bar{\Omega}} \psi_2(x), \min_{x\in\bar{\Omega}} \psi_3(x), \min_{x\in\bar{\Omega}} \psi_4(x) \big\}, \quad \psi \in \mathbb{X}^+.$$

Similar argument as in Theorem 5.3, we easily know that $\tilde{p}$ is a generalized distance function for the semiflow $\widetilde{\Phi}(t) : \mathbb{X}^+ \to \mathbb{X}^+$. From the above discussion, we know that for any $\psi \in \widetilde{M_\partial}$, $\widetilde{\omega}(\psi) = \{\widetilde{\mathcal{E}}_0\} \cup \{\widetilde{\mathcal{E}}_1\}$, namely, any forward orbit of $\widetilde{\Phi}(t)$ in $\widetilde{M_\partial}$ tends to $\{\widetilde{\mathcal{E}}_0\} \cup \{\widetilde{\mathcal{E}}_1\}$ as $t \to +\infty$. Hence, no subset of $\{\widetilde{\mathcal{E}}_0\} \cup \{\widetilde{\mathcal{E}}_1\}$ forms a cycle in $\partial\mathbb{X}_0$. Lemma 6.1 and Lemma 6.2 imply that $\{\widetilde{\mathcal{E}}_0\} \cup \{\widetilde{\mathcal{E}}_1\}$ is an isolated invariant set in $\mathbb{X}^+$ and $W^s(\{\widetilde{\mathcal{E}}_0\}) \cap \mathbb{X}_0 = \emptyset$, $W^s(\{\widetilde{\mathcal{E}}_1\}) \cap \mathbb{X}_0 = \emptyset$, where $W^s(\{\widetilde{\mathcal{E}}_0\})$ and $W^s(\{\widetilde{\mathcal{E}}_1\})$ are the stable subset of $\{\widetilde{\mathcal{E}}_0\}$ and $\{\widetilde{\mathcal{E}}_1\}$, respectively. By [44, Theorem 3], this yields that there exists a $\tilde{\varrho}_1 > 0$ such that $\min_{\phi\in\omega(\psi)} \tilde{p}(\phi) > \tilde{\varrho}_1$, for all $\psi \in \mathbb{X}_0$. Hence, $\liminf_{t\to\infty} I(\cdot,t;\psi) > \tilde{\varrho}_1$, $\liminf_{t\to\infty} V(\cdot,t;\psi) > \tilde{\varrho}_1$, and $\liminf_{t\to\infty} P(\cdot,t;\psi) > \tilde{\varrho}_1$, for $\psi \in \mathbb{X}_0$. Recall that ϱ_2 in Theorem 5.3, let $\tilde{\varrho} = \min\{\tilde{\varrho}_1, \varrho_2\}$. The uniform persistence is valid. $\square$

Finally, we discuss the global stability of $\widetilde{\mathcal{E}}^* = (\widetilde{S}^*, \widetilde{I}^*, \widetilde{V}^*, \widetilde{P}^*)$. The following assumption is necessary.

(H3) $(1 - \frac{m+\widetilde{V}^*}{m+V})(\frac{V}{\widetilde{V}^*} - \frac{P}{\widetilde{P}^*}) \le 0$ for all $V > 0$ and $P > 0$.

Theorem 6.4. *Assume* (H3) *holds. If* $\widetilde{\mathcal{R}}_0 > 1$ *and* $\widetilde{\mathcal{R}}_0^p > 1$*, then the phage-present endemic steady state* $\widetilde{\mathcal{E}}^*$ *is globally asymptotically stable.*

Proof. We choose the Lyapunov function

$$\begin{aligned}\mathcal{L}_4(t) &= \int_\Omega \widetilde{S}^* g\Big(\frac{S}{\widetilde{S}^*}\Big)\mathrm{d}x + \int_\Omega \widetilde{I}^* g\Big(\frac{I}{\widetilde{I}^*}\Big)\mathrm{d}x \\ &\quad + c_1 \int_\Omega \widetilde{V}^* g\Big(\frac{V}{\widetilde{V}^*}\Big)\mathrm{d}x + c_2 \int_\Omega \widetilde{P}^* g\Big(\frac{P}{\widetilde{P}^*}\Big)\mathrm{d}x,\end{aligned}$$

where constants the $c_1 > 0$ and $c_2 > 0$ will be determined below, and $g(x) = x - 1 - \ln x$. By calculating the derivative of $\mathcal{L}_4(t)$, we have

$$\frac{\mathrm{d}\mathcal{L}_4(t)}{\mathrm{d}t} = \int_\Omega \Big(1 - \frac{\widetilde{S}^*}{S}\Big)\Big(D_S \Delta S + \Lambda - \alpha SI - \beta \frac{SV}{k+V} - dS\Big)\mathrm{d}x$$

$$
\begin{aligned}
&+\int_\Omega \Big(1-\frac{\widetilde{I}^*}{I}\Big)\Big(D_I\Delta I+\alpha SI+\beta\frac{SV}{k+V}-(\gamma+d)I\Big)\mathrm{d}x\\
&+c_1\int_\Omega \Big(1-\frac{\widetilde{V}^*}{V}\Big)\Big(\eta I+\mu V-\mu_0 V-\frac{\xi VP}{m+V}\Big)\mathrm{d}x\\
&+c_2\int_\Omega \Big(1-\frac{\widetilde{P}^*}{P}\Big)\Big(\frac{\theta\xi VP}{m+V}-\delta P\Big)\mathrm{d}x.
\end{aligned}
$$

With the help of (6.1), we obtain

$$
\begin{aligned}
\frac{\mathrm{d}\mathcal{L}_4(t)}{\mathrm{d}t}&=-D_S\int_\Omega \widetilde{S}^*\frac{|\nabla S|^2}{S^2}\mathrm{d}x-D_I\int_\Omega \widetilde{I}^*\frac{|\nabla I|^2}{I^2}\mathrm{d}x-\int_\Omega dS\Big(1-\frac{\widetilde{S}^*}{S}\Big)^2\mathrm{d}x\\
&\quad+\int_\Omega \alpha\widetilde{S}^*\widetilde{I}^*\Big(2-\frac{\widetilde{S}^*}{S}-\frac{S}{\widetilde{S}^*}\Big)\mathrm{d}x+\int_\Omega c_1\eta\widetilde{I}^*\Big[1-\frac{V}{\widetilde{V}^*}+\frac{I}{\widetilde{I}^*}-\frac{I\widetilde{V}^*}{\widetilde{I}^*V}\Big]\mathrm{d}x\\
&\quad+\int_\Omega \beta\frac{\widetilde{S}^*\widetilde{V}^*}{k+\widetilde{V}^*}\Big[2-\frac{\widetilde{S}^*}{S}-\frac{I}{\widetilde{I}^*}+\frac{V/(k+V)}{\widetilde{V}^*/(k+\widetilde{V}^*)}-\frac{S\widetilde{I}^*V/(k+V)}{\widetilde{S}^*I\widetilde{V}^*/(k+\widetilde{V}^*)}\Big]\mathrm{d}x\\
&\quad-\int_\Omega c_1\frac{\xi\widetilde{V}^*\widetilde{P}^*}{m+\widetilde{V}^*}\Big[1-\frac{V}{\widetilde{V}^*}+\Big(1-\frac{\widetilde{V}^*}{V}\Big)\frac{VP/(m+V)}{\widetilde{V}^*\widetilde{P}^*/(m+\widetilde{V}^*)}\Big]\mathrm{d}x\\
&\quad+\int_\Omega c_2\frac{\theta\xi\widetilde{V}^*\widetilde{P}^*}{m+\widetilde{V}^*}\Big[1-\frac{P}{\widetilde{P}^*}+\Big(1-\frac{\widetilde{P}^*}{P}\Big)\frac{VP/(m+V)}{\widetilde{V}^*\widetilde{P}^*/(m+\widetilde{V}^*)}\Big]\mathrm{d}x.
\end{aligned}
$$

Noticing the fact that $1-x\le-\ln x$ for all $x>0$, we have

$$
\begin{aligned}
&2-\frac{\widetilde{S}^*}{S}-\frac{I}{\widetilde{I}^*}+\frac{V/(k+V)}{\widetilde{V}^*/(k+\widetilde{V}^*)}-\frac{S\widetilde{I}^*V/(k+V)}{\widetilde{S}^*I\widetilde{V}^*/(k+\widetilde{V}^*)}\\
&=\Big[2-\frac{\widetilde{S}^*}{S}-\frac{I}{\widetilde{I}^*}-\frac{S\widetilde{I}^*V/(k+V)}{\widetilde{S}^*I\widetilde{V}^*/(k+\widetilde{V}^*)}+1-\frac{V\widetilde{V}^*/(k+\widetilde{V}^*)}{\widetilde{V}^*V/(k+V)}+\frac{V}{\widetilde{V}^*}\Big]\\
&\quad-\frac{k(V-\widetilde{V}^*)^2}{\widetilde{V}^*(k+\widetilde{V}^*)(k+V)}\\
&\le 3-\frac{\widetilde{S}^*}{S}-\frac{I}{\widetilde{I}^*}-\frac{S\widetilde{I}^*V/(k+V)}{\widetilde{S}^*I\widetilde{V}^*/(k+\widetilde{V}^*)}-\frac{V\widetilde{V}^*/(k+\widetilde{V}^*)}{\widetilde{V}^*V/(k+V)}+\frac{V}{\widetilde{V}^*}\\
&=\Big(\frac{V}{\widetilde{V}^*}-\frac{I}{\widetilde{I}^*}\Big)+\Big(1-\frac{\widetilde{S}^*}{S}\Big)\\
&\quad+\Big(1-\frac{S\widetilde{I}^*V/(k+V)}{\widetilde{S}^*I\widetilde{V}^*/(k+\widetilde{V}^*)}\Big)+\Big(1-\frac{V\widetilde{V}^*/(k+\widetilde{V}^*)}{\widetilde{V}^*V/(k+V)}\Big)\\
&\le\Big(\frac{V}{\widetilde{V}^*}-\frac{I}{\widetilde{I}^*}\Big)-\ln\frac{\widetilde{S}^*}{S}-\ln\frac{S\widetilde{I}^*V/(k+V)}{\widetilde{S}^*I\widetilde{V}^*/(k+\widetilde{V}^*)}-\ln\frac{V\widetilde{V}^*/(k+\widetilde{V}^*)}{\widetilde{V}^*V/(k+V)}\\
&=\Big(\frac{V}{\widetilde{V}^*}-\ln\frac{V}{\widetilde{V}^*}\Big)-\Big(\frac{I}{\widetilde{I}^*}-\ln\frac{I}{\widetilde{I}^*}\Big),
\end{aligned}
\tag{6.4}
$$

and

$$
1-\frac{V}{\widetilde{V}^*}+\frac{I}{\widetilde{I}^*}-\frac{I\widetilde{V}^*}{\widetilde{I}^*V}\le\Big(\frac{I}{\widetilde{I}^*}-\ln\frac{I}{\widetilde{I}^*}\Big)-\Big(\frac{V}{\widetilde{V}^*}-\ln\frac{V}{\widetilde{V}^*}\Big).\tag{6.5}
$$

Applying (6.4) and (6.5), we have

$$\begin{aligned}
\frac{\mathrm{d}\mathcal{L}_4(t)}{\mathrm{d}t} \le& -D_S\int_\Omega \widetilde{S}^*\frac{|\nabla S|^2}{S^2}\mathrm{d}x - D_I\int_\Omega \widetilde{I}^*\frac{|\nabla I|^2}{I^2}\mathrm{d}x\\
&-\int_\Omega dS\Big(1-\frac{\widetilde{S}^*}{S}\Big)^2\mathrm{d}x + \int_\Omega \alpha\widetilde{S}^*\widetilde{I}^*\Big(2-\frac{\widetilde{S}^*}{S}-\frac{S}{\widetilde{S}^*}\Big)\mathrm{d}x\\
&+\int_\Omega \beta\frac{\widetilde{S}^*\widetilde{V}^*}{k+\widetilde{V}^*}\Big[\Big(\frac{V}{\widetilde{V}^*}-\ln\frac{V}{\widetilde{V}^*}\Big)-\Big(\frac{I}{\widetilde{I}^*}-\ln\frac{I}{\widetilde{I}^*}\Big)\Big]\mathrm{d}x\\
&+\int_\Omega c_1\eta\widetilde{I}^*\Big[\Big(\frac{I}{\widetilde{I}^*}-\ln\frac{I}{\widetilde{I}^*}\Big)-\Big(\frac{V}{\widetilde{V}^*}-\ln\frac{V}{\widetilde{V}^*}\Big)\Big]\mathrm{d}x\\
&-\int_\Omega c_1\frac{\xi\widetilde{V}^*\widetilde{P}^*}{m+\widetilde{V}^*}\Big[1-\frac{V}{\widetilde{V}^*}+\Big(1-\frac{\widetilde{V}^*}{V}\Big)\frac{VP/(m+V)}{\widetilde{V}^*\widetilde{P}^*/(m+\widetilde{V}^*)}\Big]\mathrm{d}x\\
&+\int_\Omega c_2\frac{\theta\xi\widetilde{V}^*\widetilde{P}^*}{m+\widetilde{V}^*}\Big[1-\frac{P}{\widetilde{P}^*}+\Big(1-\frac{\widetilde{P}^*}{P}\Big)\frac{VP/(m+V)}{\widetilde{V}^*\widetilde{P}^*/(m+\widetilde{V}^*)}\Big]\mathrm{d}x.
\end{aligned}$$

As the choice of c_1 and c_2 in Theorem 5.6, let $c_1=\beta\widetilde{S}^*\widetilde{V}^*/(\eta\widetilde{I}^*(k+\widetilde{V}^*))$ and $c_2=c_1/\theta$, then we have

$$\begin{aligned}
\frac{\mathrm{d}\mathcal{L}_4(t)}{\mathrm{d}t} \le& \int_\Omega \alpha\widetilde{S}^*\widetilde{I}^*\Big(2-\frac{\widetilde{S}^*}{S}-\frac{S}{\widetilde{S}^*}\Big)\mathrm{d}x - \int_\Omega dS\Big(1-\frac{\widetilde{S}^*}{S}\Big)^2\mathrm{d}x\\
&+\int_\Omega c_1\frac{\xi\widetilde{V}^*\widetilde{P}^*}{m+\widetilde{V}^*}\Big[\Big(1-\frac{m+\widetilde{V}^*}{m+V}\Big)\Big(\frac{V}{\widetilde{V}^*}-\frac{P}{\widetilde{P}^*}\Big)\Big]\mathrm{d}x.
\end{aligned}$$

Therefore, by the condition (H3), we have $\frac{\mathrm{d}\mathcal{L}_4(t)}{\mathrm{d}t}\le 0$. Obviously, $\frac{\mathrm{d}\mathcal{L}_4(t)}{\mathrm{d}t}=0$ if and only if $S=\widetilde{S}^*$, $I=\widetilde{I}^*$, $V=\widetilde{V}^*$ and $P=\widetilde{P}^*$. So the largest invariant subset of $\{(S,I,V,P)|\frac{\mathrm{d}\mathcal{L}_4(t)}{\mathrm{d}t}=0\}$ is the singleton $\{\widetilde{\mathcal{E}}^*\}$. Consequently, from the invariable principle [15], we conclude that $\widetilde{\mathcal{E}}^*$ is globally asymptotically stable if $\widetilde{\mathcal{R}}_0>1$ and $\widetilde{\mathcal{R}}_0^p>1$. The proof is complete. □

On the existence and stability of the phage-present endemic steady state of (2.1), we also propose an interesting opening question.

Conjecture 6.5. *If $\mathcal{R}_0>1$ and $\min\{V_1(x)\}>\max\{\frac{\delta(x)m(x)}{\theta(x)\xi(x)}\}$ for all $x\in\bar{\Omega}$, then* (2.1) *has a unique steady state $\mathcal{E}^*=(S^*(x),I^*(x),V^*(x),P^*(x))$ which is globally asymptotically stable.*

7. Numerical simulations

In this section, we present some numerical examples to illustrate the main results and verify two opening questions, as well as to investigate the effects of the strength of spatial heterogeneity on basic reproduction number $\mathcal{R}_0$.

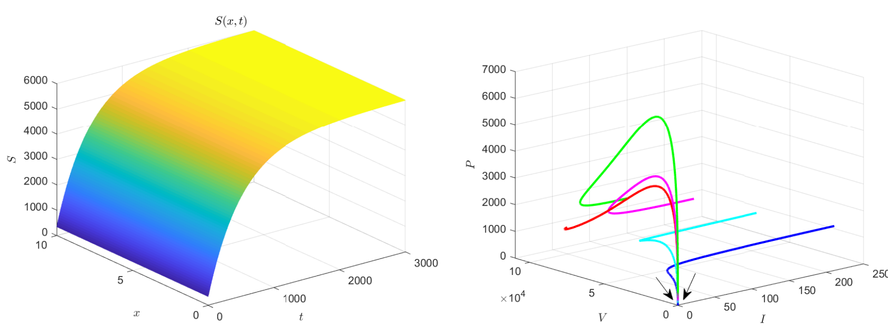
7.1. **Spatially homogeneous case.** In this subsection, we illustrate the dynamics of (2.2), that is, Theorems 4.5, 5.6, and 6.4. To simply the discussion, we choose $\Omega=[0,10]\subset\mathbb{R}$. According to the biological significance of our model and the relevant references, some main model parameters are fixed as Table 1.

Example 7.1. For the stability of the disease-free steady state $\widetilde{\mathcal{E}}_0$, we choose $\alpha=3.4286\times10^{-5}$, $\beta=0.018$, $k=1\times10^7$, $\eta=1.2$, $m=2.2\times10^6$. It follows

Table 1. Descriptions and values of parameters

Par.	Description	Value	Ref.
Λ	Recruitment rate of susceptible hosts	12	[22]
d	Natural death rate of susceptible and infected hosts	0.002	[33]
γ	Removal rate of infected hosts expect for natural death	0.205	[26]
μ	Self-growth rate of bacteria	0.001	[26]
μ_0	Natural death rate of bacteria	0.034	[26]
ξ	Adsorption rate of phage	0.02	[20, 43]
θ	Burst size of bacteria	90	[20]
δ	Loss rate of phage	0.01062	[26]
D_S	Diffusion coefficient of susceptible hosts	0.01	-
D_I	Diffusion coefficient of infected hosts	0.008	-

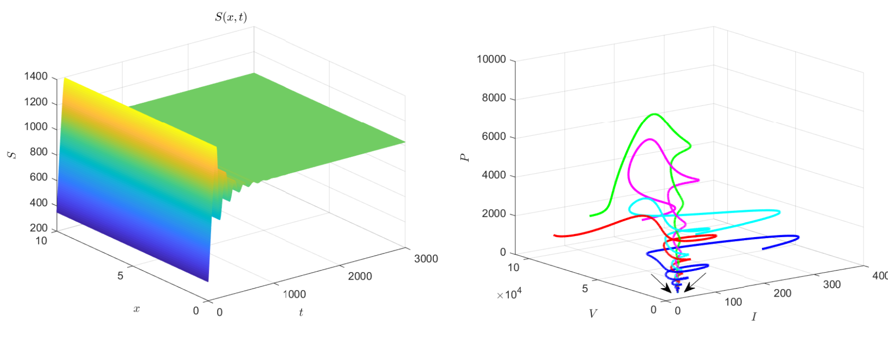
that $\widetilde{\mathcal{R}}_0 \approx 0.9957 < 1$ by direct calculations. This implies that the bacteria will be eliminated and the disease is extinct in host population. From Figure 1(a), we note that the distribution of susceptible host tend to the stable value $\frac{\Lambda}{d}$. And the plot in Figure 1(b) shows that regardless of the initial values of the infected host, environmental viruses and phages, the disease eventually converges to 0 as $t \to \infty$. This suggests that as long as the basic reproduction number is less than 1, the disease eventually disappears from the population regardless of the initial value status at the time of the outbreak.



(a) (b)

Figure 1. Global asymptotic stability of disease-free steady state $\widetilde{\mathcal{E}}_0$ of (2.2) with $\widetilde{\mathcal{R}}_0 \approx 0.9957 < 1$.

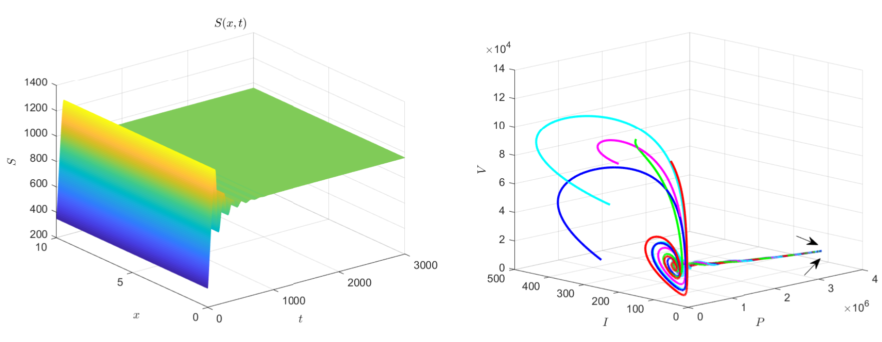
Example 7.2. For the stability of the phage-free endemic steady state $\widetilde{\mathcal{E}}_1$, we choose $\alpha = 2 \times 10^{-4}$, $\beta = 0.05$, $k = 1.1 \times 10^7$, $\eta = 6$, $m = 2.3 \times 10^6$. It follows that $\widetilde{\mathcal{R}}_0 \approx 5.8211 > 1$, and $\widetilde{V}_1 \approx 8729.4789 \le \frac{\delta m}{\theta \xi} = 13570$ by direct calculations. All conditions of Theorem 5.6 hold; therefore, the phage-free endemic steady state $\widetilde{\mathcal{E}}_1$ of (2.2) is globally asymptotically stable. This is shown in Figures 2 (a) and (b), due to the low rate of phages transformation, phages eventually become extinct regardless of their initial state. While the disease forms endemically in the host population, the susceptible, infected and environmental viruses tend to their respective steady states.



(a) (b)

FIGURE 2. Global asymptotic stability of phage-free endemic steady state $\widetilde{\mathcal{E}}_1$ of (2.2) with $\widetilde{\mathcal{R}}_0 \approx 5.8211 > 1$ and $\widetilde{V}_1 \approx 8729.4789 \leq \frac{\delta m}{\theta \xi} = 13570$.

Example 7.3. For the stability of the phage-present endemic steady state $\widetilde{\mathcal{E}}^*$, we choose $\alpha = 2.1429 \times 10^{-4}$, $\beta = 0.07$, $k = 1.2 \times 10^7$, $\eta = 18$, $m = 2.4 \times 10^6$. By direct calculations, we obtain $\widetilde{\mathcal{R}}_0 \approx 6.3034 > 1$ and $\widetilde{\mathcal{R}}_1 \approx 1.8653 > 1$. Then all conditions of Theorem 6.4 are satisfied. The phage-present endemic steady state is globally asymptotically stable, which is also shown in Figures 3(a) and (b). Numerical simulations also showed that since the phages do not have the ability to reproduce itself, it does not eliminate the spread of disease between hosts, but only reduces the distribution of infected hosts. Elimination of disease is more important to reduce the rates of horizontal transmission and environmental transmission.

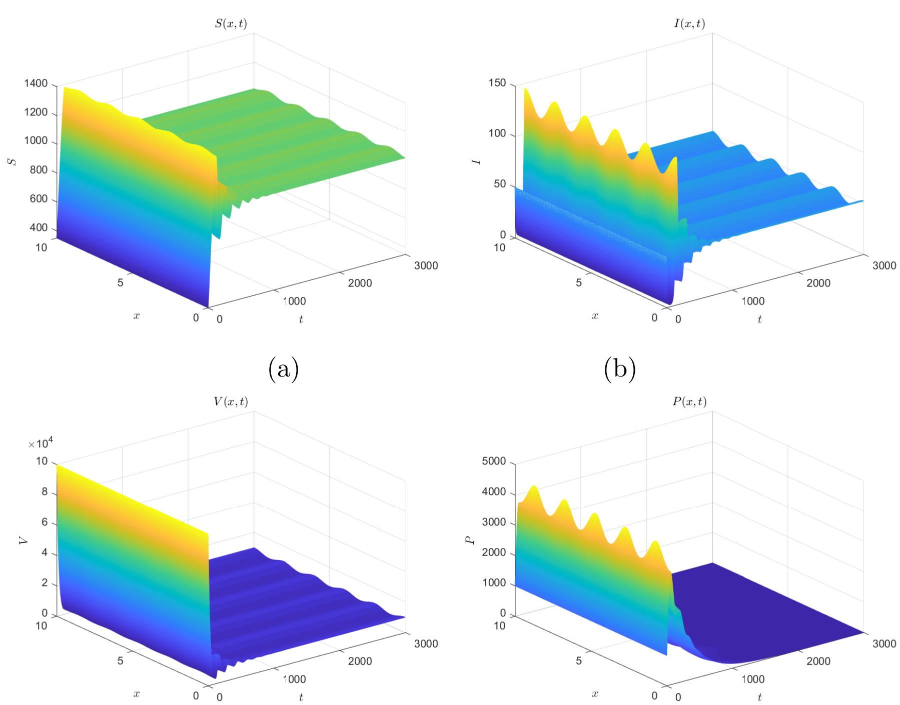


(a) (b)

FIGURE 3. Global asymptotic stability of phage-present endemic steady state $\widetilde{\mathcal{E}}^*$ of (2.2) with $\widetilde{\mathcal{R}}_0 \approx 6.3034 > 1$ and $\widetilde{\mathcal{R}}_1 \approx 1.8653 > 1$.

7.2. **Spatially heterogeneous case.** Now we explain the dynamics of spatial heterogeneity model (2.1), especially those two open problems.

Example 7.4. For the stability of phage-free steady state $\mathcal{E}_1$, we choose rhe parameters of (2.1) as follows: $D_S = 0.01$, $D_I = 0.008$, $\Lambda = 12 \times (1 + 0.05\cos(\pi x))$, $\alpha = 2 \times 10^{-4} \times (1 + 0.1\cos(\pi x))$, $\beta = 0.05 \times (1 + 0.1\cos(\pi x))$, $k = 1.1 \times 10^7 \times (1 + 0.1\cos(\pi x))$, $d = 0.002 \times (1 + 0.05\cos(\pi x))$, $\gamma = 0.205 \times (1 + 0.05\cos(\pi x))$, $\eta = 6 \times (1 + 0.1\cos(\pi x))$, $\mu = 0.001 \times (1 + 0.05\cos(\pi x))$, $\mu_0 = 0.034 \times (1 + 0.05\cos(\pi x))$, $\xi = 0.02 \times (1 + 0.05\cos(\pi x))$, $m = 2.3 \times 10^6 \times (1 + 0.1\cos(\pi x))$, $\theta = 90 \times (1 + 0.05\cos(\pi x))$ and $\delta = 0.01062 \times (1 + 0.05\cos(\pi x))$. By using the numerical scheme in [53, Appendix], we have $\mathcal{R}_0 \approx 5.8396 > 1$ and $\max\{V_1(x)\} \approx 10007.6993 < \min\{\frac{\delta(x)m(x)}{\theta(x)\xi(x)}\} \approx 12855.7895$. Thus all the conditions of Conjecture 5.8 hold. The plots in Figures 4 (a)–(d) show that the phage-free endemic steady state $\mathcal{E}_1$ is globally asymptotically stable. Further, the distributions of susceptible, infected and environmental viruses are significant different due to spatial heterogeneity.



(a) (b)

(c) (d)

FIGURE 4. Global asymptotic stability of phage-free endemic steady state $\mathcal{E}_1 = (S_1(x), I_1(x), V_1(x), 0)$ of (2.1) with $\mathcal{R}_0 \approx 5.8396 > 1$ and $\max\{V_1(x)\} \approx 10007.6993 \leq \min\{\frac{\delta(x)m(x)}{\theta(x)\xi(x)}\} \approx 12855.7895$.

Example 7.5. For the stability of the phage-present endemic steady state $\mathcal{E}^*$, we choose the parameters of model (2.1) as follows: $D_S = 0.01$, $D_I = 0.008$,

$\Lambda = 12 \times (1 + 0.05\cos(\pi x))$, $\alpha = 2.1429 \times 10^{-4} \times (1 + 0.1\cos(\pi x))$, $\beta = 0.07 \times (1 + 0.1\cos(\pi x))$, $k = 1.2 \times 10^{7} \times (1 + 0.1\cos(\pi x))$, $d = 0.002 \times (1 + 0.05\cos(\pi x))$, $\gamma = 0.205 \times (1 + 0.05\cos(\pi x))$, $\eta = 18 \times (1 + 0.1\cos(\pi x))$, $\mu = 0.001 \times (1 + 0.05\cos(\pi x))$, $\mu_0 = 0.034 \times (1 + 0.05\cos(\pi x))$, $\xi = 0.02 \times (1 + 0.05\cos(\pi x))$, $m = 1.2 \times 10^{6} \times (1 + 0.1\cos(\pi x))$, $\theta = 90 \times (1 + 0.05\cos(\pi x))$, and $\delta = 0.01062 \times (1 + 0.05\cos(\pi x))$. By direct calculation we obtain $\mathcal{R}_0 \approx 6.3231 > 1$ and $\min\{V_1(x)\} \approx 7466.4802 \geq \max\{\frac{\delta(x)m(x)}{\theta(x)\xi(x)}\} \approx 7417.1429$. Therefore, all conditions of Conjecture 6.5 are satisfy. The numerical simulations are given in Figures 5 (a)–(d), and the phage-present endemic steady state $\mathcal{E}^*$ is globally asymptotically stable.

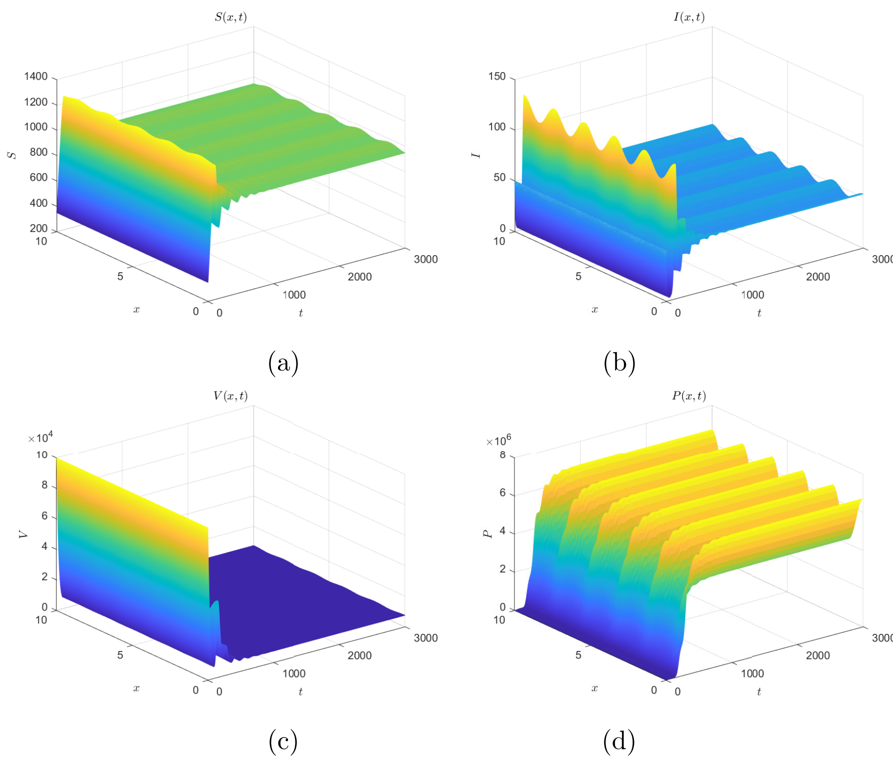


FIGURE 5. Global asymptotic stability of phage-present endemic steady state $\mathcal{E}^* = (S^*(x), I^*(x), V^*(x), P^*(x))$ of (2.1) with $\mathcal{R}_0 \approx 6.3231 > 1$ and $\min\{V_1(x)\} \approx 7466.4802 \geq \max\{\frac{\delta(x)m(x)}{\theta(x)\xi(x)}\} \approx 7417.1429$.

7.3. **Effects of the strength of spacial heterogeneity on disease risk.** It is well know that $\mathcal{R}_0$ is a crucial threshold on the risk of infection, Figure 6 illustrates the effect of different spacial heterogeneity strength $k_w \in [0,1]$ $(w = \alpha, \beta, \gamma, \eta)$ on $\mathcal{R}_0$. In this case, we choose $\alpha(x) = 3.4286 \times 10^{-5} \times (1 + k_\alpha \cos(\pi x))$, $\gamma(x) = 0.205 \times (1 + k_\gamma \cos(\pi x))$, $\beta(x) = 0.018 \times (1 + k_\beta \cos(\pi x))$ and $\eta(x) = 1.2 \times (1 + k_\eta \cos(\pi x))$, respectively. Other model parameters are the same as in Example 7.1. The plots in Figures 6 (a) and (b) show that the basic reproduction number $\mathcal{R}_0$ increases with the spatial heterogeneity of the parameters $\alpha(x)$ and $\gamma(x)$. And the Figures

6 (c) (b) show that the basic reproduction number shows oscillations as the spatial heterogeneity of parameters $\beta(x)$ and $\eta(x)$ increases. These imply that ignoring spatial heterogeneity can misestimate the underlying local basic reproduction number, with unpredictable consequences for the prevention and control of infectious disease.

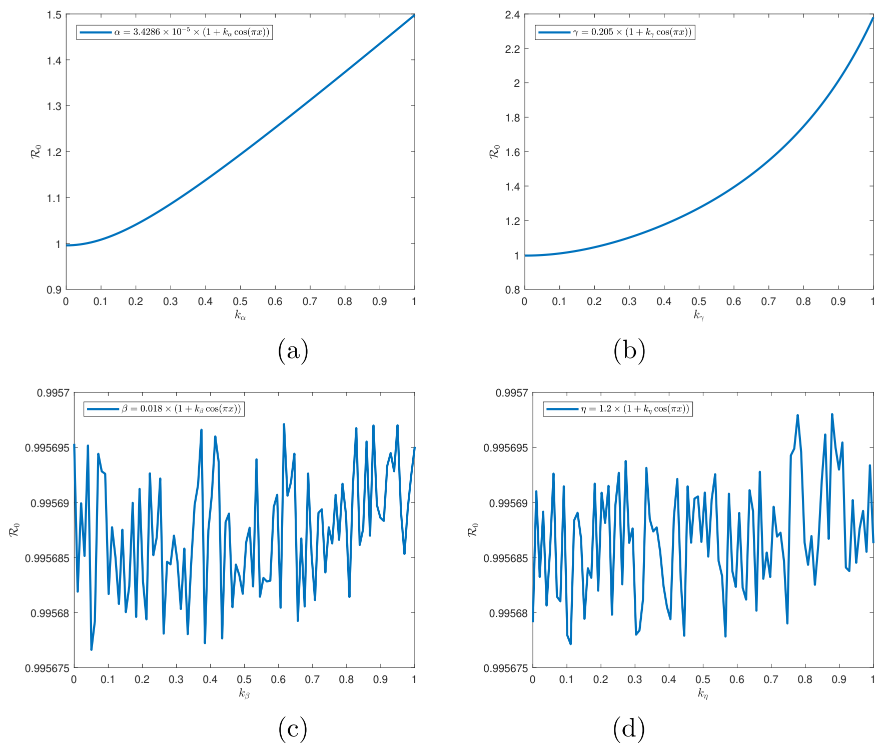


FIGURE 6. Effect of different spacial heterogeneity strength $k_w \in [0,1] (w = \alpha, \beta, \gamma, \eta)$ on the basic reproduction number $\mathcal{R}_0$.

In addition, we considered the effect of diffusion coefficient on the distribution of susceptible and infected hosts, and for this purpose, fixed the model parameters as in Figure 5. Diffusion coefficient D_S is taken as 0.01, 0.001 and 0.0001, and D_I is taken as 0.008, 0.0008 and 0.00008, respectively. Figure 7(a) shows that fixing the diffusion coefficient D_I of infected hosts (or fixing the diffusion coefficient D_S of susceptible hosts), as the distribution of susceptible hosts showed stronger spatial heterogeneity as the diffusion coefficient D_S of susceptible hosts (or as the diffusion coefficient D_I of infected hosts decreased) decreased. Figure 7(b) shows that if the diffusion coefficient D_I of infected hosts is fixed, as the diffusion coefficient D_S of susceptible hosts decreases, the spatial heterogeneity of the distribution of infected hosts decreases and tends to some constant distribution more and more; while fixing the diffusion coefficient D_S of susceptible hosts, as the diffusion coefficient D_I of infected hosts decreases, the distribution of infected hosts shows stronger spatial heterogeneity. The numerical simulations also indicate that in the case of unavoidable spread of susceptible hosts, the diffusion of infected hosts will make low-risk areas rise in risk level and will reduce the risk level of high-risk areas, making the disease epidemic on the whole region for a long time.

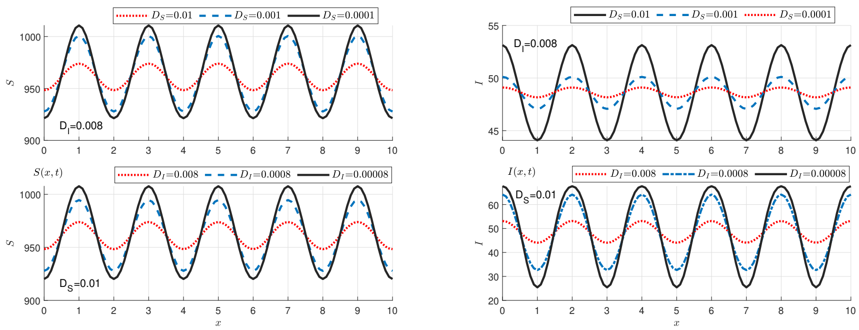


FIGURE 7. Effect of diffusion coefficient on the distribution of susceptible and infected hosts.

8. Conclusion and discussion

We developped a cholera models with coupled reaction-diffusion equations and ordinary differential equations to discuss the effects of spatial heterogeneity, environmental viruses and phages on disease transmission. Here, we consider not only the horizontal transmission of *vibrio cholerae* between hosts, but also the transmission of *vibrio cholerae* between the environment and hosts, and the interaction between *vibrio cholerae* and phages in the environment. Since the diffusion of *vibrio cholerae* and phages in the environment are not considered, this makes the solution semiflow of our model lacking compactness, while creating some difficulties in analyzing the dynamics.

By using the comparison principle, the Kuratowski measure of noncompactnes and other methodological techniques, we verify the existence of nonnegative solution, the point dissipation and the asymptotic smoothness of the solution semiflow. Further, we obtain the basic reproduction number $\mathcal{R}_0$, which is identified as the spectral radius of next generation operator. In addition, the variational formula of $\mathcal{R}_0$ for spatially heterogeneous case and the expression of $\widetilde{\mathcal{R}}_0$ for spatially homogeneous case are calculate. Of course, our basic reproduction number also perfectly portrays the persistence and extinction of the disease. Specifically, if $\mathcal{R}_0 < 1$, the disease-free steady state $\mathcal{E}_0$ is globally asymptotically stable, which indicates that the bacteria is eliminated. We also confirm the global stability of $\mathcal{E}_0$ in a critical case that $\mathcal{R}_0 = 1$, which is the novelty of this paper. Further, the global dynamics of our model for $\mathcal{R}_0 > 1$ is also analyzed in detail. This includes, the existence and global stability of phage-free endemic steady state for heterogeneous or homogeneous space, which implies that cholera becomes endemic by persisting in the host and the environment, while environmental phages tend to become extinct due to their low reproduction rate. Further, we discuss the uniform persistence of phages, bacteria, susceptible and infected hosts and the global stability of phage-present endemic steady state at $\mathcal{R}_0 > 1$ and some other technical conditions.

The numerical simulations explain the main conclusions, especially our two conjectures about the global asymptotic stability of the phage-free and phage-present endemic steady states. In addition, numerical simulations also discuss the sensitivity of the main parameters of this model with respect to the basic reproduction number and the influence of diffusion coefficients on the distribution of infectious diseases. In the era of increasingly global economy, the spread of infected hosts allows pathogens to reach all corners of the global village, resulting in increasing epidemic risk levels in some low-epidemic areas and higher epidemic levels in high-epidemic areas because of the influx of susceptible hosts. Therefore, reducing the necessary movement of people and increasing local control measures during periods of high outbreaks is one of the effective means to eliminate outbreaks throughout the region.

Notice that we have only demonstrated the global stability of phage-free and phage-present endemic steady states in a homogeneous environment, while the situation in a heterogeneous environment become two interesting open questions. In addition, the activity of *vibrio cholerae* in the environment is closely related to the time of its shedding [32, 35, 40, 50, 52], so it becomes significant to consider the effect of *vibrio cholerae* activity on disease transmission. These are all topics that deserve further consideration in the future.

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