Electronic Journal of Differential Equations, Vol. 2023 (2023), No. 11, pp. 1–41.
ISSN: 1072-6691. URL: https://ejde.math.txstate.edu or https://ejde.math.unt.edu

SMOOTHING PROPERTIES FOR A COUPLED ZAKHAROV-KUZNETSOV SYSTEM

JULIE L. LEVANDOSKY, OCTAVIO VERA

ABSTRACT. In this article we study the smoothness properties of solutions to a two-dimensional coupled Zakharov-Kuznetsov system. We show that the equations dispersive nature leads to a gain in regularity for the solution. In particular, if the initial data (u_0, v_0) possesses certain regularity and sufficient decay as $x \to \infty$, then the solution $(u(t), v(t))$ will be smoother than (u_0, v_0) for $0 < t \leq T$ where T is the existence time of the solution.

1. INTRODUCTION

The general form of the coupled Zakharov-Kuznetsov system [12] is

$$u_t + u_{xxx} + u_{yyx} - 6uu_x - v_x = 0 \tag{1.1}$$

$$v_t + \delta v_{xxx} + \lambda v_{yyx} + \eta v_x - 6\mu v v_x - \omega u_x = 0. \tag{1.2}$$

This coupled system is a model describing two interacting weakly nonlinear waves in anisotropic media. Here, x and y are the propagation and transverse coordinates respectively, η is a group velocity shift between the coupled models, δ and λ are the relative longitudinal and transverse dispersion coefficients, and μ and ω are the relative nonlinear and coupled coefficients. In the absence of the transverse variation (i.e. - $u_y = v_y = 0$), this system reduces to the set of coupled KdV equations [7] which are known to describe the interaction of nonlinear long waves in certain fluid flows. In this article, we study (1.1)-(1.2) when the dispersion coefficients, δ and λ, and the coupling coefficient ω are positive. In that case, it suffices to consider the initial-value problem

$$\begin{gathered} u_t + u_{xxx} + u_{yyx} - 6uu_x - v_x = 0 \\ bv_t + \delta v_{xxx} + \lambda v_{yyx} + \eta v_x - 6\mu v v_x - u_x = 0 \\ u(x,y,0) = u_0(x,y), \quad v(x,y,0) = v_0(x,y) \end{gathered} \tag{1.3}$$

where $b > 0$, $\delta > 0$, and $\lambda > 0$.

A number of results concerning gain of regularity for various nonlinear evolution equations have appeared. Cohen [4] considered the KdV equation, showing that "boxshaped" initial data $\phi \in L^2(\mathbb{R}^2)$ with compact support lead to a solution $u(t)$ which is smooth for $t > 0$. Kato [13] generalized this result, showing that if the initial data ϕ are in $L^2((1 + e^{\sigma x})dx)$, the unique solution $u(t) \in C^\infty(\mathbb{R}^2)$ for $t > 0$.

2020 *Mathematics Subject Classification.* 35Q53, 35Q35, 47J35.
Key words and phrases. Coupled Zakharov-Kuznetsov system; gain in regularity; weighted Sobolev space.

Submitted April 18, 2022. Published February 4, 2023.

Kruzhkov and Faminskii [15] replaced the exponential weight function with a polynomial weight function, quantifying the gain in regularity of the solution in terms of the decay at infinity of the initial data. Craig, Kappeler, and Strauss [6] expanded on the ideas from these earlier papers in their treatment of highly generalized KdV equations. Other results on gain of regularity for linear and nonlinear dispersive equations include the works of Hayashi, Nakamitsu, and Tsutsumi [9, 10], Hayashi and Ozawa [11], Constantin and Saut [5], Ponce [21], Ginibre and Velo [8], Kenig, Ponce, and Vega [14], and Vera [23]. Smoothing properties for coupled systems of nonlinear dispersive equations in one-spatial dimension were proven by Vera [22], Ceballos, Sepulveda, and Vera [3], and Alves and Vera Villagrán [1]. Here we treat a coupled system of nonlinear dispersive equations in two spatial dimensions.

In studying propagation of singularities, it is natural to consider the bicharacteristics associated with the differential operator. For the KdV equation, it is known that the bicharacteristics all point to the left for $t > 0$, and all singularities travel in that direction. Kato [13] makes use of this uniform dispersion, choosing a nonsymmetric weight function decaying as $x \to -\infty$ and growing as $x \to \infty$. In [6], Craig, Kappeler and Strauss also make use of a unidirectional propagation of singularities in their results on infinite smoothing properties for generalized KdV-type equations for which $f_{u_{xxx}} \geq c > 0$.

For the two-dimensional case, Levandosky [16] proves smoothing properties for the KP-II equation. This result makes use of the fact that the bicharacteristics all point into one half-plane. Subsequently, Levandosky [17] considers generalized KdV-type equations in two dimensions, proving that if all bicharacteristics point into one half-plane, an infinite gain in regularity will occur, assuming sufficient decay at infinity of the initial data. Levandosky Sepulveda and Vera Villagran [19] proved a smoothing property for the KP-I equation. Since the bicharacteristics do not all point into the same half-plane, singularities may travel in all of $\mathbb{R}^2$. Consequently, the same proof techniques used above do not generalize to this equation. However, they are able to prove a finite gain in regularity. Levandosky In [18] proved smoothing properties for solutions to the fifth-order Kawahara equation in two spatial dimensions.

In this paper, we extend the ideas discussed above to prove a gain in regularity result to a Zakharov-Kuznetsov system (1.3), a nonlinear dispersive system in two spatial dimensions. Specifically, we quantify the gain in regularity of the solution $(u(t), v(t))$ in relation to the decay of the initial data. In particular, we prove that if the initial data has sufficient regularity and decays sufficiently as $x \to \infty$, then the solution $(u(t), v(t)) \in C^\infty(\mathbb{R}^2) \times C^\infty(\mathbb{R}^2)$ for $0 < t \leq T$ where T is the existence time of the solution. We now state this more formally in a special case of our main theorem on the gain of regularity for the Zakharov-Kuznetsov system.

Gain of regularity theorem. Consider a coupled system of the form (1.3) where $b, \delta, \lambda > 0$. Let (u, v) be a solution of (1.3) in $\mathbb{R}^2 \times [0, T]$ such that for all integers $L \geq 1$,

$$\sup_{0 \leq t \leq T} \int_{\mathbb{R}^2} (1 + x^+)^L \sum_{|a| \leq 3} [(\partial^\alpha u)^2 + (\partial^\alpha v)^2] \, dx \, dy < +\infty.$$

Then our solution $(u(t), v(t)) \in C^\infty(\mathbb{R}^2) \times C^\infty(\mathbb{R}^2)$ for $0 < t \leq T$.

As will be shown, the assumption that

$$\sup_{0<t\le T}\int_{\mathbb{R}^2}(1+x^+)^L\sum_{|\alpha|\le 3}[(\partial^\alpha u)^2+(\partial^\alpha v)^2]\,dx\,dy<+\infty$$

for all integers $L\ge 1$ may be reduced to assuming this property holds for some integer $L\ge 1$. The smoothing phenomenon will still occur, but the amount of smoothing will depend on the size of L, thereby showing the relationship between the decay at infinity of the initial data and the gain in regularity of the solution.

The plan of the paper is the following. In Section 2, we introduce the weighted Sobolev spaces which will be used to describe the gain in regularity. In Section 3, we state and prove the main inequality used to show the gain in regularity. In Sections 4 and 5, we prove an existence result showing that if the initial data (u_0,v_0) is sufficiently smooth, there exists a unique solution $(u(t),v(t))$ of (1.3) with the same amount of regularity for a time interval $[0,T]$ depending only on a Sobolev norm involving the initial data. In Section 6, we prove that if the initial data also possesses sufficient decay at infinity, the solution $(u(t),v(t))$ possesses similar decay at infinity. Finally, in Theorem 6.4, we state and prove the main result. Using induction we show that the decay at infinity of the initial data leads to a gain in regularity for the solution $(u(t),v(t))$.

2. Preliminaries

The idea for the proof of the gain in regularity is the following. For the first step of the induction, we multiply $(1.3)_1$ by $2\xi u$ where ξ is our weight function, to be specified later, and integrate over $\mathbb{R}^2$. Upon doing so, we obtain

$$2\int\xi\,uu_t+2\int\xi\,uu_{xxx}+2\int\xi uu_{yyx}-12\int\xi\,u(u\,u_x)-2\int\xi\,uv_x=0. \tag{2.1}$$

where $\int=\int_{\mathbb{R}^2}\,dx\,dy$. Then, integrating by parts, we see that

$$\partial_t\int\xi u^2+3\int\xi_x u_x^2+\int\xi_x u_y^2=\int\xi_t u^2+\int\xi_{xxx}u^2+12\int\xi u(uu_x)+2\int\xi uv_x. \tag{2.2}$$

If our weight function ξ satisfies $0<\partial_x^j\xi\le C\partial_x^k\xi$ for all $j\ge k\ge 0$, then we arrive at the inequality

$$\partial_t\int\xi u^2+3\int\xi_x u_x^2+\int\xi_x u_y^2\le\int\xi_t u^2+C\int\xi u^2+12\int\xi u(uu_x)+2\int\xi uv_x. \tag{2.3}$$

Similarly, multiplying $(1.3)_2$ by $2\xi v$, integrating by parts, and using the same assumption on the weight function ξ, we see that

$$\begin{aligned}&\partial_t b\int\xi v^2+3\delta\int\xi_x v_x^2+\lambda\int\xi_x v_y^2\\&\le b\int\xi_t v^2+C\int\xi v^2+12\mu\int\xi v(vv_x)+2\int\xi vu_x.\end{aligned} \tag{2.4}$$

Combining these two inequalities, we have

$$\begin{aligned}&\partial_t\int\xi(u^2+bv^2)+3\int\xi_x(u_x^2+\delta v_x^2)+\int\xi_x(u_y^2+\lambda v_y^2)\\&\le\int\xi_t(u^2+bv^2)+C\int\xi(u^2+v^2)+C\Big|\int\xi u(uu_x)\Big|+C\Big|\int\xi v(vv_x)\Big|+2\int\xi[uv]_x.\end{aligned}$$

Notice the second and third terms on the left-hand side. Assuming $\xi_x > 0$, these terms have positive signs, thus, allowing us to prove a gain in regularity. We continue this procedure inductively. On each step, β, of the induction, we take α derivatives of $(1.3)_1$ and $(1.3)_2$ where $|\alpha| = \alpha_1 + \alpha_2 = \beta$. We then multiply the differentiated equations by $2\xi(\partial u)$ and $2\xi(\partial v)$, respectively, where $\partial^\alpha \equiv \partial_x^{\alpha_1}\partial_y^{\alpha_2}$ and $\xi = \xi_\beta$ is our weight function to be described below. Integrating over $\mathbb{R}^2$ and integrating by parts as described above, if our weight function ξ satisfies $0 < \partial_x^j \xi \le C\partial_x^k \xi$ for all $j \ge k \ge 0$, we arrive at the following inequality.

$$
\begin{aligned}
&\partial_t \int \xi((\partial^\alpha u)^2 + b(\partial^\alpha v)^2) + C\int \xi_x((\partial^\alpha u_x)^2 + (\partial^\alpha v_x)^2) \\
&+ C\int \xi_x((\partial^\alpha u_y)^2 + (\partial^\alpha v_y)^2) \\
&\le C\int \xi_t((\partial^\alpha u)^2 + (\partial^\alpha v)^2) + C\int \xi((\partial^\alpha u)^2 + (\partial^\alpha v)^2) + C\Big|\int \xi(\partial^\alpha u)\partial^\alpha(uu_x)\Big| \\
&\quad + C\Big|\int \xi(\partial^\alpha v)\partial^\alpha(vv_x)\Big| + 2\int \xi[(\partial^\alpha u)(\partial^\alpha v)]_x.
\end{aligned}
$$

Choice of weight function. In what follows, we will be proving that if our initial data decays sufficiently as $x \to \infty$, then the solution will experience a gain in regularity. Consequently, we will choose weight functions which behave like powers of x for $x > 1$. Since the bicharacteristics point into the left half-plane, it is natural to choose weight functions which decay as $x \to -\infty$. We will choose weight functions which behave like $e^{\sigma x}$ where $\sigma \ge 0$ for $x < -1$. We define the classes of weight functions as follows.

Definition 2.1. A function $\xi = \xi(x,t)$ belongs to the weight class $W_{\sigma\, i\, k}$ if it is a positive C^∞ function on $\mathbb{R} \times [0,T]$, $\xi_x > 0$, and there are constants c_j, $1 \le j \le 5$ such that

$$
\begin{gathered}
0 < c_1 \le t^{-k} e^{-\sigma\, x}\xi(x,t) \le c_2 \quad \forall\, x < -1,\ 0 < t < T, \\
0 < c_3 \le t^{-k}\, x^{-i}\xi(x,t) \le c_4 \quad \forall\, x > 1,\ 0 < t < T, \\
(t\,|\xi_t| + |\partial_x^j \xi|)/\xi \le c_5 \quad \forall\,(x,t) \in \mathbb{R}\times[0,T], \quad \forall\, j \in \mathbb{Z}^+.
\end{gathered} \tag{2.5}
$$

We now define weighted function spaces using the weight functions introduced above.

Definition 2.2. Let N be a positive integer. Let $H^\beta(W_{\sigma\, i\, k})$ be the space of functions with finite norm

$$
H^\beta(W_{\sigma\, i\, k}) = \big\{v : \mathbb{R}^2 \to \mathbb{R} : \|v\|^2_{H^\beta(W_{\sigma\, i\, k})} = \int_{\mathbb{R}^2} \sum_{|\alpha|\le\beta} (\partial^\alpha v)^2\, |\xi(x,y)| < \infty\big\} \tag{2.6}
$$

for any $\xi \in W_{\sigma\, i\, k}$, $\beta \ge 0$, and $0 \le t \le T$.

Remark 2.3. We note that although the norm above depends on ξ, all choices of ξ in this class lead to equivalent norms. The usual Sobolev space is $H^N(\mathbb{R}^2)$ without a weight.

Definition 2.4. For each fixed $\xi \in W_{\sigma\, i\, k}$, $\beta \ge 0$, we define the space

$$
\begin{aligned}
&L^p([0,T]:\ H^\beta(W_{\sigma\, i\, k})) \\
&= \big\{v(x,y,t) : \|v\|^p_{L^p([0,T]:H^\beta(W_{\sigma\, i\, k}))} = \int_0^T \|v(\cdot,\cdot,t)\|^p_{H^\beta(W_{\sigma\, i\, k})} dt < +\infty\big\}
\end{aligned} \tag{2.7}
$$

$$\begin{aligned} &L^\infty([0,T]:\ H^\beta(W_{\sigma\,i\,k})) \\ &= \big\{v(x,y,t):\ \|v\|^2_{L^\infty([0,T]:H^\beta(W_{\sigma\,i\,k}))} = \sup_{t\in[0,T]} \|v(\cdot,\cdot,t)\|_{H^\beta(W_{\sigma\,i\,k})} dt < +\infty\big\} \end{aligned} \tag{2.8}$$

Moreover, we define the spaces

$$\widetilde{W}_{\sigma\,i\,k} = \cup_{j<i} W_{\sigma\,i\,k}, \tag{2.9}$$

$$L^p(H^s(\widetilde{W}_{\sigma\,i\,k})) = \cup_{j<i} L^p(H^s(W_{\sigma\,i\,k})) \tag{2.10}$$

We shall use those spaces only in the case when $i = -1$.

In this article, we make use of the following Sobolev embedding estimates

$$\|u\|_{L^\infty(\mathbb{R}^2)} \le C\Big(\int_{\mathbb{R}^2} [u^2 + u_{xx}^2 + u_{yy}^2]\,dx\,dy\Big)^{1/2} \tag{2.11}$$

and (see [2, Th. 10.2, page 187])

$$\Big(\int_{\mathbb{R}^2} |u|^4\,dx\,dy\Big)^{1/4} \le \Big(\int_{\mathbb{R}^2} [u^2 + u_x^2 + u_y^2]\,dx\,dy\Big)^{1/2}. \tag{2.12}$$

In general, we have the anisotropic imbedding. For $2 \le n < 6$,

$$\Big(\int_{\mathbb{R}^2} |u|^n\,dx\,dy\Big)^{1/n} \le \Big(\int_{\mathbb{R}^2} [u^2 + u_x^2 + u_y^2]\,dx\,dy\Big)^{1/2}. \tag{2.13}$$

3. Main inequality

In this section we state and prove the main lemma that will be used in our main theorem on the gain of regularity. Specifically, we prove that if there exists a solution (u,v) of (1.3) sufficiently smooth and with sufficient decay at infinity, the weighted Sobolev norms for (u,v) are bounded above by other weighted Sobolev norms involving less derivatives of u and v.

Lemma 3.1. *, For (u,v) a solution of (1.3) sufficiently smooth and with sufficient decay at infinity,*

$$\begin{aligned} &\sup_{0\le t\le T} \int \xi_\beta[(\partial^\alpha u)^2 + (\partial^\alpha v)^2] + \int_0^T \int \chi_\beta[(\partial^\alpha u_x)^2 + (\partial^\alpha u_y)^2] \\ &+ \int_0^T \int \chi_\beta[(\partial^\alpha v_x)^2 + (\partial^\alpha v_y)^2] \le C \end{aligned} \tag{3.1}$$

for $1 \le \beta \le L$, where $\beta = |\alpha|$, $\xi_\beta \in W_{\sigma,L-\beta,\beta}$, $\chi_\beta \in W_{\sigma,L-\beta-1,\beta}$ and C depends only on $\|u\|_{H^3}$, $\|v\|_{H^3}$, and

$$\sup_{0\le t\le T} \int \xi_\nu(\partial^\gamma u)^2, \quad \sup_{0\le t\le T} \int \xi_\nu(\partial^\gamma v)^2 \tag{3.2}$$

$$\int_0^T \int \chi_\nu[(\partial^\gamma u_x)^2 + (\partial^\gamma u_y)^2], \tag{3.3}$$

$$\int_0^T \int \chi_\nu[(\partial^\gamma v_x)^2 + (\partial^\gamma v_y)^2], \tag{3.4}$$

where $\nu = |\gamma| \le \beta - 1$, $\xi_\nu \in W_{\sigma,L-\nu,\nu}$, and $\chi_\nu \in W_{\sigma,L-\nu-1,\nu}$.

The idea of the proof is the following. We would like to bound terms on the left-hand side of (3.1) in terms of integrals of the same form, but with a lower number of derivatives. In particular, we hope to bound the left-hand side of (3.1) in terms of (3.2) - (3.4). On each level of the induction, the weight function ξ behaves like a power of x for $x > 1$, an exponential $e^{\sigma x}$ where $\sigma > 0$ for $x < -1$ and a power of t. As we proceed inductively, the powers of x for $x > 1$ decrease while the powers of t increase. In particular, for $\beta = 1$, $\xi_\beta \approx tx^{L-1}$ for $x > 1$. For $\beta = 2$, $\xi_\beta \approx t^2 x^{L-2}$ for $x > 1$. We continue in this way, decreasing the power of x for $x > 1$ and increasing the power of t on each level of the induction.

Proof of Lemma 3.1. Let $\beta \geq 1$. Let $\alpha = (\alpha_1, \alpha_2)$ where $|\alpha| = \beta$. Take α derivatives of $(1.3)_1$, multiply the differentiated equation by $2\xi_\beta(\partial^\alpha u)$ where

$$\xi_\beta(x,t) = \int_{-\infty}^{x} \chi_\beta(z,t)dz \tag{3.5}$$

for $\chi_\beta \in W_{\sigma,L-\beta-1,\beta}$, and integrate over $\mathbb{R}^2 \times [0,t]$ for $0 \leq t \leq T$. Letting $\xi \equiv \xi_\beta$, we conclude that

$$\begin{aligned}
&\int \xi(\cdot,t)\,(\partial^\alpha u)^2 + 3\int_0^t\int \xi_x\,(\partial^\alpha u_x)^2 + \int_0^t\int \xi_x\,(\partial^\alpha u_y)^2 \\
&= \int \xi(\cdot,0)(\partial^\alpha u_0)^2 + \int_0^t\int [\xi_t + \xi_{xxx}]\,(\partial^\alpha u)^2 + 12\int_0^t\int \xi(\partial^\alpha u)\partial^\alpha(uu_x) \\
&\quad + 2\int_0^t\int \xi(\partial^\alpha u)(\partial^\alpha v_x).
\end{aligned} \tag{3.6}$$

Similarly, take α derivatives of $(1.3)_2$, multiply the differentiated equation by $2\xi_\beta(\partial^\alpha v)$, and integrate over $\mathbb{R}^2 \times [0,t]$ for $0 \leq t \leq T$. Doing so, we conclude that

$$\begin{aligned}
&b\int \xi(\cdot,t)\,(\partial^\alpha v)^2 + 3\delta\int_0^t\int \xi_x\,(\partial^\alpha v_x)^2 + \lambda\int_0^t\int \xi_x\,(\partial^\alpha v_y)^2 \\
&= \int \xi(\cdot,0)(\partial^\alpha v_0)^2 + \int_0^t\int [\xi_t + \xi_{xxx} + \eta\xi_x]\,(\partial^\alpha v)^2 \\
&\quad + 12\mu\int_0^t\int \xi(\partial^\alpha v)\partial^\alpha(vv_x) + 2\int_0^t\int \xi(\partial^\alpha u_x)(\partial^\alpha v).
\end{aligned} \tag{3.7}$$

Then, adding (3.6) and (3.7), we have

$$\begin{aligned}
&\int \xi(\cdot,t)[(\partial^\alpha u)^2 + b(\partial^\alpha v)^2] + 3\int_0^t\int \xi_x[(\partial^\alpha u_x)^2 + \delta(\partial^\alpha v_x)^2] \\
&+ \int_0^t\int \xi_x[(\partial^\alpha u_y)^2 + \lambda(\partial^\alpha v_y)^2] \\
&= \int \xi(\cdot,0)[(\partial^\alpha u_0)^2 + (\partial^\alpha v_0)^2] + \int_0^t\int [\xi_t + \xi_{xxx}][(\partial^\alpha u)^2 + (\partial^\alpha v)^2] \\
&\quad + \int_0^t\int \eta\xi_x(\partial^\alpha v)^2 + 12\int_0^t\int \xi(\partial^\alpha u)\partial^\alpha(uu_x) + 12\mu\int_0^t\int \xi(\partial^\alpha v)\partial^\alpha(vv_x) \\
&\quad + 2\int_0^t\int \xi\partial_x[(\partial^\alpha u)(\partial^\alpha v)].
\end{aligned}$$

Now using the fact that $\partial_x^j \xi \le C\xi$ and $\xi(\cdot,0)=0$ for $\beta \ge 1$, we obtain the identity

$$\begin{aligned}&\int \xi(\cdot,t)[(\partial^\alpha u)^2 + b(\partial^\alpha v)^2] + 3\int_0^t\int \xi_x[(\partial^\alpha u_x)^2 + \delta(\partial^\alpha v_x)^2]\\&+\int_0^t\int \xi_x[(\partial^\alpha u_y)^2 + \lambda(\partial^\alpha v_y)^2]\\&\le C\int_0^t\int [\xi_t+\xi][(\partial^\alpha u)^2 + (\partial^\alpha v)^2] + 12\int_0^t\int \xi(\partial^\alpha u)\partial^\alpha(uu_x)\\&\quad + 12\mu\int_0^t\int \xi(\partial^\alpha v)\partial^\alpha(vv_x) + 2\int_0^t\int \xi\partial_x[(\partial^\alpha u)(\partial^\alpha v)].\end{aligned}$$

The first term on the right-hand side above is bounded by terms of the form (3.3) and (3.4). Integrating by parts and using the Cauchy-Schwarz inequality, we see that the last term on the right-hand side satisfies

$$\big|\int_0^t\int \xi\partial_x[(\partial^\alpha u)(\partial^\alpha v)]\big| = \big|\int_0^t\int \xi_x(\partial^\alpha u)(\partial^\alpha v)\big| \le C\int_0^t\int \xi_x[(\partial^\alpha u)^2 + (\partial^\alpha v)^2].$$

Each of these terms is bounded by terms of the form (3.3) and (3.4). Therefore, we conclude that

$$\begin{aligned}&\int \xi(\cdot,t)[(\partial^\alpha u)^2 + b(\partial^\alpha v)^2] + 3\int_0^t\int \xi_x[(\partial^\alpha u_x)^2 + \delta(\partial^\alpha v_x)^2]\\&+\int_0^t\int \xi_x[(\partial^\alpha u_y)^2 + \lambda(\partial^\alpha v_y)^2]\\&\le C + C\big|\int_0^t\int \xi(\partial^\alpha u)\partial^\alpha(uu_x)\big| + C\big|\int_0^t\int \xi(\partial^\alpha v)\partial^\alpha(vv_x)\big|\end{aligned}\tag{3.8}$$

where C depends only on (3.3) and (3.4). Therefore, it remains to look for bounds on the remainder terms

$$C\big|\int_0^t\int \xi(\partial^\alpha u)\partial^\alpha(uu_x)\big| + C\big|\int_0^t\int \xi(\partial^\alpha v)\partial^\alpha(vv_x)\big|\tag{3.9}$$

for each $\beta \ge 1$.

Case $\beta = 1$. Let $\xi \equiv \xi_\beta = \int_{-\infty}^x \chi_\beta(z,t)dz$ where $\chi_\beta \in W_{\sigma,L-2,1}$. Therefore, $\xi \approx tx^{L-1}$ for $x > 1$ and $\xi \approx te^{\sigma x}$ for $x < -1$.

Subcase $\alpha = (1,0)$. The remainder terms satisfies

$$\begin{aligned}\big|\int_0^t\int \xi u_x(uu_x)_x\big| &= \big|\int_0^t\int \xi(u_x^3 + u_x uu_{xx})\big|\\&\le C\big|\int_0^t\int \xi u_x^3\big| + C\big|\int_0^t\int \xi u(u_x)_x^2\big|\\&\le C|u_x|_{L^\infty}\int_0^t\int \xi u_x^2 + C\int_0^t\int \big|\xi_x u(u_x)^2 + \xi u_x^3\big|\\&\le C|u_x|_{L^\infty}\int_0^t\int \xi u_x^2 + C|u|_{L^\infty}\int_0^t\int \xi_x u_x^2\\&\le C(|u|_{L^\infty} + |u_x|_{L^\infty})\int_0^t\int \xi u_x^2\end{aligned}$$

$$\leq C \int_0^t \int \xi u_x^2$$

where C depends only on $\|u\|_{H^3}$. Similarly for the remainder term involving v. Combining this bound with (3.8) and the fact that $\xi_x = \chi_\beta$ we conclude that

$$\sup_{0\leq t\leq T} C \int \xi(\cdot,t)[u_x^2 + v_x^2] + C\int_0^T \int \chi[u_{xx}^2 + v_{xx}^2] + C\int_0^T \int \chi[u_{xy}^2 + v_{xy}^2] \leq C$$

where $\chi \equiv \chi_\beta$ and C depends only on $\|u\|_{H^3}, \|v\|_{H^3}$ and terms of the form (3.3) and (3.4), as desired.

Subcase $\alpha = (0,1)$. In this case, the remainder terms in u satisfy

$$\begin{aligned}
|\int_0^t \int \xi u_y (uu_x)_y| &= |\int_0^t \int \xi u_y(u_y u_x + uu_{xy})| \\
&\leq C|u_x|_{L^\infty} \int_0^t \int \xi u_y^2 + C|\int_0^t \int \xi u (u_y^2)_x| \\
&\leq C|u_x|_{L^\infty} \int_0^t \int \xi u_y^2 + C|\int_0^t \int \xi_x u u_y^2 + \xi u_x u_y^2| \\
&\leq C|u_x|_{L^\infty} \int_0^t \int \xi u_y^2 + C|u|_{L^\infty} \int_0^t \int \xi_x u_y^2 \\
&\leq C(|u|_{L^\infty} + |u_x|_{L^\infty}) \int_0^t \int \xi u_y^2 \\
&\leq C\int_0^t \int \xi u_y^2
\end{aligned}$$

where C depends only on $\|u\|_{H^3}$. Similarly for the terms in v. Combining these estimates with (3.8), we conclude that

$$\sup_{0\leq t\leq T} C \int \xi(\cdot,t)[u_y^2 + v_y^2] + C\int_0^T \int \chi[u_{xy}^2 + v_{xy}^2] + C\int_0^T \chi[u_{yy}^2 + v_{yy}^2] \leq C$$

where C depends only on $\|u\|_{H^3}, \|v\|_{H^3}$ and terms in (3.3) and (3.4).

Case $\beta = 2$. In this case, let $\xi \equiv \xi_\beta = \int_{-\infty}^x \chi_\beta(z,t)dz$ where $\chi_\beta \in W_{\sigma,L-3,2}$. Therefore, $\xi \approx t^2 x^{L-2}$ for $x > 1$ and $\xi \approx t^2 e^{\sigma x}$ for $x < -1$.

Subcase $\alpha = (2,0)$. In this case, the remainder terms satisfy

$$\begin{aligned}
|\int_0^t \int \xi u_{xx}(uu_x)_{xx}| &= |\int_0^t \int \xi u_{xx}(3u_x u_{xx} + uu_{xxx})| \\
&\leq C(|u|_{L^\infty} + |u_x|_{L^\infty}) \int_0^t \int \xi u_{xx}^2 \\
&\leq C\int_0^t \int \xi u_{xx}^2
\end{aligned}$$

where C depends only on $\|u\|_{H^3}$.

Subcase $\alpha = (1,1)$. In this case, the remainder terms satisfy

$$|\int_0^t \int \xi u_{xy}(uu_x)_{xy}|$$

$$
\begin{aligned}
&= \Big|\int_0^t \int \xi u_{xy}(2u_x u_{xy} + u_y u_{xx} + u u_{xxy})\Big| \\
&\le |u_x|_{L^\infty} \int_0^t \int \xi u_{xy}^2 + C|u_y|_{L^\infty} \int_0^t \int \xi u_{xx}^2 + |u_y|_{L^\infty} \int_0^t \int \xi u_{xy}^2 \\
&\le C \int_0^t \int \xi (u_{xx}^2 + u_{xy}^2)
\end{aligned}
$$

where C depends only on $\|u\|_{H^3}$.

Subcase $\alpha = (0,2)$. In this case, the remainder terms satisfy

$$
\begin{aligned}
\Big|\int_0^t \int \xi u_{yy}(uu_x)_{yy}\Big| &= \Big|\int_0^t \int \xi u_{yy}(u_{yy}u_x + 2u_y u_{xy} + u u_{xyy})\Big| \\
&\le C|u_x| \int_0^t \int \xi u_{yy}^2 + C|u_y|_{L^\infty} \int_0^t \int \xi (u_{xy}^2 + u_{yy}^2) \\
&\le C \int_0^t \int \xi (u_{xy}^2 + u_{yy}^2)
\end{aligned}
$$

where C depends only on $\|u\|_{H^3}$. Similarly for v. Combining these estimates with (3.8), we conclude that

$$
\sum_{|\alpha|=2} \sup_{0\le t\le T} C \int \xi(\cdot,t)[(\partial^\alpha u)^2 + (\partial^\alpha v)^2] + C \sum_{|\alpha|=3} \int_0^T \int \chi[(\partial^\alpha u)^2 + (\partial^\alpha v)^2] \le C
$$

where C depends only on $\|u\|_{H^3(\mathbb{R}^2)}$, $\|v\|_{H^3(\mathbb{R}^2)}$ and terms in (3.3) and (3.4).

Case $\beta = 3$. In this case, let $\xi \equiv \xi_\beta = \int_{-\infty}^x \chi_\beta(z,t)dz$ where $\chi_\beta \in W_{\sigma,L-4,3}$. Therefore, $\xi \approx t^3 x^{L-3}$ for $x > 1$ and $\xi \approx t^3 e^{\sigma x}$ for $x < -1$. We consider the subcase $\alpha = (3,0)$. The other cases can be handled similarly.

Subcase $\alpha = (3,0)$. The remainder terms satisfy

$$
\begin{aligned}
&\Big|\int_0^t \int \xi u_{xxx}(uu_x)_{xxx}\Big| \\
&= \Big|\int_0^t \int \xi u_{xxx}(4u_x u_{xxx} + 3u_{xx}^2 + u u_{xxxx})\Big| \\
&= \Big|\int_0^t \int 4\xi u_x u_{xxx}^2 + 3\xi u_{xx}^2 u_{xxx} + \xi u u_{xxx} u_{xxxx}\Big| \\
&\le C|u_x|_{L^\infty} \int_0^t \int \xi u_{xxx}^2 + C \int_0^t \int \xi u_{xx}^2 u_{xxx} + C(|u|_{L^\infty} + |u_x|_{L^\infty}) \int_0^t \int \xi u_{xxx}^2 \\
&\le C(|u|_{L^\infty} + |u_x|_{L^\infty}) \int_0^t \int \xi u_{xxx}^2 + C \int_0^t \int \xi u_{xx}^2 u_{xxx} \\
&\le C \int_0^t \int \xi u_{xxx}^2 + C \int_0^t \int \xi u_{xx}^2 u_{xxx}
\end{aligned}
$$

where C depends only on $\|u\|_{H^3}$.

To handle the last term on the right-hand side above, we use (2.12). In addition, we consider the cases $x > 1$ and $x < -1$ separately. Let $A = \{x > 1\} \times \mathbb{R}$ and let $B = \{x < -1\} \times \mathbb{R}$. First, for $x > 1$,

$$
\int_0^t \int_A \xi u_{xx}^2 u_{xxx}
$$

$$
\begin{aligned}
&= C\int_0^t\int_A t^3x^{L-3}u_{xx}^2u_{xxx}\\
&\le C\Big(\int_0^t\int_A t^3x^{L-3}u_{xx}^4\Big)^{1/2}\Big(\int_0^t\int_A t^3x^{L-3}u_{xxx}^2\Big)^{1/2}\\
&\le C\Big(\int_0^t\int_A t^3[x^{(L-3)/4}u_{xx}]^4\Big)^{1/2}\Big(\int_0^t\int_A t^3x^{L-3}u_{xxx}^2\Big)^{1/2}\\
&\le C\Big(\int_0^t\int_A t^3(x^{(L-3)/4}u_{xx})^2+t^3([x^{(L-3)/4}u_{xx}]_x)^2+t^3([x^{(L-3)/4}u_{xx}]_y)^2\Big)\\
&\quad\times\Big(\int_0^t\int_A t^3x^{L-3}u_{xxx}^2\Big)^{1/2}\\
&\le C\Big(\int_0^t\int_A t^3x^{(L-3)/2}u_{xx}^2+t^3x^{(L-3)/2}u_{xxx}^2+t^3x^{(L-3)/2}u_{xxy}^2\Big)\\
&\quad\times\Big(\int_0^t\int_A t^3x^{L-3}u_{xxx}^2\Big)^{1/2}\\
&\le C\Big(\int_0^t\int_A \xi(u_{xx}^2+u_{xxx}^2+u_{xxy}^2)\Big)\Big(\int_0^t\int_A \xi u_{xxx}^2\Big)^{1/2}.
\end{aligned}
$$

Second, for $x<-1$,

$$
\begin{aligned}
\int_0^t\int_B \xi u_{xx}^2u_{xxx} &= \int_0^t\int_B t^3e^{\sigma x}u_{xx}^2u_{xxx}\\
&\le Ct^{3/2}\Big(\int_0^t\int_B t^3e^{2\sigma x}u_{xx}^4\Big)^{1/2}\Big(\int_0^t\int u_{xxx}^2\Big)^{1/2}\\
&\le C\|u\|_{H^3}\Big(\int_0^t\int_B t^3|e^{\sigma x/2}u_{xx}|^4\Big)^{1/2}\\
&\le C\Big(\int_0^t\int_B t^3[(e^{\sigma x/2}u_{xx})^2+([e^{\sigma x/2}u_{xx}]_x)^2+([e^{\sigma x/2}u_{xx}]_y)^2]\Big)^2\\
&\le C\Big(\int_0^t\int_B t^3e^{\sigma x}(u_{xx}^2+u_{xxx}^2+u_{xxy}^2)\Big)^2.
\end{aligned}
$$

Combining these estimates, we have

$$
\Big|\int_0^t\int \xi u_{xxx}(uu_x)_{xxx}\Big|\le C
$$

where C depends only on $\|u\|_{H^3}$ and terms of the form (3.3). The other terms on the level $\beta=3$ can be handled similarly. Similarly for v. Combining these estimates with (3.8), we conclude that

$$
\sum_{|\alpha|=3}\sup_{0\le t\le T}C\int\xi(\cdot,t)[(\partial^\alpha u)^2+(\partial^\alpha v)^2]+C\sum_{|\alpha|=4}\int_0^T\int\chi[(\partial^\alpha u)^2+(\partial^\alpha v)^2]\le C
$$

where C depends only on $\|u\|_{H^3(\mathbb{R}^2)}$, $\|v\|_{H^3(\mathbb{R}^2)}$ and terms in (3.3) and (3.4).

Case $\beta\ge 4$. For $\beta\ge 4$, let $\xi\equiv\xi_\beta=\int_{-\infty}^x\chi_\beta(z,t)dz$ where $\chi_\beta\in W_{\sigma,L-\beta-1,\beta}$. Therefore, $\xi\approx t^\beta x^{L-\beta}$ for $x>1$ and $\xi\approx t^\beta e^{\sigma x}$ for $x<-1$. We combine Lemma 3.2 given below, in which we estimate our remainder term (3.9) with our main inequality (3.8) to conclude that

$$\sum_{|\alpha|=\beta}\sup_{0\le t\le T} C\int \xi[(\partial^\alpha u)^2+(\partial^\alpha v)^2]+\sum_{|\alpha|=\beta+1}\int_0^T\int \chi[(\partial^\alpha u)^2+(\partial^\alpha v)^2]\le C$$

where C depends only on $\|u\|_{H^3}$, $\|v\|_{H^3}$ and terms in (3.2)-(3.4). □

We now show the bounds on the remainder (3.9) for $\beta\ge 4$.

Lemma 3.2. *For $4\le\beta\le L$, $\beta=|\alpha|$, and (u,v) a solution of (1.3) sufficiently smooth and with sufficient decay at infinity, for $0\le t\le T$, we have*

$$\Big|\int_0^t\int \xi_\beta(\partial^\alpha u)\partial^\alpha(uu_x)\Big|\le C,\quad \Big|\int_0^t\int \xi_\beta(\partial^\alpha v)\partial^\alpha(vv_x)\Big|\le C \tag{3.10}$$

where $\xi_\beta\in W_{\sigma,L-\beta,\beta}$, $\chi_\beta\in W_{\sigma,L-\beta-1,\beta}$, $\sigma>0$ arbitrary and C depends only on $\|u\|_{H^3(\mathbb{R}^2)}$, $\|v\|_{H^3(\mathbb{R}^2)}$, and on (3.2), (3.3), and (3.4).

Before proving Lemma 3.2, we describe the form of each term in (3.9).

Lemma 3.3. *Every term in the integrand of*

$$\begin{gathered}\int_0^t\int_{\mathbb{R}^2}\xi(\partial^\alpha u)\partial^\alpha(uu_x)\,dx\,dy\,dt\\ \int_0^t\int_{\mathbb{R}^2}\xi(\partial^\alpha v)\partial^\alpha(vv_x)\,dx\,dy\,dt\end{gathered}\tag{3.11}$$

is of the form

$$\begin{gathered}C\xi(\partial^\alpha u)(\partial^r u)(\partial^s u_x)\\ C\xi(\partial^\alpha v)(\partial^r v)(\partial^s v_x)\end{gathered}\tag{3.12}$$

respectively, where $r=(r_1,r_2)$, $s=(s_1,s_2)$, $r_i+s_i=\alpha_i$ for $i=1,2$.

The above lemma follows from the Leibniz formula applied to $\partial^\alpha(uu_x)$ and $\partial^\alpha(vv_x)$.

Proof of Lemma 3.2. By Lemma 3.3 we can write every term in the integrand of (3.11) in the form (3.12) where $\xi\equiv\xi_\beta\in W_{\sigma,L-\beta,\beta}$. It remains to show that each of these terms is bounded by constants depending only on (3.2)-(3.4). In this part we draw our attention to the case when $x>1$. For $x>1$, our weight function $\xi\simeq t^k x^\ell$ for $k,\ell\ge 0$. For the case $x<-1$, $\xi\simeq t^k e^{\sigma x}$ for $\sigma>0$ arbitrary. That case is even easier to handle. Let $A=\{(x,y):\ x>1\}$. Then for $x>1$, we are looking to get bounds on (3.11) and (3.12) in terms of

$$\sup_{0\le t\le T}\int_A t^\nu x^{L-\nu}[(\partial^\gamma u)^2+(\partial^\gamma v)^2],\tag{3.13}$$

$$\int_0^T\int_A t^\nu x^{L-\nu-1}[(\partial^\gamma u_x)^2+(\partial^\gamma v_x)^2],\tag{3.14}$$

$$\int_0^T\int_A t^\nu x^{L-\nu-1}[(\partial^\gamma u_y)^2+(\partial^\gamma v_y)^2]\tag{3.15}$$

where $\beta-1\ge\nu=|\gamma|\ge 0$. We need to break up each term of the form (3.12) into three parts, being sure to divide the weight function appropriately among the three terms. To do so, we combine

$$\|\partial^\gamma u\|_{L^\infty(\mathbb{R}^2)}\le\Big(\int[(\partial^\gamma u)^2+(\partial^\gamma u_{xx})^2+(\partial^\gamma u_{yy})^2]\Big)^{1/2}\tag{3.16}$$

with

$$t^k x^\ell (\partial^\gamma u) = t^k \sum_{j=0}^{\gamma_1} (-1)^j \binom{\gamma}{j} \partial_x^j [(\partial_x^{\gamma_1 - j} x^\ell)(\partial_y^{\gamma_2} u)]. \tag{3.17}$$

In fact,

$$\|t^k x^\ell (\partial^\gamma u)\|_{L^\infty(\mathbb{R}^2)} \le t^k \sum_{j=0}^{\gamma_1} \Big\| \binom{\gamma}{j} \partial_x^j [(\partial_x^{\gamma_1 - j} x^\ell)(\partial_y^{\gamma_2} u)] \Big\|_{L^\infty(\mathbb{R}^2)}. \tag{3.18}$$

Hence, combining (3.18) with (3.16) we conclude that

$$\sup_{0 \le t \le T} \|t^{(\nu+1)/2}\, x^{(L-(\nu+1))/2} (\partial^\gamma u)\|_{L^\infty(A)} \le C \tag{3.19}$$

for $\nu = |\gamma| \le \beta - 3$, and

$$\int_0^T \|t^{(\nu+1)/2}\, x^{(L-(\nu+1)-1)/2}\, (\partial^\gamma u)\|_{L^\infty(A)} \le C \tag{3.20}$$

for $\nu = |\gamma| \le \beta - 2$, where the constant C depends only (3.13), (3.14), (3.15). Similarly for v. Then using the above inequalities, we look at terms of the form (3.12). For $x > 1$, these terms can be expressed as follows

$$\Big| \int_0^t \int_A \xi (\partial^\alpha u)(\partial^r u)(\partial^s u_x) \Big| \le C \int_0^T \Big| \int_A t^\beta x^{L-\beta} (\partial^\alpha u)(\partial^r u)(\partial^s u_x) \Big|. \tag{3.21}$$

For notation, let $\nu_r = r_1 + r_2$ and $\nu_s = s_1 + s_2$. Since $\beta = \alpha_1 + \alpha_2$, it follows that $\nu_r + \nu_s = \beta$.

Case $v_s \le \beta - 4$. In this case, we bound the remainder term as follows

$$\begin{aligned}
&\int_0^T \Big| \int_A t^\beta x^{L-\beta} (\partial^\alpha u)(\partial^r u)(\partial^s u_x) \Big| \\
&\le \sup_{x>1} x^N\, T^M \sup_{0 \le t \le T} \|t^{(\nu_s+2)/2} x^{(L-\nu_s-1)/2} (\partial^s u_x)\|_{L^\infty(A)} \\
&\quad \times \Big(\int_0^T \int_A t^{(\nu_r-1)} x^{L-(\nu_r-1)-1} (\partial^r u)^2 \Big)^{1/2} \Big(\int_0^T \int_A t^{\beta-1} x^{L-(\beta-1)-1} (\partial^\alpha u)^2 \Big)^{1/2}.
\end{aligned}$$

First, we show that $M \ge 0$ and $N \le 0$, so that any extra powers of t or x can be thrown away. We see that

$$\beta = M + \frac{\nu_s + 2}{2} + \frac{\nu_r - 1}{2} + \frac{\beta - 1}{2}.$$

Therefore,

$$2\beta = 2M + \nu_s + 2 + \nu_r - 1 + \beta - 1.$$

Therefore,

$$2M = \beta - \nu_s - \nu_r = 0.$$

Therefore, $M = 0$. Next

$$L - \beta = N + \frac{L - \mu_s - 1}{2} + \frac{L - (\nu_r - 1) - 1}{2} + \frac{L - (\beta - 1) - 1}{2}.$$

Therefore,

$$2L - 2\beta = 2N + 3L - \nu_s - \nu_r - \beta - 1,$$

which implies $-L + 1 = 2N$. Therefore,

$$N = \frac{-L+1}{2} \le 0$$

as long as $L \geq 1$. The others three terms are bounded by (3.19), (3.14) and (3.15).

Case $\beta - 4 < \nu_s \leq \beta - 2$**.** In this case, we have

$$\begin{aligned}
&\int_0^T \int t^\beta x^{L-\beta}(\partial^\alpha u)(\partial^r u)(\partial^s u_x) \\
&\leq CT^M x^N \Big(\int_0^T \int t^{\beta-1} x^{L-(\beta-1)-1}(\partial^\alpha u)^2\Big)^{1/2} \Big(\int_0^T \int t^{\nu_r} x^{L-\nu_r-1}(\partial^r u)^4\Big)^{1/4} \\
&\quad \times \Big(\int_0^T \int t^{\nu_s+1} x^{L-(\nu_s+1)-1}(\partial^s u_x)^4\Big)^{1/4} \\
&\leq CT^M x^N \Big(\int_0^T \int t^{\beta-1} x^{L-(\beta-1)-1}(\partial^\alpha u)^2\Big)^{1/2} \\
&\quad \times \Big(\int_0^T \int t^{\nu_r} x^{L-\nu_r-1}[(\partial^r u_x)^2 + (\partial^r u_y)^2]\Big)^{1/2} \\
&\quad \times \Big(\int_0^T \int t^{\nu_s+1} x^{L-(\nu_s+1)-1}[(\partial^s u_{xx})^2 + (\partial^s u_{xy})^2]\Big)^{1/2}.
\end{aligned}$$

Then

$$\beta = M + \frac{\beta-1}{2} + \frac{\nu_r}{2} + \frac{\nu_s+1}{2}.$$

Therefore, $M = 0$. Also,

$$L - \beta = N + \frac{L-(\beta-1)-1}{2} + \frac{L-\nu_r-1}{2} + \frac{L-(\nu_s+1)-1}{2}.$$

Therefore,

$$2(L-\beta) = 2N + 3L - \beta - \nu_r - \nu_s - 3.$$

Therefore, $N \leq 0$ as long as $3 \leq L$. But $L \geq \beta \geq 4$. Therefore, $L \geq 3$.

Case $\nu_s = \beta - 1$**.** In this case, $\nu_r = 1$. Therefore $r = (1,0)$ or $r = (0,1)$. Consider first $r = (1,0)$. Then

$$\begin{aligned}
\int_0^T \int_A t^\beta x^{L-\beta}(\partial^r u)(\partial^s u_x)(\partial^\alpha u) &= \int_0^T t^\beta x^{L-\beta} u_x(\partial^{(\alpha_1-1,\alpha_2)} u_x)(\partial^\alpha u) \\
&= \int_0^T \int_A t^\beta x^{L-\beta} u_x (\partial^\alpha u)^2 \\
&\leq CT|u_x|_{L^\infty} \int_0^T \int_A t^{\beta-1} x^{L-\beta}(\partial^\alpha u)^2 \\
&\leq CT\|u\|_{H^3} \int_0^T \int_A t^{\beta-1} x^{L-(\beta-1)-1}(\partial^\alpha u)^2.
\end{aligned}$$

Then the term on the right-hand side is bounded as desired. Similarly, if $r = (0,1)$, we have

$$\begin{aligned}
&\int_0^T \int_A t^\beta x^{L-\beta}(\partial^r u)(\partial^s u_x)(\partial^\alpha u) \\
&= \int_0^T \int_A t^\beta x^{L-\beta} u_y(\partial^{(\alpha_1,\alpha_2-1)} u_y)(\partial^\alpha u) \\
&\leq CT|u_y|_{L^\infty} \int_0^T \int_A t^{\beta-1} x^{L-(\beta-1)-1}[(\partial^{(\alpha_1,\alpha_2-1)} u_y)^2 + (\partial^\alpha u)^2].
\end{aligned}$$

Case $\nu_s = \beta$. In this case $\nu_r = 0$. Then

$$
\begin{aligned}
&\int_0^T \big| \int_A t^\beta \, x^{L-\beta} (\partial^\alpha u)(\partial^r u)\, (\partial^s u_x)\big| \\
&= \int_0^T \big| \int_A t^\beta x^{L-\beta} u (\partial^\alpha u)(\partial^\alpha u_x)\big| \\
&= C \int_0^T \big| \int_A t^\beta [x^{L-\beta} u]_x (\partial^\alpha u)^2 \big| \\
&\le CT \left(\|u\|_{L^\infty(A)} + \|u_x\|_{L^\infty(A)} \right) \Big(\int_0^T \int_A t^{\beta-1} x^{L-\beta-1} (\partial^\alpha u)^2 \Big) \\
&\le CT |\!|\!| u |\!|\!|_{H^3} \Big(\int_0^T \int_A t^{\beta-1} x^{L-(\beta-1)-1} (\partial^\alpha u_x)^2 \Big),
\end{aligned}
$$

where again the right-hand side is bounded by (3.14) or (3.15). Lemma 3.2 follows. □

4. A priori estimates

In this section we prove two lemmas that will be used in a local-in-time existence theorem in Section 5. First we prove an a priori estimate for a linearized system related to (1.3). Second, we prove existence of a unique solution of that linearized system.

For the lemma involving the a priori estimate, we introduce the following function space Z_T^N. We define

$$
Z_T^N = \{u : \; u \in L^\infty([0,T] : H^{N+3}(\mathbb{R}^2)), \; u_t \in L^\infty([0,T] : H^N(\mathbb{R}^2))\}, \tag{4.1}
$$

with the norm

$$
\|u\|^2_{Z_T^N} = \sup_{0\le t\le T} \int_{\mathbb{R}^2} \big[u^2 + \sum_{|\alpha|=N} \{(\partial^\alpha u_{xxx})^2 + (\partial^\alpha u_{yyy})^2\} + u_t^2 + \sum_{|\alpha|=N} (\partial^\alpha u_t)^2 \big]. \tag{4.2}
$$

Lemma 4.1. *Let u, v, w, z be functions in Z_t^N for all N and all $t \ge 0$ such that u, v, w, z are solutions to*

$$
\begin{gathered}
u_t + u_{xxx} + u_{yyx} - v_x - 6 w u_x = 0, \\
b v_t + \delta v_{xxx} + \lambda v_{yyx} + \eta v_x - u_x - 6\mu z v_x = 0.
\end{gathered} \tag{4.3}
$$

Then for all $N \ge 0$, the following inequality holds

$$
\begin{aligned}
\|u\|^2_{Z_t^N} + b\|v\|^2_{Z_t^N} &\le \|u(\cdot,\cdot,0)\|^2_{H^{N+3}} + b\|v(\cdot,\cdot,0)\|^2_{H^{N+3}} + \|u_t(\cdot,\cdot,0)\|^2_{H^N} \\
&\quad + b\|v_t(\cdot,\cdot,0)\|^2_{H^N} + Ct\|w\|_{Z_t^N}\|u\|^2_{Z_t^N} + Ct\|z\|_{Z_t^N}\|v\|^2_{Z_t^N}
\end{aligned} \tag{4.4}
$$

for all $t \ge 0$.

Proof. Fix $N \ge 0$ and choose α such that $|\alpha| = N$. Applying ∂^α to $(4.3)_1$ we have

$$
\partial^\alpha u_t + \partial^\alpha u_{xxx} + \partial^\alpha u_{yyx} - \partial^\alpha v_x - 6\partial^\alpha (w u_x) = 0. \tag{4.5}
$$

Multiplying (4.5) by $2\partial^\alpha u$ and integrating over $\mathbb{R}^2$ we obtain

$$\begin{aligned}\partial_t \int (\partial^\alpha u)^2 &= 12 \int (\partial^\alpha u)\partial^\alpha (w u_x) + 2\int (\partial^\alpha u)(\partial^\alpha v_x)\\ &\le C\Big|\int (\partial^\alpha u)[(\partial^\alpha w)u_x + \dots + w(\partial^\alpha u_x)]\Big| + 2\int (\partial^\alpha u)(\partial^\alpha v_x)\\ &\le C\Big|\int (\partial^\alpha w)u_x(\partial^\alpha u)\Big| + \dots + \Big|\int_{\mathbb{R}^2} w(\partial^\alpha u_x)(\partial^\alpha u)\Big| + 2\int (\partial^\alpha u)(\partial^\alpha v_x)\end{aligned} \tag{4.6}$$

The first term on the right-hand side of (4.6) is estimated using (2.11) together with the Cauchy-Schwarz inequality

$$\begin{aligned}\Big|\int (\partial^\alpha w)u_x(\partial^\alpha u)\Big| &\le \|u_x\|_{L^\infty}\Big(\int (\partial^\alpha w)^2\Big)^{1/2}\Big(\int (\partial^\alpha u)^2\Big)^{1/2}\\ &\le C\Big(\int [u_x^2 + u_{xxx}^2 + u_{xyy}^2]\Big)^{1/2}\|w\|_{H^{|\alpha|}}\|u\|_{H^{|\alpha|}}\\ &\le C\|w\|_{Z_t^{|\alpha|}}\|u\|^2_{Z_t^{|\alpha|}}.\end{aligned}$$

For the second to last term on the right-hand side of (4.6), integrating by parts and using (2.11) we have

$$\begin{aligned}\Big|\int w(\partial^\alpha u_x)(\partial^\alpha u)\Big| &= C\int w_x(\partial^\alpha u)^2 \le C\|w_x\|_{L^\infty}\int (\partial^\alpha u)^2\\ &\le C\Big(\int [w_x^2 + w_{xxx}^2 + w_{xy}^2]\Big)^{1/2}\|u\|^2_{Z_t^{|\alpha|}}\\ &\le C\|w\|_{Z_t^{|\alpha|}}\|u\|^2_{Z_t^{|\alpha|}}.\end{aligned}$$

Therefore,

$$\partial_t \int \big(\partial^\alpha u(\cdot,\cdot,\widetilde{t})\big)^2 \le C\,\|w\|_{Z_t^{|\alpha|}}\ \|u\|^2_{Z_t^{|\alpha|}} + 2\int (\partial^\alpha u)(\partial^\alpha v_x). \tag{4.7}$$

In a similar way applying the same idea to $(4.3)_2$, we obtain

$$\partial_t b\int \big(\partial^\alpha v(\cdot,\cdot,\widetilde{t})\big)^2 \le C\|z\|_{Z_t^{|\alpha|}}\ \|v\|^2_{Z_t^{|\alpha|}} + 2\int (\partial^\alpha v)(\partial^\alpha u_x). \tag{4.8}$$

Then adding (4.7) and (4.8) we obtain

$$\partial_t \int [(\partial^\alpha u(\cdot,\cdot,\widetilde{t}))^2 + b(\partial^\alpha v(\cdot,\cdot,\widetilde{t}))^2] \le C\|w\|_{Z_t^{|\alpha|}}\ \|u\|^2_{Z_t^{|\alpha|}} + C\|z\|_{Z_t^{|\alpha|}}\ \|v\|^2_{Z_t^{|\alpha|}}. \tag{4.9}$$

Now, taking three x-derivatives of (4.5) and multiplying the result by $2\partial^\alpha u_{xxx}$, we have

$$\begin{aligned}
&\partial_t \int (\partial^\alpha u_{xxx})^2 \\
&\le \Big| \int (\partial^\alpha (w\, u_x)_{xxx})\, (\partial^\alpha u_{xxx}) \Big| + 2 \int (\partial^\alpha u_{xxx})(\partial^\alpha v_{xxxx}) \\
&\le \Big| \int \partial^\alpha (w_{xxx}\, u_x + 3\, w_{xx}\, u_{xx} + 3\, w_x\, u_{xxx} + w\, u_{xxxx})\, (\partial^\alpha u_{xxx}) \Big| \\
&\quad + 2 \int (\partial^\alpha u_{xxx})(\partial^\alpha v_{xxxx}) \\
&\le \Big| \int \partial^\alpha (w_{xxx} u_x)\, (\partial^\alpha u_{xxx}) \Big| + 3 \Big| \int \partial^\alpha (w_{xx} u_{xx})\, (\partial^\alpha u_{xxx}) \Big| \\
&\quad + 3 \Big| \int \partial^\alpha (w_x u_{xxx})\, (\partial^\alpha u_{xxx}) \Big| + \Big| \int \partial^\alpha (w u_{xxxx})\, (\partial^\alpha u_{xxx}) \Big| \\
&\quad + 2 \int_{\mathbb{R}^2} (\partial^\alpha u_{xxx})(\partial^\alpha v_{xxxx}) \\
&= I_1 + 3I_2 + 3I_3 + I_4 + 2 \int (\partial^\alpha u_{xxx})(\partial^\alpha v_{xxxx}).
\end{aligned} \tag{4.10}$$

Each term in the above expression is estimate separately. For the first term it follows that

$$\begin{aligned}
I_1 &= \Big| \int \partial^\alpha (w_{xxx} u_x)\, (\partial^\alpha u_{xxx}) \Big| \\
&= \Big| \int [(\partial^\alpha w_{xxx})\, u_x\, (\partial^\alpha u_{xxx}) + \dots + w_{xxx}\, (\partial^\alpha u_x)\, (\partial^\alpha u_{xxx})] \Big| \\
&\le C \|u_x\|_{L^\infty} \Big(\int (\partial^\alpha w_{xxx})^2 \Big)^{1/2} \Big(\int (\partial^\alpha u_{xxx})^2 \Big)^{1/2} + \dots \\
&\quad + C \|\partial^\alpha u_x\|_{L^\infty} \Big(\int w_{xxx}^2 \Big)^{1/2} \Big(\int (\partial^\alpha u_{xxx})^2 \Big)^{1/2} \\
&\le C \Big(\int \big[u_x^2 + u_{xxx}^2 + u_{xyy}^2 \big] \Big)^{1/2} \Big(\int (\partial^\alpha w_{xxx})^2 \Big)^{1/2} \Big(\int (\partial^\alpha u_{xxx})^2 \Big)^{1/2} \\
&\quad + \dots + C \Big(\int \big[(\partial^\alpha u_x)^2 + (\partial^\alpha u_{xxx})^2 + (\partial^\alpha u_{xyy})^2 \big] \Big)^{1/2} \\
&\quad \times \Big(\int w_{xxx}^2 \Big)^{1/2} \Big(\int_{\mathbb{R}^2} (\partial^\alpha u_{xxx})^2 \Big)^{1/2} \\
&\le C \|u\|_{Z_t^0} \|w\|_{Z_t^{|\alpha|}} \|u\|_{Z_t^{|\alpha|}} + \dots + C \|u\|_{Z_t^{|\alpha|}} \|w\|_{Z_t^0} \|u\|_{Z_t^{|\alpha|}} \\
&\le C \|u\|_{Z_t^{|\alpha|}} \|w\|_{Z_t^{|\alpha|}} \|u\|_{Z_t^{|j|}} + \dots + C \|u\|_{Z_t^{|\alpha|}} \|w\|_{Z_t^{|\alpha|}} \|u\|_{Z_t^{|\alpha|}} \\
&\le C \|w\|_{Z_t^{|\alpha|}} \|u\|^2_{Z_t^{|\alpha|}}.
\end{aligned}$$

Next, we consider I_2.

$$\begin{aligned}
I_2 &= \Big| \int \partial^\alpha (w_{xx} u_{xx})\, (\partial^\alpha u_{xxx}) \Big| \\
&= \Big| \int [(\partial^\alpha w_{xx})\, u_{xx}\, (\partial^\alpha u_{xxx}) + \dots + w_{xx}\, (\partial^\alpha u_{xx})\, (\partial^\alpha u_{xxx})] \Big|.
\end{aligned} \tag{4.11}$$

Using the Cauchy-Schwarz inequality and (2.12), for the first term in (4.11), we have

$$
\begin{aligned}
&\Big|\int (\partial^\alpha w_{xx}) u_{xx}\,(\partial^\alpha u_{xxx})\Big| \\
&\le \Big(\int (\partial^{|\alpha|} w_{xx})^4\Big)^{1/4}\Big(\int u_{xx}^4\,dx\,dy\Big)^{1/4}\Big(\int (\partial^\alpha u_{xxx})^2\Big)^{1/2} \\
&\le \Big(\int [(\partial^\alpha w_{xx})^2 + (\partial^\alpha w_{xxx})^2 + (\partial^\alpha w_{xxy})^2]\Big)^{1/2}\Big(\int \big[u_{xx}^2 + u_{xxx}^2 + u_{xxy}^2\big]\Big)^{1/2} \\
&\quad \times \Big(\int (\partial^\alpha u_{xxx})^2\Big)^{1/2} \\
&\le c\|w\|_{Z_t^{|\alpha|}}\|u\|^2_{Z_t^{|\alpha|}}.
\end{aligned}
$$

Integrating by parts the last term in (4.11), using the Cauchy-Schwarz inequality, and (2.12), we have

$$
\begin{aligned}
\Big|\int w_{xx}\,(\partial^\alpha u_{xx})\,(\partial^\alpha u_{xxx})\Big| &= \frac{1}{2}\Big|\int w_{xxx}\,(\partial^\alpha u_{xx})^2\Big| \\
&\le \frac{1}{2}\Big(\int w_{xxx}^2\Big)^{1/2}\Big(\int (\partial^\alpha u_{xx})^4\Big)^{1/2} \\
&\le \frac{1}{2}\|w\|_{Z_t^{|0|}}\Big(\int (\partial^\alpha u_{xx})^2 + (\partial^\alpha u_{xxx})^2 + (\partial^\alpha u_{xxy})^2\Big)^{1/2} \\
&\le C\|w\|_{Z_t^{|\alpha|}}\|u\|^2_{Z_t^{|\alpha|}}.
\end{aligned}
$$

For the third term in (4.10) we have

$$
\begin{aligned}
I_3 &= \Big|\int \partial^\alpha\,(w_x u_{xxx})\,(\partial^\alpha u_{xxx})\Big| \\
&= \Big|\int [(\partial^\alpha w_x)\,u_{xxx}\,(\partial^\alpha u_{xxx}) + \dots + w_x\,(\partial^\alpha u_{xxx})\,(\partial^\alpha u_{xxx})]\Big| \\
&\le C\|\partial^\alpha w_x\|_{L^\infty}\Big(\int u_{xxx}^2\Big)^{1/2}\Big(\int (\partial^\alpha u_{xxx})^2\Big)^{1/2} + \dots + C\|w_x\|_{L^\infty}\int (\partial^\alpha u_{xxx})^2 \\
&\le C\|w\|_{Z_t^{|\alpha|}}\|u\|_{Z_t^0}\|v\|_{Z_t^{|\alpha|}} + \dots + C\|w\|_{Z_t^0}\|u\|^2_{Z_t^{|\alpha|}} \\
&\le C\|w\|_{Z_t^{|\alpha|}}\|u\|^2_{Z_t^{|\alpha|}}.
\end{aligned}
$$

Now we estimate the last term in (4.10) as follows

$$
\begin{aligned}
I_4 &= \Big|\int \partial^\alpha\,(w u_{xxxx})\,(\partial^\alpha u_{xxx})\Big| \\
&= \Big|\int [(\partial^\alpha w)\,u_{xxxx}\,(\partial^\alpha u_{xxx}) + \dots + w\,(\partial^\alpha u_{xxxx})\,\big(\partial^j u_{xxx}\big)]\Big|.
\end{aligned}
\tag{4.12}
$$

If $\alpha = (0,0)$, then integrating by parts and using (2.11),

$$
\begin{aligned}
\Big|\int \partial^\alpha\,(w\,u_{xxxx})\,(\partial^\alpha u_{xxx})\Big| &= \Big|\int w u_{xxxx} u_{xxx}\Big| \\
&= \frac{1}{2}\Big|\int_{\mathbb{R}^2} w_x\,u_{xxx}^2\Big| \\
&\le C\|w_x\|_{L^\infty}\int u_{xxx}^2
\end{aligned}
$$

$$
\begin{aligned}
&\le C\Big(\int \big[w_x^2+w_{xxx}^2+w_{xyy}^2\big]\Big)^{1/2}\int_{\mathbb{R}^2} u_{xxx}^2\\
&\le C\|u\|_{Z_t^0}^2\,\|w\|_{Z_t^{|\alpha|}}.
\end{aligned}
$$

For $|\alpha|>0$, using (2.11) for the first term in (4.12), we have

$$
\begin{aligned}
&\Big|\int (\partial^\alpha w)\,u_{xxxx}\,(\partial^\alpha u_{xxx})\Big|\\
&\le C\|\partial^\alpha w\|_{L^\infty}\Big(\int_{\mathbb{R}^2} u_{xxxx}^2\Big)^{1/2}\Big(\int (\partial^\alpha u_{xxx})^2\Big)^{1/2}\\
&\le \Big(\int \Big[(\partial^\alpha w)^2+(\partial^\alpha w_{xx})^2+(\partial^\alpha w_{yy})^2\Big]\Big)^{1/2}\Big(\int u_{xxxx}^2\Big)^{1/2}\Big(\int (\partial^\alpha u_{xxx})^2\Big)^{1/2}\\
&\le C\|w\|_{Z_t^{|\alpha|}}\,\|u\|_{Z_t^{|\alpha|}}^2.
\end{aligned}
$$

For the last term in (4.12), integrating by parts and using (2.11) we have

$$
\begin{aligned}
\Big|\int w\,(\partial^\alpha u_{xxxx})\,(\partial^\alpha u_{xxx})\Big| &= \frac{1}{2}\Big|\int w_x\,(\partial^\alpha u_{xxx})^2\Big|\\
&\le C\|w_x\|_{L^\infty}\int (\partial^\alpha u_{xxx})^2\\
&\le C\Big(\int \big[w_x^2+w_{xxx}^2+w_{xyy}^2\big]\Big)^{1/2}\int_{\mathbb{R}^2}(\partial^\alpha u_{xxx})^2\\
&\le C\|w\|_{Z_t^{|\alpha|}}\|u\|_{Z_t^{|\alpha|}}^2.
\end{aligned}
$$

Consequently,

$$
\partial_t\int (\partial^\alpha u_{xxx})^2 \le C\|w\|_{Z_t^{|\alpha|}}\|u\|_{Z_t^{|\alpha|}}^2 + 2\int (\partial^\alpha u_{xxx})(\partial^\alpha v_{xxxx}). \tag{4.13}
$$

In a similar way, applying the same idea to $(4.3)_2$, we obtain

$$
\partial_t b\int (\partial^\alpha v_{xxx})^2 \le C\|z\|_{Z_t^{|\alpha|}}\|v\|_{Z_t^{|\alpha|}}^2 + 2\int (\partial^\alpha v_{xxx})(\partial^\alpha u_{xxxx}). \tag{4.14}
$$

Then adding (4.13) and (4.14) we obtain

$$
\partial_t\int \Big[(\partial^\alpha u_{xxx})^2 + b\,(\partial^\alpha v_{xxx})^2\Big] \le C\|w\|_{Z_t^{|\alpha|}}\|u\|_{Z_t^{|\alpha|}}^2 + C\|z\|_{Z_t^{|\alpha|}}\|v\|_{Z_t^{|\alpha|}}^2. \tag{4.15}
$$

Similarly, applying $\partial^\alpha\partial_y^3$ to each equation in (4.3), and using similar analysis, we obtain

$$
\partial_t\int [(\partial^\alpha u_{yyy})^2 + b(\partial^\alpha v_{yyy})^2] \le C\|w\|_{Z_t^{|\alpha|}}\|u\|_{Z_t^{|\alpha|}}^2 + C\|z\|_{Z_t^{|\alpha|}}\|v\|_{Z_t^{|\alpha|}}^2. \tag{4.16}
$$

On the other hand, applying one t-derivative to (4.5), multiplying by $2\,(\partial^\alpha u_t)$ and integrating over $\mathbb{R}^2$ we arrive at the inequality

$$
\begin{aligned}
\partial_t\int (\partial^\alpha u_t)^2 &\le C\Big|\int (\partial^\alpha\,(wu_x)_t)\,(\partial^\alpha u_t)\Big| + 2\int (\partial^\alpha u_t)(\partial^\alpha v_{xt})\\
&\le C\Big|\int \partial^\alpha\,(w_tu_x)\,(\partial^\alpha u_t)\Big| + C\Big|\int \partial^\alpha\,(wu_{xt})\,(\partial^\alpha u_t)\Big|\\
&= K_1 + K_2 + 2\int (\partial^\alpha u_t)(\partial^\alpha v_{xt}).
\end{aligned}
$$

For the first term on the right hand side, we use (2.11) and the Cauchy-Schwarz inequality

$$\begin{aligned}K_1 &= \Big|\int \partial^\alpha (w_t u_x)(\partial^\alpha u_t)\Big| \\ &\le C\Big|\int (\partial^\alpha w_t)\, u_x\, (\partial^\alpha u_t)\Big| + \dots + C\Big|\int w_t\, (\partial^\alpha u_x)(\partial^\alpha u_t)\Big| \\ &\le C\|u_x\|_{L^\infty}\Big(\int (\partial^\alpha w_t)^2\Big)^{1/2}\Big(\int (\partial^\alpha u_t)^2\Big)^{1/2} + \dots \\ &\quad + C\|\partial^\alpha u_x\|_{L^\infty}\Big(\int_{\mathbb{R}^2} w_t^2\Big)^{1/2}\Big(\int (\partial^\alpha u_t)^2\Big)^{1/2} \\ &\le C\|u\|_{Z_t^0}\|w\|_{Z_t^{|\alpha|}}\|u\|_{Z_t^{|\alpha|}} + \dots + c\|u\|_{Z_t^{|\alpha|}}\|w\|_{Z_t^0}\|u\|_{Z_t^{|\alpha|}} \\ &\le C\|w\|_{Z_t^{|\alpha|}}\|u\|^2_{Z_t^{|\alpha|}}.\end{aligned}$$

We look at the second term on the right-hand side. If $\alpha = (0,0)$ we have

$$K_2 = \Big|\int w u_{xt}\, u_t\Big| = \frac{1}{2}\Big|\int w_x u_t^2\Big| \le C\|w_x\|_{L^\infty}\int_{\mathbb{R}^2} u_t^2 \le C\|w\|_{Z_t^{|\alpha|}}\|u\|^2_{Z_t^{|\alpha|}}.$$

If $\alpha \neq (0,0)$, we have

$$\begin{aligned}&\Big|\int \partial^\alpha (w u_{xt})(\partial^\alpha u_t)\Big| \\ &= C\Big|\int (\partial^\alpha w)\, u_{xt}\, (\partial^\alpha u_t)\Big| + \dots + C\Big|\int w\, (\partial^\alpha u_{xt})(\partial^\alpha u_t)\Big|.\end{aligned} \tag{4.17}$$

The first term in (4.17) is estimated as

$$\begin{aligned}\Big|\int (\partial^\alpha w)\, u_{xt}\, (\partial^\alpha u_t)\Big| &\le C\|\partial^\alpha w\|_{L^\infty}\Big(\int u_{xt}^2\Big)^{1/2}\Big(\int (\partial^\alpha u_t)^2\Big)^{1/2} \\ &\le C\|w\|_{Z_t^{|\alpha|}}\|u\|^2_{Z_t^{|\alpha|}}.\end{aligned}$$

Using integration by parts in the last term in (4.17) along with (2.11) and the Cauchy-Schwarz inequality, we have

$$\begin{aligned}\Big|\int w\, (\partial^\alpha u_{xt})(\partial^\alpha u_t)\Big| &= \frac{1}{2}\Big|\int w_x\, (\partial^\alpha u_t)^2\Big| \\ &\le C\|w_x\|_{L^\infty}\int (\partial^\alpha u_t)^2 \\ &\le C\|w\|_{Z_t^{|\alpha|}}\|u\|^2_{Z_t^{|\alpha|}}.\end{aligned}$$

Thus

$$\partial_t \int (\partial^\alpha u_t)^2 \le C\|w\|_{Z_t^{|\alpha|}}\|u\|^2_{Z_t^{|\alpha|}} + 2\int (\partial^\alpha u_t)(\partial^\alpha v_{xt}). \tag{4.18}$$

In a similar way, we obtain

$$\partial_t b \int (\partial^\alpha v_t)^2 \le C\|z\|_{Z_t^{|\alpha|}}\|v\|^2_{Z_t^{|\alpha|}} + 2\int (\partial^\alpha v_t)(\partial^\alpha u_{xt}). \tag{4.19}$$

Then adding (4.18) and (4.19) we obtain

$$\partial_t \int [(\partial^\alpha u_t)^2 + b\,(\partial^\alpha v_t)^2] \le C\|w\|_{Z_t^{|\alpha|}}\|u\|^2_{Z_t^{|\alpha|}} + C\|z\|_{Z_t^{|\alpha|}}\|v\|^2_{Z_t^{|\alpha|}}. \tag{4.20}$$

Then, for $0 \le \widetilde{t} \le t$, it follows that

$$\begin{aligned}
&\partial_t \int \Big[\big(\partial^\alpha u(\cdot,\cdot,\widetilde{t})\big)^2 + b\,\big(\partial^\alpha v(\cdot,\cdot,\widetilde{t})\big)^2 + \big(\partial^\alpha u_{xxx}(\cdot,\cdot,\widetilde{t})\big)^2 + b\,\big(\partial^\alpha v_{xxx}(\cdot,\cdot,\widetilde{t})\big)^2 \\
&\quad + \big(\partial^\alpha u_{yyy}(\cdot,\cdot,\widetilde{t})\big)^2 + b\,\big(\partial^\alpha v_{yyy}(\cdot,\cdot,\widetilde{t})\big)^2 + \big(\partial^\alpha u_t(\cdot,\cdot,\widetilde{t})\big)^2 + b\,\big(\partial^\alpha v_t(\cdot,\cdot,\widetilde{t})\big)^2 \Big] \\
&\le c\|w\|_{Z_t^{|\alpha|}}\|u\|^2_{Z_t^{|\alpha|}} + C\|z\|_{Z_t^{|\alpha|}}\|v\|^2_{Z_t^{|\alpha|}}.
\end{aligned}$$

Integrating with respect to t, and using the fact that this estimate is true for all α such that $|\alpha| = N$, we obtain

$$\begin{aligned}
\|u\|^2_{Z_t^N} + b\|v\|^2_{Z_t^N} &\le \|u(\cdot,\cdot,0)\|^2_{H^{N+3}} + b\|v(\cdot,\cdot,0)\|^2_{H^{N+3}} + \|u_t(\cdot,\cdot,0)\|^2_{H^N} \\
&\quad + b\|v_t(\cdot,\cdot,0)\|^2_{H^N} + Ct\|w\|_{Z_t^N}\|u\|^2_{Z_t^N} + Ct\|z\|_{Z_t^N}\|v\|^2_{Z_t^N},
\end{aligned}$$

as claimed. □

Next we prove an existence result for a linearized version of (1.3). Consider the linear system

$$\begin{gathered}
u_t^{(n)} + u_{xxx}^{(n)} + u_{yyx}^{(n)} - v_x^{(n)} - 6u^{(n-1)}\,u_x^{(n)} = 0, \\
bv_t^{(n)} + \delta v_{xxx}^{(n)} + \lambda v_{yyx}^{(n)} + \eta v_x^{(n)} - u_x^{(n)} - 6\mu v^{(n-1)} v_x^{(n)} = 0
\end{gathered} \tag{4.21}$$

where the initial conditions are $u^{(n)}(x,y,0) = u_0(x,y)$ and $v^{(n)}(x,y,0) = v_0(x,y)$, and the first approximations are $u^{(0)}(x,y,t) = u_0(x,y)$ and $v^{(0)}(x,y,t) = v_0(x,y)$.

Lemma 4.2. *Given initial data $u_0, v_0 \in \cap_{N\ge 0} H^N(\mathbb{R}^2)$, there exists a unique solution of system (4.21). The solution is defined in any time interval in which the coefficients are defined.*

Proof. The linear system (4.21) which is to be solved at each iteration has the form

$$u_t + u_{xxx} + u_{yyx} - v_x - hu_x = 0 \tag{4.22}$$

$$bv_t + \delta v_{xxx} + \lambda v_{yyx} + \eta v_x - u_x - \widetilde{h} v_x = 0 \tag{4.23}$$

where h, and $\widetilde{h}$ are smooth bounded coefficients. Fix a time $T > 0$ and a constant $M > 0$. Define

$$\mathcal{L} = \partial_t F + A\partial_{xxx} + B\partial_{yyx} + C\partial_x + D\partial_x \tag{4.24}$$

where

$$\begin{gathered}
F = \begin{pmatrix} 1 & 0 \\ 0 & b \end{pmatrix}, \quad A = \begin{pmatrix} 1 & 0 \\ 0 & \delta \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 \\ 0 & \lambda \end{pmatrix}, \\
C = \begin{pmatrix} 0 & -1 \\ -1 & \eta \end{pmatrix}, \quad D = \begin{pmatrix} -h & 0 \\ 0 & -\widetilde{h} \end{pmatrix}, \quad \Psi = \begin{pmatrix} u \\ v \end{pmatrix}.
\end{gathered} \tag{4.25}$$

Let $\mathcal{L}$ be defined on those functions $(u,v) \in H^3(\mathbb{R}^2) \times H^3(\mathbb{R}^2)$. For functions $\Phi_i = (\nu_i, \zeta_i) \in C\big([0,T] : L^2(\mathbb{R}^2)\big) \times C\big([0,T] : L^2(\mathbb{R}^2)\big)$ which vanish at $t = 0$, we introduce the bilinear form

$$\begin{aligned}
\mathcal{B}(\Phi_1, \Phi_2) = \langle \Phi_1, \Phi_2 \rangle &= \int_0^T \int_{\mathbb{R}^2} e^{-Mt}\Phi_1 \cdot \Phi_2 \,dx\,dy\,dt \\
&= \int_0^T \int_{\mathbb{R}^2} e^{-Mt}\,(\nu_1\nu_2 + \zeta_1\zeta_2)\,dx\,dy\,dt.
\end{aligned}$$

Integrating by parts, we have

$$\begin{aligned}\int_{\mathbb{R}^2} \mathcal{L}\Psi \cdot \Psi \, dx\, dy &\geq \frac{1}{2}\partial_t \int_{\mathbb{R}^2} |\Psi|^2 \, dx\, dy - \frac{1}{2}\int_{\mathbb{R}^2} D_x \Psi \cdot \Psi \, dx\, dy \\ &\geq \frac{1}{2}\partial_t \int_{\mathbb{R}^2} |\Psi|^2 \, dx\, dy - \frac{c}{2}\int_{\mathbb{R}^2} |\Psi|^2 \, dx\, dy\end{aligned} \tag{4.26}$$

for some constant c large enough. Multiplying (4.26) by e^{-Mt} and integrating in time from $t=0$ to $t=T$, we obtain for $\Psi \in C([0,T]: H^3(\mathbb{R}^2)) \times C([0,T]: H^3(\mathbb{R}^2))$ with $\Psi(x,y,0) = (\nu(x,y,0), \eta(x,y,0)) = (0,0)$,

$$\langle \mathcal{L}\Psi, \Psi\rangle \geq \frac{1}{2} e^{-MT} \int |\Psi(x,y,T)|^2 \, dx\, dy + \frac{1}{2}(M-c)\int_0^T \int e^{-Mt} |\Psi|^2 \, dx \, dy \, dt. \tag{4.27}$$

Therefore, $\langle \mathcal{L}\Psi, \Psi\rangle \geq \langle \Psi, \Psi\rangle$ provided M is large enough. Similarly, $\langle \mathcal{L}^*\Phi, \Phi\rangle \geq \langle \Phi, \Phi\rangle$ for all $\Phi \in C([0,T]: H^3(\mathbb{R}^2)) \times C([0,T]: H^3(\mathbb{R}^2))$ with $\Phi(x,y,T) \equiv (0,0)$ where $\mathcal{L}^*$ denotes the formal adjoint of $\mathcal{L}$. Therefore, $\langle \mathcal{L}^*\Phi, \mathcal{L}^*\Psi\rangle$ is an inner product on $\mathcal{D} = \{\Phi \in C([0,T]: H^3(\mathbb{R}^2)) \times C([0,T]: H^3(\mathbb{R}^2)) : \Phi(x,y,T) \equiv (0,0)\}$. We denote by Y the completion of $\mathcal{D}$ with respect to this inner product. By the Riesz representation theorem, there exists a unique solution $V \in Y$ such that for any $\Phi \in \mathcal{D}$,

$$\langle \mathcal{L}^* V, \mathcal{L}^*\Phi\rangle = (\Psi(0), \Phi(x,y,0)) \tag{4.28}$$

where we have used that $(\Psi(0), \Phi(x,y,0))$ is a bounded linear functional on $\mathcal{D}$. Then $\Psi = \mathcal{L}^* V$ is a weak solution of $\mathcal{L}\Psi = 0$, with $\Psi \in L^2(\mathbb{R}^2 \times [0,T]) \times L^2(\mathbb{R}^2 \times [0,T])$. □

Remark 4.3. To obtain higher regularity of the solution, we repeat the proof with higher derivatives included in the inner product.

5. Uniqueness and local existence

In this section, we prove that for initial data $(u_0, v_0) \in H^N(\mathbb{R}^2) \times H^N(\mathbb{R}^2)$, for $N \geq 3$ there exists a unique local solution (u,v) of (1.3) such that $(u,v) \in L^\infty([0,T]: H^N(\mathbb{R}^2)) \times L^\infty([0,T]: H^N(\mathbb{R}^2))$ where the time T of existence depends only on $\|u_0\|_{H^3}$ and $\|v_0\|_{H^3}$. First we address the question of uniqueness.

Theorem 5.1. *Let $0 < T < \infty$. Assume that $(u_0, v_0) \in H^3(\mathbb{R}^2) \times H^3(\mathbb{R}^2)$. Then there is at most one solution $(u,v) \in L^\infty([0,T]: H^3(\mathbb{R}^2)) \times L^\infty([0,T]: H^3(\mathbb{R}^2))$ of (1.3) with initial data (u_0, v_0).*

Proof. Assume that (u,v) and $(\widetilde{u}, \widetilde{v})$ are two solutions of (1.3) in $L^\infty([0,T]: H^3(\mathbb{R}^2)) \times L^\infty([0,T]: H^3(\mathbb{R}^2))$ with the same initial data $(u_0(x,y), v_0(x,y))$. From (1.3), u_t, v_t, $\widetilde{u}_t$, $\widetilde{v}_t \in L^\infty([0,T]: L^2(\mathbb{R}^2))$, so the integrations below are justified. Therefore, the differences $(u - \widetilde{u})$ and $(v - \widetilde{v})$ satisfy

$$(u-\widetilde{u})_t + (u-\widetilde{u})_{xxx} + (u-\widetilde{u})_{yyx} - (v-\widetilde{v})_x - 6(uu_x - \widetilde{u}\widetilde{u}_x) = 0. \tag{5.1}$$

Now, multiplying (5.1) by $2(u-\widetilde{u})$ and integrating for $(x,y) \in \mathbb{R}^2$, we have

$$\begin{aligned}&2\int (u-\widetilde{u})(u-\widetilde{u})_t + 2\int (u-\widetilde{u})(u-\widetilde{u})_{xxx} + 2\int (u-\widetilde{u})(u-\widetilde{u})_{yyx} \\ &- 2\int (u-\widetilde{u})(v-\widetilde{v})_x - 12\int (u-\widetilde{u})(uu_x - \widetilde{u}\widetilde{u}_x) = 0.\end{aligned} \tag{5.2}$$

Then the 2nd and 3rd terms in (5.2) are shown to be identically zero. Hence

$$\partial_t \int (u-\widetilde{u})^2 - 2\int (u-\widetilde{u})\,(v-\widetilde{v})_x = 12\int (u-\widetilde{u})(uu_x - \widetilde{u}\widetilde{u}_x). \tag{5.3}$$

On the other hand,

$$\begin{aligned} 12\int (u-\widetilde{u})(u\,u_x - \widetilde{u}\widetilde{u}_x) &= 6\int (u-\widetilde{u})(u^2-\widetilde{u}^2)_x = -\,6\int (u-\widetilde{u})_x(u^2-\widetilde{u}^2) \\ &= -\,6\int (u-\widetilde{u})_x(u-\widetilde{u})(u+\widetilde{u}) = 3\int (u-\widetilde{u})^2(u_x+\widetilde{u}_x) \\ &\le 3\,(\|u_x\|_{L^\infty} + \|\widetilde{u}_x\|_{L^\infty})\int_{\mathbb{R}^2}(u-\widetilde{u})^2 \le C\int (u-\widetilde{u})^2. \end{aligned}$$

Combining this estimate with (5.3), we have

$$\partial_t \int_{\mathbb{R}^2}(u-\widetilde{u})^2 \le C\int_{\mathbb{R}^2}(u-\widetilde{u})^2 + 2\int_{\mathbb{R}^2}(u-\widetilde{u})(v-\widetilde{v})_x. \tag{5.4}$$

Similarly we have

$$\begin{aligned} &b(v-\widetilde{v})_t + \delta(v-\widetilde{v})_{xxx} + \lambda(v-\widetilde{v})_{yyx} + \eta(v-\widetilde{v})_x \\ &-\omega(u-\widetilde{u})_x - 6\mu(vv_x - \widetilde{v}\widetilde{v}_x) = 0. \end{aligned} \tag{5.5}$$

Now, multiplying (5.1) by $2(v-\widetilde{v})$ and integrating over $\mathbb{R}^2$ we have

$$\begin{aligned} &2b\int (v-\widetilde{v})(v-\widetilde{v})_t + 2\delta\int (v-\widetilde{v})(v-\widetilde{v})_{xxx} \\ &+\ 2\lambda\int (v-\widetilde{v})(v-\widetilde{v})_{xyy} + 2\eta\int (v-\widetilde{v})(v-\widetilde{v})_x \\ &-\ 2\int (v-\widetilde{v})(u-\widetilde{u})_x - 12\mu\int (v-\widetilde{v})(v\,v_x - \widetilde{v}\widetilde{v}_x) = 0. \end{aligned} \tag{5.6}$$

Then the 2nd, 3rd, and 4th terms in (5.6) are shown to be identically zero. Hence

$$\partial_t \int b(v-\widetilde{v})^2 - 2\int (v-\widetilde{v})(u-\widetilde{u})_x = 12\mu\int (v-\widetilde{v})(v\,v_x - \widetilde{v}\widetilde{v}_x). \tag{5.7}$$

On the other hand,

$$\begin{aligned} 12\mu\int (v-\widetilde{v})(vv_x - \widetilde{v}\widetilde{v}_x) &= 6\mu\int (v-\widetilde{v})(v^2-\widetilde{v}^2)_x = -\,6\mu\int (v-\widetilde{v})_x(v^2-\widetilde{v}^2) \\ &= -\,6\mu\int (v-\widetilde{v})_x(v-\widetilde{v})(v+\widetilde{v}) = 3\mu\int (v-\widetilde{v})^2(v_x+\widetilde{v}_x) \\ &\le 3\mu\,(\|v_x\|_{L^\infty} + \|\widetilde{v}_x\|_{L^\infty})\int_{\mathbb{R}^2}(v-\widetilde{v})^2 \le C\int (v-\widetilde{v})^2. \end{aligned}$$

Combining this estimate with (5.3), we have

$$\partial_t b\int (v-\widetilde{v})^2 \le C\int (v-\widetilde{v})^2 + 2\int (v-\widetilde{v})(u-\widetilde{u})_x. \tag{5.8}$$

Adding (5.4) with (5.8) we obtain

$$\begin{aligned} &\partial_t \int (u-\widetilde{u})^2 + \partial_t b \int (v-\widetilde{v})^2 \\ &\le C\int (u-\widetilde{u})^2 + C\int (v-\widetilde{v})^2 + 2\int (u-\widetilde{u})(v-\widetilde{v})_x \\ &\quad + 2\int (v-\widetilde{v})(u-\widetilde{u})_x. \end{aligned} \tag{5.9}$$

The last two terms satisfy

$$2\int (u-\widetilde{u})\,(v-\widetilde{v})_x + 2\int (v-\widetilde{v})(u-\widetilde{u})_x = \int [(u-\widetilde{u})(v-\widetilde{v})]_x = 0.$$

Therefore,

$$\partial_t \int \left[(u-\widetilde{u})^2 + b(v-\widetilde{v})^2\right] \le C\int \left[(u-\widetilde{u})^2 + b(v-\widetilde{v})^2\right]. \tag{5.10}$$

Using that $u(x,y,0)-\widetilde{u}(x,y,0)\equiv 0$, $v(x,y,0)-\widetilde{v}(x,y,0)\equiv 0$ and Gronwall's inequality it follows that

$$\int (u-\widetilde{u})^2 + \int b(v-\widetilde{v})^2 \le 0.$$

We conclude that $u\equiv\widetilde{u}$ and $v\equiv\widetilde{v}$. This proves the uniqueness of the solution. $\square$

We now prove the existence of a local solution for (1.3). We show that for each $(u_0,v_0)\in H^{N+3}\left(\mathbb{R}^2\right)\times H^{N+3}\left(\mathbb{R}^2\right)$ there exists a solution (u,v) in the space $L^\infty\left([0,T]:H^{N+3}(\mathbb{R}^2)\right)\times L^\infty\left([0,T];H^{N+3}(\mathbb{R}^2)\right)$ for a time T depending only on $\|u_0\|_{H^3(\mathbb{R}^2)}$ and $\|v_0\|_{H^3(\mathbb{R}^2)}$.

Theorem 5.2. *Let $\kappa_0,\widetilde{\kappa}_0>0$ and N be an integer ≥ 0. Then there exists a time $0<T<\infty$, depending only on κ_0 and $\widetilde{\kappa}_0$ such that for all $u_0,v_0\in H^{N+3}(\mathbb{R}^2)$, with $\|u_0\|_{H^3(\mathbb{R}^2)}\le\kappa_0$ and $\|v_0\|_{H^3(\mathbb{R}^2)}\le\widetilde{\kappa}_0$, there exists a solution of (1.3) with $(u,v)\in L^\infty\left([0,T]:H^{N+3}(\mathbb{R}^2)\right)\times L^\infty\left([0,T]:H^{N+3}(\mathbb{R}^2)\right)$ such that $u(x,y,0)=u_0(x,y)$ and $v(x,y,0)=v_0(x,y)$.*

The method of proof is as follows. We begin by approximating (1.3) by a sequence of linear equations. We then show that the sequence of solutions to our linear equations is bounded in $L^\infty([0,T];H^3(\mathbb{R}^2))\times L^\infty([0,T]:H^3(\mathbb{R}^2))$ for a time T depending only on $\|u_0\|_{H^3},\|v_0\|_{H^3}$. Third, we prove that a subsequence of solutions to our approximate equations converges to a solution

$$(u,v)\in L^\infty([0,T];H^3(\mathbb{R}^2))\times L^\infty([0,T];H^3(\mathbb{R}^2))$$

of (1.3). Lastly, we show that if $(u_0,v_0)\in H^{N+3}(\mathbb{R}^2)\times H^{N+3}(\mathbb{R}^2)$ for $N>0$, then our solution (u,v) is in $L^\infty([0,T];H^{N+3}(\mathbb{R}^2))\times L^\infty([0,T];H^{N+3}(\mathbb{R}^2))$, where the time T depends only on $\|u_0\|_{H^3},\|v_0\|_{H^3}$.

Proof of Theorem 5.2. It suffices to prove this result for $u_0,v_0\in\cap_{N\ge 0}H^N(\mathbb{R}^2)$. We can use the same approximation procedure as before to prove the result for general initial data. We begin by approximating (1.3) by the linear system (4.21) with initial data $u^{(n)}(x,y,0)=u_0(x,y)$, $v^{(n)}(x,y,0)=v_0(x,y)$, and where the first approximations are given by $u^{(0)}(x,y,t)=u_0(x,y)$ and $v^{(0)}(x,y,t)=v_0(x,y)$. By

Lemma 4.2, this system can be solved at each iteration. In particular, for each n there exists a unique solution $(u^{(n)}, v^{(n)})$ and by Lemma 4.1 for $N = 0$ we have

$$\begin{aligned} &\|u^{(n)}\|^2_{Z^0_t} + b\|v^{(n)}\|^2_{Z^0_t} \\ &\le \|u^{(n)}(\cdot,\cdot,0)\|^2_{H^3} + b\|v^{(n)}(\cdot,\cdot,0)\|^2_{H^3} + \|u^{(n)}_t(\cdot,\cdot,0)\|^2_{L^2} + b\|v^{(n)}_t(\cdot,\cdot,0)\|^2_{L^2} \\ &\quad + Ct\|u^{(n-1)}\|_{Z^0_t}\|u^{(n)}\|^2_{Z^0_t} + Ct\|v^{(n-1)}\|_{Z^0_t}\|v^{(n)}\|^2_{Z^0_t} \end{aligned} \tag{5.11}$$

for all $t \ge 0$. By assumption, $\kappa_0 \ge \|u_0\|_{H^3(\mathbb{R}^2)}$ and $\widetilde{\kappa}_0 \ge \|v_0\|_{H^3(\mathbb{R}^2)}$. On the other hand,

$$\begin{aligned} &\|u^{(n)}(\cdot,\cdot,0)\|^2_{H^3} + \|u^{(n)}_t(\cdot,\cdot,0)\|^2_{L^2} = \|u^{(n)}(\cdot,\cdot,0)\|^2_{H^3} \\ &+ \int \Big[u^{(n)}_{xxx}(\cdot,\cdot,0) - v^{(n)}_x(\cdot,\cdot,0) + u^{(n)}_{xyy}(\cdot,\cdot,0) - 6\,u^{(n-1)}\,u^{(n)}_x(\cdot,\cdot,0)\Big]^2 \\ &\le \|u_0\|^2_{H^3} + C\int \big[u^2_{0xxx} - v^2_{0x} + (u_{0xyy})^2 - (u_0\,u_{0x})^2\big] \\ &\le K\left(\|u_0\|^2_{H^3} + \|v_0\|^2_{H^3}\right) \le K\left(\kappa_0^2 + \widetilde{\kappa}_0^2\right). \end{aligned} \tag{5.12}$$

In a similar way we have

$$\begin{aligned} &b\|v^{(n)}(\cdot,\cdot,0)\|^2_{H^3} + b\|v^{(n)}_t(\cdot,\cdot,0)\|^2_{L^2} \\ &= b\|v^{(n)}(\cdot,\cdot,0)\|^2_{H^3} + b\int_{\mathbb{R}^2} \big[\delta\, v^{(n)}_{xxx}(\cdot,\cdot,0) + \eta v^{(n)}_x(\cdot,\cdot,0) - u^{(n)}_x(\cdot,\cdot,0) \\ &\quad + \lambda\partial_x^{-1} v^{(n)}_{yy}(\cdot,\cdot,0) - 6\,\mu\, v^{(n-1)}\, v^{(n)}_x(\cdot,\cdot,0)\big]^2 \\ &\le b\|v_0\|^2_{H^3} + C\int_{\mathbb{R}^2} [\delta\, v^2_{0xxx} + \eta\, v^2_{0x} - u^2_{0x} + \lambda\, v_{0xyy})^2 - (v_0\, v_{0x})^2] \\ &\le \widetilde{K}\left(\|u_0\|^2_{H^3} + \|v_0\|^2_{H^3}\right) \\ &\le \widetilde{K}\left(\kappa_0^2 + \widetilde{\kappa}_0^2\right), \end{aligned} \tag{5.13}$$

where K and $\widetilde{K}$ are independent of n. Without loss of generality, suppose $K \ge \widetilde{K}$. Let $c_0^2 = \big(2K(\kappa_0^2 + \widetilde{\kappa}_0^2) + 1\big)$. Let

$$\begin{gathered} T_0^{(n)} = \sup\{t :\, \|u^{(j)}\|_{Z^0_t} \le c_0 \text{ for } 0 \le j \le n\} \\ \widetilde{T}_0^{(n)} = \sup\{t :\, \|v^{(j)}\|_{Z^0_t} \le c_0 \text{ for } 0 \le j \le n\}. \end{gathered}$$

Let $T_0^{*(n)} = \min\{T_0^{(n)}, \widetilde{T}_0^{(n)}\}$. Then, for t in the interval $[0, T_0^{*(n)}]$, from (5.11), (5.12), and (5.13), it follows that

$$\begin{aligned} &\|u^{(n)}\|^2_{Z^0_t} + b\|v^{(n)}\|^2_{Z^0_t} \\ &\le \|u^{(n)}(\cdot,\cdot,0)\|^2_{H^3} + b\|v^{(n)}(\cdot,\cdot,0)\|^2_{H^3} + \|u^{(n)}_t(\cdot,\cdot,0)\|^2_{L^2} + b\|v^{(n)}_t(\cdot,\cdot,0)\|^2_{L^2} \\ &\quad + Ct\|u^{(n-1)}\|_{Z^0_t}\,\|u^{(n)}\|^2_{Z^0_t} + Ct\|v^{(n-1)}\|_{Z^0_t}\|v^{(n)}\|^2_{Z^0_t} \\ &\le K\left(\kappa_0^2 + \widetilde{\kappa}_0^2\right) + \widetilde{K}\left(\kappa_0^2 + \widetilde{\kappa}_0^2\right) + Ctc_0^3 + Ct\widetilde{c}_0^3 \\ &\le 2K(\kappa_0^2 + \widetilde{\kappa}_0^2) + Ctc_0^3. \end{aligned} \tag{5.14}$$

Now choose $T > 0$ such that $CTc_0^3 = 1$. We claim that $T_0^{*(n)} \ge T$ for all n and therefore, the sequence of approximate solutions $\{(u^{(n)}, v^{(n)})\}$ is bounded for the time T which is independent of n. If $T_0^{*(n)} = \infty$ for all n, then clearly $T_0^{*(n)} \ge T$ for all n. So, assume there exists n such that $T_0^{*(n)} < \infty$. Suppose $T > T_0^{*(n)}$. Then, by the continuity of $\|u^{(n)}\|_{Z_t^0}$, $\|v^{(n)}\|_{Z_t^0}$ with respect to t, we have $c_0^2 = \|u^{(j)}\|^2_{Z^0_{T_0^{(n)}}}$ for some $j \in [0,n]$ and $c_0^2 = \|v^{(\widetilde{j})}\|^2_{Z^0_{\widetilde{T}_0^{(n)}}}$ for some integer $\widetilde{j} \in [0,n]$. Without loss of generality, suppose $T_0^{*(n)} = T_0^{(n)}$. Therefore, by (5.14),

$$\begin{aligned} c_0^2 &\le \|u^{(j)}\|^2_{Z^0_{T_0^{(n)}}} + \|v^{(j)}\|^2_{Z^0_{T_0^{(n)}}} \\ &\le 2K(\kappa_0^2 + \widetilde{\kappa}_0^2) + CT_0^{(n)} c_0^3 \\ &< 2K(\kappa_0^2 + \widetilde{\kappa}_0^2) + CTc_0^3 = c_0^2. \end{aligned}$$

However, this implies $c_0^2 < c_0^2$ and we have a contradiction. Thus, we conclude that $T_0^{*(n)} \ge T$ for all n, and, therefore,

$$\begin{gathered} \sup_{0\le t\le T} \int ((u^{(n)}(\cdot,t))^2 + (u^{(n)}_{xxx})^2 + (u^{(n)}_{yyy})^2 + (u^{(n)}_t)^2 \le c_0^2, \\ \sup_{0\le t\le T} \int ((v^{(n)}(\cdot,t))^2 + (v^{(n)}_{xxx})^2 + (v^{(n)}_{yyy})^2 + (v^{(n)}_t)^2 \le c_0^2 \end{gathered}$$

for all n. Consequently, there exists a bounded sequence of solutions $\{(u^{(n)}, v^{(n)})\} \in Z_T^0 \times Z_T^0$. Therefore,

$$\begin{gathered} u^{(n)} \rightharpoonup u \quad \text{weak}^* \text{ in } L^\infty\left([0,T] : H^3(\mathbb{R}^2)\right) \\ u_t^{(n)} \rightharpoonup u_t \quad \text{weak}^* \text{ in } L^\infty\left([0,T] : L^2(\mathbb{R}^2)\right) \end{gathered} \tag{5.15}$$

and

$$\begin{gathered} v^{(n)} \rightharpoonup v \quad \text{weak}^* \text{ in } L^\infty\left([0,T] : H^3(\mathbb{R}^2)\right) \\ v_t^{(n)} \rightharpoonup v_t \quad \text{weak}^* \text{ in } L^\infty\left([0,T] : L^2(\mathbb{R}^2)\right). \end{gathered} \tag{5.16}$$

On the other hand, $H^3_{\rm loc}(\mathbb{R}^2) \overset{c}{\hookrightarrow} H^1_{\rm loc}(\mathbb{R}^2) \hookrightarrow L^2(\mathbb{R}^2)$. Then by the Lions-Aubin compactness Theorem [20] there are subsequences $u^{(n_j)} := u^{(n)}$ and $v^{(n_j)} := v^{(n)}$ such that

$$\begin{gathered} u^{(n)} \to u \quad \text{strongly in } L^\infty\left([0,T] : H^1_{\rm loc}(\mathbb{R}^2)\right) \\ v^{(n)} \to v \quad \text{strongly in } L^\infty\left([0,T] : H^1_{\rm loc}(\mathbb{R}^2)\right) \end{gathered} \tag{5.17}$$

Hence for subsequences $u^{(n_j)} := u^{(n)}$ and $v^{(n_j)} := v^{(n)}$, we have

$$\begin{gathered} u^{(n)} \to u \quad \text{a.e. in } L^\infty\left([0,T] : H^1_{\rm loc}(\mathbb{R}^2)\right) \\ v^{(n)} \to v \quad \text{a.e. in } L^\infty\left([0,T] : H^1_{\rm loc}(\mathbb{R}^2)\right). \end{gathered} \tag{5.18}$$

Moreover, from (5.15) we have

$$\begin{gathered} u^{(n)}_{xxx} \rightharpoonup u_{xxx}, \quad u_{xyy} \rightharpoonup u_{xyy} \quad \text{weakly}^* \text{ in } L^\infty\left([0,T] : L^2(\mathbb{R}^2)\right) \\ v^{(n)}_{xxx} \rightharpoonup v_{xxx}, \quad v^{(n)}_{xyy} \rightharpoonup v_{xyy} \quad \text{weakly}^* \text{ in } L^\infty\left([0,T] : L^2(\mathbb{R}^2)\right). \end{gathered} \tag{5.19}$$

Now we show that the nonlinear term converges to its correct limit. From (5.17),

$$\begin{gathered} u^{(n-1)} \to u \quad \text{strongly in } L^\infty\left([0,T] : H^1_{\rm loc}(\mathbb{R}^2)\right) \hookrightarrow L^\infty\left([0,T] : L^2_{\rm loc}(\mathbb{R}^2)\right) \\ v^{(n-1)} \to v \quad \text{strongly in } L^\infty\left([0,T] : H^1_{\rm loc}(\mathbb{R}^2)\right) \hookrightarrow L^\infty\left([0,T] : L^2_{\rm loc}(\mathbb{R}^2)\right). \end{gathered} \tag{5.20}$$

Moreover,

$$\begin{aligned} u_x^{(n-1)} &\rightharpoonup u_x \quad \text{weak}^* \text{ in } L^\infty\left([0,T]:L^2(\mathbb{R}^2)\right) \\ v_x^{(n-1)} &\rightharpoonup v_x \quad \text{weak}^* \text{ in } L^\infty\left([0,T]:L^2(\mathbb{R}^2)\right). \end{aligned} \tag{5.21}$$

Therefore,

$$\begin{aligned} u^{(n-1)}u_x^{(n)} &\rightharpoonup uu_x \quad \text{weakly}^* \text{ in } L^\infty\left([0,T]:L^1_{\rm loc}(\mathbb{R}^2)\right) \\ v^{(n-1)}v_x^{(n)} &\rightharpoonup vv_x \quad \text{weakly}^* \text{ in } L^\infty\left([0,T]:L^1_{\rm loc}(\mathbb{R}^2)\right). \end{aligned} \tag{5.22}$$

Therefore, $(u,\ v)$ is a solution to (1.3).

Now, we prove that if $(u_0,v_0)\in H^{N+3}(\mathbb{R}^2)\times H^{N+3}(\mathbb{R}^2)$ for some integer $N>0$, then the solution (u,v) satisfies

$$(u,v)\in L^\infty\left([0,T]:H^{N+3}(\mathbb{R}^2)\right)\times L^\infty\left([0,T]:H^{N+3}(\mathbb{R}^2)\right)$$

for the time T chosen above. We already know that there is a solution $(u,v)\in L^\infty\left([0,T]:H^3(\mathbb{R}^2)\right)\times L^\infty([0,T]:H^3(\mathbb{R}^2))$. Therefore, we only need to show that the approximating sequence $(u^{(n)},v^{(n)})$ is bounded in $Z_T^N\times Z_T^N$ and thus, by the convergence arguments above, our solution (u,v) is in $L^\infty\left([0,T]:H^{N+3}(\mathbb{R}^2)\right)\times L^\infty([0,T]:H^{N+3}(\mathbb{R}^2))$. We use the same argument as before. By Lemma 4.2, we know our linearized equation can be solved in any interval of time in which the coefficients are defined. Therefore, for each iterate, $\|u^{(n)}\|_{Z_t^N}$ and $\|v^{(n)}\|_{Z_t^N}$ are continuous in $t\in[0,T]$. Using Lemma 4.1 it follows that

$$\begin{aligned} \|u^{(n)}\|^2_{Z_t^N}+b\|v^{(n)}\|^2_{Z_t^N} &\le \|u^{(n)}(\cdot,\cdot,0)\|^2_{H^{N+3}(\mathbb{R}^2)}+b\|v^{(n)}(\cdot,\cdot,0)\|^2_{H^{N+3}(\mathbb{R}^2)} \\ &\quad +\|u_t^{(n)}(\cdot,\cdot,0)\|^2_{H^N(\mathbb{R}^2)}+b\|v_t^{(n)}(\cdot,\cdot,0)\|^2_{H^N(\mathbb{R}^2)} \\ &\quad +ct\|u^{(n-1)}\|_{Z_t^N}\|u^{(n)}\|^2_{Z_t^N}+ct\|v^{(n-1)}\|_{Z_t^N}\|v^{(n)}\|^2_{Z_t^N}. \end{aligned}$$

As before, we have

$$\|u^{(n)}(\cdot,\cdot,0)\|^2_{H^{N+3}(\mathbb{R}^2)}+\|u_t^{(n)}(\cdot,\cdot,0)\|^2_{H^N(\mathbb{R}^2)}\le K\kappa_N^2, \tag{5.23}$$

$$b\|v^{(n)}(\cdot,\cdot,0)\|^2_{H^{N+3}(\mathbb{R}^2)}+b\|v_t^{(n)}(\cdot,\cdot,0)\|^2_{H^N(\mathbb{R}^2)}\le \widetilde{K}\widetilde{\kappa}_N^2 \tag{5.24}$$

where κ_N and $\widetilde{\kappa}_N$ are independent of n. Without loss of generality, assume $K\ge\widetilde{K}$ and define $c_N^2=\left(2K(\kappa_N^2+\widetilde{\kappa}_N^2)+1\right)$. Let

$$\begin{aligned} T_N^{(n)} &= \sup\{t:\|u^{(j)}\|_{Z_t^N}\le c_N \text{ for } 0\le j\le n\} \\ \widetilde{T}_N^{(n)} &= \sup\{t:\|v^{(j)}\|_{Z_t^N}\le c_N \text{ for } 0\le j\le n\}. \end{aligned}$$

Let $T_N^{*(n)}=\min\{T_N^{(n)},\widetilde{T}_N^{(n)}\}$. Then, for t in the interval $[0,T_N^{*(n)}]$, it follows that

$$\|u^{(n)}\|^2_{Z_t^N}+b\|v^{(n)}\|^2_{Z_t^N}\le 2K_N(\kappa_N^2+\widetilde{\kappa}_N^2)+Ctc_N^3.$$

Now choosing T_N such that

$$CT_Nc_N^3=1,$$

by the same arguments as in the case $N=0$, we conclude that $T_N^{(n)}\ge T_N$, and, therefore,

$$\|u^{(n)}\|^2_{Z^N_{T_N}}\le c_N^2,\quad \|v^{(n)}\|^2_{Z^N_{T_N}}\le c_N^2.$$

Now, let

$$T_N^*\equiv\sup\{t:u,v\in Z_t^N\}.$$

We claim that $T_N^* \geq T$, and, therefore, a time of existence can be chosen depending only on $\|u_0\|_{H^3}, \|v_0\|_{H^3}$. By Lemma 4.2 the linear equation (4.21) can be solved in any interval of time in which the coefficients are defined, and, thus $T_N^* \geq T$. □

Corollary 5.3. *Let $u_0, v_0 \in H^{N+3}(\mathbb{R}^2)$ for some $N \geq 0$ and let $u_0^{(n)}$ be a sequence converging to u_0 in $H^{N+3}(\mathbb{R}^2)$, $v_0^{(n)}$ be a sequence converging to v_0. Let (u, v) and $(u^{(n)}, v^{(n)})$ be the corresponding unique solutions, given by Theorems 5.1 and 5.2 in $L^\infty([0,T] : H^{N+3}(\mathbb{R}^2)) \times L^\infty([0,T] : H^{N+3}(\mathbb{R}^2))$ for a time T depending only on $\sup_n \|u_0^{(n)}\|_{H^3(\mathbb{R}^2)}$ and $\sup_n \|v_0^{(n)}\|_{H^3(\mathbb{R}^2)}$. Then*

$$\begin{aligned} u^{(n)} &\rightharpoonup u \quad \text{weak* in } L^\infty\left([0,T] : H^{N+3}(\mathbb{R}^2)\right) \\ v^{(n)} &\rightharpoonup v \quad \text{weak* in } L^\infty\left([0,T] : H^{N+3}(\mathbb{R}^2)\right). \end{aligned} \tag{5.25}$$

Proof. By assumption $u^{(n)}, v^{(n)} \in L^\infty([0,T] : H^{N+3}(\mathbb{R}^2))$, then there exist weak* convergent subsequences, still denoted $\{u^{(n)}\}$ and $\{v^{(n)}\}$ such that

$$\begin{aligned} u^{(n)} &\rightharpoonup \widetilde{u} \quad \text{weak* in } L^\infty\left([0,T] : H^{N+3}(\mathbb{R}^2)\right) \\ v^{(n)} &\rightharpoonup \widetilde{v} \quad \text{weak* in } L^\infty\left([0,T] : H^{N+3}(\mathbb{R}^2)\right). \end{aligned}$$

Moreover, by equation (1.3), $u^{(n)}, v^{(n)} \in L^\infty\left([0,T] : H^{N+3}(\mathbb{R}^2)\right)$ implies

$$u_t^{(n)}, v_t^{(n)} \in L^\infty\left([0,T] : L^2(\mathbb{R}^2)\right).$$

By the Lions-Aubin Compactness theorem [20] we have

$$\begin{aligned} u^{(n)} &\to \widetilde{u} \quad \text{strongly in } L^\infty([0,T] : H^{1/2}_{\mathrm{loc}}(\mathbb{R}^2)) \\ v^{(n)} &\to \widetilde{v} \quad \text{strongly in } L^\infty([0,T] : H^{1/2}_{\mathrm{loc}}(\mathbb{R}^2)). \end{aligned}$$

Now we just to show that each term in (1.3) converges to its correct limit, and thus $u_t^{(n)} \to \widetilde{u}_t$ and $v_t^{(n)} \to \widetilde{v}_t$ for $\widetilde{u}, \widetilde{v} \in L^\infty\left([0,T] : H^{N+3}(\mathbb{R}^2)\right)$. The only thing we need to show is that the nonlinear term converges to its correct limit, namely that $u^{(n)}\, u_x^{(n)} \to \widetilde{u}\, \widetilde{u}_x$. We know that $u_x^{(n)} \overset{*}{\rightharpoonup} \widetilde{u}_x$ weakly in $L^\infty([0,T] : H^1(\mathbb{R}^2))$ and $u^{(n)} \to \widetilde{u}$ strongly in $L^\infty([0,T] : H^{1/2}_{\mathrm{loc}}(\mathbb{R}^2))$. Therefore, their product converges in $L^2([0,T] : L^1_{\mathrm{loc}}(\mathbb{R}^2))$. Clearly the linear terms also converge in $L^2([0,T] : L^1_{\mathrm{loc}}(\mathbb{R}^2))$ and therefore, we conclude that $u_t^{(n)} \to \widetilde{u}_t$ in $L^2([0,T] : L^1_{\mathrm{loc}}(\mathbb{R}^2))$. In a similar way we conclude that $v_t^{(n)} \to \widetilde{v}_t$ in $L^2([0,T] : L^1_{\mathrm{loc}}(\mathbb{R}^2))$. We also know that $(\widetilde{u}, \widetilde{v}) \in L^\infty([0,T] : H^{N+3}(\mathbb{R}^2)) \times L^\infty([0,T] : H^{N+3}(\mathbb{R}^2))$. By the uniqueness theorem, Theorem 5.1, $(\widetilde{u}, \widetilde{v}) = (u, v)$. □

6. Weighted estimates and main estimates of error terms

At the end of this section, we state and prove our main theorem, Theorem 6.4. First, however, as a starting point for the a priori gain of regularity results that will be discussed in Theorem 6.4, we need to develop some estimates for solutions of the coupled system (1.3) in weighted Sobolev spaces. The existence of these weighted estimates is often called a persistence property of the initial data (u_0, v_0). Indeed, we prove that if our initial data $(u_0, v_0) \in H^3(\mathbb{R}^2) \times H^3(\mathbb{R}^2)$ also lies in some weighted space $H^K(W_{0\,i\,0}) \times H^K(W_{0\,i\,0})$, for integers $K \geq 0$ and $i \geq 1$, then our solution (u, v) also lies in $L^\infty\left([0,T] : H^K(W_{0\,i\,0})\right) \times L^\infty\left([0,T] : H^K(W_{0\,i\,0})\right)$.

Theorem 6.1. *Assume* $(u,\ v)$ *is the solution to* (1.3) *in* $L^\infty([0,T] : H^3(\mathbb{R}^2)) \times L^\infty([0,T] : H^3(\mathbb{R}^2))$ *with initial data* $(u_0,v_0) \in H^3(\mathbb{R}^2)\times H^3(\mathbb{R}^2)$ *such that* (u_0,v_0) *also lie in the weighted space* $H^K(W_{0\,i\,0}) \times H^K(W_{0\,i\,0})$ *for some integers* $K \geq 0$, $i \geq i$. *Then*

$$\begin{gathered} u \in L^\infty([0,T] : H^3(\mathbb{R}^2) \cap H^K(W_{0\,i\,0})),\\ v \in L^\infty([0,T] : H^3(\mathbb{R}^2) \cap H^K(W_{0\,i\,0})) \end{gathered} \tag{6.1}$$

and

$$\begin{gathered} \int_0^T \int \chi(\partial^\gamma u_x)^2 + \int_0^T \int \chi(\partial^\gamma u_y)^2 \leq C,\\ \int_0^T \int \chi(\partial^\gamma v_x)^2 + \int_0^T \int \chi(\partial^\gamma v_y)^2 \leq C \end{gathered} \tag{6.2}$$

for $|\gamma| \leq K$, *where* χ *is a weight function in* $W_{\sigma,i-1,0}$ *for* $\sigma > 0$ *arbitrary, and* C *depends only on* T *and the norms of* $u_0, v_0 \in H^3(\mathbb{R}^2) \cap H^K(W_{0\,i\,0})$.

Proof. We will prove this result by induction on β, for $0 \leq \beta \leq K$. As before, we need to derive *a priori* estimates for smooth solutions (u,v) which depend only on the noms of $u,v \in L^\infty([0,T]; H^3(\mathbb{R}^2))$ and $u_0, v_0 \in H^K(W_{0\,i\,0})$. Then, we can apply convergence arguments to show that the result holds true for general solutions. In order to do so, we need to approximate general solutions $u, v \in H^3(\mathbb{R}^2)$ by smooth solutions and approximate general weight functions $\xi \in W_{0\,i\,0}$ by smooth, bounded weight functions. We have discussed approximating solutions in the previous section, so we will concentrate on the approximation of the weight function here.

We begin by taking a sequence of bounded weight functions χ_ν, which decay as $|x| \to \infty$ and which approximate $\chi \in W_{\sigma,i-1,0}$ from below, uniformly on any half-line $(-\infty, c)$. Let

$$\xi_\nu = 1 + \int_{-\infty}^x \chi_\nu(z,t)dz. \tag{6.3}$$

Hence, the functions ξ_ν are bounded weight functions which approximate a weight function $\xi \in W_{0\,i\,0}$ from below, uniformly on compact sets.

Now we will follow the same methodology as in the development of the Lemma 3.1. Indeed, for the β^{th} induction step, we take α derivatives of $(1.3)_1$, where $|\alpha| = \beta$, multiply the result by $2\xi_\nu\,(\partial^\alpha u)$, and integrate over $\mathbb{R}^2$. Performing straightforward calculations and using $(\xi_\nu)_t,\ (\xi_\nu)_x \leq C\xi_\nu$ we obtain the following estimate

$$\begin{aligned} &\partial_t \int \xi_\nu\,(\partial^\alpha u)^2 + 3\int (\xi_\nu)_x\,(\partial^\alpha u_x)^2 + \int (\xi_\nu)_x\,(\partial^\alpha u_y)^2\\ &\leq C\int \xi_\nu\,(\partial^\alpha u)^2 + 2\int \xi_\nu\,(\partial^\alpha u)\,\partial^\alpha\,(uu_x) + 2\int \xi_\nu(\partial^\alpha u)(\partial^\alpha v_x). \end{aligned} \tag{6.4}$$

Similarly, we take α derivatives of $(1.3)_2$, where $|\alpha| = \beta$, multiply the result by $2\xi_\nu\,(\partial^\alpha v)$, and integrate over $\mathbb{R}^2$. Performing straightforward calculations as in (6.4) we obtain the estimate

$$\begin{aligned} &\partial_t b\int \xi_\nu(\partial^\alpha v)^2 + 3\delta\int (\xi_\nu)_x\,(\partial^\alpha v_x)^2 + \lambda\int_{\mathbb{R}^2} (\xi_\nu)_x\,(\partial^\alpha v_y)^2\\ &\leq C\int \xi_\nu(\partial^\alpha v)^2 + 2\int \xi_\nu\,(\partial^\alpha v)\,\partial^\alpha\,(vv_x) + 2\int \xi_\nu(\partial^\alpha u_x)(\partial^\alpha v). \end{aligned} \tag{6.5}$$

Adding (6.4) and (6.5) and using the Cauchy-Schwarz inequality we have

$$\begin{aligned}
&\partial_t \int \xi_\nu \left[(\partial^\alpha u)^2 + b(\partial^\alpha v)^2\right] + 3\int (\xi_\nu)_x \left[\left(\partial^\alpha u_x\right)^2 + \delta\left(\partial^\alpha v_x\right)^2\right] \\
&+ \int (\xi_\nu)_x \left[\left(\partial^\alpha u_y\right)^2 + \lambda\left(\partial^\alpha v_y\right)^2\right] \\
&\le \int \xi_\nu \left(\partial^\alpha u\right)^2 + \int \xi_\nu \left(\partial^\alpha v\right)^2 + \int 2\left(\xi_\nu\right)\left(\partial^\alpha u\right)\partial^\alpha\left(u u_x\right) \\
&\quad + \int 2\left(\xi_\nu\right)\left(\partial^\alpha v\right)\partial^\alpha\left(v v_x\right) + 2\int \xi_\nu \partial_x[(\partial^\alpha u)(\partial^\alpha v)] \\
&\le \int \xi_\nu \left(\partial^\alpha u\right)^2 + \int \xi_\nu \left(\partial^\alpha v\right)^2 + \int 2\xi_\nu \left(\partial^\alpha u\right)\partial^\alpha\left(u u_x\right) \\
&\quad + \int 2\xi_\nu \left(\partial^\alpha v\right)\partial^\alpha\left(v v_x\right) + C\Big|\int (\xi_\nu)_x \left(\partial^\alpha u\right)(\partial^\alpha v)\Big| \\
&\le C\int \xi_\nu \left[\left(\partial^\alpha u\right)^2 + \left(\partial^\alpha v\right)^2\right] + \int 2\xi_\nu \left(\partial^\alpha u\right)\partial^\alpha\left(u u_x\right) \\
&\quad + \int 2\xi_\nu \left(\partial^\alpha v\right)\partial^\alpha\left(v v_x\right).
\end{aligned} \tag{6.6}$$

Case $\beta = 0$**.** We need to estimate the terms

$$\Big|\int \xi_\nu \left(\partial^\alpha u\right)\partial^\alpha\left(u u_x\right)\Big|, \quad \Big|\int \xi_\nu \left(\partial^\alpha v\right)\partial^\alpha\left(v v_x\right)\Big|. \tag{6.7}$$

For $(6.7)_1$, we have

$$\begin{aligned}
\Big|\int \xi_\nu \left(\partial^\alpha u\right)\partial^\alpha\left(u u_x\right)\Big| = \Big|\int \xi_\nu\, u^2 u_x\Big| &\le \|u_x\|_{L^\infty(\mathbb{R}^2)} \int \xi_\nu\, u^2 \\
&\le C\|u\|_{H^3} \int \xi_\nu u^2 \le C\int \xi_\nu u^2
\end{aligned}$$

where C depends only on the norm of $u \in L^\infty([0,T] : H^3(\mathbb{R}^2))$ (which depends only on the norms of $u_0, v_0 \in H^3(\mathbb{R}^2)$). Similarly,

$$\Big|\int \xi_\nu v^2 v_x\Big| \le C\int \xi_\nu v^2$$

Combining these estimates with (6.6), we conclude that

$$\begin{aligned}
&\partial_t \int \xi_\nu (u^2 + bv^2) + 3\int (\xi_\nu)_x (u_x^2 + \delta v_x^2) + \int (\xi_\nu)_x (u_y^2 + \lambda v_y^2) \\
&\le C\int (\xi_\nu)_x (u^2 + v^2) + \int 2\xi_\nu u^2 u_x + \int 2\xi_\nu v^2 v_x \\
&\le C\int \xi_\nu (u^2 + bv^2)
\end{aligned} \tag{6.8}$$

where C depends only on $\|u_0\|_{H^3}$ and $\|v_0\|_{H^3}$. Integrating (6.8) on $t \in [0,T]$ we obtain

$$\begin{aligned} &\int \xi_\nu (u^2+bv^2) + 3\int_0^T \int (\xi_\nu)_x (u_x^2+\delta v_x^2) + \int_0^T \int (\xi_\nu)_x (u_y^2+\lambda v_y^2) \\ &\le \int \xi_\nu(\cdot,\cdot,0)(u_0^2+v_0^2) + C\int_0^t \int \xi_\nu (u^2+v^2) \\ &\le C + C\Big(\int_0^t \int \xi_\nu (u^2+bv^2)\Big). \end{aligned} \tag{6.9}$$

Therefore, using Gronwall's inequality,

$$\sup_{0\le t\le T} C\int \xi_\nu (u^2+v^2) + 3\int_0^T \int (\xi_\nu)_x (u_x^2+\delta v_x^2) + \int_0^T \int (\xi_\nu)_x (u_y^2+\lambda v_y^2) \le C$$

where C does not depend on ν, but only on T and the norm of $u_0,\ v_0 \in H^3(\mathbb{R}^2)\cap H^0(W_{0\,i\,0})$. Passing to the limit,

$$\sup_{0\le t\le T} C\int \xi (u^2+v^2) + \ 3\int_0^T \int \chi (u_x^2+\delta v_x^2) + \int_0^T \int \chi (u_y^2+\lambda v_y^2) \le C. \tag{6.10}$$

Case $\beta = 1$. Consider $\alpha = (1,0)$. In fact, for the case $\alpha=(1,0)$ we have

$$\begin{aligned} \big|\int \xi_\nu u_x (uu_x)_x\big| &= \big|\int \xi_\nu u_x (u_x^2 + u\,u_{xx})\big| = \big|\int \xi_\nu (u_x^3 + uu_x\,u_{xx})\big| \\ &\le \big|\int_{\mathbb{R}^2} \xi_\nu u_x^3\big| + \big|\int \xi_\nu u u_x u_{xx}\big| \\ &= \big|\int \xi_\nu u_x^3\big| + \big|\frac{1}{2}\int \xi_\nu u (u_x^2)_x\big| \\ &\le C\big|\int \xi_\nu u_x^3\big| + C\big|\int (\xi_\nu)_x u u_x^2\big| \\ &\le C(|u|_{L^\infty} + |u_x|_{L^\infty})\int \xi_\nu u_x^2 \le C\int \xi_\nu u_x^2 \end{aligned} \tag{6.11}$$

where C depends only on the norms of $u_0, v_0 \in H^3(\mathbb{R}^2)$. Performing similar calculations as in the case above, along with (6.6), and Gronwall's inequality we conclude that

$$\begin{aligned} &\sup_{0\le t\le T} C\int \xi_\nu (u_x^2+v_x^2) + 3\int_0^T \int (\xi_\nu)_x (u_{xx}^2+\delta v_{xx}^2) \\ &+ \int_0^T \int (\xi_\nu)_x (u_{xy}^2+\lambda v_{xy}^2) \le C \end{aligned} \tag{6.12}$$

where C does not depend on ν, but only on T and the norm of $u_0, v_0 \in H^3(\mathbb{R}^2)\cap H^1(W_{0\,i\,0})$. Passing to the limit,

$$\sup_{0\le t\le T} C\int \xi\,(u_x^2+v_x^2) + 3\int_0^T \int \chi (u_{xx}^2+\delta v_{xx}^2) + \int_0^T \int \chi (u_{xy}^2+\lambda v_{xy}^2) \le C$$

where C depends only on the norms of $u_0, v_0 \in H^3(\mathbb{R}^2)\cap H^1(W_{0\,i\,0})$.

Next, we consider the case $\alpha=(0,1)$. For the case $\alpha=(0,1)$, we have

$$\big|\int \xi_\nu u_y (uu_x)_y\big| = \big|\int \xi_\nu u_y (u_y u_x + uu_{xy})\big|$$

$$
\begin{aligned}
&\leq \Big|\int \xi_\nu u_x u_y^2\Big| + \Big|\int \xi_\nu u u_y u_{xy}\Big| \\
&\leq C(|u|_{L^\infty} + |u_x|_{L^\infty})\int \xi_\nu u_y^2 \leq C\int \xi_\nu u_y^2.
\end{aligned}
$$

Therefore, using the same idea as above, we conclude that

$$
\sup_{0\leq t\leq T} C\int \xi(u_y^2+v_y^2) + 3\int_0^T\int \chi(u_{xy}^2+\delta v_{xy}^2) + \int_0^T\int \chi(u_{yy}^2+\lambda v_{yy}^2) \leq C.
$$

Case $\beta = 2$**.** We have $\alpha = (2,0)$, $\alpha = (1,1)$, and $\alpha = (0,2)$. First, for the case $\alpha = (2,0)$ we have

$$
\begin{aligned}
\Big|\int \xi_\nu u_{xx}(uu_x)_{xx}\Big| &= \Big|\int \xi_\nu u_{xx}(3u_x u_{xx} + u u_{xxx})\Big| \\
&= \Big|\int \xi_\nu (3u_x u_{xx}^2 + u\, u_{xx} u_{xxx})\Big| \\
&= \Big|3\int_{\mathbb{R}^2} \xi_\nu u_x u_{xx}^2\Big| + \Big|\frac{1}{2}\int \xi_\nu u (u_{xx}^2)_x\Big| \\
&\leq C(|u|_{L^\infty} + |u_x|_{L^\infty})\int \xi_\nu u_{xx}^2 \leq C\int \xi_\nu u_{xx}^2.
\end{aligned}
\tag{6.13}
$$

Performing similar calculations as in the cases given above together with (6.6) and using Gronwall's inequality we have

$$
\sup_{0\leq t\leq T} C\int_{\mathbb{R}^2} \xi_\nu (u_{xx}^2 + v_{xx}^2) + 3\int_0^T\int_{\mathbb{R}^2} (\xi_\nu)_x (u_{xxx}^2 + \delta v_{xxx}^2) \tag{6.14}
$$

$$
+ \int_0^T\int_{\mathbb{R}^2} (\xi_\nu)_x (u_{xxy}^2 + \lambda v_{xxy}^2) \leq C \tag{6.15}
$$

where C depends on T and the norm of $u_0, v_0 \in H^3(\mathbb{R}^2) \cap H^2(W_{0\,i\,0})$, and does not depend on ν. Passing to the limit,

$$
\sup_{0\leq t\leq T} C\int \xi(u_{xx}^2+v_{xx}^2) + 3\int_0^T\int \chi(u_{xxx}^2+\delta v_{xxx}^2) + \int_0^T\int \chi(u_{xxy}^2+\lambda v_{xxy}^2) \leq C.
$$

For the case $\alpha = (1,1)$ we have

$$
\begin{aligned}
&\Big|\int \xi_\nu u_{xy}(uu_x)_{xy}\Big| \\
&= \Big|\int \xi_\nu u_{xy}(2u_x u_{xy} + u_y u_{xx} + u u_{xxy})\Big| \\
&= \Big|\int \xi_\nu (2u_x u_{xy}^2 + u_y u_{xy} u_{xx} + u u_{xy} u_{xxy})\Big| \\
&\leq \Big|2\int \xi_\nu u_x u_{xy}^2\Big| + \Big|\int \xi_\nu u_y u_{xy} u_{xx}\Big| + \Big|\int \xi_\nu u u_{xy} u_{xxy}\Big| \\
&\leq C|u_x|_{L^\infty}\int \xi_\nu u_{xy}^2 + C|u_y|_{L^\infty}\int \xi_\nu u_{xx}^2\,dx\,dy + C|u_y|_{L^\infty}\int \xi_\nu u_{xy}^2 \\
&\quad + (|u|_{L^\infty} + |u_x|_{L^\infty})\int \xi_\nu u_{xy}^2 \\
&\leq C + C\int \xi_\nu u_{xy}^2
\end{aligned}
$$

where C depends only on $u_0, v_0 \in H^3(\mathbb{R}^2) \cap H^2(W_{0\,i\,0})$. Consequently, using the same ideas as above,

$$\sup_{0\le t\le T} C\int \xi(u_{xy}^2 + v_{xy}^2) + 3\int_0^T\int \chi(u_{xxy}^2 + \delta v_{xxy}^2) + \int_0^T\int \chi(u_{xyy}^2 + \lambda v_{xyy}^2) \le C$$

where C depends only on $u_0, v_0 \in H^3(\mathbb{R}^2) \cap H^2(W_{0\,i\,0})$.

For the case $\alpha = (0,2)$, our remainder term satisfies

$$\begin{aligned}
\Big|\int \xi_\nu u_{yy}(uu_x)_{yy}\Big| &= \Big|\int \xi_\nu u_{yy}(u_{yy}u_x + 2u_y u_{xy} + uu_{xyy})\Big| \\
&\le C|u_x|_{L^\infty}\int \xi_\nu u_{yy}^2 + C|u_y|_{L^\infty}\int \xi_\nu u_{xy}^2 + C|u_y|_{L^\infty}\int \xi_\nu u_{yy}^2 \\
&\quad + C(|u|_{L^\infty} + |u_x|_{L^\infty})\int \xi_\nu u_{yy}^2.
\end{aligned}$$

Therefore,

$$\sup_{0\le t\le T} C\int \xi(u_{yy}^2 + v_{yy}^2) + 3\int_0^T\int \chi(u_{xyy}^2 + \delta v_{xyy}^2) + \int_0^T\int \chi(u_{yyy}^2 + \lambda v_{yyy}^2) \le C$$

where C depends only on $u_0, v_0 \in H^3(\mathbb{R}^2) \cap H^2(W_{0\,i\,0})$.

Case $\beta = 3$. For $\beta = 3$, we consider the case $\alpha = (3,0)$. The other cases can be handled similarly. For $\alpha = (3,0)$, our remainder terms satisfy

$$\begin{aligned}
\Big|\int \xi_\nu\, u_{xxx}(uu_x)_{xxx}\Big| &= \Big|\int \xi_\nu u_{xxx}(3u_{xx}^2 + 4u_x u_{xxx} + uu_{xxxx})\Big| \\
&\le \Big|3\int \xi_\nu u_{xx}^2 u_{xxx}\Big| + \Big|4\int \xi_\nu u_x u_{xxx}^2\Big| + \Big|\int_{\mathbb{R}^2} \xi_\nu u u_{xxx} u_{xxxx}\Big| \\
&\equiv I_1 + I_2 + I_3.
\end{aligned}$$

First, we consider I_1. We consider the case $x > 1$ and $x < -1$ separately. For $x > 1$, we use the fact that $\xi_\nu^{1/2} \le C\xi_\nu$ for $i \ge 1$. In addition, we will use (2.13).

$$\begin{aligned}
&\Big|3\int_A \xi_\nu u_{xx}^2 u_{xxx}\Big| \\
&= C\Big|\int_A \xi_\nu (u_{xx}^3)_x\Big| = C\Big|\int_A (\xi_\nu)_x u_{xx}^3\Big| \\
&\le C\Big(\int_A \xi_\nu u_{xx}^4\Big)^{1/2}\Big(\int_A \xi_\nu u_{xx}^2\Big)^{1/2} \\
&\le C\Big(\int_A ([\xi_\nu^{1/4} u_x]_x)^4]\Big)^{1/2}\Big(\int_A \xi_\nu u_{xx}^2\Big)^{1/2} \\
&\le C\Big(\int_A ([\xi_\nu^{1/4} u_x]_x)^2 + ([\xi_\nu^{1/4} u_x]_{xx})^2 + ([\xi_\nu^{1/4} u_x]_{xy})^2\Big)\Big(\int_A \xi_\nu u_{xx}^2\Big)^{1/2} \\
&\le C\Big(\int_A \xi_\nu^{1/2}(u_x^2 + u_{xxx}^2 + u_{xxy}^2\Big)\Big(\int_A \xi_\nu u_{xx}^2\Big)^{1/2} \\
&\le C\Big(\int_A \xi_\nu(u_x^2 + u_{xxx}^2 + u_{xxy}^2)\Big)\Big(\int_A \xi_\nu u_{xx}^2\Big)^{1/2}.
\end{aligned}$$

Further, we note that

$$\int_0^T \Big(\int_A \xi_\nu(u_x^2 + u_{xxx}^2 + u_{xxy}^2)\Big)\Big(\int_A \xi_\nu u_{xx}^2\Big)^{1/2} dt$$

$$
\begin{aligned}
&\le C \sup_{0\le t\le T} \Big(\int_A \xi_\nu u_{xx}^2\Big)^{1/2} \int_0^T \int_A \xi_\nu (u_x^2 + u_{xxx}^2 + u_{xxy}^2)\\
&\le C \int_0^T \int_A \xi_\nu (u_x^2 + u_{xxx}^2 + u_{xxy}^2)
\end{aligned}
$$

where C depends only on the norms of $u_0, v_0 \in H^3(\mathbb{R}^2) \cap H^2(W_{0\,i\,0})$ by the previous step of the induction. On the other hand, for $x < -1$, we use the fact that $\xi_\nu \simeq C$ to show

$$
\begin{aligned}
|\int_B \xi_\nu u_{xx}^2 u_{xxx}| &\le C|\int_B u_{xx}^2 u_{xxx}|\\
&\le C\Big(\int_B u_{xx}^4\Big)^{1/2}\Big(\int_B u_{xxx}^2\Big)^{1/2}\\
&\le C\Big(\int_B [u_{xx}^2 + u_{xxx}^2 + u_{xxy}^2]\Big)\Big(\int_B u_{xxx}^2\Big)^{1/2}\\
&\le C\|u\|_{H^3} \int_B u_{xxx}^2 \le C\int_B \xi_\delta u_{xxx}^2.
\end{aligned}
\tag{6.16}
$$

For the term I_2, we have

$$
|\int \xi_\nu u_x u_{xxx}^2| \le C\|u_x\|_{L^\infty} \int \xi_\nu u_x\, u_{xxx}^2 \le C\|u\|_{H^3(\mathbb{R}^2)} \int \xi_\nu u_{xxx}^2. \tag{6.17}
$$

Lastly for I_3, we have

$$
\begin{aligned}
|\int \xi_\nu\, u\, u_{xxx}\, u_{xxxx}| &\le |\int [\xi_\nu\, u]_x\, u_{xxx}^2|\\
&\le C\,(|u|_{L^\infty} + |u_x|_{L^\infty}) \int \xi_\nu u_{xxx}^2\\
&\le C\|u\|_{H^3(\mathbb{R}^2)} \int \xi_\nu u_{xxx}^2.
\end{aligned}
\tag{6.18}
$$

Combining these estimates with (6.4) and using similar estimates for v, we conclude that

$$
\begin{aligned}
&\sup_{0\le t\le T} C\int \xi_\nu(\cdot,\cdot,t)(u_{xxx}^2 + v_{xxx}^2) + 3\int_0^T \int (\xi_\nu)_x (u_{xxxx}^2 + u_{xxxy}^2)\\
&\le C\int \xi_\nu(\cdot,\cdot,t) u_{0xxx}^2 + C\int \xi_\nu(\cdot,\cdot,t) v_{0xxx}^2\\
&\quad + C\int_0^T \int \xi_\nu (u_{xxx}^2 + v_{xxx}^2 + u_{xxy}^2 + v_{xxy}^2)
\end{aligned}
\tag{6.19}
$$

for $0 \le t \le T$. Using similar estimates for other derivatives on the level $\beta = 3$, we conclude that

$$
\begin{aligned}
&\sum_{|\alpha|=3} \sup_{0\le t\le T} C\int \xi_\nu(\cdot,\cdot,t)[(\partial^\alpha u)^2 + (\partial^\alpha v)^2] + C\sum_{|\alpha|=4} \int_0^T \int (\xi_\nu)_x [(\partial^\alpha u)^2 + (\partial^\alpha v)^2]\\
&\le \sum_{|\alpha|=3} C\int \xi_\nu(\cdot,\cdot,t)[(\partial^\alpha u_0)^2 + (\partial^\alpha v_0)^2] + C\sum_{|\alpha|=3} \int_0^T \int \xi_\nu [(\partial^\alpha u)^2 + (\partial^\alpha v)^2].
\end{aligned}
$$

Therefore, by Gronwall's inequality,

$$\begin{aligned}&\sum_{|\alpha|=3}\sup_{0\le t\le T}\int C\xi_\nu(\cdot,\cdot,t)[(\partial^\alpha u)^2+(\partial^\alpha v)^2]\\&+C\sum_{|\alpha|=4}\int_0^T\int(\xi_\nu)_x[(\partial^\alpha u)^2+(\partial^\alpha v)^2]\le C\end{aligned}$$

where C does not depend on ν, but only on T and the norms of $u_0,v_0\in H^3(\mathbb{R}^2)\cap H^3(W_{0\,i\,0})$. Passing to the limit, we conclude that

$$\sum_{|\alpha|=3}\sup_{0\le t\le T}C\int\xi(\cdot,\cdot,t)[(\partial^\alpha u)^2+(\partial^\alpha v)^2]+C\sum_{|\alpha|=4}\int_0^T\int\chi[(\partial^\alpha u)^2+(\partial^\alpha v)^2]\le C.$$

Case $\beta\ge 4$**.** For $K\ge\beta\ge 4$, $0\le t\le T$, we use Lemma 6.2 below, where we prove that

$$\begin{aligned}&\sum_{|\alpha|=\beta}\Big|\int_0^t\int\xi_\nu(\partial^\alpha u)\partial^\alpha(uu_x)\Big|+\Big|\int_0^t\int\xi_\nu(\partial^\alpha v)\partial^\alpha(vv_x)\Big|\\&\le C+C\sum_{|\alpha|=\beta}\Big(\int_0^t\int\xi_\nu(\partial^\alpha u)^2\Big)\end{aligned}\tag{6.20}$$

where C depends only on terms bounded in previous steps of the induction. Consequently,

$$\sup_{0\le t\le T}\sum_{|\alpha|=\beta}\int\xi_\nu[(\partial^\alpha u)^2+(\partial^\alpha v)^2]+C\sum_{|\alpha|=\beta+1}\int_0^T\int(\xi_\nu)_x[(\partial^\alpha u)^2+(\partial^\alpha v)^2]\le C$$

where C does not depend on ν, but only on T and the norms of $u_0,v_0\in H^3(\mathbb{R}^2)\cap H^\beta(W_{0\,i\,0})$. Passing to the limit, we obtain the desired estimates. □

Lemma 6.2. *For* ξ_ν *as defined in* (6.3), $\alpha=(\alpha_1,\alpha_2)$ *such that* $|\alpha|=\beta$, $4\le\beta\le K$, *the following holds:*

$$\begin{aligned}&\sum_{|\alpha|=\beta}\Big|\int_0^t\int\xi_\nu(\partial^\alpha u)\partial^\alpha(uu_x)\Big|+\Big|\int_0^t\int\xi_\nu(\partial^\alpha v)\partial^\alpha(vv_x)\Big|\\&\le C+C\sum_{|\alpha|=\beta}\Big(\int_0^t\int\xi_\nu(\partial^\alpha u)^2\Big)+C\sum_{|\alpha|=\beta}\Big(\int_0^t\int\xi_\nu(\partial^\alpha v)^2\Big)\end{aligned}\tag{6.21}$$

for $0\le t\le T$, *where* C *depends only on*

$$\sup_{0\le t\le T}\int\xi_\nu(\partial^\gamma u)^2,\quad\sup_{0\le t\le T}\int\xi_\nu(\partial^\gamma v)^2,\tag{6.22}$$

$$\int_0^T\int(\xi_\nu)_x(\partial^\gamma u_x)^2,\quad\int_0^T\int(\xi_\nu)_x(\partial^\gamma v_x)^2,\tag{6.23}$$

$$\int_0^T\int(\xi_\nu)_x(\partial^\gamma u_y)^2,\quad\int_0^T\int(\xi_\nu)_x(\partial^\gamma v_y)^2\tag{6.24}$$

for $\gamma=(\gamma_1,\gamma_2)$ *where* $|\gamma|\le\beta-1$.

The proof uses the same ideas as in the proof of Lemma 3.2. The primary difference is in our weight function ξ_ν. First, our weight function here, ξ_ν is approximately constant for $x < -1$, whereas in Lemma 3.2, the weight function decayed exponentially for $x < -1$. Consequently, in our inductive proof of Lemma 6.2 below, we are not able to use the estimates we obtained on

$$\int_0^T \int_B \int (\xi_\nu)_x \{(\partial^\gamma u_x)^2 + (\partial^\gamma u_y)^2\} \le C \int_0^T \int_B e^{\sigma x} \{(\partial^\gamma u_x)^2 + (\partial^\gamma u_y)^2\}$$

from the previous step of the induction. In addition, for $x > 1$, the weight function $\xi_\nu \approx x^i$ at all levels of the induction.

Proof. We estimate only the terms

$$\Big| \int_0^T \int \xi_\nu \, (\partial^\alpha u) \, \partial^\alpha \, (u u_x) \Big|. \tag{6.25}$$

The terms

$$\Big| \int_0^T \int \xi_\nu \, (\partial^\alpha v) \, \partial^\alpha \, (v v_x) \Big|$$

are bounded in the same way. Each term in (6.25) is of the form

$$\Big| \int_0^T \int \xi_\nu \, (\partial^\alpha u) \, (\partial^r u) \, (\partial^s u_x) \Big|$$

where $r_i + s_i = \alpha_i$ for $i = 1, 2$. We use the notation $q_r = r_1 + r_2$, $q_s = s_1 + s_2$, With this notation, it follows that $\beta = q_r + q_s$.

Remark 6.3. In what follows, we combine the fact that

$$\xi_\nu(\partial^\gamma u) = \sum_{j=0}^{\gamma_1} (-1)^j \binom{\gamma}{j} \partial_x^j ((\partial_x^{\gamma_1 - j} \xi_\nu)(\partial_y^{\gamma_2} u)).$$

with (2.11) to conclude that

$$\sup_{0 \le t \le T} \|\xi_\nu (\partial^\gamma u)^2\|_{L^\infty(\mathbb{R}^2)} \le C, \quad \text{for } q \le \beta - 3, \tag{6.26}$$

$$\int_0^T \|\xi_\nu (\partial^\gamma u)^2\|_{L^\infty(\mathbb{R}^2)} \le C, \quad \text{for} q \le \beta - 2, \tag{6.27}$$

where $\gamma_1 + \gamma_2 = q$ and C depends only on (6.22)-(6.24).

We will use estimates (6.26) and (6.27) below in bounding each term in the integrand.

Case $q_s \le \beta - 4$. For this case we have

$$\begin{aligned}
&\Big| \int_0^t \int \xi_\nu \, (\partial^\alpha u) \, (\partial^r u) \, (\partial^s u_x) \Big| \\
&= \Big| \int_0^t \int \xi_\nu^{1/2} \, (\partial^\alpha u) \, (\partial^r u) \, \xi_\nu^{1/2} \, (\partial^s u_x) \Big| \\
&\le \sup_{0 \le t \le T} \|\xi_\nu^{1/2} \, (\partial^s u_x) \, \|_{L^\infty} \Big(\int_0^t \int_{\mathbb{R}^2} (\partial^r u)^2 \Big)^{1/2} \Big(\int_0^t \int \xi_\nu \, (\partial^\alpha u)^2 \Big)^{1/2}.
\end{aligned}$$

The first term is bounded by (6.26). If $q_r \le \beta-1$, then the second term is bounded by (6.22). If $q_r = \beta$, then the third term is bounded by

$$\sum_{\alpha_1+\alpha_2=\beta} \Big(\int_0^t \int \xi_\nu \big(\partial^\alpha u\big)^2 \Big)^{1/2}.$$

In either case, we obtain

$$\Big| \int_0^t \int \xi_\nu \,(\partial^\alpha u)\,(\partial^r u)\,(\partial^s u_x) \Big| \le C + C \sum_{|\alpha|=\beta} \Big(\int_0^t \int \xi_\nu \,(\partial^\alpha u)^2 \Big) \quad \text{for } q_s \le \beta - 4.$$

Case $q_s = \beta - 3$. If $q_s = \beta - 3$, then $q_r = 3$. For this case, we consider $x > 1$ and $x < -1$ separately. For $x > 1$, $\xi_\nu \le C x^i$, while for $x < -1$, $\xi_\nu \simeq C(1 + e^{\sigma\, x})$. Again, let $A = \{x > 1\} \times \mathbb{R}$ and $B = \{x < -1\} \times \mathbb{R}$.

First, we consider $x > 1$. For $x > 1$, if $\beta \ge 5$, we use the estimate

$$\begin{aligned}
&\Big| \int_0^t \int_A \xi_\nu \,(\partial^\alpha u)\,(\partial^r u)\,(\partial^s u_x) \Big| \\
&\le \Big(\int_0^t \|(\xi_\nu)_x \,(\partial^r u)^2 \,\|_{L^\infty(A)}\, dt \Big)^{1/2} \Big(\sup_{0\le t\le T} \int_A \xi_\nu \,(\partial^s u_x)^2 dx\, dy \Big)^{1/2} \\
&\quad \times \Big(\int_0^t \int_A \xi_\nu \,(\partial^\alpha u)^2 \Big)^{1/2}.
\end{aligned}$$

The first term is bounded by (6.27) because $q_r \le \beta-2$. The second term is bounded by (6.22) because $q_s + 1 = \beta - 2$. If $\beta = 4$ and $q_s = \beta - 3$, we have $q_s = 1$, $q_r = 3$, in which case,

$$\begin{aligned}
&\Big| \int_0^t \int_A \xi_\nu \,(\partial^\alpha u)\,(\partial^r u)\,(\partial^s u_x) \Big| \\
&\le \Big(\int_0^t \|(\xi_\nu)_x \,(\partial^s u_x)^2 \,\|_{L^\infty(A)} dt \Big)^{1/2} \Big(\sup_{0\le t\le T} \int_A \xi_\nu (\partial^r u)^2 \Big)^{1/2} \Big(\int_0^t \int_A \xi_\nu (\partial^\alpha u)^2 \Big)^{1/2}.
\end{aligned}$$

The first time is bounded by (6.27) because $q_s + 1 = 2 \le \beta - 2$. The second term is bounded by (6.22) because $q_r \le \beta - 1$.

We now consider $x < -1$. In that case $\xi_\nu \approx 1 + e^{\sigma x} \le C$. Since $q_s = \beta - 3$, we know that $q_r = 3 \le \beta - 1$. Therefore,

$$\begin{aligned}
&\Big| \int_0^t \int_B \xi_\nu (\partial^\alpha u)(\partial^t u)(\partial^s u_x) \Big| \\
&\le C \int_0^t |\partial^s u_x|_{L^\infty(B)} \Big(\int_B (\partial^r u)^2 \Big)^{1/2} \Big(\int_B (\partial^\alpha u)^2 \Big)^{1/2} \\
&\le \sup_{0\le t\le T} \Big(\int_B (\partial^r u)^2 \Big)^{1/2} \Big(\int_0^t |\partial^s u_x|^2_{L^\infty(B)} \Big)^{1/2} \Big(\int_0^t \int_B (\partial^\alpha u)^2 \Big)^{1/2} \\
&\le C \sup_{0\le t\le T} \Big(\int_B \xi_\nu (\partial^r u)^2 \Big)^{1/2} \Big(\int_0^t \int_B \xi_\nu (\partial^\alpha u)^2 \Big)^{1/2} \\
&\quad \times \Big(\int_0^t \int_B \xi_\nu \{(\partial^s u_x)^2 + (\partial^s u_{xxx})^2 + (\partial^s u_{xyy})^2\} \Big)^{1/2}.
\end{aligned}$$

The first term is bounded by (6.22) since $q_r = 3 \le \beta - 1$. The other two terms are bounded by

$$\begin{aligned}&C\int_0^t \int_B \xi_\nu \{(\partial^s u_x)^2 + (\partial^s u_{xxx})^2 + (\partial^s u_{xyy})^2\} + C\int_0^t \int \xi_\nu (\partial^\alpha u)^2\\ &\le C\sum_{|\alpha|=\beta} \int_0^t \int_B \xi_\nu (\partial^\alpha u)^2,\end{aligned}$$

as desired.

Case $q_s = \beta - 2$**.** If $q_s = \beta - 2$, then $q_r = 2 = 4 - 2 \le \beta - 2$. We consider $x > 1$ and $x < -1$ separately.

First, for $x > 1$, as in the case $q_s = \beta - 3$, we have

$$\begin{aligned}&\Big|\int_0^t \int_A \xi_\nu (\partial^\alpha u)(\partial^r u)(\partial^s u_x)\Big|\\ &\le \int_0^t |\partial^r u|_{L^\infty(A)} \Big(\int_A \xi_\nu (\partial^s u_x)^2\Big)^{1/2} \Big(\int_A \xi_\nu (\partial^\alpha u)^2\Big)^{1/2}\\ &\le \sup_{0\le t\le T} \Big(\int_A \xi_\nu (\partial^s u_x)^2\Big)^{1/2} \Big(\int_0^t |(\xi_\nu)_x (\partial^r u)^2|_{L^\infty(A)}\Big)^{1/2} \Big(\int_0^t \int \xi_\nu (\partial^\alpha u)^2\Big)^{1/2}.\end{aligned}$$

The first term is bounded by (6.22). The second term is bounded by (6.27). Therefore,

$$\Big|\int_0^t \int \xi_\nu (\partial^\alpha u)(\partial^r u)(\partial^s u_x)\Big| \le C + C\int_0^t \int \xi_\nu (\partial^\alpha u)^2$$

where C depends only on (6.22), (6.23), and (6.24).

Next, we consider $x < -1$. In this case, $\xi_\nu \approx 1 + e^{\sigma x} \le C$. In the case when $q_s = \beta - 2$ and $\beta \ge 5$, we can bound it as follows:

$$\begin{aligned}&\Big|\int_0^t \int_B \xi_\nu (\partial^\alpha u)(\partial^r u)(\partial^s u_x)\Big|\\ &\le \int_0^t |\partial^r u|_{L^\infty(B)} \Big(\int_B (\partial^s u_x)^2\Big)^{1/2} \Big(\int_B (\partial^\alpha u)^2\Big)^{1/2}\\ &\le C \sup_{0\le t\le T} |\partial^r u|_{L^\infty(B)} \sup_{0\le t\le T} \Big(\int_B (\partial^s u_x)^2\Big)^{1/2} \int_0^T \Big(\int_B (\partial^\alpha u)^2\Big)^{1/2}.\end{aligned}$$

Since $q_s + 1 = \beta - 1$, the first two terms on the right-hand side are bounded by (6.22).

It remains to consider $x < -1$ when $q_s = \beta - 2$ and $\beta = 4$. In that case, $q_s = 2$ and $q_r = 2$. Then

$$\begin{aligned}&\Big|\int_0^t \int_B \xi_\nu (\partial^\alpha u)(\partial^r u)(\partial^s u_x)\Big|\\ &\le C\Big(\sup_{0\le t\le T} \int_B (\partial^s u_x)^2\Big)^{1/2} \Big(\int_0^t |\partial^r u|_{L^\infty(B)}\Big)^{1/2} \Big(\int_0^t \int_B (\partial^\alpha u)^2\Big)^{1/2}.\end{aligned}$$

Since $q_s = 2$, it follows that $q_s + 1 = 3 = \beta - 1$. Therefore, the first term is bounded by (6.22). Since $q_r = 2$, the second term satisfies

$$\int_0^t |\partial^r u|^2_{L^\infty(B)} \le C\int_0^t \int_B (\partial^r u)^2 + (\partial^r u_{xx})^2 + (\partial^r u_{yy})^2$$

$$\begin{aligned}&\le C\int_0^t\int_B \xi_\nu\left\{(\partial^r u)^2+(\partial^r u_{xx})^2+(\partial^r u_{yy})^2\right\}\\&\le C\sum_{\alpha_1+\alpha_2=\beta}\int_0^t\int_B \xi_\nu(\partial^\alpha u)^2.\end{aligned}$$

Therefore, we conclude that for $q_s=\beta-2$,

$$\Big|\int_0^t\int_B \xi_\nu(\partial^\alpha u)(\partial^r u)(\partial^s u_x)\Big|\le C\sum_{|\alpha|=\beta}\int_0^t\int_B \xi_\nu(\partial^\alpha u)^2$$

where C depends only on (6.22).

Case $q_s=\beta-1$. If $q_s=\beta-1$, then $q_r=1$. Therefore,

$$\begin{aligned}&\Big|\int_0^t\int_B \xi_\nu(\partial^\alpha u)(\partial^r u)(\partial^s u_x)\Big|\\&\le C\sup_{0\le t\le T}|\partial^r u|_{L^\infty(B)}\Big(\int_0^t\int_B \xi_\nu(\partial^s u_x)^2\Big)^{1/2}\Big(\int_0^t\int_B \xi_\nu(\partial^\alpha u)^2\Big)^{1/2}\\&\le C\sup_{0\le t\le T}\Big(\int_B(\partial^r u)^2+(\partial^r u_{xx})^2+(\partial^r u_{yy})^2\Big)\Big(\int_0^t\int_B \xi_\nu(\partial^s u_x)^2\Big)^{1/2}\\&\quad\times\Big(\int_0^t\int_B \xi_\nu(\partial^\alpha u)^2\Big)^{1/2}\\&\le C\sum_{|\alpha|=\beta}\int_0^t\int_B \xi_\nu(\partial^\alpha u)^2.\end{aligned}$$

Case $q_s=\beta$. If $q_s=\beta$, then $q_r=0$ and $s=\alpha$. Therefore,

$$\begin{aligned}\Big|\int_0^t\int_B \xi_\nu(\partial^\alpha u)(\partial^r u)(\partial^s u_x)\Big|&=C\Big|\int_0^t\int_B \xi_\nu u[(\partial^\alpha u)^2]_x\Big|\\&\le C(\|u\|_{L^\infty}+\|u_x\|_{L^\infty})\int_0^t\int_B \xi_\nu(\partial^\alpha u)^2\\&\le C\int_0^t\int_B \xi_\nu(\partial^\alpha u)^2.\end{aligned}$$

□

We now state and prove our main theorem, that if our initial data (u_0,v_0) has minimal regularity and sufficient decay as $x\to\infty$, then the solution (u,v) is smoother than (u_0,v_0). For simplicity, we introduce the following space which will be used in the proof. Let

$$\mathcal{Z}_L=H^3(\mathbb{R}^2)\cap H^0(W_{0\,L\,0})$$

with the accompanying norm

$$\|f\|^2_{\mathcal{Z}_L}=\int(1+\xi)f^2+\sum_{|\alpha|=3}(\partial^\alpha f)^2$$

where $\xi\in W_{0\,L\,0}$.

Theorem 6.4. *Let $T>0$ and (u,v) be a solution of (1.3) in the region $\mathbb{R}^2\times[0,T]$ such that*

$$(u,v)\in L^\infty([0,T]:\mathcal{Z}_L)\times L^\infty([0,T]:\mathcal{Z}_L) \tag{6.28}$$

for some $L \geq 1$. *Then*

$$\sup_{0\leq t\leq T}\int \xi_\beta[(\partial^\alpha u)^2+(\partial^\alpha v)^2]<\infty,$$
$$\int_0^T\int \chi_\beta[(\partial^\alpha u_x)^2+(\partial^\alpha u_y)^2]<\infty,$$
$$\int_0^T\int \chi_\beta[(\partial^\alpha v_x)^2+(\partial^\alpha v_y)^2]<\infty,$$

for $0 \leq \beta \leq L$ *where* $\beta = \alpha_1 + \alpha_2$, $\xi_\beta \in W_{\sigma,L-\beta,\beta}$, $\chi_\beta \in W_{\sigma,L-\beta-1,\beta}$, $\sigma > 0$ *arbitrary, with the exception that for* $\beta = L$, $W_{\sigma,-1,L}$ *is replaced by* $\widetilde{W}_{\sigma,-1,L}$.

Remark 6.5. If assumption (6.28) holds for all $L \geq 1$, then the solution is infinitely differentiable in the x and y variables. In that case, from (1.3), the solution is C^∞ in all of its variables.

Proof of Theorem 6.4. By assumption, $u, v \in L^\infty([0,T]; \mathcal{Z}_L)$. Recall this means $u, v \in H^3(\mathbb{R}^2)$ and $\int \xi(u^2+v^2) < \infty$ for $\xi \in W_{0\,L\,0}$. The equations imply $u_t, v_t \in L^\infty([0,T]; L^2(\mathbb{R}^2))$. Therefore, u, v are weakly continuous functions of t with values in $\mathcal{Z}_L$, and, in particular, $u(\cdot,\cdot,t), v(\cdot,\cdot,t)$ are in $\mathcal{Z}_L$ for every t. Let $\{u_0^{(n)}\}$, $\{v_0^{(n)}\}$ be sequences of functions in $C_0^\infty(\mathbb{R}^2)$ which converge to $u(\cdot,\cdot,t_0)$, $v(\cdot,\cdot,t_0)$ strongly in $\mathcal{Z}_L$, for $0 \leq t_0 < T$. Let $(u^{(n)}(x,y,t), v^{(n)}(x,y,t))$ be the unique solution of (1.3) with initial data $(u_0^{(n)}(x,y), v_0^{(n)}(x,y))$ at time $t = t_0$. By Theorem 5.2, the solution is guaranteed to exist in a time interval $[t_0, t_0+\delta]$ where δ does not depend on n. By Theorem 6.1, $u^{(n)}, v^{(n)} \in L^\infty([t_0, t_0+\delta]; \mathcal{Z}_L)$ and

$$\begin{aligned}&\int_{t_0}^{t_0+\delta}\int \chi[(u_x^{(n)})^2+(u_y^{(n)})^2]\,dx\,dy\,dt\\&+\int_{t_0}^{t_0+\delta}\int \chi[(v_x^{(n)})^2+(v_y^{(n)})^2]\,dx\,dy\,dt\leq C,\end{aligned}\tag{6.29}$$

where $\chi \in W_{\sigma,L-1,0}$ and C depends only on the norms of $u_0^{(n)}, v_0^{(n)} \in \mathcal{Z}_L$. Also by Theorem 6.1, we have (non-uniform) bounds on

$$\sup_{t\in[t_0,t_0+\delta]}\ \sup_{(x,y)\in\mathbb{R}^2}(1+|x^+|^k)\left|\partial^\alpha u^{(n)}(x,y,t)\right|<+\infty,\tag{6.30}$$

$$\sup_{t\in[t_0,t_0+\delta]}\ \sup_{(x,y)\in\mathbb{R}^2}(1+|x^+|^k)\left|\partial^\alpha v^{(n)}(x,y,t)\right|<+\infty\tag{6.31}$$

for each n, k, and α. Therefore, the main estimates in Lemma 3.1 are justified for each $u^{(n)}$ and $v^{(n)}$ in the interval $[t_0, t_0+\delta]$. The multiplier χ may be chosen arbitrarily in its weight class and ξ is defined by (3.5).

We start our induction with $\beta = 1$, in which case $\alpha = (1,0)$ or $\alpha = (0,1)$. Take $\chi \in W_{\sigma,L-2,1}$ and let $\xi = \int_{-\infty}^x \chi(z,t)dz$. As shown in Lemma 3.1, we have the following bounds on the higher derivatives of $u^{(n)}$, $v^{(n)}$,

$$\begin{aligned}&\sup_{[t_0,t_0+\delta]}\int \xi[(u_x^{(n)})^2+(v_x^{(n)})^2]+\int_{t_0}^{t_0+\delta}\int \chi[(u_{xx}^{(n)})^2+(u_{xy}^{(n)})^2]\\&+\int_{t_0}^{t_0+\delta}\int \chi[(v_{xx}^{(n)})^2+(v_{xy}^{(n)})^2]\leq C,\end{aligned}\tag{6.32}$$

$$
\begin{aligned}
&\sup_{[t_0,t_0+\delta]}\int \xi[(u_y^{(n)})^2+(v_y^{(n)})^2]+\int_{t_0}^{t_0+\delta}\int \chi[(u_{xy}^{(n)})^2+(u_{yy}^{(n)})^2]\\
&+\int_{t_0}^{t_0+\delta}\int \chi[(v_{xy}^{(n)})^2+(v_{yy}^{(n)})^2]\le C,
\end{aligned}
\tag{6.33}
$$

where C depends only on the norms of $u^{(n)}, v^{(n)} \in L^\infty([0,T];\mathcal{Z}_L)$ and the terms in (6.29). We conclude, therefore, that the constants C in (6.32) and (6.33) depend only on $\|u_0^{(n)}\|_{\mathcal{Z}_L}$ and $\|v_0^{(n)}\|_{\mathcal{Z}_L}$.

We continue this process inductively. For the β^{th} step, let

$$
\chi \in W_{\sigma,L-\beta-1,\beta},
$$

and define $\xi = \int_{-\infty}^x \chi(z,t)dz$. The non-uniform bounds on $u^{(n)}$ and $v^{(n)}$ in (6.30) and (6.31) allow us to use Lemma 3.1 and our inductive hypothesis to conclude that

$$
\begin{aligned}
&\sup_{[t_0,t_0+\delta]}\int \xi[(\partial^\alpha u^{(n)})^2+(\partial^\alpha v^{(n)})^2]+\int_{t_0}^{t_0+\delta}\int \chi[(\partial^\alpha u_x^{(n)})^2+(\partial^\alpha u_y^{(n)})^2]\\
&+\int_{t_0}^{t_0+\delta}\int \chi[(\partial^\alpha v_x^{(n)})^2+(\partial^\alpha v_y^{(n)})^2]\le C,
\end{aligned}
$$

where again C does not depend on n, but only on the norms of $u_0^{(n)}, v_0^{(n)} \in \mathcal{Z}_L$. By Corollary 5.3, $u^{(n)} \rightharpoonup u$ weak* in $L^\infty([t_0,t_0+\delta];H^3(\mathbb{R}^2))$ and $v^{(n)} \rightharpoonup v$ weak* in $L^\infty([t_0,t_0+\delta];H^3(\mathbb{R}^2))$. Therefore, we can pass to the limit and conclude that

$$
\begin{aligned}
&\sup_{[t_0,t_0+\delta]}\int \xi[(\partial^\alpha u)^2+(\partial^\alpha v)^2]+\int_{t_0}^{t_0+\delta}\int \chi[(\partial^\alpha u_x)^2+(\partial^\alpha u_y)^2]\\
&+\int_{t_0}^{t_0+\delta}\int \chi[(\partial^\alpha v_x)^2+(\partial^\alpha v_y)^2]\le C.
\end{aligned}
$$

We continued the process inductively up to $\beta = L$, with the exception that on the last level, $\beta = L$, we replace $W_{\sigma,-1,L}$ with $\widetilde{W}_{\sigma,-1,L}$. Since δ is fixed, this result is valid over the whole interval $[0,T]$. □

7. Concluding remarks

In this article, we proved that if the initial data decays faster than any polynomial as $x \to \infty$, the solution of a coupled Zakharov-Kuznetsov system lies in $C^\infty(\mathbb{R}^2) \times C^\infty(\mathbb{R}^2)$. We quantified the gain in regularity of the solution depending on the amount of decay of the initial data as $x \to \infty$. In particular, we showed that if the initial data (u_0, v_0) lies in $H^3(\mathbb{R}^2) \times H^3(\mathbb{R}^2)$ and lies in a weighted $L^2 \times L^2$ space with a weight function that behaves like x^L as $x \to \infty$, then the solution (u,v) lies in a weighted Sobolev space $H^L \times H^L$ for $0 < t \le T$ where T is the existence time of the solution.

Acknowledgments. This research was partially supported by project Fondecyt 1191137 and UTA MAYOR 2022-2023, 4764-22.

References

[1] M. S. Alves, O. Vera Villagrán; *Smoothing properties for a coupled system of nonlinear evolution dispersive equations*, Indag. Math., 20 no. 2 (2009), 285-327.

[2] O. V. Besov, V. P. II'in, S. M. Nikolskii; *Integral representations of functions and imbedding theorem.* Vol. I. J. Wiley. 1978.

[3] J. C. Ceballos, M. Sepulveda, O. Vera; *Gain in regularity for a coupled nonlinear Schrödinger system*, Bol. Soc. Parana. Mat., 24 (2006), 41-68.

[4] A. Cohen; *Solutions of the Korteweg-de Vries equation from irregular data*, Duke Math. J., 45 (1978) 149-181. ,

[5] P. Constantin, J. C. Saut; *Local smoothing properties of dispersive equations*, J. Amer. Math. Soc., 1 (1988), 413-439.

[6] W. Craig, T. Kappeler, W. Strauss; *Gain of regularity for equations of KdV type*, Ann. Inst. H. Poincaré, 9 (1992), 147-186.

[7] J. Gear, R. Grimshaw; *Weak and strong interaction between internal solitarywaves.* Stud. Appl. Math., Vol. 70, 3 (1984), 235-258.

[8] J. Ginibre, G. Velo; *Propriétés de lissage et existence de solutions pour l'équation de Benjamin-Ono généralisée*, C. R. Acad. Sci. Paris 308 (1989) 309-314.

[9] N. Hayashi, K. Nakamitsu, M. Tsutsumi; *On solutions of the initial value problem for the nonlinear Schödinger equation in one space dimension*, Math. Z. 192 (1986), 637-650.

[10] N. Hayashi, K. Nakamitsu, M. Tsutsumi; *On solutions of the initial value problem for the nonlinear Schödinger equation*, J. Funct. Anal. 71 (1987) 218-245.

[11] N. Hayashi, T. Ozawa; *Smoothing effect for some Schödinger equations*, J. Funct. Anal. 85 (1989) 307-348.

[12] W. Huang; *A polynomial expansion method and its applicationin the coupled Zakharov-Kuznetsov equations*, Chaos, Solitons and Fractals, 29 (2006) no. 2, 365-371.

[13] T. Kato; *On the Cauchy problem for the (generalized) Korteweg-de Vries equation*, Adv. Math. (Suppl. Studies: Studies in Appl. Math.) 8 (1983), 93-128.

[14] C. Kenig, G. Ponce, L. Vega; *Oscillatory integrals and regularity of dispersive equations*, Indiana Univ. Math., J. 40 (1) (1991), 33-69.

[15] S. Kruzhkov, A. V. Faminskii; *Generalized solutions of the Cauchy problem for the Korteweg-de Vries equation*, Math. USSR Sb., 48 (1984), 93-138.

[16] J. Levandosky; *Gain of regularity for the KP-II equation*, Indiana Univ. Math. J., 49 (2000), 353-403.

[17] J. Levandosky; *Smoothing properties of nonlinear dispersive equations in two spatial dimensions*, J. Differential Equations, 175 (2001), 275-352.

[18] J. Levandosky; *Smoothing properties for a two-dimensional Kawahara equation*, J. Differential Equations, 316 (2022), 158-196.

[19] J. Levandosky, M. Sepulveda, O. Vera Villagran; *Gain of regularity for the KP-I equation*, J. Differential Equations, 245, 3 (2008), 762-808.

[20] J. L. Lions; *Quelque méthodes de résolution des problemes aux limites non linéaires*, Gauthiers-Villars.

[21] G. Ponce; *Regularity of solutions to nonlinear dispersive equations*, J. Differential Equations, 78 (1989), 122-135.

[22] O. Vera; *Gain of regularity for a generalized coupled system of nonlinear evolution dispersive equations type*, PhD thesis, UFRJ, Rio de Janeiro, Brazil, 2001.

[23] O. Vera; *Gain of regularity for a Korteweg-de Vries-Kawahara type equation*, Electron. J. Differential Equations, 2004, 71 (2004), 1-24.

[24] Y. Zhu, R. Grimshaw; *Oblique interactions between internal solitary waves.*, Stud. Appl. math. Vol. 92, 3 (1994), 249-270.

Julie L. Levandosky

Department of Mathematics, Framingham State University, Framingham, MA 01701 USA

Email address: `jlevandosky@framingham.edu`

Octavio Vera

Departamento de Matemáticas, Universidad de Tarapaca, Casilla 7-D, Arica, Chile

Email address: `opverav@academicos.uta.cl`