Electronic Journal of Differential Equations, Vol. 2023 (2023), No. 16, pp. 1–10.
ISSN: 1072-6691. URL: https://ejde.math.txstate.edu or https://ejde.math.unt.edu

NONEXITENCE OF NONTRIVIAL SOLUTIONS TO DIRICHLET PROBLEMS FOR THE FRACTIONAL LAPLACIAN

JOSÉ CARMONA, ALEXIS MOLINO

Abstract. In this article we prove that there are no nontrivial solutions to the Dirichlet problem for the fractional Laplacian

$$\begin{gathered}(-\Delta)^s u = f(u) \quad \text{in } \Omega,\\ u = 0 \quad \text{in } \mathbb{R}^N \setminus \Omega,\end{gathered}$$

where $\Omega \subset \mathbb{R}^N$ $(N \geq 1)$ is a bounded domain, and f is locally Lipschitz with non-positive primitive $F(t) = \int_0^t f(\tau)d\tau$.

1. Introduction

In this work, we investigate the nonexistence of nontrivial bounded solutions for the Dirichlet problem for the fractional Laplacian

$$\begin{gathered}(-\Delta)^s u = f(u) \quad \text{in } \Omega,\\ u = 0 \quad \text{in } \mathbb{R}^N \setminus \Omega,\end{gathered} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$ $(N \geq 1)$ is a bounded domain with $\mathcal{C}^{1,1}$ regular boundary, $\partial\Omega$, and $f : \mathbb{R} \to \mathbb{R}$ is a locally Lipschitz sign-changing function.

Throughout this article, the fractional Laplacian operator $(-\Delta)^s$ (also called, Riesz fractional Laplacian) is formally defined by

$$(-\Delta)^s u = C(N,s)\,\mathrm{P.V.}\int_{\mathbb{R}^N} \frac{u(x)-u(y)}{|x-y|^{N+2s}}dy, \quad s \in (0,1),$$

where $C(N,s)$ is the positive constant given by

$$C(N,s) = \frac{s2^{2s}\Gamma\left(\frac{2s+N}{2}\right)}{\pi^{N/2}\Gamma(1-s)}, \tag{1.2}$$

Γ denotes the Gamma function, and P.V. stands for the principal value of the integral

$$P.V.\int_{\mathbb{R}^N} \frac{u(x)-u(y)}{|x-y|^{N+2s}}dy = \lim_{\varepsilon\to 0}\int_{\mathbb{R}^N\setminus B_\varepsilon(x)} \frac{u(x)-u(y)}{|x-y|^{N+2s}}dy,$$

where $B_\varepsilon(x)$ is the ball of radius ε centered at x.

2020 *Mathematics Subject Classification.* 35J05, 35J15, 35J25.
Key words and phrases. Fractional Laplacian; Dirichlet problem; nonexistence of solutions.

Submitted March 30, 2022 Published February 17, 2023.

To establish the concept of solution to problem (1.1) we consider the usual Sobolev fractional spaces

$$H^s(\mathbb{R}^N) = \Big\{u \in L^2(\mathbb{R}^N) : \frac{u(x)-u(y)}{|x-y|^{N/2+s}} \in L^2\big(\mathbb{R}^N \times \mathbb{R}^N\big)\Big\},$$
$$H_0^s(\Omega) = \{u \in H^s(\mathbb{R}^N) : u \equiv 0, \text{ a.e. } \mathbb{R}^N \setminus \Omega\}.$$

Let us recall that $H_0^s(\Omega)$ is a Hilbert space with the scalar product

$$\langle u, v\rangle_{H_0^s(\Omega)} = \frac{C(N,s)}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+2s}}\,dx\,dy, \quad u,v \in H_0^s(\Omega).$$

We denote the induced norm by

$$\|u\|^2_{H_0^s(\Omega)} = \frac{C(N,s)}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{(u(x)-u(y))^2}{|x-y|^{N+2s}}\,dx\,dy, \quad u \in H_0^s(\Omega).$$

For a detailed study on the different approaches to fractional Sobolev spaces, see [3]. In addition, a more extensively study of non-local operators, of which the fractional Laplacian is a particular case, can be found in the survey [16].

Multiplying equation (1.1) by $v \in H_0^s(\Omega)$ and integrating in $\mathbb{R}^N$, we obtain

$$\frac{C(N,s)}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+2s}}\,dy\,dx = \int_{\mathbb{R}^N} f(u(x))v(x)dx.$$

Therefore, we say that $u \in H_0^s(\Omega)$ is a *weak* solution to problem (1.1) if

$$\langle u, v\rangle_{H_0^s(\Omega)} = \int_{\mathbb{R}^N} f(u(x))v(x)dx, \tag{1.3}$$

for every test function $v \in H_0^s(\Omega)$.

We deal with *bounded solutions* (solutions from now on) of problem (1.1). A direct consequence of [17, Corollary 1.6] ensures that weak bounded solutions to problem (1.1) belong to $\mathcal{C}^s(\mathbb{R}^N) \cap \mathcal{C}^{2s+\varepsilon}(\Omega)$, whenever $\partial\Omega$ is $\mathcal{C}^{1,1}$. As a consequence, weak bounded solutions are classical solutions to problem (1.1) in the sense that the fractional Laplacian operator can be pointwise evaluated in Ω.

It follows immediately, taking u as a test function in (1.3), that a necessary condition for the existence of a solution u to (1.1) is

$$\|u\|^2_{H_0^s(\Omega)} = \int_{\mathbb{R}^N} f(u)u. \tag{1.4}$$

In particular, if the hypothesis

$$f(t)t \le 0, \quad \text{for all } t \in \mathbb{R}, \tag{1.5}$$

holds, then the unique solution to problem (1.1) is the trivial one.

The main motivation for this work comes from the interest in finding sufficient conditions in the nonlinear term f, beyond (1.5), that guarantee the nonexistence of a nontrivial solution to (1.1). This represents a challenging problem even in the case of the Laplace operator, where (1.1) becomes

$$\begin{gathered} -\Delta u = f(u) \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega. \end{gathered} \tag{1.6}$$

Nonexistence results for (1.6) are usually deduced from the Pohozaev identity [12]

$$\int_\Omega (2NF(u) - (N-2)uf(u))\,dx = \int_{\partial\Omega} \Big(\frac{\partial u}{\partial \nu}\Big)^2 (x\cdot\nu)\,d\sigma(x),$$

where ν is the unit outward normal to $\partial\Omega$ at x and $F(t) = \int_0^t f(\tau)d\tau$. Indeed, for star-shaped domains with respect to 0 (i.e., $x \cdot \nu(x) > 0$ on $\partial\Omega$) and $f(s) = |s|^{p-1}s$ with $p > 1$, Pohozaev identity leads to nonexistence of nontrivial solutions for supercritical values of p, i.e. $p \geq \frac{N+2}{N-2}$, $N > 2$. However, existence of solution is known in the supercritical regime when Ω is not star-shaped [9].

A second nonexistence of nontrivial solution result is deduced from the Pohozaev identity when Ω is star-shaped and F satisfies

$$F(t) = \int_0^t f(\tau)d\tau \leq 0, \quad \text{for all } t \in \mathbb{R}. \tag{1.7}$$

A typical example satisfying (1.7) is $f(t) = \lambda \sin t$, with $\lambda < 0$. By similarity with the power case, in [15], the author conjectured that, when Ω is not star-shaped and $-\lambda$ is large enough, a nontrivial solution may exist. Some partial results were obtained in [4, 5, 6, 7, 8, 14]. Finally, it was shown in [10] that this conjecture is false by proving, for general domains Ω, that (1.6) admits only the trivial solution when (1.7) is satisfied.

To the best of our knowledge, there are no such results in the case of non-local operators. This is the main goal of this paper and the main result we obtain is the following.

Theorem 1.1. *Let $\Omega \subset \mathbb{R}^N$ $(N \geq 1)$ be a $\mathcal{C}^{1,1}$ bounded domain, $f : \mathbb{R} \to \mathbb{R}$ a locally Lipschitz function satisfying* (1.7). *Then, $u \equiv 0$ is the unique solution to problem* (1.1).

Let us recall the Pohožaev identity for solutions to (1.1) due to [18] which states that, for $N > 2s$,

$$(2s - N)\int_\Omega uf(u)\,dx + 2N\int_\Omega F(u)\,dx = \Gamma(1+s)^2 \int_{\partial\Omega} \Big(\frac{u}{\delta^s}\Big)^2 (x\cdot\nu)\,d\sigma, \tag{1.8}$$

where $\delta(x) = \operatorname{dist}(x, \partial\Omega)$. Observe that, as in the local case, this equality implies Theorem 1.1 when Ω is starshaped and $N \geq 2s$. Indeed, in this case, the right hand side is non-negative, and taking into account (1.4) we obtain the following inequality

$$\int_\Omega F(u)\,dx = \frac{\Gamma(1+s)^2}{2N}\int_{\partial\Omega}\Big(\frac{u}{\delta^s}\Big)^2 (x\cdot\nu)\,d\sigma + \frac{N-2s}{2N}\int_\Omega uf(u)\,dx \geq 0.$$

Moreover, the equality holds only when $u \equiv 0$. In the present work, we prove that condition (1.7) ensures the nonexistence of nontrivial solutions to problem (1.1) with no additional hypotheses on the geometry of Ω or the dimension N.

For the proof of Theorem 1.1 we rely mostly on two results: on one hand, we are inspired in the result for problem (1.6) carried out by the second author in [10] and, on the other hand, on [2] for the existence of an increasing solution for a certain type of non-local ordinary differential equation.

This articleis organized as follows. In Section 2, we establish Maximum Principles to the fractional Laplacian. In Section 3, we prove Theorem 1.1, and in Section 4, we summarize in conclusion the main findings though some examples. Finally, in the Appendix we prove a technical Lemma used in the previous section.

2. Maximum Principle

This section is devoted to the Maximum Principle for the fractional Laplacian operator. Specifically, we prove a version of the well-known *Serrin's Sweeping Principle* (see e.g. [13]).

We denote by $\lambda_1 > 0$ the first eigenvalue with associated nonnegative eigenfunction φ_1, i.e. it is satisfied that

$$\begin{gathered}(-\Delta)^s\varphi_1(x) = \lambda_1\varphi_1(x) \quad \text{in } \Omega,\\ \varphi_1(x) = 0 \quad \text{in } \mathbb{R}^N \setminus \Omega.\end{gathered}$$

First, we state and prove the Strong Maximum Principle for fractional Laplacian operators, [1], with the intention of making this section self-contained.

Proposition 2.1 (Strong Maximum Principle)**.** *Let $\Omega \subset \mathbb{R}^N$ $(N \geq 1)$ be a bounded domain. Consider the function $m : \Omega \to (-\lambda_1, \infty)$ and $u \in H^s(\mathbb{R}^N)$ satisfying the following inequality pointwise*

$$\begin{gathered}(-\Delta)^s u(x) + m(x)u(x) \geq 0 \quad \text{in } \Omega,\\ u \geq 0 \quad \text{in } \mathbb{R}^N \setminus \Omega.\end{gathered} \tag{2.1}$$

Then $u \geq 0$ in $\mathbb{R}^N$. Furthermore, either $u \equiv 0$ in $\mathbb{R}^N$ or $u > 0$ in Ω.

Remark 2.2. As a direct consequence, it follows that if u satisfies the hypotheses of Proposition 2.1 and there exists $x_0 \in \Omega$ such that $u(x_0) = 0$, then $u \equiv 0$ in $\mathbb{R}^N$.

Proof of Proposition 2.1. To prove that $u \geq 0$ in $\mathbb{R}^N$ we observe that (2.1) is true for $u^-(x) = \min\{u(x), 0\}$. Thus, multiplying by the first eigenfunction φ_1 and integrating in $\mathbb{R}^N$ we obtain

$$\begin{aligned}0 &\leq \int_{\mathbb{R}^N} \big((-\Delta)^s u^-(x) + m(x)u^-(x)\big)\,\varphi_1(x)dx\\ &= \frac{C(N,s)}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{(u^-(x)-u^-(y))(\varphi_1(x)-\varphi_1(y))}{|x-y|^{N+2s}}\,dy\,dx\\ &\quad + \int_{\mathbb{R}^N} m(x)u^-(x)\varphi_1(x)dx\\ &= \int_{\mathbb{R}^N} (\lambda_1 + m(x))u^-(x)\varphi_1(x)dx \leq 0.\end{aligned}$$

Thus, $u^- \equiv 0$ and we have that u is non-negative. We assume now that u is non-trivial in $\mathbb{R}^N$. Then, the set $A = \{x \in \mathbb{R}^N : u(x) > 0\}$ has non-zero measure.

To prove that $u > 0$ in Ω we argue by contradiction. Suppose that there exists $x_0 \in \Omega$ such that $u(x_0) = 0$. Evaluating x_0 in inequality (2.1) we obtain the following contradiction

$$\begin{aligned}0 &\leq (-\Delta)^s u(x_0) + m(x_0)u(x_0)\\ &= C(N,s)\int_{\mathbb{R}^N} \frac{-u(y)}{|x_0-y|^{N+2s}}dy\\ &= C(N,s)\int_{A} \frac{-u(y)}{|x_0-y|^{N+2s}}dy < 0.\end{aligned}$$

□

Next result is known as *Serrin's Sweeping Principle* and it has been shown to hold for the Laplacian operator (and other uniformly elliptic operators), see the pioneering works [11] and [19]. To our knowledge, this result is new in the field of fractional Laplacian operators.

Proposition 2.3 (Sweeping Principle). *Let $\Omega \subset \mathbb{R}^N$ $(N \geq 1)$ be a bounded domain, $a : \Omega \to \mathbb{R}$ and f be a globally Lipschitz function, with Lipschitz constant $L > 0$ such that $a(x) + L > -\lambda_1$. Assume that $\{v_\lambda\}$, $\lambda \in \mathbb{R}$, is a one-parameter family of lower semicontinuous functions in $H^s(\mathbb{R}^N)$ such that the application $\lambda \to v_\lambda$ is continuous (uniformly in $x \in \mathbb{R}^N$) and, for every $\lambda \in \mathbb{R}$, v_λ satisfies pointwise the inequalities*

$$(-\Delta)^s v_\lambda(x) + a(x) v_\lambda(x) \geq f(v_\lambda(x)) \quad \text{in } \Omega,$$
$$v_\lambda(x) \geq 0 \quad \text{in } \mathbb{R}^N \setminus \Omega.$$

Assume also that $u \in H^s(\mathbb{R}^N)$ is an upper semicontinuous function satisfying pointwise the inequalities

$$(-\Delta)^s u(x) + a(x) u(x) \leq f(u(x)) \quad \text{in } \Omega,$$
$$u(x) \leq 0 \quad \text{in } \mathbb{R}^N \setminus \Omega.$$

Moreover, let us suppose that $u(x) \not\equiv v_\lambda(x)$ for every $\lambda \in \mathbb{R}$ and $x \in \mathbb{R}^N \setminus \Omega$, and that there exists $\lambda_0 \in \mathbb{R}$ such that $u(x) \leq v_{\lambda_0}(x)$ in $\mathbb{R}^N$. Then

$$u(x) \leq v_\lambda(x), \quad \text{for all } \lambda \in \mathbb{R}, \quad x \in \mathbb{R}^N.$$

Remark 2.4. This result shows that for a subsolution u to be $u \leq v_\lambda$ for every $\lambda \in \mathbb{R}$, being v_λ a familiy of supersolutions continuous respect to λ, it suffices that $u \leq v_{\lambda_0}$, for some value $\lambda_0 \in \mathbb{R}$.

Proof of Proposition 2.3. First, let us define the following subset

$$\mathcal{A} = \{\lambda \in \mathbb{R} : u(x) \leq v_\lambda(x), \text{for all } x \in \mathbb{R}^N\}.$$

Since $\lambda_0 \in \mathcal{A}$, this subset is not empty. Obviously as v_λ is continuous with respect to λ, $\mathcal{A}$ is closed. In order to prove that $\mathcal{A}$ is open, consider $\bar{\lambda} \in \mathcal{A}$ and, by hypotheses, we have that $(-\Delta)^s v_{\bar{\lambda}}(x) + a(x) v_{\bar{\lambda}}(x) \geq f(v_{\bar{\lambda}}(x))$ in Ω.

Now we define $w_{\bar{\lambda}} = v_{\bar{\lambda}} - u \geq 0$, which satisfies

$$(-\Delta)^s w_{\bar{\lambda}}(x) + a(x) w_{\bar{\lambda}}(x) \geq f(v_{\bar{\lambda}}(x)) - f(u(x)), \quad x \in \Omega.$$

We add $L(v_{\bar{\lambda}} - u)$ to both sides of this inequality and we obtain

$$(-\Delta)^s w_{\bar{\lambda}} + (a(x) + L) w_{\bar{\lambda}} \geq f(v_{\bar{\lambda}}(x)) - f(u(x)) + L(v_{\bar{\lambda}} - u) \geq 0.$$

Therefore, $w_{\bar{\lambda}} \in H^s(\mathbb{R}^N)$ is a lower semicontinuous function which satisfies inequality (2.1) with $m(x) \equiv a(x) + L$ for all $x \in \mathbb{R}^N$. By the Strong Maximum Principle (Proposition 2.1), this implies that either $w_{\bar{\lambda}} > 0$ in Ω or $w_{\bar{\lambda}} \equiv 0$ in $\mathbb{R}^N$ (which is not possible since by hypothesis $w_{\bar{\lambda}} \not\equiv 0$ in $\mathbb{R}^N \setminus \Omega$). In particular

$$u(x) < v_{\bar{\lambda}}(x), \quad \text{for all } x \in \bar{\Omega}.$$

As a consequence, since $v_{\bar{\lambda}} - u$ is lower semicontinuous, there is $x^* \in \bar{\Omega}$ such that

$$\inf_{x \in \bar{\Omega}} |v_{\bar{\lambda}}(x) - u(x)| \geq |v_{\bar{\lambda}}(x^*) - u(x^*)| > \varepsilon > 0.$$

Thus, since the application $\lambda \to v_\lambda$ is uniformly continuous respect to x, there exists $\delta > 0$ such that

$$u(x) < v_\kappa(x), \quad \text{for all } x \in \mathbb{R}^N \quad \text{and} \quad \kappa \in (\bar{\lambda} - \delta, \bar{\lambda} + \delta).$$

This proves that $\mathcal{A}$ is open and this finally leads us to confirm that $\mathcal{A} = \mathbb{R}$. $\square$

3. Proof of the main result

To prove the main result we use the Sweeping Principle for the fractional Laplacian operator (Proposition 2.3), and the following result from [2, Theorem 2.4, Remark 2.5].

Theorem 3.1. *Let $\tilde{f}$ be any Lipschitz function in $[-1,1]$ such that $\tilde{f}(-1)=\tilde{f}(1)=0$ and $\tilde{F}(t)<\tilde{F}(-1)=\tilde{F}(1)$ for all $t\in(-1,1)$, where $\tilde{F}(t)=\int_0^t \tilde{f}(\tau)d\tau$. Then there exists $\tilde{v}$ solution of*

$$(-\partial_{tt})^s\tilde{v}(t)=\tilde{f}(\tilde{v}(t)),\quad t\in\mathbb{R},$$

with $\tilde{v}'(t)>0$ for all $t\in\mathbb{R}$ and $\lim_{t\to\pm\infty}\tilde{v}(t)=\pm 1$.

Here, $(-\partial_{tt})^s$ is the one dimensional fractional Laplacian $(-\Delta)^s$. Next we prove Theorem 1.1. It should be noticed that, for this purpose, Theorem 3.1 will allow us to overcame the main difficulties in adapting the proof used in [10] to the fractional Laplacian operator framework.

Proof of Theorem 1.1. Obviously $u\equiv 0$ is solution to problem (1.1) since (1.7) implies that $f(0)=0$. Thus, arguing by contradiction, we suppose that there exists $u\in H_0^s(\Omega)$ being a nontrivial solution to (1.1). In this case, $v=-u$ satisfies the equation

$$\begin{gathered}(-\Delta)^s v=-f(-v)\quad \text{in } \Omega,\\ v=0\quad \text{in } \mathbb{R}^N\setminus\Omega,\end{gathered}$$

and the function $-f(-\tau)$ satisfies the same hypotheses of Theorem 1.1. Therefore, since the maximum value of either u or $-u$ is positive, without loss of generality, we may assume that

$$u_\infty:=\max_{x\in\bar{\Omega}}u(x)>0.$$

Since the value of $f(\tau)$ is irrelevant for $\tau>u_\infty$, we also assume that $\lim_{\tau\to\infty}f(\tau)=-\infty$ and that function f is globally Lipschitz, with Lipschitz constant $L>0$.

On the other hand, we claim that $f(u_\infty)>0$. Indeed, otherwise $f(u_\infty)\le 0$ and, since $(-\Delta)^s u_\infty=0$, we have the inequality

$$(-\Delta)^s u_\infty+Lu_\infty\ge f(u_\infty)+Lu_\infty,\quad \text{in }\Omega. \tag{3.1}$$

Moreover, since u solves (1.1), we also have that

$$(-\Delta)^s u+Lu=f(u)+Lu,\quad \text{in }\Omega. \tag{3.2}$$

Subtracting (3.2) from (3.1), and using that f is L-Lipschitz, we obtain

$$(-\Delta)^s(u_\infty-u)+L(u_\infty-u)\ge f(u_\infty)+Lu_\infty-f(u)-Lu\ge 0,\quad \text{in }\Omega.$$

Since $u_\infty-u>0$ in $\mathbb{R}^N\setminus\Omega$, we deduce, from the Strong Maximum Principle (Proposition 2.1), that $u_\infty>u(x)$ in Ω, which is a contradiction.

Therefore, as $f(u_\infty)>0$, we can assume that there are τ_1 and τ_2 positive constants such that $\tau_1<u_\infty<\tau_2$ and

$$f(\tau)>0,\quad \text{for all } \tau\in(\tau_1,\tau_2),\quad \text{and}\quad f(\tau_1)=0. \tag{3.3}$$

Even more, since $\lim_{\tau\to\infty}f(\tau)=-\infty$ and the value of $f(\tau)$ is irrelevant for $s>u_\infty$, we can modify the function f, being still L-Lipschitz, and choose τ_2 such that

$$f(\tau_2)=F(\tau_2)=0\quad \text{and}\quad f(\tau)<0,\quad \text{for } \tau>\tau_2. \tag{3.4}$$

Now, we set

$$\bar{\tau} = \max\{\tau \in \mathbb{R} : \tau < \tau_1,\, F(\tau) = 0\},$$

and we observe that $\bar{\tau} \geq 0$ and, since F satisfies (1.7), $f(\bar{\tau}) = 0$.

We define now $g(t) = \frac{\bar{\tau}+\tau_2}{2} + \frac{\tau_2-\bar{\tau}}{2}t$ and the auxiliary function

$$\tilde{f}(t) = \frac{2}{\tau_2 - \bar{\tau}} f\big(g(t)\big).$$

Note that $\tilde{f}(-1) = \tilde{f}(1) = 0$ and $\tilde{f}$ is a Lipschitz function in $[-1, 1]$. Even more, since $F(\tau_2) = F(\bar{\tau}) = 0$, it follows that $\tilde{F}(t) < \tilde{F}(-1) = \tilde{F}(1)$, for all $t \in (-1, 1)$. Then, by using Theorem 3.1, there exists a function $\tilde{v}$ which is solution to problem

$$(-\partial_{tt})^s \tilde{v}(t) = \tilde{f}(\tilde{v}(t)), \quad t \in \mathbb{R},$$

with $\tilde{v}' > 0$ and $\lim_{t\to\pm\infty} \tilde{v}(t) = \pm 1$.

Let us define

$$\tilde{w}(t) = g(\tilde{v}(t)), \quad t \in \mathbb{R},$$

which satisfies the equation

$$(-\partial_{tt})^s \tilde{w}(t) = f(\tilde{w}(t)), \quad t \in \mathbb{R}.$$

Furthermore, $\tilde{w}$ is increasing ($\tilde{w}' > 0$), since $\tilde{v}$ and g are also increasing functions. In addition, $\lim_{t\to-\infty} \tilde{w}(t) = \bar{\tau} \geq 0$ and $\lim_{t\to\infty} \tilde{w}(t) = \tau_2$. In particular, $\tilde{w}$ is uniformly continuous.

For every $\lambda \in \mathbb{R}$, consider the family of parametric functions

$$v_\lambda(x) = \tilde{w}(x_1 + \lambda) > 0, \quad \text{for all } x = (x_1, \dots, x_N) \in \mathbb{R}^N.$$

Clearly, the application $\lambda \to v_\lambda$ is continuous, since $\tilde{w}$ is uniformly continuous and

$$v_\lambda(x) \to \tau_2, \quad \text{as } \lambda \to \infty, \quad \text{for all } x \in \Omega. \tag{3.5}$$

Also, it satisfies (see Lemma 5.1)

$$\begin{gathered} (-\Delta)^s v_\lambda(x) = f(v_\lambda(x)) \quad \text{in } \Omega, \\ v_\lambda(x) > 0 \quad \text{in } \mathbb{R}^N \setminus \Omega, \end{gathered}$$

for every $\lambda \in \mathbb{R}$. Furthermore, $v_\lambda > u(x) = 0$ in $\mathbb{R}^N \setminus \Omega$ and, due to (3.5), there exists $\lambda_0 >> 0$ such that $u(x) \leq v_{\lambda_0}(x)$ in Ω (since $u_\infty < \tau_2$). Then, by using the Sweeping Principle (Proposition 2.3), $u(x) \leq v_\lambda(x)$ for every $\lambda \in \mathbb{R}$ and every $x \in \mathbb{R}^N$. In particular,

$$u(x) \leq \inf_{\lambda\in\mathbb{R}} v_\lambda(x) = \inf_{t\in\mathbb{R}} \tilde{w}(t) = \bar{\tau}, \quad \text{for all } x \in \mathbb{R}^N,$$

which contradicts $\bar{\tau} < \tau_1 < u_\infty$. □

4. Conclusion

In this section we summarize the main consequences of Theorem 1.1 through a series of corollaries.

On the one hand we are concerned with nonlinear eigenvalue problems

$$\begin{gathered} (-\Delta)^s u = \lambda u - g(u) \quad \text{in } \Omega, \\ u = 0 \quad \text{in } \mathbb{R}^N \setminus \Omega. \end{gathered} \tag{4.1}$$

Existing methods for proving nonexistence of nontrivial solutions for some values of the parameter λ requires to multiply by a convenient test function and integrate.

Theorem 1.1 provides an alternative by imposing conditions on λ to assure that (1.7) is satisfied with $f(t) = \lambda t - g(t)$.

Corollary 4.1. *Let $\Omega \subset \mathbb{R}^N$ ($N \geq 1$) be a $\mathcal{C}^{1,1}$ bounded domain, $g : \mathbb{R} \to \mathbb{R}$ a locally Lipschitz function and assume that*

$$\lambda \leq \frac{2\int_0^t g(\tau)d\tau}{t^2}, \quad \text{for all } t \in \mathbb{R}.$$

Then, $u \equiv 0$ is the unique solution to problem (4.1).

We include here some particular choice of functions $g(t)$ leading to a simpler condition on λ in the above result.

Corollary 4.2. *Let $\Omega \subset \mathbb{R}^N$ ($N \geq 1$) be a $\mathcal{C}^{1,1}$ bounded domain and $\alpha, \lambda \in \mathbb{R}$ with $\lambda \leq \min\{0, -\alpha\}$. Then, $u \equiv 0$ is the unique solution to problem*

$$\begin{gathered}
(-\Delta)^s u = \lambda u + \alpha \sin(u) \quad \text{in } \Omega, \\
u = 0 \quad \text{in } \mathbb{R}^N \setminus \Omega.
\end{gathered}$$

Corollary 4.3. *Let $\Omega \subset \mathbb{R}^N$ ($N \geq 1$) be a $\mathcal{C}^{1,1}$ bounded domain and $\lambda \in \mathbb{R}$ with $\lambda \leq 1$. Then $u \equiv 0$ is the unique solution to the problem*

$$\begin{gathered}
(-\Delta)^s u = \lambda u - 2ue^{u^2} \quad \text{in } \Omega, \\
u = 0 \quad \text{in } \mathbb{R}^N \setminus \Omega.
\end{gathered}$$

On the other hand we include some applications to obtain a priori bounds for positive solutions to (1.1). We observe that when $f(0) = 0$ then nonnegative solutions to problem (1.1) are solutions to

$$\begin{gathered}
(-\Delta)^s u = f(u^+) \quad \text{in } \Omega, \\
u = 0 \quad \text{in } \mathbb{R}^N \setminus \Omega,
\end{gathered}$$

where $u^+ = \max\{u, 0\}$. Thus, the following corollaries show how Theorem 1.1 also provides a priori estimates of positive solutions to (1.1) when (1.7) is satisfied in a positive interval.

Corollary 4.4. *Let $\Omega \subset \mathbb{R}^N$ ($N \geq 1$) be a $\mathcal{C}^{1,1}$ bounded domain, $f : \mathbb{R} \to \mathbb{R}$ a locally Lipschitz function with $f(0) = 0$ and*

$$F(t) = \int_0^t f(\tau)d\tau \leq 0, \quad \text{for all } t > 0.$$

Then, $u \equiv 0$ is the unique nonnegative solution to problem (1.1).

Corollary 4.5. *Let $\Omega \subset \mathbb{R}^N$ ($N \geq 1$) be a $\mathcal{C}^{1,1}$ bounded domain, $f : \mathbb{R} \to \mathbb{R}$ a locally Lipschitz function with $f(0) = 0$ and $t_0 > 0$ such that*

$$F(t) = \int_0^t f(\tau)d\tau \leq 0, \quad \text{for all } t \in (0, t_0),$$

and assume that there exists a positive solution u to problem (1.1). *Then*

$$\|u\|_\infty \geq t_0.$$

5. Appendix

Lemma 5.1. *Let $u \in H^s(\mathbb{R}^N)$ defined as $u(x) := v(x_1)$ for a certain function v, for all $x = (x_1, \dots, x_N) \in \mathbb{R}^N$. Then*

$$(-\Delta)^s u(x) = (-\partial_{x_1x_1})^s v(x_1).$$

Proof. By definition,

$$\begin{aligned}
&(-\Delta)^s u(x) \\
&= C(N,s)\,\mathrm{P.V.}\int_{\mathbb{R}^N} \frac{u(x)-u(y)}{|x-y|^{N+2s}}\,dy \\
&= C(N,s)\,\mathrm{P.V.}\int_{\mathbb{R}^N} \frac{v(x_1)-v(y_1)}{|x-y|^{N+2s}}\,dy \\
&= C(N,s)\,\mathrm{P.V.}\int_{\mathbb{R}} \Big(\frac{v(x_1)-v(y_1)}{|x_1-y_1|^{1+2s}} \int_{\mathbb{R}^{N-1}} \frac{|x_1-y_1|^{1+2s}}{|x-y|^{N+2s}} dy_N \cdots dy_2\Big)\,dy_1,
\end{aligned} \tag{5.1}$$

with $C(N,s)$ defined by (1.2). Now, relabeling the last integral expression as I_N and computing the integral

$$\begin{aligned}
&\int_{\mathbb{R}} \frac{|x_1-y_1|^{1+2s}}{|x-y|^{N+2s}} dy_N \\
&= \frac{\sqrt{\pi}\,\Gamma\big(\frac{N-1}{2}+s\big)}{\Gamma\left(\frac{N}{2}+s\right)} \cdot \frac{|x_1-y_1|^{1+2s}}{((x_1-y_1)^2+\cdots+(x_{N-1}-y_{N-1})^2)^{\frac{N-1}{2}+s}},
\end{aligned}$$

we obtain the recursive sequence

$$I_N = \frac{\sqrt{\pi}\,\Gamma\big(\frac{N-1}{2}+s\big)}{\Gamma\left(\frac{N}{2}+s\right)} I_{N-1}.$$

A simple computation leads us to the explicit expression

$$I_N = \frac{\pi^{\frac{N-1}{2}}\Gamma(\frac{1}{2}+s)}{\Gamma(\frac{N}{2}+s)} = \frac{C(1,s)}{C(N,s)}.$$

Hence, replacing in (5.1), we obtain that

$$(-\Delta)^s u(x) = C(1,s)\,\mathrm{P.V.}\int_{\mathbb{R}} \frac{v(x_1)-v(y_1)}{|x_1-y_1|^{1+2s}} dy_1 = (-\partial_{x_1x_1})^s v(x_1). \qquad \square$$

Acknowledgements. The authors were partially supported by Grant PID2021-122122NB-I00 funded by MCIN/AEI/10.13039/501100011033 by the "ERDF A way of making Europe", and by grant P18-FR-667 funded by Junta de Andalucía, Consejería de Transformación Económica, Industria, Conocimiento y Universidades Unión Europea. J. Carmona was supported by Junta de Andalucía FQM194 and CDTIME. A. Molino was supported by grant UAL2020-FQM-B2046 (UAL/ CTE-ICU/FEDER) and FQM-116.

References

[1] X. Cabré, Y. Sire; Nonlinear equations for fractional Laplacians, I: Regularity, maximum principles, and Hamiltonian estimates, *Ann. Inst. H. Poincaré Anal. Non Linéaire*, **31** (2014), No. 1, 23-53.

[2] X. Cabré, Y. Sire; Nonlinear equations for fractional Laplacians II: Existence, uniqueness, and qualitative properties of solutions, *Trans. Amer. Math. Soc.*, **367** (2015), No. 2, 911-941.

[3] E. Di Nezza, G. Palatucci, E. Valdinoci; Hitchhiker's guide to the fractional Sobolev spaces, *Bull. Sci. Math.*, **136** (2012), No. 5, 521-573.
[4] X. Fan; A remark on Ricceri's conjecture for a class of nonlinear eigenvalue problems, *Journal of Mathematical Analysis and Applications*, **349** (2009), No. 2, 436-442.
[5] X. Fan; On Ricceri's conjecture for a class of nonlinear eigenvalue problems, *Appl. Math. Lett.*, **22** (2009), No. 9, 1386-1389.
[6] X. Fan, B. Ricceri; On the Dirichlet problem involving non-linearities with non-positive primitive: a problem and a remark, *Applicable Analysis*, **89** (2010), No. 2, 189-192.
[7] O. Goubet; Remarks on some dissipative sine-gordon equations, *Complex Variables and Elliptic Equations*, **65** (2020), No. 8, 1336–1342.
[8] O. Goubet, B. Ricceri; Non-existence results for an eigenvalue problem involving Lipschitzian non-linearities with non-positive primitive, *Bull. Lond. Math. Soc.*, **51** (2019), No. 3, 531-538.
[9] J. L. Kazdan, F. W. Warner; Remarks on some quasilinear elliptic equations, *Comm. Pure Appl. Math.*, **28** (1975), No. 5, 567-597.
[10] S. López-Martínez, A. Molino; Nonexistence result of nontrivial solutions to the equation $-\Delta u = f(u)$, *Complex Variables and Elliptic Equations*, **67** (2022), No. 1, 239-245.
[11] A. McNabb; Strong comparison theorems for elliptic equations of second order, *J. Math. Mech.*, **10** (1961), 431–440.
[12] S. I. Pohozaev; On the eigenfunctions of the equation $\Delta u + \lambda f(u) = 0$, *Dokl. Akad. Nauk SSSR*, **165** (1965), 1408-1411.
[13] P. Pucci, J. Serrin; *The maximum principle*, volume 73 of *Progress in Nonlinear Differential Equations and their Applications.* Birkhäuser Verlag, Basel, 2007. ISBN 978-3-7643-8144-8.
[14] B. Ricceri; A remark on a class of nonlinear eigenvalue problems, *Nonlinear Analysis: Theory, Methods & Applications*, **69** (2008), No. 9, 2964- 2967.
[15] B. Ricceri; *Four Conjectures in Nonlinear Analysis*, pages 681–710. Springer International Publishing, Cham, 2018. ISBN 978-3-319-89815-5.
[16] X. Ros-Oton; Nonlocal elliptic equations in bounded domains: a survey, *Publ. Mat.*, **60** (2016), No. 1, 3-26.
[17] X. Ros-Oton, J. Serra; The Dirichlet problem for the fractional Laplacian: regularity up to the boundary, *J. Math. Pures Appl.*, **101** (2014), No. 3, 275-302.
[18] X. Ros-Oton, J. Serra; The Pohozaev identity for the fractional Laplacian, *Arch. Ration. Mech. Anal.*, **213** (2014), No. 2, 587-628.
[19] D. H. Sattinger; *Topics in stability and bifurcation theory.* Lecture Notes in Mathematics, Vol. 309. Springer-Verlag, Berlin-New York, 1973.

José Carmona
Departamento de Matemáticas, Universidad de Almería, Facultad de Ciencias Experimentales, Ctra. de Sacramento sn. 04120 La Cañada de San Urbano. Almería, Spain
Email address: jcarmona@ual.es

Alexis Molino
Departamento de Matemáticas, Universidad de Almería, Facultad de Ciencias Experimentales, Ctra. de Sacramento sn. 04120 La Cañada de San Urbano. Almería, Spain
Email address: amolino@ual.es