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NULL CONTROLLABILITY OF COUPLED SYSTEMS OF DEGENERATE PARABOLIC INTEGRO-DIFFERENTIAL EQUATIONS

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Abstract. This article concerns the null controllability of a coupled system of two degenerate parabolic integro-differential equations with one locally distributed control force. Since the memory terms do not allow applying the standards Carleman estimates directly, we start by proving a null controllability result for an associated nonhomogeneous degenerate coupled system employing new Carleman estimates with appropriate weight functions. As a consequence, we deduce the null controllability result for the initial memory system by using the Kakutani's fixed point Theorem.

1. Introduction

This article studies the null controllability of a coupled system of two degenerate parabolic equations involving memory terms, by means of a single distributed control force. More precisely, we consider the system

$$\begin{gathered} y_{1t} - (a(x)y_{1x})_x + b_{11}y_1 + b_{12}y_2 = H_1(t,y_1) + 1_\omega u, \quad (t,x) \in Q, \\ y_{2t} - (a(x)y_{2x})_x + b_{21}y_1 + b_{22}y_2 = H_2(t,y_2), \quad (t,x) \in Q, \\ y_1(t,1) = y_2(t,1) = 0, \quad t \in (0,T), \\ \begin{cases} y_1(t,0) = y_2(t,0) = 0, & \text{if } a \text{ is weakly degenerate,} \\ (ay_{1x})(t,0) = (ay_{2x})(t,0) = 0, & \text{if } a \text{ is strongly degenerate,} \end{cases} \quad t \in (0,T), \\ y_1(0,x) = y_1^0(x), \quad y_2(0,x) = y_2^0(x), \quad x \in (0,1), \end{gathered} \tag{1.1}$$

where $Q = (0,T)\times(0,1)$, $\omega \Subset (0,1)$ is a non-empty open set, 1_ω is the corresponding characteristic function, $b_{ij} := b_{ij}(t,x) \in L^\infty(Q)$ and $u = u(t,x)$ is the distributed control function. By $H_k(t,y_k)$ we denote the following quantity

$$H_k(t,y_k) = \int_0^t h_k(t,r,x)y_k(r,x)\,dr, \quad k = 1,2, \tag{1.2}$$

where $h_k = h_k(t,r,x) \in L^\infty((0,T)\times Q)$, $k = 1,2$, are memory kernels. Moreover, the diffusion coefficient a degenerates at $x = 0$ and we say that

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- a is weakly degenerate (WD) if $a \in C[0,1] \cap C^1(0,1]$ is such that $a(0)=0$, $a>0$ on $(0,1]$ and there exists $\alpha \in [0,1)$, such that $xa'(x) \le \alpha a(x)$ for all $x \in [0,1]$.
- a is strongly degenerate (SD) if $a \in C^1[0,1]$ is such that $a(0)=0$, $a>0$ on $(0,1]$ and there exists $\alpha \in [1,2)$, such that $xa'(x) \le \alpha a(x)$ for all $x \in [0,1]$; moreover,

$$\exists\, \beta \in (1,\alpha],\ x \mapsto \frac{a(x)}{x^\beta} \text{ is nondecreasing near 0, if } \alpha > 1,$$
$$\exists\, \beta \in (0,1),\ x \mapsto \frac{a(x)}{x^\beta} \text{ is nondecreasing near 0, if } \alpha = 1.$$

The study of controllability properties for (1.1) is motivated by numerous real world applications. Indeed, degenerate partial differential equations play a major role in modeling many processes coming from physics, biology and finance. However, in several complex problems, the history of the phenomena under investigation is of relevance and must be incorporated in the mathematical model. As it is by now classical, standard PDEs models cannot provide a good description of such processes. For this reason, PDEs have been replaced by partial integro-differential equations that take into account this memory effect, and that have been largely investigated in previous decades.

Up to now, the controllability of degenerate parabolic equations with distributed controls has been largely developed in several recent papers, see [2, 7, 8, 11] and the references therein. Moreover, in the last recent years an increasing interest has been devoted to the study of controllability properties for parabolic equations involving memory terms, see [9, 15, 16, 19, 20, 21]. But, very little is known for the controllability analysis of parabolic equations that couple a degenerate diffusion coefficient with a nonlocal reaction term. We refer to [4, 6, 22] for some related results. See also [5] for a similar work on this theme.

In this work, we aim to extend those known results to coupled systems of kind (1.1). More precisely, we seek for suitable conditions on the kernels h_1 and h_2 so that the coupled system (1.1) is null controllable, that is to say, for any initial data (y_1^0, y_2^0), there exists a control function u such that the associated solution to (1.1) vanishes at the end of the time horizon $[0,T]$. To our knowledge, this is the first paper dealing with a coupled system of degenerate parabolic equations in presence of memory terms.

The starting point for proving the null controllability for the integro-differential system (1.1) is to show the null controllability for the nonhomogeneous degenerate parabolic system without memory

$$\begin{gathered}
y_{1t} - (a(x)y_{1x})_x + b_{11}y_1 + b_{12}y_2 = F_1 + 1_\omega u, \quad (t,x) \in Q,\\
y_{2t} - (a(x)y_{2x})_x + b_{21}y_1 + b_{22}y_2 = F_2, \quad (t,x) \in Q,\\
y_1(t,1) = y_2(t,1) = 0, \quad t \in (0,T),\\
\begin{cases} y_1(t,0) = y_2(t,0) = 0, & (WD),\\ (ay_{1x})(t,0) = (ay_{2x})(t,0) = 0, & (SD), \end{cases} \quad t \in (0,T),\\
y_1(0,x) = y_1^0(x), \quad y_2(0,x) = y_2^0(x), \quad x \in (0,1)
\end{gathered} \tag{1.3}$$

for arbitrary functions $F_1, F_2 \in L^2(Q)$.

The proof of this result relies on a new modified Carleman inequality for the associated adjoint problem with some weight functions that blow up as $t \to T$. The new Carleman inequality is the key point to derive the null controllability result for an intermediate problem similar to the integro-differential system (1.1). At the end, we deduce the desired controllability result for the original problem using a classical fixed point argument.

This article is organized in the following way: in Section 2, we first consider the nonhomogeneous degenerate system (1.3) studying its well posedness, the Carleman estimates for the associated adjoint problem and, finally, its null controllability. As a consequence, in Section 3, by means of Kakutani's fixed point Theorem, we prove that system (1.1) is null contrallable under a decaying condition on the kernels h_1 and h_2 only at $t = T$. In the last section, we show the same controllability result for kernels vanishing in a neighborhood of the initial time.

2. Null controllability of a nonhomogeneous degenerate system

As stated in the introduction, we first study system (1.3).

2.1. **Well-posedness.**

To study the well-posedness of the degenerate system (1.3), we first recall the following weighted Sobolev spaces (in the sequel, a.c. means absolutely continuous):

In the (WD) case we use

$$H_a^1(0,1) := \big\{ y \in L^2(0,1) : y \text{ a.c. in } [0,1],\ \sqrt{a} y_x \in L^2(0,1) \text{ and } y(1) = y(0) = 0 \big\},$$
$$H_a^2(0,1) := \big\{ y \in H_a^1(0,1) : a y_x \in H^1(0,1) \big\}.$$

In the (SD) case we use

$$\begin{aligned} H_a^1(0,1) &:= \big\{ y \in L^2(0,1) : y \quad \text{locally a.c. in } (0,1],\ \sqrt{a} y_x \in L^2(0,1) \text{ and } y(1) = 0 \big\}, \\ H_a^2(0,1) &:= \big\{ y \in H_a^1(0,1) : a y_x \in H^1(0,1) \big\} \\ &= \Big\{ y \in L^2(0,1) : y \text{ locally a.c. in } (0,1], ay \in H_0^1(0,1), \\ &\qquad a y_x \in H^1(0,1) \text{ and } (a y_x)(0) = 0 \Big\}. \end{aligned}$$

In both cases, the norms are defined as

$$\|y\|_{H_a^1}^2 := \|y\|_{L^2(0,1)}^2 + \|\sqrt{a} y_x\|_{L^2(0,1)}^2, \quad \|y\|_{H_a^2}^2 := \|y\|_{H_a^1}^2 + \|(a y_x)_x\|_{L^2(0,1)}^2.$$

Now we recall a well-posedness result for system (1.3) (see, for instance, [1]).

Proposition 2.1. *Assume that $(y_1^0, y_2^0) \in L^2(0,1)^2$, $(F_1, F_2) \in L^2(Q)^2$, and $u \in L^2(Q)$. Then, system (1.3) admits a unique weak solution*

$$(y_1, y_2) \in W_T := L^2(0,T; H_a^1(0,1)^2) \cap C([0,T]; L^2(0,1)^2) \tag{2.1}$$

such that

$$\begin{aligned} &\|(y_1, y_2)\|_{L^2(0,T;H_a^1(0,1)^2)} + \sup_{t\in[0,T]} \|(y_1(t), y_2(t))\|_{L^2(0,1)^2} \\ &\le C\Big(\|(y_1^0, y_2^0)\|_{L^2(0,1)^2} + \|(F_1, F_2)\|_{L^2(Q)^2} + \|1_\omega u\|_{L^2(Q)^2} \Big), \end{aligned} \tag{2.2}$$

for some positive constant C. Moreover, if $(y_1^0, y_2^0) \in H_a^1(0,1)^2$, then

$$(y_1, y_2) \in Z_T := L^2(0,T; H_a^2(0,1)^2) \cap H^1(0,T; L^2(0,1)^2)$$

and

$$\begin{aligned} &\|(y_1,y_2)\|_{L^2(0,T;H_a^2(0,1)^2)} + \|(y_1,y_2)\|_{H^1(0,T;L^2(0,1)^2)} \\ &\le C\Big(\|(y_1^0,y_2^0)\|_{H_a^1(0,1)^2} + \|(F_1,F_2)\|_{L^2(Q)^2} + \|1_\omega u\|_{L^2(Q)^2}\Big), \end{aligned} \tag{2.3}$$

for some positive constant C.

2.2. **Carleman estimates.** In this subsection, we establish a Carleman type estimate for the nonhomogeneous adjoint system

$$\begin{gathered} -v_{1t} - (a(x)v_{1x})_x + b_{11}v_1 + b_{21}v_2 = g_1, \quad (t,x)\in Q, \\ -v_{2t} - (a(x)v_{2x})_x + b_{12}v_1 + b_{22}v_2 = g_2, \quad (t,x)\in Q, \\ v_1(t,1) = v_2(t,1) = 0, \quad t\in(0,T), \\ \begin{cases} v_1(t,0) = v_2(t,0) = 0, & (WD), \\ (av_{1x})(t,0) = (av_{2x})(t,0) = 0, & (SD), \end{cases} \quad t\in(0,T), \\ v_1(T,x) = v_1^T(x), \quad v_2(T,x) = v_2^T(x), \quad x\in(0,1), \end{gathered} \tag{2.4}$$

where $v_1^T, v_2^T \in L^2(0,1)$ and $g_1, g_2 \in L^2(Q)$.

To develop a Carleman estimate for (2.4), some suitable weight functions are needed. As in [2], we introduce the weight functions

$$\psi(x) := \gamma\Big(\int_0^x \frac{y}{a(y)}\,dy - d\Big), \quad \theta(t) := \frac{1}{\big(t(T-t)\big)^4}, \quad \varphi(t,x) := \theta(t)\psi(x). \tag{2.5}$$

Now, let $\tilde\omega$ be an arbitrary open subset of ω and $\rho \in C^2([0,1])$ be such that

$$\rho > 0, \text{ in } (0,1), \quad \rho(0) = \rho(1) = 0, \quad \text{and} \quad \rho_x \neq 0, \text{ in } [0,1]\backslash\tilde\omega,$$

and define

$$\Psi(x) := e^{\lambda\rho(x)} - e^{2\lambda\|\rho\|_\infty}, \quad \Phi(t,x) := \theta(t)\Psi(x). \tag{2.6}$$

We also define

$$\sigma := 4\Phi - 3\varphi \quad \text{and} \quad \sigma_1 := 2\Phi - \varphi. \tag{2.7}$$

By taking the parameters λ, d such that

$$d > 4d^\star := 4\int_0^1 \frac{y}{a(y)}\,dy \quad \text{and} \quad \lambda > \frac{1}{\|\rho\|_\infty}\ln\Big(\frac{4(d-d^*)}{d-4d^*}\Big), \tag{2.8}$$

one can show that the interval $\big(\frac{e^{2\lambda\|\rho\|_\infty}}{d-d^\star}, \frac{4(e^{2\lambda\|\rho\|_\infty} - e^{\lambda\|\rho\|_\infty})}{3d}\big)$ is nonempty. This permits to choose the constant γ (see (2.5)) in such a way that

$$\frac{e^{2\lambda\|\rho\|_\infty}}{d-d^*} < \gamma < \frac{4\big(e^{2\lambda\|\rho\|_\infty} - e^{\lambda\|\rho\|_\infty}\big)}{3d}. \tag{2.9}$$

With this choice of the parameters d, λ and γ one can readily show that the above weight functions satisfy the following inequalities which will play a crucial role in the sequel.

Lemma 2.2. (1) *$\max_{x\in[0,1]}\psi(x) \le \min_{x\in[0,1]}\Psi(x)$;*
(2) *$\frac{4}{3}\max_{x\in[0,1]}\Psi(x) \le \min_{x\in[0,1]}\psi(x)$;*
(3) *$\frac{4}{3}\Phi(t,x) \le \varphi(t,x) \le \Phi(t,x)$, for all $(t,x)\in Q$;*
(4) *$\varphi(t,x) \le \Phi(t,x) \le \sigma_1(t,x) \le \sigma(t,x) < 0$, for all $(t,x)\in Q$.*

From the definition of the function θ, we observe that

$$|\theta'(t)| \le C\theta^{3/2}(t),\ \forall\, t\in[0,T], \quad \text{and} \quad \theta(t)\to+\infty \text{ as } t\to 0^-, T^+. \tag{2.10}$$

Then, the next Carleman estimate holds (see [3, Theorem 3.3]).

Theorem 2.3. *Assume that a is (WD) or (SD) and let $T>0$. Then, there exist two positive constants C and s_0, such that the solution $(v_1,v_2)\in Z_T$ of (2.4) satisfies*

$$\begin{aligned} &\iint_Q \big(s\theta a(x)(v_{1x}^2+v_{2x}^2)+s^3\theta^3\frac{x^2}{a(x)}(v_1^2+v_2^2)\big)e^{2s\varphi}\,dt\,dx \\ &\le C\Big(\iint_Q (g_1^2+g_2^2)e^{2s\Phi}\,dt\,dx+\iint_{Q_\omega} s^3\theta^3(v_1^2+v_2^2)e^{2s\Phi}\,dtdx\Big), \end{aligned} \tag{2.11}$$

for all $s\ge s_0$. Here $Q_\omega=(0,T)\times\omega$.

To obtain the controllability for the degenerate nonlocal system (1.1) with only one control force, we need to show the following Carleman estimate with a single locally distributed observation.

Theorem 2.4. *Assume that a is (WD) or (SD) and let $T>0$. Suppose that for some open subset $\widehat{\omega}\Subset\omega$*

$$b_{21}\ge b_0>0, \quad \text{in } (0,T)\times\widehat{\omega}. \tag{2.12}$$

Then, there exist two positive constants C and s_0, such that the solution $(v_1,v_2)\in Z_T$ of (2.4) satisfies

$$\begin{aligned} &\iint_Q \Big(s\theta a(x)(v_{1x}^2+v_{2x}^2)+s^3\theta^3\frac{x^2}{a(x)}(v_1^2+v_2^2)\Big)e^{2s\varphi}\,dt\,dx \\ &\le C\Big(\iint_Q s^3\theta^3(g_1^2+g_2^2)e^{2s\sigma_1}\,dt\,dx+\iint_{Q_\omega} s^7\theta^7 v_1^2 e^{2s\sigma}\,dt\,dx\Big), \end{aligned} \tag{2.13}$$

for all $s\ge s_0$.

Proof. Let us consider a nonnegative smooth cut-off function $\zeta\in C^\infty([0,1])$ such that

$$0\le\zeta(x)\le 1, \quad \zeta(x)=\begin{cases}1, & x\in\widehat{\omega},\\ 0, & x\in(0,1)\setminus\omega.\end{cases} \tag{2.14}$$

Multiplying the first equation in (2.4) by $s^3\theta^3\zeta e^{2s\Phi}v_2$ and integrating on Q, we have

$$\begin{aligned} \iint_Q \zeta b_{21}s^3\theta^3e^{2s\Phi}v_2^2\,dt\,dx &= \iint_Q \zeta s^3\theta^3e^{2s\Phi}\left(v_2\left(av_{1x}\right)_x+v_2v_{1t}\right)\,dt\,dx \\ &\quad -\iint_Q \zeta b_{11}s^3\theta^3e^{2s\Phi}v_2v_1\,dt\,dx \\ &\quad +\iint_Q \zeta s^3\theta^3e^{2s\Phi}v_2g_1\,dt\,dx. \end{aligned} \tag{2.15}$$

Integrating by parts and using the second equation of (2.4), we obtain

$$\begin{aligned}\iint_Q \zeta s^3\theta^3 e^{2s\Phi} v_2 \left(a v_{1x}\right)_x \,dt\,dx = &-\iint_Q \zeta a s^3\theta^3 e^{2s\Phi} v_{1x} v_{2x}\,dt\,dx \\ &+\iint_Q s^3\theta^3 a\left(\zeta e^{2s\Phi}\right)_x v_1 v_{2x}\,dt\,dx \\ &+\iint_Q s^3\theta^3 \left(a\left(\zeta e^{2s\Phi}\right)_x\right)_x v_1 v_2\,dt\,dx\end{aligned} \tag{2.16}$$

and

$$\begin{aligned}&\iint_Q \zeta s^3\theta^3 e^{2s\Phi} v_2 v_{1t}\,dt\,dx \\ &= -\iint_Q \zeta a s^3\theta^3 e^{2s\Phi} v_{1x} v_{2x}\,dt\,dx - \iint_Q \zeta b_{12} s^3\theta^3 e^{2s\Phi} v_1^2\,dt\,dx \\ &\quad -\iint_Q a s^3\theta^3 \left(\zeta e^{2s\Phi}\right)_x v_1 v_{2x}\,dt\,dx - \iint_Q \zeta b_{22} s^3\theta^3 e^{2s\Phi} v_1 v_2\,dt\,dx \\ &\quad -\iint_Q \zeta s^3 \left(\theta^3 e^{2s\Phi}\right)_t v_1 v_2\,dt\,dx + \iint_Q \zeta s^3\theta^3 e^{2s\Phi} v_1 g_2\,dt\,dx.\end{aligned} \tag{2.17}$$

Combining the identities (2.15)-(2.17), it follows that

$$\begin{aligned}&\iint_Q \zeta b_{21} s^3\theta^3 e^{2s\Phi} v_2^2\,dt\,dx \\ &= -\overbrace{2\iint_Q \zeta a s^3\theta^3 e^{2s\Phi} v_{1x} v_{2x}\,dt\,dx}^{I_1} - \overbrace{\iint_Q \zeta b_{12} s^3\theta^3 e^{2s\Phi} v_1^2\,dt\,dx}^{I_2} \\ &\quad + \overbrace{\iint_Q \Big(s^3\theta^3\left(a\left(\zeta e^{2s\Phi}\right)_x\right)_x - \zeta(b_{11}+b_{22}) s^3\theta^3 e^{2s\Phi} - \zeta s^3\left(\theta^3 e^{2s\Phi}\right)_t\Big) v_2 v_1\,dt\,dx}^{I_3} \\ &\quad + \overbrace{\iint_Q \zeta s^3\theta^3 e^{2s\Phi} v_1 g_2\,dt\,dx}^{I_4} + \overbrace{\iint_Q \zeta s^3\theta^3 e^{2s\Phi} v_2 g_1\,dt\,dx}^{I_5}.\end{aligned} \tag{2.18}$$

Now, we estimate the integrals I_1, I_2, I_3, I_4 and I_5. Applying the Young's inequality, one has

$$\begin{aligned}|I_1| &= |2\iint_Q \zeta a s^3\theta^3 e^{2s\Phi} v_{1x} v_{2x}\,dt\,dx| \\ &= |2\iint_Q \left(s^{1/2}\theta^{1/2} a^{1/2} e^{s\varphi} v_{2x}\right)\left(s^{\frac{5}{2}}\theta^{\frac{5}{2}}\zeta a^{1/2} e^{s(2\Phi-\varphi)} v_{1x}\right)\,dt\,dx| \\ &\le \varepsilon \iint_Q s\theta a e^{2s\varphi} v_{2x}^2\,dt\,dx + \frac{1}{\varepsilon}\overbrace{\iint_Q s^5\theta^5\zeta^2 a e^{2s(2\Phi-\varphi)} v_{1x}^2\,dt\,dx}^{J}\end{aligned} \tag{2.19}$$

for every $\varepsilon > 0$.

The term J should be estimated by an integral of v_1^2. For this, we multiply the first equation in (2.4) by $s^5\theta^5\zeta^2 e^{2s(2\Phi-\varphi)}v_1$ and we integrate by parts to obtain

$$\begin{aligned} J = &-\overbrace{\frac{1}{2}\iint_Q s^5\zeta^2\big(\theta^5 e^{2s(2\Phi-\varphi)}\big)_t v_1^2\,dt\,dx}^{J_1} \\ &+\overbrace{\frac{1}{2}\iint_Q s^5\theta^5\,\Big(a\big(\zeta^2 e^{2s(2\Phi-\varphi)}\big)_x\Big)_x v_1^2\,dt\,dx}^{J_2} - \overbrace{\iint_Q \zeta^2 b_{11}s^5\theta^5 e^{2s(2\Phi-\varphi)}v_1^2\,dt\,dx}^{J_3} \\ &-\overbrace{\iint_Q \zeta^2 b_{21}s^5\theta^5 e^{2s(2\Phi-\varphi)}v_1 v_2\,dt\,dx}^{J_4} + \overbrace{\iint_Q \zeta^2 s^5\theta^5 e^{2s(2\Phi-\varphi)}g_1 v_1\,dt\,dx}^{J_5}. \end{aligned} \tag{2.20}$$

Since $|\dot\theta| \le C\theta^2$ and $\operatorname{supp}\zeta \Subset \omega$, we obtain

$$|J_k| \le C\iint_{Q_\omega} s^7\theta^7 e^{2s(2\Phi-\varphi)}v_1^2\,dt\,dx, \quad k\in\{1,2,3\}.$$

Moreover, using the Young's inequality, the boundedness of a/x^2 in ω and again the fact that $\operatorname{supp}\zeta \Subset \omega$, the term J_4 can be estimated in the following way

$$\begin{aligned} |J_4| &= \Big|\iint_Q \Big(s^{3/2}\theta^{3/2}\big(\frac{x^2}{a}\big)^{1/2}e^{s\varphi}v_2\Big)\Big(s^{\frac72}\theta^{\frac72}b_{21}\zeta^2\,\Big(\frac{a}{x^2}\Big)^{1/2}\,e^{s(4\Phi-3\varphi)}v_1\Big)\,dt\,dx\Big| \\ &\le \varepsilon^2\iint_Q s^3\theta^3\frac{x^2}{a}e^{2s\varphi}v_2^2\,dt\,dx + C_\varepsilon\iint_{Q_\omega} s^7\theta^7 e^{2s(4\Phi-3\varphi)}v_1^2\,dt\,dx. \end{aligned}$$

Similarly,

$$|J_5| \le C\iint_{Q_\omega} s^3\theta^3 e^{2s(2\Phi-\varphi)}g_1^2\,dt\,dx + C\iint_{Q_\omega} s^7\theta^7 e^{2s(2\Phi-\varphi)}v_1^2\,dt\,dx.$$

On the other hand, thanks to Lemma 2.2, one can check that

$$2\Phi-\varphi \le 4\Phi-3\varphi. \tag{2.21}$$

Hence,

$$\begin{aligned} |J| \le{}& \varepsilon^2\iint_Q s^3\theta^3\frac{x^2}{a}e^{2s\varphi}v_2^2\,dt\,dx + C_\varepsilon\iint_{Q_\omega} s^7\theta^7 e^{2s(4\Phi-3\varphi)}v_1^2\,dt\,dx \\ &+ C\iint_{Q_\omega} s^3\theta^3 e^{2s(2\Phi-\varphi)}g_1^2\,dt\,dx. \end{aligned} \tag{2.22}$$

Putting together inequalities (2.19) and (2.22), we obtain

$$\begin{aligned} |I_1| \le{}& \varepsilon\iint_Q s\theta a e^{2s\varphi}v_{2x}^2\,dt\,dx + \varepsilon\iint_Q s^3\theta^3\frac{x^2}{a}e^{2s\varphi}v_2^2\,dt\,dx \\ &+ C_\varepsilon\iint_{Q_\omega} s^7\theta^7 e^{2s(4\Phi-3\varphi)}v_1^2\,dt\,dx + C\iint_{Q_\omega} s^3\theta^3 e^{2s(2\Phi-\varphi)}g_1^2\,dt\,dx. \end{aligned} \tag{2.23}$$

In view of Lemma 2.2, we also have

$$|I_2| \le C\iint_{Q_\omega} s^3\theta^3 e^{2s\Phi}v_1^2\,dt\,dx \le C\iint_{Q_\omega} s^3\theta^3 e^{2s(4\Phi-3\varphi)}v_1^2\,dt\,dx. \tag{2.24}$$

Proceeding as before, we obtain

$$\begin{aligned}|I_3| &\le C\iint_{Q_\omega} s^5\theta^5 e^{2s\Phi} v_1 v_2\,dt\,dx\\ &\le \varepsilon\iint_Q s^3\theta^3\frac{x^2}{a}e^{2s\varphi}v_2^2\,dt\,dx + C_\varepsilon\iint_{Q_\omega} s^7\theta^7 e^{2s(2\Phi-\varphi)}v_1^2\,dt\,dx\end{aligned} \tag{2.25}$$

and

$$|I_5| \le \varepsilon\iint_Q s^3\theta^3\frac{x^2}{a}e^{2s\varphi}v_2^2\,dt\,dx + C_\varepsilon\iint_{Q_\omega} s^3\theta^3 e^{2s(2\Phi-\varphi)}g_1^2\,dt\,dx. \tag{2.26}$$

Finally, using once again the Young's inequality and Lemma 2.2, it follows that

$$\begin{aligned}|I_4| &\le C\iint_{Q_\omega} s^3\theta^3 e^{2s\Phi}v_1^2\,dt\,dx + C\iint_{Q_\omega} s^3\theta^3 e^{2s\Phi}g_2^2\,dt\,dx\\ &\le C\iint_{Q_\omega} s^3\theta^3 e^{2s(4\Phi-3\varphi)}v_1^2\,dt\,dx + C\iint_{Q_\omega} s^3\theta^3 e^{2s\Phi}g_2^2\,dt\,dx.\end{aligned} \tag{2.27}$$

Combining the estimates (2.18), (2.23)-(2.27) together with (2.12) and (2.21), we obtain

$$\begin{aligned}&b_0\iint_{Q_\omega} s^3\theta^3 e^{2s\Phi}v_2^2\,dt\,dx\\ &\le \iint_Q \zeta b_{21} s^3\theta^3 e^{2s\Phi}v_2^2\,dt\,dx\\ &\le 3\varepsilon\Big(\iint_Q s\theta a e^{2s\varphi}v_{2x}^2\,dt\,dx + \iint_Q s^3\theta^3\frac{x^2}{a}e^{2s\varphi}v_2^2\,dt\,dx\Big)\\ &+ C_\varepsilon\iint_{Q_\omega} s^7\theta^7 e^{2s(4\Phi-3\varphi)}v_1^2\,dt\,dx + C_\varepsilon\iint_{Q_\omega} s^3\theta^3 e^{2s(2\Phi-\varphi)}(g_1^2+g_2^2)\,dt\,dx.\end{aligned}$$

Hence, using the Carleman estimate (2.11) together with the previous inequality with $\varepsilon = \frac{b_0}{6C}$, where C is the positive constant in (2.11), we readily deduce the desired result. □

Next, using (2.13), we are going to establish a new Carleman inequality with a modified weight time function that blows up only as $t \to T$. This will give the null controllability result for system (1.1) imposing a decaying condition on the kernels h_1 and h_2 only at $t = T$. Thus, as in [12], we introduce the weight function

$$\beta(t) := \begin{cases}\theta(\frac{T}{2}) = \big(\frac{2}{T}\big)^8, & \text{for } t \in [0, \frac{T}{2}],\\ \theta(t), & \text{for } t \in [\frac{T}{2}, T],\end{cases}$$

and the associated weight functions

$$\begin{gathered}\tilde\varphi(t,x) = \beta(t)\psi(x), \quad \tilde\Phi(t,x) := \beta(t)\Psi(x),\\ \tilde\sigma = 4\tilde\Phi - 3\tilde\varphi, \quad \tilde\sigma_1 = 2\tilde\Phi - \tilde\varphi.\end{gathered} \tag{2.28}$$

In what follows we will use the notation

$$\begin{gathered}\widehat\Phi(t) := \max_{x\in[0,1]}\Phi(t,x), \quad \widehat\varphi(t) := \max_{x\in[0,1]}\varphi(t,x) = \gamma(d^*-d)\beta(t),\\ \varphi^*(t) := \min_{x\in[0,1]}\varphi(t,x) = -\gamma d\beta(t), \quad \Phi^*(t) := \min_{x\in[0,1]}\Phi(t,x).\end{gathered} \tag{2.29}$$

Using Lemma 2.2, one can easily check that the next inequalities hold.

Lemma 2.5. (1) $\frac{4}{3}\widehat{\Phi}(t) \le \varphi^*(t)$ *and* $\widehat{\varphi}(t) \le \Phi^*(t)$, *for all* $t \in (0,T)$;
(2) $\frac{4}{3}\tilde{\Phi} \le \tilde{\varphi} \le \tilde{\Phi}$ *in* Q;
(3) $\tilde{\varphi} \le \tilde{\Phi} \le \tilde{\sigma}_1 \le \tilde{\sigma} < 0$, *in* Q.

Now, we are ready to state our main modified Carleman inequality.

Lemma 2.6. *Assume that the conditions of Theorem 2.4 hold and let* $T^* \in (\frac{T}{2}, T)$. *Then, there exist two positive constants* C *and* s_0 *such that every solution* $(v_1, v_2) \in Z_T$ *of system* (2.4) *satisfies*

$$
\begin{aligned}
&e^{2s\widehat{\varphi}(0)} \int_0^1 \left(v_1^2(0) + v_2^2(0)\right)\, dx + \iint_Q (v_1^2 + v_2^2) e^{2s\varphi}\, dt\, dx \\
&\le C e^{2s[\widehat{\varphi}(0) - \varphi^*(T^*)]} \Big(\iint_Q s^3\theta^3 (g_1^2 + g_2^2) e^{2s\sigma_1}\, dt\, dx \\
&\quad + \iint_{Q_\omega} s^7\theta^7 v_1^2 e^{2s\sigma}\, dt\, dx \Big),
\end{aligned} \tag{2.30}
$$

for all $s \ge s_0$.

Proof. Let us first prove that

$$
\begin{aligned}
&\int_{\frac{T}{2}}^T \int_0^1 (v_1^2 + v_2^2) e^{2s\tilde{\varphi}}\, dt\, dx \\
&\le C \Big(\iint_Q s^3\theta^3 (g_1^2 + g_2^2) e^{2s\sigma_1}\, dt\, dx + \iint_{Q_\omega} s^7\theta^7 v_1^2 e^{2s\sigma}\, dt\, dx \Big),
\end{aligned} \tag{2.31}
$$

for some positive constant C.

Using the monotonicity of $\frac{x^2}{a(x)}$ and the Hardy-Poincaré inequality given in [2, Proposition 2.1], we have

$$
\int_0^1 v_1^2 e^{2s\varphi}\, dx \le \frac{1}{a(1)} \int_0^1 \frac{a(x)}{x^2} (v_1 e^{s\varphi})^2\, dx \le C \int_0^1 a(x) (v_1 e^{s\varphi})_x^2\, dx. \tag{2.32}
$$

Since $\varphi_x(t,x) = \gamma\theta(t)\frac{x}{a(x)}$, we have

$$
\int_0^1 v_1^2 e^{2s\varphi}\, dx \le C \int_0^1 \Big(a(x) v_{1x}^2 + s^2\theta^2 \frac{x^2}{a(x)} v_1^2 \Big) e^{2s\varphi}\, dx. \tag{2.33}
$$

Proceeding in a similar way, one can easily obtain

$$
\int_0^1 (v_1^2 + v_2^2) e^{2s\varphi}\, dx \le C \int_0^1 \Big(a(x)(v_{1x}^2 + v_{2x}^2) + s^2\theta^2 \frac{x^2}{a(x)} (v_1^2 + v_2^2) \Big) e^{2s\varphi}\, dx. \tag{2.34}
$$

Therefore, observing that $\tilde{\varphi} = \varphi$ in $[\frac{T}{2}, T]$ and applying the Carleman inequality (2.13), we obtain

$$
\begin{aligned}
&\int_{\frac{T}{2}}^T \int_0^1 (v_1^2 + v_2^2) e^{2s\tilde{\varphi}}\, dt\, dx \\
&= \int_{\frac{T}{2}}^T \int_0^1 (v_1^2 + v_2^2) e^{2s\varphi}\, dt\, dx \\
&\le \int_{\frac{T}{2}}^T \int_0^1 \Big(s\theta a(x)(v_{1x}^2 + v_{2x}^2) + s^3\theta^3 \frac{x^2}{a(x)} (v_1^2 + v_2^2) \Big) e^{2s\varphi}\, dt\, dx
\end{aligned}
$$

$$\le C\Big(\iint_Q s^3\theta^3(g_1^2+g_2^2)e^{2s\sigma_1}\,dt\,dx+\iint_{Q_\omega} s^7\theta^7 v_1^2 e^{2s\sigma}\,dt\,dx\Big),$$

which gives (2.31).

On the other hand, let $\xi\in C^\infty([0,T])$ be a cut-off function such that

$$0\le\xi\le 1,\quad \xi(t):=\begin{cases}1, & \text{for } t\in[0,T/2],\\ 0, & \text{for } t\in[T^*,T],\end{cases} \tag{2.35}$$

and define $w_i=\tilde\xi v_i$, $i=1,2$, where $\tilde\xi=\xi e^{s\widehat{\varphi}(0)}$ and (v_1,v_2) satisfies the adjoint system (2.4). Thus, (w_1,w_2) solves

$$\begin{gathered}-w_{1t}-(a(x)w_{1x})_x+b_{11}w_1+b_{21}w_2=-\tilde\xi' v_1+\tilde\xi g_1,\quad (t,x)\in Q,\\ -w_{2t}-(a(x)w_{2x})_x+b_{12}w_1+b_{22}w_2=-\tilde\xi' v_2+\tilde\xi g_2,\quad (t,x)\in Q,\\ w_1(t,1)=w_2(t,1)=0,\quad t\in(0,T),\\ \begin{cases}w_1(t,0)=w_2(t,0)=0, & (WD),\\ (aw_{1x})(t,0)=(aw_{2x})(t,0)=0, & (SD),\end{cases}\quad t\in(0,T),\\ w_1(T,x)=w_2(T,x)=0,\quad x\in(0,1).\end{gathered} \tag{2.36}$$

Thanks to the energy estimate (2.2), one has

$$\begin{aligned}&(\|w_1(0)\|^2_{L^2(0,1)}+\|w_2(0)\|^2_{L^2(0,1)})+(\|w_1\|^2_{L^2(Q)}+\|w_2\|^2_{L^2(Q)})\\ &\le C\iint_Q\Big((-\tilde\xi' v_1+\tilde\xi g_1)^2+(-\tilde\xi' v_2+\tilde\xi g_2)^2\Big)\,dt\,dx,\end{aligned}$$

which yields

$$\begin{aligned}&e^{2s\widehat{\varphi}(0)}\Big(\|v_1(0)\|^2_{L^2(0,1)}+\|v_2(0)\|^2_{L^2(0,1)}+\|\xi v_1\|^2_{L^2(Q)}+\|\xi v_2\|^2_{L^2(Q)}\Big)\\ &\le C\iint_Q\big((\xi')^2(v_1^2+v_2^2)+(\xi)^2(g_1^2+g_2^2)\big)e^{2s\widehat{\varphi}(0)}\,dt\,dx\\ &= Ce^{2s\widehat{\varphi}(0)}\int_{\frac{T}{2}}^{T^*}\int_0^1(\xi')^2(v_1^2+v_2^2)\,dt\,dx\\ &\quad+Ce^{2s\widehat{\varphi}(0)}\int_0^{T^*}\int_0^1(\xi)^2(g_1^2+g_2^2)\,dt\,dx.\end{aligned} \tag{2.37}$$

Using that $\xi'(t)=0$ in $[0,\frac{T}{2}]$, $\xi(t)=0$ in $[T^*,T]$ and $\varphi\le\widehat{\varphi}(0)$, one has

$$\begin{aligned}&e^{2s\widehat{\varphi}(0)}\Big(\|v_1(0)\|^2_{L^2(0,1)}+\|v_2(0)\|^2_{L^2(0,1)}\Big)+\int_0^{\frac{T}{2}}\int_0^1(v_1^2+v_2^2)e^{2s\tilde\varphi}\,dt\,dx\\ &\le e^{2s\widehat{\varphi}(0)}\Big(\|v_1(0)\|^2_{L^2(0,1)}+\|v_2(0)\|^2_{L^2(0,1)}\Big)+\iint_Q\xi^2(v_1^2+v_2^2)e^{2s\widehat{\varphi}(0)}\,dt\,dx\\ &\le C\Big(\int_{\frac{T}{2}}^{T^*}\int_0^1(v_1^2+v_2^2)e^{2s\widehat{\varphi}(0)}\,dt\,dx+\int_0^{T^*}\int_0^1(g_1^2+g_2^2)e^{2s\widehat{\varphi}(0)}\,dt\,dx\Big)\\ &\le Ce^{2s[\widehat{\varphi}(0)-\varphi^*(T^*)]}\Big(\int_{\frac{T}{2}}^{T^*}\int_0^1(v_1^2+v_2^2)e^{2s\tilde\varphi}\,dt\,dx\\ &\quad+\int_0^{T^*}\int_0^1(g_1^2+g_2^2)e^{2s\tilde\varphi}\,dt\,dx\Big),\end{aligned} \tag{2.38}$$

since $\varphi^*(T^*) \le \tilde{\varphi}$ in $(0,T^*)\times(0,1)$.

By (2.31), we have

$$
\begin{aligned}
&\int_{\frac{T}{2}}^{T^*}\int_0^1 (v_1^2+v_2^2)e^{2s\tilde{\varphi}}\,dt\,dx\\
&\le \int_{\frac{T}{2}}^{T}\int_0^1 (v_1^2+v_2^2)e^{2s\tilde{\varphi}}\,dt\,dx\\
&\le C\Big(\iint_Q s^3\theta^3(g_1^2+g_2^2)e^{2s\sigma_1}\,dt\,dx+\iint_{Q_\omega} s^7\theta^7 v_1^2 e^{2s\sigma}\,dt\,dx\Big).
\end{aligned}
$$

Plugging the above inequality in (2.38), we obtain

$$
\begin{aligned}
&e^{2s\widehat{\varphi}(0)}\Big(\|v_1(0)\|^2_{L^2(0,1)}+\|v_2(0)\|^2_{L^2(0,1)}\Big)+\int_0^{\frac{T}{2}}\int_0^1 (v_1^2+v_2^2)e^{2s\tilde{\varphi}}\,dt\,dx\\
&\le Ce^{2s[\widehat{\varphi}(0)-\varphi^*(T^*)]}\Big(\iint_Q s^3\theta^3(g_1^2+g_2^2)e^{2s\sigma_1}\,dt\,dx+\iint_{Q_\omega} s^7\theta^7 v_1^2 e^{2s\sigma}\,dt\,dx \qquad (2.39)\\
&\quad+\int_0^{T^*}\int_0^1 (g_1^2+g_2^2)e^{2s\tilde{\varphi}}\,dt\,dx\Big).
\end{aligned}
$$

Using the definition of the modified weights, in particular the fact that $\tilde{\varphi}\le\tilde{\sigma}_1$ in Q, together with (2.31) and (2.39), it follows that

$$
\begin{aligned}
&e^{2s\widehat{\varphi}(0)}\int_0^1 \big(v_1^2(0)+v_2^2(0)\big)\,dx+\iint_Q (v_1^2+v_2^2)e^{2s\tilde{\varphi}}\,dt\,dx\\
&\le Ce^{2s[\widehat{\varphi}(0)-\varphi^*(T^*)]}\Big(\iint_Q s^3\theta^3(g_1^2+g_2^2)e^{2s\sigma_1}\,dt\,dx \qquad (2.40)\\
&\quad+\iint_Q (g_1^2+g_2^2)e^{2s\tilde{\sigma}_1}\,dt\,dx+\iint_{Q_\omega} s^7\theta^7 v_1^2e^{2s\sigma}\,dt\,dx\Big).
\end{aligned}
$$

Finally, observe that for $c>0$ and $n\ge 0$, the function $x\mapsto x^ne^{-cx}$ is non-increasing for x sufficiently large. Thus, using the fact that $\beta(t)\le\theta(t)$, one has

$$
(s\theta)^ne^{2s\sigma}\le(s\beta)^ne^{2s\tilde{\sigma}},\quad (s\theta)^ne^{2s\sigma_1}\le(s\beta)^ne^{2s\tilde{\sigma}_1}
$$

for s large enough. This, together with (2.40), gives the estimate (2.30). This completes the proof of Lemma 2.6. □

2.3. **Null controllability result.** In this subsection, as a consequence of Lemma 2.6, we will show the null controllability for the nonhomogeneous system (1.3) with more regular solution. This result will be the key tool in the proof of the null controllability for the memory system (1.1). To this purpose, we introduce the following weighted space where the controllability will be solved:

$$
E_s:=\big\{(y_1,y_2)\in Z_T \mid \quad (s\beta)^{-\frac{3}{2}}e^{-s\tilde{\sigma}_1}(y_1,y_2)\in L^2(Q)^2\big\}
$$

endowed with the associated norm

$$
\|y\|^2_{E_s}:=\iint_Q (s\beta)^{-3}e^{-2s\tilde{\sigma}_1}(y_1^2+y_2^2)\,dt\,dx.
$$

Remark 2.7. If (y_1,y_2) belongs to E_s, then $(y_1,y_2)\in C([0,T];L^2(0,1)^2)$ and

$$
\iint_Q (s\beta)^{-3}e^{-2s\tilde{\sigma}_1}(y_1^2+y_2^2)\,dt\,dx<+\infty.
$$

Since $\tilde{\sigma}_1 < 0$, one has

$$y_1(T,\cdot) = y_2(T,\cdot) = 0 \quad \text{in } (0,1).$$

From the modified Carleman inequality, we can obtain the following null controllability result for (1.3).

Theorem 2.8. *Assume that the conditions of Theorem 2.4 hold. Let $T > 0$, $T^* \in (\frac{T}{2}, T)$ and suppose that $e^{-s\tilde{\varphi}}(F_1, F_2) \in L^2(Q)^2$ with $s \geq s_0$. Then, for any $(y_1^0, y_2^0) \in H_a^1(0,1)^2$, there exists $u \in L^2(Q)$ such that the associated solution (y_1, y_2) of system* (1.3) *belongs to E_s.*

Moreover, there exists a positive constant C such that

$$\begin{aligned} &\iint_Q (s\beta)^{-3} e^{-2s\tilde{\sigma}_1}(y_1^2 + y_2^2)\,dt\,dx + \iint_{Q_\omega} (s\beta)^{-7} e^{-2s\tilde{\sigma}} u^2\,dtdx \\ &\leq C e^{2s[\widehat{\varphi}(0) - \varphi^*(T^*)]}\Big(\iint_Q e^{-2s\tilde{\varphi}}(F_1^2 + F_2^2)\,dt\,dx \\ &\quad + e^{-2s\widehat{\varphi}(0)}(\|y_1^0\|^2_{L^2(0,1)} + \|y_2^0\|^2_{L^2(0,1)})\Big). \end{aligned} \tag{2.41}$$

Proof. Let us introduce the functional

$$J(y_1, y_2, u) = \iint_Q (s\beta)^{-3} e^{-2s\tilde{\sigma}_1}(y_1^2 + y_2^2)\,dt\,dx + \iint_{Q_\omega} (s\beta)^{-7} e^{-2s\tilde{\sigma}} u^2\,dtdx, \tag{2.42}$$

where $u \in L^2(Q)$ and (y_1, y_2) satisfies the system

$$\begin{gathered} y_{1t} - (a(x)y_{1x})_x + b_{11}y_1 + b_{12}y_2 = F_1 + 1_\omega u, \quad (t,x) \in Q, \\ y_{2t} - (a(x)y_{2x})_x + b_{21}y_1 + b_{22}y_2 = F_2, \quad (t,x) \in Q, \\ y_1(t,1) = y_2(t,1) = 0, \quad t \in (0,T), \\ \begin{cases} y_1(t,0) = y_2(t,0) = 0, & (WD), \\ (ay_{1x})(t,0) = (ay_{2x})(t,0) = 0, & (SD), \end{cases} \quad t \in (0,T), \\ y_1(0,x) = y_1^0(x), \quad y_2(0,x) = y_2^0(x), \quad x \in (0,1) \\ y_1(T,x) = y_2(T,x) = 0, \quad x \in (0,1). \end{gathered} \tag{2.43}$$

By standard arguments (see for instance [17]), J attains its minimum at a unique point $(\bar{y}_1, \bar{y}_2, \bar{u})$.

We are going to prove the existence of a dual variable $\bar{z} = (\bar{z}_1, \bar{z}_2)$ such that

$$\begin{gathered} (\bar{y}_1, \bar{y}_2) = (s\beta)^3 e^{2s\tilde{\sigma}_1}\mathcal{L}^*(\bar{z}_1, \bar{z}_2), \quad \text{in } Q, \\ \bar{u} = -1_\omega (s\beta)^7 e^{2s\tilde{\sigma}}\bar{z}_1, \quad \text{in } Q, \end{gathered}$$

where $\mathcal{L}^*\bar{z} = -\bar{z}_t - (a(x)\bar{z}_x)_x + B^*\bar{z}$, with $B = (b_{ij})_{1 \leq i,j \leq 2}$ such that

$$\bar{z}(\cdot,1) = 0 \quad \text{and} \quad \begin{cases} \bar{z}(\cdot,0) = 0, & (WD) \\ (a\bar{z}_x)(\cdot,0) = 0, & (SD) \end{cases} \quad \text{on } (0,T). \tag{2.44}$$

Let us define the linear space

$$X_a = \big\{w \in C^\infty(\overline{Q})^2 : w \text{ satisfies } (2.44)\big\}.$$

In addition, we set

$$\beta(z,w) = \iint_Q (s\beta)^3 e^{2s\tilde{\sigma}_1}(\mathcal{L}^*z \cdot \mathcal{L}^*w)\,dt\,dx + \iint_{Q_\omega} (s\beta)^7 e^{2s\tilde{\sigma}} z_1 w_1\,dtdx, \tag{2.45}$$

for all $z, w \in X_a$, and

$$\ell(w) = \iint_Q F \cdot w \, dt \, dx + \int_0^1 y_0 \cdot w(0) dx, \quad \forall w \in X_a, \tag{2.46}$$

where $F = (F_1, F_2)$ and $y_0 = (y_1^0, y_2^0)$ are the functions in (1.3).

Observe that the Carleman inequality (2.30) holds for all $w \in X_a$. Notably, we have

$$\begin{aligned} &e^{2s\widehat{\varphi}(0)} \int_0^1 \left(w_1^2(0) + w_2^2(0)\right) \, dx + \iint_Q (w_1^2 + w_2^2) e^{2s\tilde{\varphi}} \, dt \, dx \\ &\le C e^{2s[\widehat{\varphi}(0) - \varphi^*(T^*)]} \beta(w, w), \end{aligned}$$

for all $w \in X_a$.

Now, let us denote by $\widetilde{X}_a$ the completion of X_a with the norm $\|w\|_{\widetilde{X}_a} = (\beta(w, w))^{1/2}$. Thus, $\widetilde{X}_a$ is a Hilbert space with this norm.

Clearly, β is a strictly positive, symmetric and continuous bilinear form in $\widetilde{X}_a$. Moreover, in view of the above inequality, one can see that the linear form ℓ is continuous in $\widetilde{X}_a$. Indeed, employing the Cauchy-Schwarz inequality, one has

$$\begin{aligned} |\ell(w)| &= \iint_Q (F \cdot w) \, dt \, dx + \int_0^1 y_0 \cdot w(0) dx \\ &\le C e^{s[\widehat{\varphi}(0) - \varphi^*(T^*)]} \Big(\Big(\iint_Q e^{-2s\tilde{\varphi}} (F_1^2 + F_2^2) \, dt \, dx \Big)^{1/2} \\ &\quad + e^{-s\widehat{\varphi}(0)} (\|y_1^0\|_{L^2(0,1)} + \|y_2^0\|_{L^2(0,1)}) \Big) \|w\|_{\widetilde{X}_a}, \end{aligned} \tag{2.47}$$

for all $w \in \widetilde{X}_a$.

Hence, in view of Lax-Milgram's Lemma, there exists one and only one $\bar{z} \in \widetilde{X}_a$ satisfying

$$\beta(\bar{z}, w) = \ell(w), \quad \forall w \in \widetilde{X}_a. \tag{2.48}$$

Moreover,

$$\begin{aligned} \|\bar{z}\|_{\widetilde{X}_a} &\le C e^{s[\widehat{\varphi}(0) - \varphi^*(T^*)]} \Big(\Big(\iint_Q e^{-2s\tilde{\varphi}} (F_1^2 + F_2^2) \, dt \, dx \Big)^{1/2} \\ &\quad + e^{-s\widehat{\varphi}(0)} (\|y_1^0\|_{L^2(0,1)} + \|y_2^0\|_{L^2(0,1)}) \Big). \end{aligned} \tag{2.49}$$

Let us set

$$(\bar{y}_1, \bar{y}_2) = (s\beta)^3 e^{2s\tilde{\sigma}_1} \mathcal{L}^*(\bar{z}_1, \bar{z}_2) \quad \text{and} \quad \bar{u} = -1_\omega (s\beta)^7 e^{2s\tilde{\sigma}} \bar{z}_1. \tag{2.50}$$

Using these definitions together with (2.49), it is not difficult to check that $(\bar{y}_1, \bar{y}_2)$ and $\bar{u}$ satisfy

$$\begin{aligned} &\iint_Q (s\beta)^{-3} e^{-2s\tilde{\sigma}_1} (y_1^2 + y_2^2) \, dt \, dx + \iint_{Q_\omega} (s\beta)^{-7} e^{-2s\tilde{\sigma}} u^2 \, dt dx \\ &\le C e^{2s[\widehat{\varphi}(0) - \varphi^*(T^*)]} \Big(\iint_Q e^{-2s\tilde{\varphi}} (F_1^2 + F_2^2) \, dt \, dx \\ &\quad + e^{-2s\widehat{\varphi}(0)} (\|y_1^0\|_{L^2(0,1)}^2 + \|y_2^0\|_{L^2(0,1)}^2) \Big) \end{aligned} \tag{2.51}$$

which yields (2.41).

To complete the proof, it suffices to check that $(\bar{y}_1, \bar{y}_2, \bar{u})$ satisfies the system (2.43). First of all, notice that $(\bar{y}_1, \bar{y}_2) \in E_s$ and $\bar{u} \in L^2(Q)$. Denote by $(\tilde{y}_1, \tilde{y}_2)$ the (weak) solution of (1.3) associated to the control function $u = \bar{u}$. Then $\tilde{y} = (\tilde{y}_1, \tilde{y}_2)$ is also the unique solution of (1.3) defined by transposition. Therefore, $\tilde{y}$ is the unique function in $L^2(Q)^2$ satisfying

$$\iint_Q \tilde{y} \cdot G \, dt \, dx = \iint_Q 1_\omega \bar{u} z_1 \, dt \, dx + \iint_Q F \cdot z \, dt \, dx + \int_0^1 y_0 \cdot z(0) \, dx, \tag{2.52}$$

for all $G = (G_1, G_2) \in L^2(Q)^2$, where $z := (z_1, z_2)$ solves

$$\begin{gathered} -z_{1t} - (a(x) z_{1x})_x + b_{11} z_1 + b_{21} z_2 = G_1, \quad (t,x) \in Q, \\ -z_{2t} - (a(x) z_{2x})_x + b_{12} z_1 + b_{22} z_2 = G_2, \quad (t,x) \in Q, \\ z_1(t,1) = z_2(t,1) = 0, \quad t \in (0,T), \\ \begin{cases} z_1(t,0) = z_2(t,0) = 0, & (WD), \\ (a z_{1x})(t,0) = (a z_{2x})(t,0) = 0, & (SD), \end{cases} \quad t \in (0,T), \\ z_1(T,x) = z_2(T,x) = 0, \quad x \in (0,1). \end{gathered}$$

Now, using the expressions of $(\bar{y}_1, \bar{y}_2)$ and $\bar{u}$ (see (2.50)) in (2.52), we easily obtain

$$\begin{aligned} &\iint_Q \bar{y} \cdot G \, dt \, dx \\ &= \iint_Q 1_\omega \bar{u} z_1 \, dt \, dx + \iint_Q F \cdot z \, dt \, dx + \int_0^1 y_0 \cdot z(0) \, dx, \quad \forall\, G \in L^2(Q)^2. \end{aligned}$$

This together with (2.52), implies that $\bar{y} = \tilde{y}$. Thus, the control $\bar{u} \in L^2(\omega \times (0,T))$ drives the state $(\bar{y}_1, \bar{y}_2) \in E_s$ to zero at time T. □

3. Null controllability for the integro-differential system

In this section, we establish our main null controllability result for the integro-differential system (1.1).

At first, we recall that proceeding as in [14], thanks to a fixed point argument and invoking Proposition 2.1, one can show that the following well-posedness result holds.

Proposition 3.1. *Assume that $(y_1^0, y_2^0) \in L^2(0,1)^2$ and $u \in L^2(Q)$. Then system (1.1) admits a unique solution $(y_1, y_2) \in W_T$.*

Before presenting our main result, in what follows we start by proving some technical results.

Lemma 3.2. *Let $\tilde{\varphi}$ be the function in (2.28). Then*

$$-\tilde{\varphi}(t,x) \le \frac{\gamma d}{(T/4)^4 (T-t)^4}, \quad \forall\, (t,x) \in Q, \tag{3.1}$$

where γ and d are the constants in (2.5).

Proof. By the definition of $\tilde{\varphi}$, we see that

$$-\tilde{\varphi}(t,x) \le \gamma d \beta(t), \quad \forall\, (t,x) \in Q. \tag{3.2}$$

We next observe that, when $t \in (0, \frac{T}{2})$, we have

$$\frac{1}{T^4} \le \frac{1}{(T-t)^4},$$

which yields

$$\beta(t) = \Big(\frac{4}{T^2}\Big)^4 \le \Big(\frac{4}{T}\Big)^4 \frac{1}{(T-t)^4}, \quad \forall\, t \in \big(0, \frac{T}{2}\big). \tag{3.3}$$

On the other hand,

$$\beta(t) \le \Big(\frac{2}{T}\Big)^4 \frac{1}{(T-t)^4}, \quad \forall\, t \in \big(\frac{T}{2}, T\big).$$

This together with (3.3) gives

$$\beta(t) \le \Big(\frac{4}{T}\Big)^4 \frac{1}{(T-t)^4}, \quad \forall\, t \in (0, T). \tag{3.4}$$

Then, putting (3.4) in (3.2), we finally deduce (3.1). □

Lemma 3.3. *Let $T^* = (1+\varepsilon)T/2$. Assume that $d > 5d^*$ and*

$$\varepsilon \in \Big(0, \sqrt{1 - \sqrt[4]{\frac{4}{5}\big(\frac{d}{d-d^*}\big)}}\Big).$$

Then

$$\frac{5}{2}\widehat{\varphi}(0) - 2\varphi^*(T^*) < 0. \tag{3.5}$$

Proof. From the definitions of $\widehat{\varphi}$ and φ^*, one has

$$\begin{aligned} \frac{5}{2}\widehat{\varphi}(0) - 2\varphi^*(T^*) &= \frac{5}{2}\gamma(d^* - d)\beta(0) + 2\gamma d\beta(T^*) \\ &= \gamma\Big(\frac{2}{T}\Big)^8 \Big[\frac{5}{2}(d^* - d) + \frac{2d}{(1-\varepsilon^2)^4}\Big] \\ &= \frac{d\gamma}{2}\Big(\frac{2}{T}\Big)^8 \Big[-5\frac{(d-d^*)}{d} + \frac{4}{(1-\varepsilon^2)^4}\Big]. \end{aligned} \tag{3.6}$$

On the other hand, using the fact that $d > 5d^*$, we immediately have

$$\frac{4}{5}\frac{d}{(d-d^*)} < 1.$$

Hence, taking $\varepsilon \in \Big(0, \sqrt{1 - \sqrt[4]{\frac{4}{5}(\frac{d}{d-d^*})}}\Big)$, it results

$$\varepsilon^2 < 1 - \sqrt[4]{\frac{4}{5}\big(\frac{d}{d-d^*}\big)}$$

and, in particular,

$$(1-\varepsilon^2)^4 > \frac{4}{5}\frac{d}{(d-d^*)}.$$

This is equivalent to

$$-\frac{5(d-d^*)}{d} + \frac{4}{(1-\varepsilon^2)^4} < 0$$

and, by (3.6), the claim follows. □

Next, we make the following assumption on the kernels h_1 and h_2.

Hypothesis 3.4. *Assume that a is (WD) or (SD) and h_1, h_2 satisfy*

$$e^{\frac{c_0 s}{(T-t)^4}} h_k \in L^\infty((0,T)\times Q), \quad k=1,2, \tag{3.7}$$

with $c_0 := \gamma d\left(\frac{4}{T}\right)^4$. Fix $s \geq s_0$ such that

$$2Ce^{s(\frac{5}{2}\hat{\varphi}(0)-2\varphi^*(T^*))} < 1, \tag{3.8}$$

where C and s_0 are the constants in (2.41) *and Theorem 2.8, respectively.*

Thanks to the previous hypothesis, we are able to prove the main result of this paper.

Theorem 3.5. *Assume the conditions of Theorem 2.4 and Hypothesis 3.4. Then for any $(y_1^0, y_2^0) \in H_a^1(0,1)^2$, there exists a control function $u \in L^2(Q)$ such that the associated solution $(y_1,y_2)\in Z_T$ of* (1.1) *satisfies*

$$y_1(T,\cdot) = y_2(T,\cdot) = 0 \quad \text{in } (0,1). \tag{3.9}$$

The proof of this theorem is based on the following generalized version of Kakutani's fixed point Theorem, due to Glicksberg [13].

Theorem 3.6. *Let B be a non-empty convex, compact subset of a locally convex topological vector space X. If $\Lambda : B \to B$ is a convex set-valued mapping with closed graph and $\Lambda(B)$ is closed, then Λ has a fixed point.*

Proof of Theorem 3.5. To prove the desired result, we begin by showing the null controllability for the system

$$\begin{gathered}
y_{1t} - (a(x)y_{1x})_x + b_{11}y_1 + b_{12}y_2 = \int_0^t h_1(t,r,x)w_1(r,x)\,dr + 1_\omega u, \quad (t,x)\in Q,\\
y_{2t} - (a(x)y_{2x})_x + b_{21}y_1 + b_{22}y_2 = \int_0^t h_2(t,r,x)w_2(r,x)\,dr, \quad (t,x)\in Q,\\
y_1(t,1) = y_2(t,1) = 0, \quad t\in(0,T),\\
\begin{cases} y_1(t,0) = y_2(t,0) = 0, & (WD),\\ (ay_{1x})(t,0) = (ay_{2x})(t,0) = 0, & (SD), \end{cases} \quad t\in(0,T),\\
y_1(0,x) = y_1^0(x), \quad y_2(0,x) = y_2^0(x), \quad x\in(0,1),
\end{gathered} \tag{3.10}$$

for each $(w_1,w_2)\in E_{s,M} = \big\{(w_1,w_2)\in E_s : \|(s\beta)^{-3/2}e^{-s\tilde{\sigma}_1}(w_1,w_2)\|_{L^2(Q)} \leq M\big\}$, where M and s are two arbitrary positive constants to be fixed later. More precisely, as a first step we prove that this system is null controllable under Hypothesis 3.4. As consequence, we obtain the null controllability result for the original memory system through a fixed point technique.

Notice that $E_{s,M}$ is a non empty, bounded, closed, and convex subset of $L^2(Q)^2$. Now, let $(w_1,w_2)\in E_{s,M}$. By (3.1), we obtain

$$\begin{aligned}
&\iint_Q e^{-2s\tilde{\varphi}}\Big(\int_0^t h_k(t,r,x)w_k(r,x)\,dr\Big)^2\,dt\,dx\\
&\leq T\iint_Q \int_0^t e^{-2s\tilde{\varphi}} h_k^2(t,r,x)w_k^2(r,x)\,dr\,dt\,dx
\end{aligned}$$

(by (3.1))

$$\le T \iint_Q \int_0^t e^{\frac{2s\gamma d}{(T/4)^4 (T-t)^4}} h_k^2(t,r,x) w_k^2(r,x)\,dr\,dt\,dx, \quad k=1,2.$$

Next, using the condition (3.7), it follows that

$$\iint_Q e^{-2s\tilde{\varphi}} \Big(\int_0^t h_k(t,r,x) w_k(r,x)\,dr \Big)^2 dt\,dx \le CT \iint_Q w_k^2\,dt\,dx, \tag{3.11}$$

for $k=1,2$ and some positive constant C.

Applying Hölder's inequality and using that $\sup_{(t,x)\in\overline{Q}} \big((s\beta(t))^3 e^{2s\tilde{\sigma}_1(t,x)} \big) < +\infty$ and $(w_1,w_2)\in E_{s,M}$, we deduce that

$$\begin{aligned}
&\iint_Q e^{-2s\tilde{\varphi}} \Big(\Big(\int_0^t h_1(t,r,x) w_1(r,x)\,dr \Big)^2 + \Big(\int_0^t h_2(t,r,x) w_2(r,x)\,dr \Big)^2 \Big) dt\,dx \\
&\le CT \sup_{(t,x)\in\overline{Q}} \Big((s\beta(t))^3 e^{2s\tilde{\sigma}_1(t,x)} \Big) \iint_Q (s\beta)^{-3} e^{-2s\tilde{\sigma}_1} (w_1^2+w_2^2)\,dt\,dx \\
&\le CTM^2 < +\infty.
\end{aligned}$$

Therefore, setting $F_k := \int_0^t h_k(t,r,x) w_k(r,x)\,dr$, $k=1,2$, we have $e^{-s\tilde{\varphi}}(F_1,F_2)\in L^2(Q)^2$. It follows from Theorem 2.8 that the system (3.10) is null controllable, that is, for any $(y_1^0,y_2^0)\in H_a^1(0,1)^2$ and $(w_1,w_2)\in E_{s,M}$, there exists a control function $u\in L^2(Q)$ such that the solution of (3.10) fulfills $y_1(T,\cdot)=y_2(T,\cdot)=0$ in $(0,1)$. Furthermore, in this case, the control u satisfies the estimate

$$\begin{aligned}
&\iint_{Q_\omega} (s\beta)^{-7} e^{-2s\tilde{\sigma}} u^2\,dtdx \\
&\le Ce^{2s[\widehat{\varphi}(0)-\varphi^*(T^*)]} \Big(M^2 + e^{-2s\widehat{\varphi}(0)} (\|y_1^0\|^2_{L^2(0,1)} + \|y_2^0\|^2_{L^2(0,1)}) \Big).
\end{aligned} \tag{3.12}$$

In the following, we extend this controllability result to the memory system (1.1). First, we introduce the mapping $\Lambda : E_{s,M} \to 2^{E_s}$ defined by

$$\begin{aligned}
\Lambda(w_1,w_2) = \Big\{ &(y_1,y_2)\in E_s : (y_1,y_2) \text{ is a solution of (3.10), such that} \\
&y_1(T,\cdot)=y_2(T,\cdot)=0, \text{ for a control } u\in L^2(Q) \text{ satisfying (3.12)} \Big\}.
\end{aligned}$$

Here, $X=L^2(Q)^2$ and $B=E_{s,M}$.

Clearly, $\Lambda(w_1,w_2)$ is a convex set of $L^2(Q)^2$. Moreover, thanks to the null controllability of the system (3.10), $\Lambda(w_1,w_2)$ is non empty. Let us now prove that Λ is compact and has closed graph. This will be done in the next few steps.

• $\Lambda(E_{s,M})\subset E_{s,M}$ for a sufficiently large M. Indeed, using the inequality (2.41), condition (3.7) and proceeding as in (3.11), we obtain

$$\begin{aligned}
&\iint_Q (s\beta)^{-3} e^{-2s\tilde{\sigma}_1} (y_1^2+y_2^2)\,dt\,dx + \iint_{Q_\omega} (s\beta)^{-7} e^{-2s\tilde{\sigma}} u^2\,dtdx \\
&\le Ce^{2s[\widehat{\varphi}(0)-\varphi^*(T^*)]} \Big(\iint_Q e^{-2s\tilde{\varphi}} \Big(\int_0^t h_1(t,r,x) w_1(r,x)\,dr \Big)^2 dt\,dx \\
&\quad + \iint_Q e^{-2s\tilde{\varphi}} \Big(\int_0^t h_2(t,r,x) w_2(r,x)\,dr \Big)^2 dt\,dx + e^{-2s\widehat{\varphi}(0)} (\|y_1^0\|^2_{L^2(0,1)} \\
&\quad + \|y_2^0\|^2_{L^2(0,1)}) \Big)
\end{aligned}$$

$$
\begin{aligned}
&\le Ce^{2s[\widehat{\varphi}(0)-\varphi^*(T^*)]}\Big(\iint_Q (w_1^2+w_2^2)\,dt\,dx + e^{-2s\widehat{\varphi}(0)}(\|y_1^0\|^2_{L^2(0,1)}+\|y_2^0\|^2_{L^2(0,1)})\Big)\\
&\le Ce^{2s[\widehat{\varphi}(0)-\varphi^*(T^*)]}\Big(\sup_{(t,x)\in\overline{Q}}\Big((s\beta(t))^3 e^{2s\tilde{\sigma}_1(t,x)}\Big)\iint_Q (s\beta)^{-3}e^{-2s\tilde{\sigma}_1}(w_1^2+w_2^2)\,dt\,dx\\
&\quad + e^{-2s\widehat{\varphi}(0)}(\|y_1^0\|^2_{L^2(0,1)}+\|y_2^0\|^2_{L^2(0,1)})\Big).
\end{aligned}
$$

Using that $\sup_{(t,x)\in\overline{Q}}(s\beta(t))^3e^{s\tilde{\sigma}_1(t,x)} < +\infty$ and Lemma 2.5, it is not difficult to show that

$$
\sup_{(t,x)\in\overline{Q}} e^{s\tilde{\sigma}_1(t,x)} \le e^{s(2\widehat{\Phi}(0)-\varphi^*(0))} \le e^{\frac{s}{2}\varphi^*(0)} \le e^{\frac{s}{2}\widehat{\varphi}(0)}. \tag{3.13}
$$

The above estimate together with the fact that $(w_1,w_2)\in E_{s,M}$ implies that

$$
\begin{aligned}
&\iint_Q (s\beta)^{-3}e^{-2s\tilde{\sigma}_1}(y_1^2+y_2^2)\,dt\,dx + \iint_{Q_\omega}(s\beta)^{-7}e^{-2s\tilde{\sigma}}u^2\,dtdx\\
&\le CM^2e^{s[\frac{5}{2}\widehat{\varphi}(0)-2\varphi^*(T^*)]} + Ce^{-2s\varphi^*(T^*)}\Big(\|y_1^0\|^2_{L^2(0,1)}+\|y_2^0\|^2_{L^2(0,1)}\Big).
\end{aligned}
$$

On the other hand, from (3.5) and (3.8), we obtain

$$
Ce^{s[\frac{5}{2}\widehat{\varphi}(0)-2\varphi^*(T^*)]} \le \frac{1}{2}. \tag{3.14}
$$

Hence, for M sufficiently large, we deduce that

$$
\begin{aligned}
&\iint_Q (s\beta)^{-3}e^{-2s\tilde{\sigma}_1}(y_1^2+y_2^2)\,dt\,dx + \iint_{Q_\omega}(s\beta)^{-7}e^{-2s\tilde{\sigma}}u^2\,dtdx\\
&\le \frac{M^2}{2} + Ce^{-2s\widehat{\varphi}(0)}(\|y_1^0\|^2_{L^2(0,1)}+\|y_2^0\|^2_{L^2(0,1)}) \le M^2,
\end{aligned} \tag{3.15}
$$

which yields

$$
\iint_Q (s\beta)^{-3}e^{-2s\tilde{\sigma}_1}(y_1^2+y_2^2)\,dt\,dx \le M^2. \tag{3.16}
$$

Thus, Λ maps $E_{s,M}$ into itself, i.e., $\Lambda(E_{s,M})\subset E_{s,M}$.

- $\Lambda(w_1,w_2)$ is a closed subset of $L^2(Q)^2$. Let (w_1,w_2) fixed and $(y_1^n,y_2^n)\in\Lambda(w_1,w_2)$ such that $(y_1^n,y_2^n)\to(y_1,y_2)$. Let us show that $(y_1,y_2)\in\Lambda(w_1,w_n)$. In fact, by definition we have that (y_1^n,y_2^n) is, together with a control function u_n the solution of the system

$$
\begin{gathered}
y_{1t}^n-(a(x)y_{1x}^n)_x+b_{11}y_1^n+b_{12}y_2^n=\int_0^t h_1(t,r,x)w_1(r,x)\,dr+1_\omega u_n, \quad (t,x)\in Q,\\
y_{2t}^n-(a(x)y_{2x}^n)_x+b_{21}y_1^n+b_{22}y_2^n=\int_0^t h_2(t,r,x)w_2(r,x)\,dr, \quad (t,x)\in Q,\\
y_1^n(t,1)=y_2^n(t,1)=0, \quad t\in(0,T),\\
\begin{cases} y_1^n(t,0)=y_2^n(t,0)=0, & (WD),\\ (ay_{1x}^n)(t,0)=(ay_{2x}^n)(t,0)=0, & (SD), \end{cases} \quad t\in(0,T),\\
y_1^n(0,x)=y_1^0(x), \quad y_2^n(0,x)=y_2^0(x), \quad x\in(0,1),
\end{gathered} \tag{3.17}
$$

with

$$
\iint_Q (s\beta)^{-3}e^{-2s\tilde{\sigma}_1}(y_1^2+y_2^2)\,dt\,dx + \iint_{Q_\omega}(s\beta)^{-7}e^{-2s\tilde{\sigma}}u^2\,dtdx \le M^2. \tag{3.18}
$$

Furthermore, in view of Proposition 2.1, the solution (y_1^n, y_2^n) is bounded in Z_T. Thus, thanks to the Aubin-Lions Theorem, this implies that $\Lambda(E_{s,M})$ is relatively compact in $L^2(Q)^2$.

Hence, by Proposition 2.1 and (3.18), we infer that, on a subsequence (denoted by the same index n) we have the convergences:

$$
\begin{gathered}
1_\omega u_n \to 1_\omega u \quad \text{weakly in } L^2(Q), \\
(y_1^n, y_2^n) \to (y_1, y_2) \quad \text{weakly in } Z_T, \\
(y_1^n, y_2^n) \to (y_1, y_2) \quad \text{strongly in } C(0,T; L^2(0,1)^2).
\end{gathered}
$$

By passing to the limit in (3.17), it follows that (y_1, y_2) is a controlled solution of (3.10) associated to the control u. Consequently, $(y_1, y_2) \in \Lambda(w_1, w_2)$ and $\Lambda(E_{s,M})$ is closed and compact of $L^2(Q)^2$.

• $\Lambda(w_1, w_2)$ has closed graph in $L^2(Q)^2$. We need to prove that if $(w_1^n, w_2^n) \to (w_1, w_2)$ and $(y_1^n, y_2^n) \to (y_1, y_2)$ with $(y_1^n, y_2^n) \in \Lambda(w_1, w_2)$, then $(y_1, y_2) \in \Lambda(w_1, w_2)$. Using the last two steps, one can easily prove that $(y_1, y_2) \in \Lambda(w_1, w_2)$. Therefore, we can apply the fixed point theorem (see Theorem 3.6) in the $L^2(Q)^2$ topology for the mapping Λ to conclude that there is at least one $(y_1, y_2) \in E_{s,M}$ such that $(y_1, y_2) \in \Lambda(w_1, w_2)$. This completes the proof. □

As a consequence of Theorem 3.5 and arguing as in scalar case (see [4]), one can show the following result.

Theorem 3.7. *Assume the conditions of Theorem 2.4 and Hypothesis 3.4. Then for any $(y_1^0, y_2^0) \in L^2(0,1)^2$, there exists a control function $u \in L^2(Q)$ such that the associated solution $(y_1, y_2) \in W_T$ of (1.1) satisfies*

$$
y_1(T, \cdot) = y_2(T, \cdot) = 0 \quad \textit{in } (0,1).
$$

4. Concluding remarks

We are interested in proving that assumption (3.7) on the decay in time of the kernels h_1 and h_2 as t approaches T^- can be substituted by the following assumption:

Hypothesis 4.1. Assume that a is (WD) or (SD) and suppose that there exists $t_0 \in (0,T)$ such that

$$
\operatorname{supp} h_k(t, \cdot, x) \Subset (t_0, T), \quad k = 1, 2, \quad \forall\, (t,x) \in Q. \tag{4.1}
$$

Observe that in this case we do not require condition (3.8). Then the following null controllability result holds.

Theorem 4.2. *Assume Hypothesis 4.1. Then for any $(y_1^0, y_2^0) \in L^2(0,1)^2$, there exists a control function $u \in L^2(Q)$ such that the associated solution $(y_1, y_2) \in W_T$ of (1.1) satisfies*

$$
y_1(T, \cdot) = y_2(T, \cdot) = 0 \quad \textit{in} \quad (0,1).
$$

Moreover,

$$
\|u\|_{L^2(Q)} \le C_{t_0} \|y_0\|_{L^2(0,1)^2},
$$

for some positive constant C_{t_0} depending on t_0.

Proof. Consider the controlled parabolic system

$$
\begin{gathered}
w_{1t} - (a(x)w_{1x})_x + b_{11}w_1 + b_{12}w_2 = 1_\omega v, \quad (t,x) \in (0,t_0) \times (0,1), \\
w_{2t} - (a(x)w_{2x})_x + b_{21}w_1 + b_{22}w_2 = 0, \quad (t,x) \in (0,t_0) \times (0,1), \\
w_1(t,1) = w_2(t,1) = 0, \quad t \in (0,t_0), \\
\begin{cases} w_1(t,0) = w_2(t,0) = 0, & (WD), \\ (aw_{1x})(t,0) = (aw_{2x})(t,0) = 0, & (SD), \end{cases} \quad t \in (0,t_0), \\
w_1(0,x) = y_1^0(x), \quad w_2(0,x) = y_2^0(x), \quad x \in (0,1),
\end{gathered} \tag{4.2}
$$

where (y_1^0, y_2^0) is the initial condition in (1.1).

Thanks to [1, Theorem 4.2] (see also [10, Theorem 3.10]), there exists $v \in L^2((0,t_0)\times(0,1))$ such that the associated solution $(w_1,w_2) \in L^2\left(0,t_0; H_a^1(0,1)^2\right) \cap C\left([0,t_0]; L^2(0,1)^2\right)$ satisfies

$$
w_1(t_0,\cdot) = w_2(t_0,\cdot) = 0 \quad \text{in } (0,1).
$$

Moreover, there exists a positive constant C_{t_0} depending on t_0 such that

$$
\|v\|_{L^2((0,t_0)\times(0,1))} \le C_{t_0}\|y_0\|_{L^2(0,1)^2}. \tag{4.3}
$$

Now, we consider the uncontrolled integro-differential system

$$
\begin{gathered}
\begin{aligned}
& z_{1t} - (a(x)z_{1x})_x + b_{11}z_1 + b_{12}z_2 \\
& = \int_{t_0}^t h_1(t,r,x)z_1(r,x)\,dr, \quad (t,x) \in (t_0,T) \times (0,1), \\
& z_{2t} - (a(x)z_{2x})_x + b_{21}z_1 + b_{22}z_2 \\
& = \int_{t_0}^t h_2(t,r,x)z_2(r,x)\,dr, \quad (t,x) \in (t_0,T) \times (0,1),
\end{aligned} \\
z_1(t,1) = z_2(t,1) = 0, \quad t \in (t_0,T), \\
\begin{cases} z_1(t,0) = z_2(t,0) = 0, & (WD), \\ (az_{1x})(t,0) = (az_{2x})(t,0) = 0, & (SD), \end{cases} \quad t \in (t_0,T), \\
z_1(t_0,x) = w_1(t_0,x) = 0, \quad z_2(t_0,x) = w_2(t_0,x) = 0, \quad x \in (0,1).
\end{gathered} \tag{4.4}
$$

Using Proposition 3.1, we infer that $(z_1,z_2) = (0,0)$ is the unique solution of (4.4). Finally, we set

$$
(y_1,y_2) := \begin{cases} (w_1,w_2), & \text{in } [0,t_0], \\ (z_1,z_2), & \text{in } [t_0,T] \end{cases} \quad \text{and} \quad u := \begin{cases} v, & \text{in } [0,t_0], \\ 0, & \text{in } [t_0,T]. \end{cases}
$$

Note that, according to Hypothesis 4.1 and the previous definition, one has

$$
\int_0^t h_k(t,r,x)y_k(r,x)\,dr = \int_{t_0}^t h_k(t,r,x)z_k(r,x)\,dr = 0, \tag{4.5}
$$

for $k = 1,2$, and $t \in (t_0,T)$.

We can readily show that $(y_1,y_2) \in L^2\left(0,T; H_a^1(0,1)^2\right) \cap C\left([0,T]; L^2(0,1)^2\right)$ solves the system (1.1) associated to u and is such that

$$
y_1(T,\cdot) = y_2(T,\cdot) = 0 \quad \text{in } (0,1).
$$

Furthermore, using (4.3), we have that u satisfies the estimate

$$
\|u\|_{L^2(Q)} = \|v\|_{L^2((0,t_0)\times(0,1))} \le C_{t_0}\|y_0\|_{L^2(0,1)^2}.
$$

This completes the proof. □

Observe that with this technique we obtain also an estimate on the control function through the norm of the initial data and thus we can estimate the cost for controlling the solution of the system to zero.

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