Electronic Journal of Differential Equations, Vol. 2023 (2023), No. 27, pp. 1–14.
ISSN: 1072-6691. URL: https://ejde.math.txstate.edu or https://ejde.math.unt.edu

GROWTH PROPERTIES OF SOLUTIONS OF COMPLEX DIFFERENTIAL EQUATIONS WITH ENTIRE COEFFICIENTS OF FINITE (α, β, γ)-ORDER

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Abstract. In this article, we investigate the complex higher order linear differential equations in which the coefficients are entire functions of (α, β, γ)-order and obtain some results which improve and generalize some previous results of Tu et al. [29] as well as Belaïdi [1, 2, 3].

1. Introduction

Throughout this article, we assume that the reader is familiar with the fundamental results and the standard notations of the Nevanlinna value distribution theory of entire and meromorphic functions and the theory of complex linear differential equations which are available in [12, 21, 34] and therefore we do not explain those in details. To study the generalized growth properties of entire and meromorphic functions, the concepts of different growth indicators such as the iterated p-order (see [20, 26]), the (p, q)-th order (see [17, 18]), (p, q)-φ order (see [27]) etc. are very useful and during the past decades, several authors made close investigations on the generalized growth properties of entire and meromorphic functions related to the above growth indicators in some different directions. The theory of complex linear equations has been developed since 1960s. Many authors have investigated the complex linear differential equations

$$f^{(k)}(z) + A_{k-1}(z)f^{(k-1)}(z) + \cdots + A_0(z)f(z) = 0, \tag{1.1}$$

$$f^{(k)}(z) + A_{k-1}(z)f^{(k-1)}(z) + \cdots + A_0(z)f(z) = F(z) \tag{1.2}$$

and achieved many valuable results when the coefficients $A_0(z), \dots, A_{k-1}(z)$, $F(z)$ $(k \geq 2)$ in (1.1) or (1.2) are entire functions of finite order or finite iterated p-order or (p, q)-th order or (p, q)-φ order; see [1, 2, 3, 7, 8, 10, 15, 21, 22, 23, 24, 26, 27, 29, 30, 31, 33].

In [9], Chyzhykov and Semochko showed that both definitions of iterated p-order and the (p, q)-th order have the disadvantage that they do not cover arbitrary growth (see [9, Example 1.4]). They used more general scale, called the φ-order (see [9]). In recent times, the concept of φ-order is used to study the growth of solutions of complex differential equations which extend and improve many previous results (see [4, 5, 9, 19]).

2020 *Mathematics Subject Classification.* 30D35, 34M10.
Key words and phrases. Complex differential equations; (α, β, γ)-order; growth of solutions.

Submitted July 22, 2023. Published March 11, 2023.

In [25], Mulyava *et al.* have used the concept of (α,β)-order or generalized order of an entire function in order to investigate the properties of solutions of a heterogeneous differential equation of the second order and obtained several interesting results. For details about (α,β)-order one may see [25, 28].

In this paper, we investigate the complex higher order linear differential equations in which the coefficients are entire functions of (α,β,γ)-order and obtain some results which improve and generalize some previous results of Tu et al. [29] as well as Belaïdi [1, 2, 3].

2. Definitions and notation

First of all, let L be a class of continuous non-negative on $(-\infty,+\infty)$ function α such that $\alpha(x)=\alpha(x_0)\geq 0$ for $x\leq x_0$ and $\alpha(x)\uparrow+\infty$ as $x_0\leq x\to+\infty$. We say that $\alpha\in L_1$, if $\alpha\in L$ and $\alpha(a+b)\leq\alpha(a)+\alpha(b)+c$ for all $a,b\geq R_0$ and fixed $c\in(0,+\infty)$. Further we say that $\alpha\in L_2$, if $\alpha\in L$ and $\alpha(x+O(1))=(1+o(1))\alpha(x)$ as $x\to+\infty$. Finally, $\alpha\in L_3$, if $\alpha\in L$ and $\alpha(a+b)\leq\alpha(a)+\alpha(b)$ for all $a,b\geq R_0$, i.e., α is subadditive. Clearly $L_3\subset L_1$.

Particularly, when $\alpha\in L_3$, then one can easily verify that $\alpha(mr)\leq m\alpha(r)$, $m\geq 2$ is an integer. Up to a normalization, subadditivity is implied by concavity. Indeed, if $\alpha(r)$ is concave on $[0,+\infty)$ and satisfies $\alpha(0)\geq 0$, then for $t\in[0,1]$,

$$\alpha(tx)=\alpha(tx+(1-t)\cdot 0)\geq t\alpha(x)+(1-t)\alpha(0)\geq t\alpha(x),$$

so that by choosing $t=\frac{a}{a+b}$ or $t=\frac{b}{a+b}$, we obtain

$$\begin{aligned}\alpha(a+b)&=\frac{a}{a+b}\alpha(a+b)+\frac{b}{a+b}\alpha(a+b)\\&\leq\alpha\Big(\frac{a}{a+b}(a+b)\Big)+\alpha\Big(\frac{b}{a+b}(a+b)\Big)\\&=\alpha(a)+\alpha(b),\quad a,b\geq 0.\end{aligned}$$

As a non-decreasing, subadditive and unbounded function, $\alpha(r)$ satisfies

$$\alpha(r)\leq\alpha(r+R_0)\leq\alpha(r)+\alpha(R_0)$$

for any $R_0\geq 0$. This yields that $\alpha(r)\sim\alpha(r+R_0)$ as $r\to+\infty$.

Now we add two conditions on α, β and γ: (i) Always $\alpha\in L_1$, $\beta\in L_2$ and $\gamma\in L_3$; and (ii) $\alpha(\log^{[p]}x)=o(\beta(\log\gamma(x)))$, $p\geq 2$, $\alpha(\log x)=o(\alpha(x))$ and $\alpha^{-1}(kx)=o(\alpha^{-1}(x))$ $(k<1)$ as $x\to+\infty$.

Throughout this paper, we assume that α, β and γ always satisfy the above two conditions unless otherwise specifically stated.

Heittokangas et al. [16] introduced a new concept of φ-order of entire and meromorphic function considering φ as subadditive function. For details one may see [16]. Extending this notion, recently Belaïdi and Biswas [6] introduce the definition of the (α,β,γ)-order of a meromorphic function in the following way:

Definition 2.1 ([6])**.** *The (α,β,γ)-order denoted by $\sigma_{(\alpha,\beta,\gamma)}[f]$ of an entire function $f(z)$ is defined by*

$$\sigma_{(\alpha,\beta,\gamma)}[f]=\limsup_{r\to+\infty}\frac{\alpha(\log^{[2]}M(r,f))}{\beta(\log\gamma(r))}.$$

By the inequality $T(r,f) \le \log^+ M(r,f) \le \frac{R+r}{R-r} T(R,f)$ $(0<r<R)$ [12] for an entire function $f(z)$, one can easily verify that [6]

$$\sigma_{(\alpha,\beta,\gamma)}[f] = \limsup_{r\to+\infty} \frac{\alpha(\log T(r,f))}{\beta(\log\gamma(r))} = \limsup_{r\to+\infty} \frac{\alpha(\log^{[2]} M(r,f))}{\beta(\log\gamma(r))}.$$

Proposition 2.2 ([6])**.** *If $f(z)$ is an entire function, then*

$$\sigma_{(\alpha(\log),\beta,\gamma)}[f] = \limsup_{r\to+\infty} \frac{\alpha(\log^{[2]} T(r,f))}{\beta(\log\gamma(r))} = \limsup_{r\to+\infty} \frac{\alpha(\log^{[3]} M(r,f))}{\beta(\log\gamma(r))}.$$

Similar to Definition 2.1, one can define the (α,β,γ)-exponent convergence of the zero-sequence of a meromorphic the following way:

Definition 2.3 ([6])**.** The (α,β,γ)-exponent convergence of the zero-sequence denoted by $\lambda_{(\alpha,\beta,\gamma)}[f]$ of a meromorphic function $f(z)$ is defined by

$$\lambda_{(\alpha,\beta,\gamma)}[f] = \limsup_{r\to+\infty} \frac{\alpha(\log n(r,1/f))}{\beta(\log\gamma(r))}.$$

Analogously, the (α,β,γ)-exponent convergence of the distinct zero-sequence denoted by $\overline{\lambda}_{(\alpha,\beta,\gamma)}[f]$ of $f(z)$ is defined by

$$\overline{\lambda}_{(\alpha,\beta,\gamma)}[f] = \limsup_{r\to+\infty} \frac{\alpha(\log \overline{n}(r,1/f))}{\beta(\log\gamma(r))}.$$

Accordingly, the values

$$\lambda_{(\alpha(\log),\beta,\gamma)}[f] = \limsup_{r\to+\infty} \frac{\alpha(\log^{[2]} n(r,1/f))}{\beta(\log\gamma(r))},$$
$$\overline{\lambda}_{(\alpha(\log),\beta,\gamma)}[f] = \limsup_{r\to+\infty} \frac{\alpha(\log^{[2]} \overline{n}(r,1/f))}{\beta(\log\gamma(r))}$$

are respectively called as $(\alpha(\log),\beta,\gamma)$-exponent convergence of the zero-sequence and $(\alpha(\log),\beta,\gamma)$ -exponent convergence of the distinct zero-sequence of a meromorphic function $f(z)$.

The linear measure of a set $E \subset [0,+\infty)$ is defined as $m(E) = \int_0^{+\infty} \chi_E(t)dt$. The logarithmic measure of a set $E \subset [1,+\infty)$ is defined by $lm(E) = \int_1^{+\infty} \frac{\chi_E(t)}{t}dt$, where $\chi_E(t)$ is the characteristic function of E. The upper and lower densities of E are

$$\overline{\text{dens}}E = \limsup_{r\to+\infty} \frac{m(E\cap[0,r])}{r}, \quad \underline{\text{dens}}E = \liminf_{r\to+\infty} \frac{m(E\cap[0,r])}{r}.$$

Proposition 2.4 ([6])**.** *If $f(z)$ is a meromorphic function, then*

$$\lambda_{(\alpha,\beta,\gamma)}[f] = \limsup_{r\to+\infty} \frac{\alpha(\log n(r,1/f))}{\beta(\log\gamma(r))} = \limsup_{r\to+\infty} \frac{\alpha(\log N(r,1/f))}{\beta(\log\gamma(r))}$$

and

$$\overline{\lambda}_{(\alpha,\beta,\gamma)}[f] = \limsup_{r\to+\infty} \frac{\alpha(\log \overline{n}(r,1/f))}{\beta(\log\gamma(r))} = \limsup_{r\to+\infty} \frac{\alpha(\log \overline{N}(r,1/f))}{\beta(\log\gamma(r))}.$$

Proposition 2.5 ([6])**.** *If $f(z)$ is a meromorphic function, then*

$$\lambda_{(\alpha(\log),\beta,\gamma)}[f] = \limsup_{r\to+\infty} \frac{\alpha(\log^{[2]} n(r,1/f))}{\beta(\log\gamma(r))} = \limsup_{r\to+\infty} \frac{\alpha(\log^{[2]} N(r,1/f))}{\beta(\log\gamma(r))}$$

and

$$\overline{\lambda}_{(\alpha(\log),\beta,\gamma)}[f] = \limsup_{r\to+\infty} \frac{\alpha(\log^{[2]} \overline{n}(r,1/f))}{\beta(\log\gamma(r))} = \limsup_{r\to+\infty} \frac{\alpha(\log^{[2]} \overline{N}(r,1/f))}{\beta(\log\gamma(r))}.$$

Proposition 2.6 ([6]). *Let $f_1(z)$, $f_2(z)$ be non-constant meromorphic functions with $\sigma_{(\alpha(\log),\beta,\gamma)}[f_1]$ and $\sigma_{(\alpha(\log),\beta,\gamma)}[f_2]$ as their $(\alpha(\log),\beta,\gamma)$-order. Then*

(i) $\sigma_{(\alpha(\log),\beta,\gamma)}[f_1 \pm f_2] \le \max\{\sigma_{(\alpha(\log),\beta,\gamma)}[f_1],\ \sigma_{(\alpha(\log),\beta,\gamma)}[f_2]\}$;
(ii) $\sigma_{(\alpha(\log),\beta,\gamma)}[f_2 \cdot f_2] \le \max\{\sigma_{(\alpha(\log),\beta,\gamma)}[f_1],\ \sigma_{(\alpha(\log),\beta,\gamma)}[f_2]\}$;
(iii) *If* $\sigma_{(\alpha(\log),\beta,\gamma)}[f_1] \ne \sigma_{(\alpha(\log),\beta,\gamma)}[f_2]$, *then* $\sigma_{(\alpha(\log),\beta,\gamma)}[f_1 \pm f_2] = \max\{\sigma_{(\alpha(\log),\beta,\gamma)}[f_1],\ \sigma_{(\alpha(\log),\beta,\gamma)}[f_2]\}$;
(iv) *If* $\sigma_{(\alpha(\log),\beta,\gamma)}[f_1] \ne \sigma_{(\alpha(\log),\beta,\gamma)}[f_2]$, *then* $\sigma_{(\alpha(\log),\beta,\gamma)}[f_2 \cdot f_2] = \max\{\sigma_{(\alpha(\log),\beta,\gamma)}[f_1],\ \sigma_{(\alpha(\log),\beta,\gamma)}[f_2]\}$.

3. Main Results

In this section we present our main results which considerably extend the results by Tu et al. [29] as well as those by Belaïdi [1, 2, 3].

Theorem 3.1. *Let $A_0(z)$, $A_1(z), \dots, A_{k-1}(z)$ be entire functions with $A_0(z) \not\equiv 0$ such that for real constants a, b, μ, θ_1, θ_2 with $0 \le b < a, \mu > 0$, $\theta_1 < \theta_2$, we have*

$$|A_0(z)| \ge \exp\{a \exp(\alpha^{-1}(\mu\beta(\log\gamma(|z|))))\} \tag{3.1}$$

and

$$|A_j(z)| \le \exp\{b \exp(\alpha^{-1}(\mu\beta(\log\gamma(|z|))))\}, \quad j = 1, \dots, k-1, \tag{3.2}$$

as $z \to \infty$ with $\theta_1 \le \arg z \le \theta_2$. Then $\sigma_{(\alpha(\log),\beta,\gamma)}[f] \ge \mu$ holds for all non-trivial solutions of (1.1).

Theorem 3.2. *Let H be a set of complex numbers satisfying $\overline{\operatorname{dens}}\{|z| : z \in H\} > 0$, and let $A_0(z), A_1(z), \dots, A_{k-1}(z)$ be entire functions and satisfy* (3.1) *and* (3.2) *as $z \to \infty$ for $z \in H$, where $0 \le b < a, \mu > 0$. Then every solution $f(z) \not\equiv 0$ of* (1.1) *satisfies $\sigma_{(\alpha(\log),\beta,\gamma)}[f] \ge \mu$.*

Theorem 3.3. *Let H be a set of complex numbers satisfying $\overline{\operatorname{dens}}\{|z| : z \in H\} > 0$, and let $A_0(z)$, $A_1(z), \dots, A_{k-1}(z)$ be entire functions of (α,β,γ)-order with $\max\{\sigma_{(\alpha,\beta,\gamma)}[A_j] : j = 1, \dots, k-1\} \le \sigma_{(\alpha,\beta,\gamma)}[A_0] = \sigma < +\infty$ such that for some constants $0 \le b < a$ and for any given $\varepsilon > 0$, we have*

$$|A_0(z)| \ge \exp\{a \exp(\alpha^{-1}((\sigma-\varepsilon)\beta(\log\gamma(|z|))))\} \tag{3.3}$$

and

$$|A_j(z)| \le \exp\{b \exp(\alpha^{-1}((\sigma-\varepsilon)\beta(\log\gamma(|z|))))\}, \quad j = 1, \dots, k-1, \tag{3.4}$$

as $z \to \infty$ for $z \in H$. Then every solution $f(z) \not\equiv 0$ of (1.1) *satisfies $\sigma_{(\alpha(\log),\beta,\gamma)}[f] = \sigma_{(\alpha,\beta,\gamma)}[A_0] = \sigma$.*

Theorem 3.4. *Let H, $A_0(z), A_1(z), \dots, A_{k-1}(z)$ satisfy the hypotheses of Theorem 3.3, and let $F(z) \not\equiv 0$ be an entire function of (α,β,γ)-order.*

(i) *If $\sigma_{(\alpha(\log),\beta,\gamma)}[F] < \sigma_{(\alpha,\beta,\gamma)}[A_0]$, then every solution $f(z)$ of* (1.2) *satisfies $\overline{\lambda}_{(\alpha(\log),\beta,\gamma)}[f] = \lambda_{(\alpha(\log),\beta,\gamma)}[f] = \sigma_{(\alpha(\log),\beta,\gamma)}[f] = \sigma$, with at most one exceptional solution $f_0(z)$ satisfying $\sigma_{(\alpha(\log),\beta,\gamma)}[f_0] < \sigma$.*
(ii) *If $\sigma_{(\alpha,\beta,\gamma)}[A_0] \le \sigma_{(\alpha(\log),\beta,\gamma)}[F] < +\infty$, then every solution $f(z)$ of* (1.2) *satisfies $\sigma_{(\alpha(\log),\beta,\gamma)}[f] = \sigma_{(\alpha(\log),\beta,\gamma)}[F]$.*

4. Some Lemmas

In this section we present some lemmas which will be needed in the sequel.

Lemma 4.1 ([11]). *Let $f(z)$ be a nontrivial entire function, and let $\kappa > 1$ and $\varepsilon > 0$ be given constants. Then there exist a constant $c > 0$ and a set $E_1 \subset [0, +\infty)$ having finite linear measure such that for all z satisfying $|z| = r \notin E_1$, we have*

$$\Big|\frac{f^{(k)}(z)}{f(z)}\Big| \le c[T(\kappa r, f) r^{\varepsilon} \log T(\kappa r, f)]^k \quad (k \in \mathbb{N}). \tag{4.1}$$

Lemma 4.2 ([13, 14, 21, 32]). *Let $f(z)$ be a transcendental entire function, and let z be a point with $|z| = r$ at which $|f(z)| = M(r, f)$. Then, for all $|z|$ outside a set E_2 of r of finite logarithmic measure, we have*

$$\frac{f^{(k)}(z)}{f(z)} = \Big(\frac{\nu(r, f)}{z}\Big)^k (1 + o(1)) \quad (k \in \mathbb{N},\ r \notin E_2), \tag{4.2}$$

where $\nu(r, f)$ is the central index of $f(z)$.

Lemma 4.3 ([6]). *Let $f(z)$ be an entire function satisfying $\sigma_{(\alpha(\log),\beta,\gamma)}[f] = \sigma_1$, and let $\nu(r, f)$ be the central index of $f(z)$. Then*

$$\limsup_{r\to+\infty} \frac{\alpha(\log^{[2]} \nu(r, f))}{\beta(\log \gamma(r))} = \sigma_1.$$

Lemma 4.4. *Let $f(z)$ be a transcendental entire function. Then $\sigma_{(\alpha(\log),\beta,\gamma)}[f] = \sigma_{(\alpha(\log),\beta,\gamma)}[f']$.*

Proof. By Cauchy's integral formula, we have

$$f'(z) = \frac{1}{2\pi i} \oint_{\Gamma} \frac{f(\zeta)}{(\zeta - z)^2} d\zeta,$$

where $\Gamma = \{\zeta : |\zeta - z| = R - r\}$, $|z| = r < R$. Set $\zeta - z = (R - r)e^{i\theta}$ $(0 \le \theta \le 2\pi)$, $d\zeta = (R - r)ie^{i\theta} d\theta$. Since $\max\{|f(\zeta)| : \zeta \in \Gamma\} \le M(R, f)$, then we obtain

$$M(r, f') = |f'(z)| \le \frac{1}{2\pi} \int_0^{2\pi} \frac{|f(\zeta)|}{|\zeta - z|^2} (R - r) d\theta \le \frac{M(R, f)}{R - r}.$$

Setting $R = r + 1$, it follows that

$$M(r, f') \le M(r + 1, f).$$

Since $\gamma(r + R_0) \sim \gamma(r)$ as $r \to +\infty$, it follows that

$$\begin{aligned}
\sigma_{(\alpha(\log),\beta,\gamma)}[f'] &= \limsup_{r\to+\infty} \frac{\alpha(\log^{[3]} M(r, f'))}{\beta(\log \gamma(r))} \\
&\le \limsup_{r\to+\infty} \Big(\frac{\alpha(\log^{[3]} M(r + 1, f))}{\beta(\log \gamma(r + 1))} \cdot \frac{\beta(\log \gamma(r + 1))}{\beta(\log \gamma(r))}\Big) \\
&= \limsup_{r\to+\infty} \Big(\frac{\alpha(\log^{[3]} M(r + 1, f))}{\beta(\log \gamma(r + 1))} \cdot \frac{\beta(\log \gamma(r))}{\beta(\log \gamma(r))}\Big) \\
&= \limsup_{r\to+\infty} \frac{\alpha(\log^{[3]} M(r + 1, f))}{\beta(\log \gamma(r + 1))}.
\end{aligned}$$

Thus, from the above we obtain

$$\sigma_{(\alpha(\log),\beta,\gamma)}[f'] \le \sigma_{(\alpha(\log),\beta,\gamma)}[f]. \tag{4.3}$$

On the other hand, for an entire function $f(z)$, we have $f(z)-f(0)=\int_0^z f'(t)dt$, where the integral being taken along the straight line from 0 to z, so we obtain that

$$M(r,f)\le \Big|\int_0^z f'(t)dt\Big| + |f(0)| \le rM(r,f') + |f(0)|.$$

Therefore from above we have

$$\log^{[3]} M(r,f)\le \log^{[3]} M(r,f') + \log^{[3]} r + \log^{[3]} |f(0)| + O(1).$$

Since $\alpha(a+b)\le\alpha(a)+\alpha(b)+c$, $c>0$ and $\alpha(\log^{[3]} x)=o(\beta(\log\gamma(x)))$, so from above we get that

$$\sigma_{(\alpha(\log),\beta,\gamma)}[f]\le\sigma_{(\alpha(\log),\beta,\gamma)}[f']. \tag{4.4}$$

Hence the lemma follows from (4.3) and (4.4). □

Remark 4.5. In the line of Lemma 4.4 one can easily deduce that $\sigma_{(\alpha,\beta,\gamma)}[f]=\sigma_{(\alpha,\beta,\gamma)}[f']$, where $f(z)$ is an entire transcendental function.

Lemma 4.6. *Let $f(z)$ be an entire function of (α,β,γ)-order that satisfies $\sigma_{(\alpha,\beta,\gamma)}[f]=\sigma$. Then there exists a set $E_3\subset(1,+\infty)$ having infinite logarithmic measure such that for all $r\in E_3$, we have*

$$\lim_{r\to+\infty}\frac{\alpha(\log T(r,f))}{\beta(\log\gamma(r))}=\sigma \quad (r\in E_3).$$

Proof. By Definition 2.1, there exists an increasing sequence $\{r_n\}_{n=1}^{+\infty}$ tending to $+\infty$ that satisfying $(1+\frac{1}{n})r_n<r_{n+1}$ and

$$\lim_{r_n\to+\infty}\frac{\alpha(\log T(r_n,f))}{\beta(\log\gamma(r_n))}=\sigma_{(\alpha,\beta,\gamma)}[f]=\sigma.$$

So, there exists an $n_1\in\mathbb{N}$ such that for $n\ge n_1$ and for any $r\in E_3=\cup_{n=n_1}^{+\infty}[r_n,(1+\frac{1}{n})r_n]$, we have

$$\frac{\alpha(\log T(r_n,f))}{\beta(\log\gamma((1+\frac{1}{n})r_n))}\le\frac{\alpha(\log T(r,f))}{\beta(\log\gamma(r))}\le\frac{\alpha(\log T((1+\frac{1}{n})r_n,f))}{\beta(\log\gamma(r_n))}. \tag{4.5}$$

From this inequality and $\gamma((1+\frac{1}{n})r_n)\le\gamma(2r_n)\le2\gamma(r_n)$, we have

$$\begin{aligned}
&\lim_{r\to+\infty,\, r\in E_3}\frac{\alpha(\log T(r,f))}{\beta(\log\gamma(r))}\\
&\ge\lim_{r_n\to+\infty}\Big(\frac{\alpha(\log T(r_n,f))}{\beta(\log\gamma(r_n))}\frac{\beta\big(\log\gamma(r_n)\big)}{\beta\big(\log\gamma((1+\frac{1}{n})r_n)\big)}\Big)\\
&\ge\lim_{r_n\to+\infty}\Big(\frac{\alpha(\log T(r_n,f))}{\beta\big(\log\gamma(r_n)\big)}\frac{\beta(\log\gamma(r_n))}{\beta\big(\log(2\gamma(r_n))\big)}\Big)\\
&=\lim_{r_n\to+\infty}\Big(\frac{\alpha(\log T(r_n,f))}{\beta(\log\gamma(r_n))}\frac{\beta(\log\gamma(r_n))}{\beta((1+\frac{\log 2}{\log\gamma(r_n)})\log\gamma(r_n))}\Big)\\
&=\lim_{r_n\to+\infty}\Big(\frac{\alpha(\log T(r_n,f))}{\beta\big(\log\gamma(r_n)\big)}\frac{\beta(\log\gamma(r_n))}{\beta\big((1+o(1))\log\gamma(r_n)\big)}\Big).
\end{aligned} \tag{4.6}$$

From this inequality and $\beta(x+o(1)) = (1+o(1))\beta(x)$ as $x \to +\infty$, we obtain that

$$\begin{aligned} &\lim_{r\to+\infty,\, r\in E_3} \frac{\alpha(\log T(r,f))}{\beta(\log\gamma(r))} \\ &\geq \lim_{r_n\to+\infty} \Big(\frac{\alpha(\log T(r_n,f))}{\beta(\log\gamma(r_n))} \frac{\beta(\log\gamma(r_n))}{(1+o(1))\beta(\log\gamma(r_n))} \Big) \\ &= \lim_{r_n\to+\infty} \frac{\alpha(\log T(r_n,f))}{\beta(\log\gamma(r_n))} = \sigma. \end{aligned} \tag{4.7}$$

On the other hand, by (4.5), $\gamma((1+\frac{1}{n})r_n) \leq \gamma(2r_n) \leq 2\gamma(r_n)$ and $\beta(x+o(1)) = (1+o(1))\beta(x)$ as $x \to +\infty$, we have

$$\begin{aligned} &\lim_{r\to+\infty,\, r\in E_3} \frac{\alpha(\log T(r,f))}{\beta(\log\gamma(r))} \\ &\leq \lim_{r_n\to+\infty} \Big(\frac{\alpha(\log T((1+\frac{1}{n})r_n,f))}{\beta(\log\gamma(1+\frac{1}{n})r_n)} \frac{\beta(\log\gamma((1+\frac{1}{n})r_n))}{\beta(\log\gamma(r_n))} \Big) \\ &\leq \lim_{r_n\to+\infty} \Big(\frac{\alpha(\log T((1+\frac{1}{n})r_n,f))}{\beta(\log\gamma(1+\frac{1}{n})r_n)} \frac{\beta(\log(2\gamma(r_n)))}{\beta(\log\gamma(r_n))} \Big) \\ &= \lim_{r_n\to+\infty} \Big(\frac{\alpha(\log T((1+\frac{1}{n})r_n,f))}{\beta(\log\gamma(1+\frac{1}{n})r_n)} \frac{\beta((1+o(1))\log\gamma(r_n))}{\beta(\log\gamma(r_n))} \Big) \\ &= \lim_{r_n\to+\infty} \Big(\frac{\alpha(\log T((1+\frac{1}{n})r_n,f))}{\beta(\log\gamma(1+\frac{1}{n})r_n)} \frac{(1+o(1))\beta(\log\gamma(r_n))}{\beta(\log\gamma(r_n))} \Big) = \sigma. \end{aligned} \tag{4.8}$$

Therefore, by (4.7) and (4.8), we obtain

$$\lim_{r\to+\infty,\, r\in E_3} \frac{\alpha(\log T(r,f))}{\beta(\log\gamma(r))} = \sigma,$$

where $lm(E_3) = \sum_{n=n_1}^{+\infty} \int_{r_n}^{(1+\frac{1}{n})} \frac{1}{t}dt = \sum_{n=n_1}^{+\infty} \log(1+\frac{1}{n}) = +\infty$. This completes the proof. □

Lemma 4.7. *Let $f(z)$ be an entire function of (α,β,γ)-order with $\sigma_{(\alpha,\beta,\gamma)}[f] = \sigma > 0$, and let $f_1(z)$ be an entire function of $(\alpha_1,\beta_1,\gamma_1)$-order with $\sigma_{(\alpha_1,\beta_1,\gamma_1)}[f_1] = \sigma_1 < +\infty$. If $\sigma_{(\alpha,\beta,\gamma)}[f]$ and $\sigma_{(\alpha_1,\beta_1,\gamma_1)}[f_1]$ satisfy one of the following conditions:*

(i) *$\alpha(r) = \alpha_1(r)$, $\beta(r) = \beta_1(r)$, $\gamma(r) = \gamma_1(r)$. and $\sigma_{(\alpha_1,\beta_1,\gamma_1)}[f_1] < \sigma_{(\alpha,\beta,\gamma)}[f]$;*

(ii) *$\lim_{r\to+\infty} \frac{\alpha_1^{-1}(r)}{\alpha^{-1}(r)} = 0$, $\beta(r) = \beta_1(r)$, $\gamma(r) = \gamma_1(r)$ and $\sigma_{(\alpha_1,\beta_1,\gamma_1)}[f_1] < \sigma_{(\alpha,\beta,\gamma)}[f]$;*

then there exists a set $E_4 \subset (1,+\infty)$ having infinite logarithmic measure such that for all $r \in E_4$, we have

$$\lim_{r\to+\infty} \frac{T(r,f_1)}{T(r,f)} = 0 \quad (r \in E_4).$$

Proof. (i) By definition, for all sufficiently large values of r, we obtain

$$T(r,f_1) \leq \exp\{\alpha^{-1}((\sigma_1+\varepsilon)\beta(\log\gamma(r)))\}. \tag{4.9}$$

From $\sigma_{(\alpha,\beta,\gamma)}[f] = \sigma$ and Lemma 4.6, there exists a set E_4 of infinite logarithmic measure satisfying

$$\lim_{r\to+\infty} \frac{\alpha(\log T(r,f))}{\beta(\log\gamma(r))} = \sigma \quad (r \in E_4).$$

Then

$$T(r,f) \geq \exp\{\alpha^{-1}((\sigma-\varepsilon)\beta(\log\gamma(r)))\} \quad (r\in E_4), \tag{4.10}$$

where $0<2\varepsilon<\sigma-\sigma_1$. Now by (4.9) and (4.10), we obtain that

$$\begin{aligned}
\frac{T(r,f_1)}{T(r,f)} &\leq \frac{\exp\{\alpha^{-1}((\sigma_1+\varepsilon)\beta(\log\gamma(r)))\}}{\exp\{\alpha^{-1}((\sigma-\varepsilon)\beta(\log\gamma(r)))\}} \\
&= \exp\Big\{\alpha^{-1}((\sigma_1+\varepsilon)\beta(\log\gamma(r))) - \alpha^{-1}((\sigma-\varepsilon)\beta(\log\gamma(r)))\Big\} \\
&= \exp\Big\{\alpha^{-1}((\sigma-\varepsilon)\beta(\log\gamma(r)))\Big(\frac{\alpha^{-1}((\sigma_1+\varepsilon)\beta(\log\gamma(r)))}{\alpha^{-1}((\sigma-\varepsilon)\beta(\log\gamma(r)))}-1\Big)\Big\} \\
&= \exp\Big\{\alpha^{-1}((\sigma-\varepsilon)\beta(\log\gamma(r)))\Big(\frac{\alpha^{-1}(\frac{\sigma_1+\varepsilon}{\sigma-\varepsilon}(\sigma-\varepsilon)\beta(\log\gamma(r)))}{\alpha^{-1}((\sigma-\varepsilon)\beta(\log\gamma(r)))}-1\Big)\Big\} \\
&= \exp\Big\{\alpha^{-1}((\sigma-\varepsilon)\beta(\log\gamma(r)))\Big(\frac{\alpha^{-1}(k(\sigma-\varepsilon)\beta(\log\gamma(r)))}{\alpha^{-1}((\sigma-\varepsilon)\beta(\log\gamma(r)))}-1\Big)\Big\} \\
&\to 0, \quad r\to+\infty \ (r\in E_4),\ k=\frac{\sigma_1+\varepsilon}{\sigma-\varepsilon}<1.
\end{aligned}$$

From the above inequality we obtain

$$\lim_{r\to+\infty}\frac{T(r,f_1)}{T(r,f)}=0 \quad (r\in E_4).$$

(ii) By definition, we obtain for all sufficiently large values of r that

$$T(r,f_1) \leq \exp\{\alpha_1^{-1}((\sigma_1+\varepsilon)\beta(\log\gamma(r)))\}. \tag{4.11}$$

Now by (4.10) and (4.11), for any given ε with $0<2\varepsilon<\sigma-\sigma_1$. we obtain that

$$\begin{aligned}
\frac{T(r,f_1)}{T(r,f)} &\leq \frac{\exp\{\alpha_1^{-1}((\sigma_1+\varepsilon)\beta(\log\gamma(r)))\}}{\exp\{\alpha^{-1}((\sigma-\varepsilon)\beta(\log\gamma(r)))\}} \\
&= \frac{\exp\{\alpha_1^{-1}((\sigma_1+\varepsilon)\beta(\log\gamma(r)))\}}{\exp\{\alpha^{-1}((\sigma_1+\varepsilon)\beta(\log\gamma(r)))\}}\frac{\exp\{\alpha^{-1}((\sigma_1+\varepsilon)\beta(\log\gamma(r)))\}}{\exp\{\alpha^{-1}((\sigma-\varepsilon)\beta(\log\gamma(r)))\}} \\
&= \exp\Big\{\alpha^{-1}((\sigma_1+\varepsilon)\beta(\log\gamma(r)))\Big(\frac{\alpha_1^{-1}((\sigma_1+\varepsilon)\beta(\log\gamma(r)))}{\alpha^{-1}((\sigma_1+\varepsilon)\beta(\log\gamma(r)))}-1\Big)\Big\} \\
&\quad\times \frac{\exp\{\alpha^{-1}((\sigma_1+\varepsilon)\beta(\log\gamma(r)))\}}{\exp\{\alpha^{-1}((\sigma-\varepsilon)\beta(\log\gamma(r)))\}}.
\end{aligned}$$

Since $\lim_{r\to+\infty}\frac{\alpha_1^{-1}(r)}{\alpha^{-1}(r)}=0$ and $\lim_{r\to+\infty}\frac{\alpha^{-1}(kr)}{\alpha^{-1}(r)}=0$ $(k<1)$, then by the inequality above, we obtain

$$\lim_{r\to+\infty}\frac{T(r,f_1)}{T(r,f)}=0 \quad (r\in E_4).$$

□

Lemma 4.8. *Let $F(z)\not\equiv 0$, $A_j(z)$ $(j=0,\dots,k-1)$ be entire functions. Also let $f(z)$ be a solution of (1.2) satisfying*

$$\max\{\sigma_{(\alpha(\log),\beta,\gamma)}[A_j],\sigma_{(\alpha(\log),\beta,\gamma)}[F]: j=0,1,\dots,k-1\}<\sigma_{(\alpha(\log),\beta,\gamma)}[f].$$

Then

$$\overline{\lambda}_{(\alpha(\log),\beta,\gamma)}[f]=\lambda_{(\alpha(\log),\beta,\gamma)}[f]=\sigma_{(\alpha(\log),\beta,\gamma)}[f].$$

Proof. By (1.2) we have

$$\frac{1}{f}=\frac{1}{F}\Big(\frac{f^{(k)}}{f}+A_{k-1}(z)\frac{f^{(k-1)}}{f}+\dots+A_1(z)\frac{f'}{f}+A_0\Big). \tag{4.12}$$

Now it is easy to see that if $f(z)$ has a zero at z_0 of order a $(a>k)$, and $A_0,\dots,A_{k-1}$ are analytic at z_0, then $F(z)$ must have a zero at z_0 of order $a-k$, hence

$$n\Big(r,\frac{1}{f}\Big)\le k\overline{n}\Big(r,\frac{1}{f}\Big)+n\Big(r,\frac{1}{F}\Big) \tag{4.13}$$

and

$$N\Big(r,\frac{1}{f}\Big)\le k\overline{N}\Big(r,\frac{1}{f}\Big)+N\Big(r,\frac{1}{F}\Big). \tag{4.14}$$

By the lemma on logarithmic derivative and (4.12), we have

$$m\Big(r,\frac{1}{f}\Big)\le m\Big(r,\frac{1}{F}\Big)+\sum_{j=0}^{k-1}m(r,A_j)+O(\log T(r,f)+\log r)\quad (r\notin E_5), \tag{4.15}$$

where E_5 is a set of r of finite linear measure. By (4.14) and (4.15), we obtain that

$$\begin{aligned} T(r,f)&=T\Big(r,\frac{1}{f}\Big)+O(1)\\ &\le k\overline{N}\Big(r,\frac{1}{f}\Big)+T(r,F)+\sum_{j=0}^{k-1}T(r,A_j)+O(\log(rT(r,f)))\end{aligned} \tag{4.16}$$

for $r\notin E_5$. Since $\max\{\sigma_{(\alpha(\log),\beta,\gamma)}[A_j],\sigma_{(\alpha(\log),\beta,\gamma)}[F]:j=0,1,\dots,k-1\}<\sigma_{(\alpha(\log),\beta,\gamma)}[f]$, by Lemma 4.7, there exists a set E_4 having infinite logarithmic measure such that

$$\max\Big\{\frac{T(r,F)}{T(r,f)},\frac{T(r,A_j)}{T(r,f)}\Big\}\to 0,\quad r\to+\infty\ (r\in E_4, j=0,\dots,k-1). \tag{4.17}$$

Since $f(z)$ is transcendental, we have

$$O(\log(rT(r,f)))=o(T(r,f)) \text{ as } r\to+\infty. \tag{4.18}$$

Therefore, by substituting (4.17) and (4.18) into (4.16), for all $|z|=r\in E_4\backslash E_5$, we obtain

$$T(r,f)\le O\Big(\overline{N}\Big(r,\frac{1}{f}\Big)\Big).$$

Hence from above we have

$$\sigma_{(\alpha(\log),\beta,\gamma)}[f]\le\overline{\lambda}_{(\alpha(\log),\beta,\gamma)}[f].$$

Therefore,

$$\overline{\lambda}_{(\alpha(\log),\beta,\gamma)}[f]=\lambda_{(\alpha(\log),\beta,\gamma)}[f]=\sigma_{(\alpha(\log),\beta,\gamma)}[f].$$

Hence the lemma follows. □

Lemma 4.9. *Let f be a meromorphic function. If $\sigma_{(\alpha,\beta,\gamma)}[f]=\sigma<+\infty$, then $\sigma_{(\alpha(\log),\beta,\gamma)}[f]=0$.*

Proof. Suppose that $\sigma_{(\alpha,\beta,\gamma)}[f]=\sigma<+\infty$. Then, for any given $\varepsilon>0$ and sufficiently large r, we have

$$T(r,f)\le\exp\{\alpha^{-1}((\sigma+\varepsilon)\beta(\log\gamma(r)))\}.$$

Then, we immediately obtain

$$\begin{aligned}\sigma_{(\alpha(\log),\beta,\gamma)}[f] &= \limsup_{r\to+\infty}\frac{\alpha(\log^{[2]}T(r,f))}{\beta(\log\gamma(r))}\\ &\le \limsup_{r\to+\infty}\frac{\alpha(\log^{[2]}(\exp\{\alpha^{-1}((\sigma+\varepsilon)\beta(\log\gamma(r)))\}))}{\beta(\log\gamma(r))}\\ &= \limsup_{r\to+\infty}\frac{\alpha(\log\alpha^{-1}((\sigma+\varepsilon)\beta(\log\gamma(r))))}{\beta(\log\gamma(r))}\\ &= \limsup_{x\to+\infty}\frac{\alpha(\log\alpha^{-1}((\sigma+\varepsilon)x))}{x}\\ &= (\sigma+\varepsilon)\limsup_{x\to+\infty}\frac{\alpha(\log x)}{\alpha(x)} = 0. \qquad \square\end{aligned}$$

5. Proof of main results

Proof of Theorem 3.1. Let $f(z) \not\equiv 0$ be a solution of (1.1) and rewrite (1.1) as

$$A_0(z) = -\Big(\frac{f^{(k)}(z)}{f(z)} + A_{k-1}(z)\frac{f^{(k-1)}(z)}{f(z)} + \dots + A_1(z)\frac{f'(z)}{f(z)}\Big).$$

Therefore,

$$|A_0(z)| \le \Big|\frac{f^{(k)}(z)}{f(z)}\Big| + |A_{k-1}(z)|\Big|\frac{f^{(k-1)}(z)}{f(z)}\Big| + \dots + |A_1(z)|\Big|\frac{f'(z)}{f(z)}\Big|. \tag{5.1}$$

By Lemma 4.1, there exist a constant $c > 0$ and a set $E_1 \subset [0,+\infty)$ having finite linear measure such that $|z| = r \notin E_1$ for all $z = re^{i\theta}$, we have

$$\Big|\frac{f^{(j)}(z)}{f(z)}\Big| \le c[rT(2r,f)]^{2k}, \quad j = 1,\dots,k. \tag{5.2}$$

By (5.1), (5.2), and the hypotheses of Theorem 3.1, we have

$$\begin{aligned}&\exp\{a\exp(\alpha^{-1}(\mu\beta(\log\gamma(|z|))))\}\\ &\le |A_0(z)|\\ &\le k\exp\{b\exp(\alpha^{-1}(\mu\beta(\log\gamma(|z|))))\}c[rT(2r,f)]^{2k}\end{aligned} \tag{5.3}$$

as $z \to \infty$ with $|z| = r \notin E_1$, $\theta_1 \le \arg z = \theta \le \theta_2$. Now from (5.3) we have

$$\begin{gathered}\exp\{(a-b)\exp(\alpha^{-1}(\mu\beta(\log\gamma(|z|))))\} \le kc[rT(2r,f)]^{2k},\\ (a-b)\exp(\alpha^{-1}(\mu\beta(\log\gamma(|z|)))) \le 2k(\log r + \log T(2r,f)) + \log(kc),\\ \exp(\alpha^{-1}(\mu\beta(\log\gamma(|z|)))) \le \frac{2k}{a-b}(\log r + \log T(2r,f)) + \frac{\log(kc)}{a-b}.\end{gathered}$$

By using $\alpha(a+b) \le \alpha(a) + \alpha(b) + c$ for all $x, y \ge R_0$ and fixed $c \in (0,+\infty)$, from the above we obtain

$$\begin{gathered}\alpha^{-1}(\mu\beta(\log\gamma(|z|))) \le \log^{[2]}T(2r,f) + \log^{[2]}r + O(1),\\ \mu\beta(\log\gamma(r)) \le \alpha((\log^{[2]}T(2r,f) + \log^{[2]}r + O(1))),\\ \mu\beta(\log\gamma(r)) \le \alpha(\log^{[2]}T(2r,f)) + \alpha(\log^{[2]}r) + c.\end{gathered} \tag{5.4}$$

By using $\gamma(2r)\le 2\gamma(r)$, $\beta(r+o(1))=(1+o(1))\beta(r)$ as $r\to+\infty$, and $\frac{\alpha(\log^{[2]} r)}{\beta(\log\gamma(r))}\to 0$ as $r\to+\infty$, then by (5.4) and Proposition 2.2, we have $\sigma_{(\alpha(\log),\beta)}[f]\ge\mu$. This completes the proof. □

Proof of Theorem 3.2. Let $f(z)\not\equiv 0$ be a solution of (1.1). By the hypotheses of Theorem 3.2, there exists a set H with $\overline{\operatorname{dens}}\{|z|: z\in H\}>0$ such that for all z satisfying $z\in H$, we have

$$|A_0(z)|\ge\exp\{a\exp(\alpha^{-1}(\mu\beta(\log\gamma(|z|))))\}, \tag{5.5}$$

$$|A_j(z)|\le\exp\{b\exp(\alpha^{-1}(\mu\beta(\log\gamma(|z|))))\},\quad j=1,\dots,k-1\,, \tag{5.6}$$

as $z\to\infty$. We set $H_1=\{|z|=r: z\in H\}$, since $\overline{\operatorname{dens}}\{|z|: z\in H\}>0$, it follows that H_1 is a set with $\int_{H_1}dr=+\infty$. Therefore from, by substituting (5.2), (5.5) and (5.6) into (5.1), it follows that for all z satisfying $|z|=r\in H_1\setminus E_1$, we have

$$\exp\{a\exp(\alpha^{-1}(\mu\beta(\log\gamma(|z|))))\}\le k\exp\{b\exp(\alpha^{-1}(\mu\beta(\log\gamma(|z|))))\}c[rT(2r,f)]^{2k}$$

as $|z|=r\to+\infty$. Thus

$$\exp\{(a-b)\exp(\alpha^{-1}(\mu\beta(\log\gamma(|z|))))\}\le kc[rT(2r,f)]^{2k} \tag{5.7}$$

as $|z|=r\in H_1\setminus E_1$, $r\to+\infty$. Since $\alpha(a+b)\le\alpha(a)+\alpha(b)+c$ for all $x,y\ge R_0$ and fixed $c\in(0,+\infty)$, $\gamma(2r)\le 2\gamma(r)$, $\beta(r+o(1))=(1+o(1))\beta(r)$ as $r\to+\infty$, and $\frac{\alpha(\log^{[2]} r)}{\beta(\log\gamma(r))}\to 0$ as $r\to+\infty$, then by (5.7) and Proposition 2.2, we obtain $\sigma_{(\alpha(\log),\beta,\gamma)}[f]\ge\mu$. □

Proof of Theorem 3.3. By Theorem 3.2, we have $\sigma_{(\alpha(\log),\beta,\gamma)}[f]\ge\sigma-\varepsilon$, since $\varepsilon>0$ is arbitrary, we obtain $\sigma_{(\alpha(\log),\beta,\gamma)}[f]\ge\sigma_{(\alpha,\beta,\gamma)}[A_0]=\sigma$. On the other hand, by Lemma 4.2, there exists a set $E_2\subset[1,+\infty)$ having finite logarithmic measure such that (4.2) holds for all z satisfying $|z|=r\notin[0,1]\cup E_2$ and $|f(z)|=M(r,f)$. Now for any given $\varepsilon>0$ and for sufficiently large r, we obtain

$$|A_j(z)|\le\exp^{[2]}\{\alpha^{-1}((\sigma+\varepsilon)\beta(\log\gamma(r)))\},\quad j=0,1,\dots,k-1. \tag{5.8}$$

Substituting (4.2) and (5.8) into (1.1), for all z satisfying $|z|=r\notin[0,1]\cup E_2$ and $|f(z)|=M(r,f)$, we have

$$\Big(\frac{\nu(r,f)}{|z|}\Big)^k|1+o(1)|\le k\Big(\frac{\nu(r,f)}{|z|}\Big)^{k-1}|1+o(1)|\exp^{[2]}\{\alpha^{-1}((\sigma+\varepsilon)\beta(\log\gamma(r)))\}.$$

It follows that

$$\nu(r,f)\le kr|1+o(1)|\exp^{[2]}\{\alpha^{-1}((\sigma+\varepsilon)\beta(\log\gamma(r)))\}. \tag{5.9}$$

Therefore in view of (5.9), $\alpha(a+b)\le\alpha(a)+\alpha(b)+c$ for all $x,y\ge R_0$ and fixed $c\in(0,+\infty)$ and $\frac{\alpha(\log^{[2]} r)}{\beta(\log\gamma(r))}\to 0$ as $r\to+\infty$, we obtain

$$\limsup_{r\to+\infty}\frac{\alpha(\log^{[2]}\nu(r,f))}{\beta(\log\gamma(r))}\le\sigma+\varepsilon. \tag{5.10}$$

Since $\varepsilon>0$ is arbitrary, by (5.10) and Lemma 4.3, we obtain that $\sigma_{(\alpha(\log),\beta,\gamma)}[f]\le\sigma$. This and the fact that $\sigma_{(\alpha(\log),\beta,\gamma)}[f]\ge\sigma$ yield $\sigma_{(\alpha(\log),\beta,\gamma)}[f]=\sigma$. The proof is complete. □

Proof of Theorem 3.4. (i) Suppose that $\sigma_{(\alpha(\log),\beta,\gamma)}[F] < \sigma_{(\alpha,\beta,\gamma)}[A_0]$. First, we show that (1.2) can possess at most one solution $f_0(z)$ satisfying $\sigma_{(\alpha(\log),\beta,\gamma)}[f_0] < \sigma$. In fact, if $f^*(z)$ is a second solution with $\sigma_{(\alpha(\log),\beta,\gamma)}[f^*] < \sigma$, then $\sigma_{(\alpha(\log),\beta,\gamma)}[f_0 - f^*] < \sigma$. But $f_0(z) - f^*(z)$ is a solution of the corresponding homogeneous equation (1.1) of (1.2), this contradicts Theorem 3.3. We assume that $f(z)$ is a solution with $\sigma_{(\alpha(\log),\beta,\gamma)}[f] \geq \sigma$ and $f_1(z)$, $f_2(z)$,..., $f_k(z)$ is a solution base of the corresponding homogeneous equation (1.1). Then, $f(z)$ can be expressed in the form

$$f(z) = B_1(z)f_1(z) + B_2(z)f_2(z) + \cdots + B_k(z)f_k(z), \tag{5.11}$$

where $B_1(z)$, $B_2(z)$, ..., $B_k(z)$ are determined by

$$\begin{gathered} B_1'(z)f_1(z) + B_2'(z)f_2(z) + \cdots + B_k'(z)f_k(z) = 0, \\ B_1'(z)f_1'(z) + B_2'(z)f_2'(z) + \cdots + B_k'(z)f_k'(z) = 0, \\ \cdots \\ B_1'(z)f_1^{(k-1)}(z) + B_2'(z)f_2^{(k-1)}(z) + \cdots + B_k'(z)f_k^{(k-1)}(z) = F(z). \end{gathered} \tag{5.12}$$

As the Wronskian $W(f_1, f_2, \dots, f_k)$ is a differential polynomial in $f_1, f_2, \dots, f_k$ with constant coefficients, it is easy to deduce that

$$\sigma_{(\alpha(\log),\beta,\gamma)}[W] \leq \sigma_{(\alpha(\log),\beta,\gamma)}[f_j] = \sigma_{(\alpha,\beta,\gamma)}[A_0] = \sigma. \tag{5.13}$$

From (5.12) we obtain

$$B_j' = F \cdot G_j(f_1, f_2, \dots, f_k) \cdot W(f_1, f_2, \dots, f_k)^{-1}, \quad j = 1, \dots, k, \tag{5.14}$$

where $G_j(f_1, f_2, \dots, f_k)$ are differential polynomials in $f_1, f_2, \dots, f_k$ with constant coefficients. Therefore,

$$\sigma_{(\alpha(\log),\beta)}[G_j] \leq \sigma_{(\alpha(\log),\beta)}[f_j] = \sigma_{(\alpha,\beta,\gamma)}[A_0] = \sigma, \quad j = 1, \dots, k. \tag{5.15}$$

Since $\sigma_{(\alpha(\log),\beta,\gamma)}[F] < \sigma_{(\alpha,\beta,\gamma)}[A_0]$, by Lemma 4.4, (5.13)-(5.15), for $j = 1, \dots, k$, we obtain

$$\begin{aligned} \sigma_{(\alpha(\log),\beta,\gamma)}[B_j] &= \sigma_{(\alpha(\log),\beta,\gamma)}[B_j'] \\ &\leq \max\{\sigma_{(\alpha(\log),\beta,\gamma)}[F], \sigma_{(\alpha,\beta,\gamma)}[A_0]\} \\ &= \sigma_{(\alpha,\beta,\gamma)}[A_0] = \sigma. \end{aligned} \tag{5.16}$$

Now, from (5.11) and (5.16), we have

$$\begin{aligned} \sigma_{(\alpha(\log),\beta,\gamma)}[f] &\leq \max\left\{\sigma_{(\alpha(\log),\beta,\gamma)}[f_j], \sigma_{(\alpha(\log),\beta,\gamma)}[B_j] \ (j = 1, \dots, k)\right\} \\ &= \sigma_{(\alpha,\beta,\gamma)}[A_0] = \sigma. \end{aligned} \tag{5.17}$$

This and the assumption $\sigma_{(\alpha(\log),\beta,\gamma)}[f] \geq \sigma$ yield $\sigma_{(\alpha(\log),\beta,\gamma)}[f] = \sigma$. By Lemma 4.9, we have

$$\begin{aligned} &\max\{\sigma_{(\alpha(\log),\beta,\gamma)}[F], \sigma_{(\alpha(\log),\beta,\gamma)}[A_j] \ (j = 0, 1, \dots, k-1)\} \\ &= \sigma_{(\alpha(\log),\beta,\gamma)}(F) \\ &< \sigma_{(\alpha,\beta,\gamma)}[A_0] = \sigma_{(\alpha(\log),\beta,\gamma)}[f]. \end{aligned}$$

So, if $f(z)$ is a solution of equation (1.2) satisfying $\sigma_{(\alpha(\log),\beta,\gamma)}[f] = \sigma$, then by Lemma 4.8, we have

$$\overline{\lambda}_{(\alpha(\log),\beta,\gamma)}[f] = \lambda_{(\alpha(\log),\beta,\gamma)}[f] = \sigma_{(\alpha(\log),\beta,\gamma)}[f] = \sigma.$$

(ii) Suppose that $\sigma_{(\alpha,\beta,\gamma)}[A_0] \le \sigma_{(\alpha(\log),\beta)}[F] < +\infty$. Then, by (5.16), for $j = 1,\dots,k$, we obtain

$$\begin{aligned} \sigma_{(\alpha(\log),\beta,\gamma)}[B_j] &= \sigma_{(\alpha(\log),\beta,\gamma)}[B_j'] \\ &\le \max\{\sigma_{(\alpha(\log),\beta,\gamma)}[F], \sigma_{(\alpha,\beta,\gamma)}[A_0]\} \\ &= \sigma_{(\alpha(\log),\beta,\gamma)}[F]. \end{aligned} \tag{5.18}$$

Now from (5.11) and (5.18), we obtain

$$\begin{aligned} \sigma_{(\alpha(\log),\beta,\gamma)}[f] &\le \max\{\sigma_{(\alpha(\log),\beta,\gamma)}[f_j], \sigma_{(\alpha(\log),\beta,\gamma)}[B_j]\ (j=1,\dots,k)\} \\ &\le \sigma_{(\alpha(\log),\beta,\gamma)}[F] \end{aligned}. \tag{5.19}$$

From (1.2), a simple consideration of $(\alpha(\log),\beta,\gamma)$-order implies that

$$\sigma_{(\alpha(\log),\beta,\gamma)}[f] \ge \sigma_{(\alpha(\log),\beta,\gamma)}[F].$$

By the above inequality and (5.19), we have $\sigma_{(\alpha(\log),\beta,\gamma)}[f] = \sigma_{(\alpha(\log),\beta,\gamma)}[F]$ which completes the proof. □

Acknowledgements. The authors are grateful to the referee for the careful reading of the manuscript and the valuable comments. The first author is supported by University of Mostaganem (UMAB) (PRFU Project Code C00L03UN270120220007).

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