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A WEIGHTED $(p,2)$-EQUATION WITH DOUBLE RESONANCE

ZHENHAI LIU, NIKOLAOS S. PAPAGEORGIOU

Abstract. We consider a Dirichlet problem driven by a weighted $(p,2)$-Laplacian with a reaction which is resonant both at $\pm\infty$ and at zero (double resonance). We prove a multiplicity theorem producing three nontrivial smooth solutions with sign information and ordered. In the appendix we develop the spectral properties of the weighted r-Laplace differential operator.

1. Introduction

Let $\Omega \subseteq \mathbb{R}^N$ be a bounded domain with a C^2-boundary $\partial\Omega$. In this article we study the weighted $(p,2)$-equation:

$$\begin{gathered} -\Delta_p^{a_1} u(z) - \Delta^{a_2} u(z) = f(z,u(z)) \quad \text{in } \Omega, \\ u|_{\partial\Omega} = 0, \quad 2 < p. \end{gathered} \tag{1.1}$$

Given $a \in L^\infty(\Omega)$ with $0 < \hat{c} \le \operatorname{ess\,inf}_\Omega a$ and $r \in (1,\infty)$, we denote by Δ_r^a the weighted r-Laplace differential operator defined by

$$\Delta_r^a u = \operatorname{div}(a(z)|Du|^{r-2}Du), \quad \forall u \in W_0^{1,r}(\Omega).$$

In problem (1.1) we have the sum of two such operators with different exponents. So,the differential operator driving the equation in (1.1) is not homogeneous and of course is space dependent. The reaction (right-hand side) of (1.1), is a Carathéodory function $f(z,x)$ (that is, for all $x \in \mathbb{R}, z \to f(z,x)$ is measurable and for a.a.$z \in \Omega, x \to f(z,x)$ is continuous) which exhibits $(p-1)$ sublinear growth as $x \to \pm\infty$ and resonance can occur with respect to the principal eigenvalue of $(-\Delta_p^{a_1}, W_0^{1,p}(\Omega))$ (see the apendix). Also at zero, we can have resonance with respect to some nonprincipal eigenvalue of $(-\Delta^{a_2}, H_0^1(\Omega))$. So, our problem has double resonance. Using variational tools from the critical point theory together with truncation techniques and critical groups, we prove a multiplicity theorem for problem (1.1), producing three nontrivial smooth solutions, all with sign information and ordered.

Recently a three solutions theorem for a superlinear weignted (p,q)-equation without resonance at zero, was proved by Liu-Papageorgiou [12], extending the well-known semilinear work of Wang [20]. Here we complement the aforementioned work of Liu-Papageorgiou [12], by examining the sublinear, double resonance case.

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Our hypotheses allow for resonance to occur as $x \to +\infty$ with respect to the principal eigenvalue of $(-\Delta_p^{a_1}, W_0^{1,p}(\Omega))$ and as $x \to 0^+$ with respect to a higher eigenvalue of $(-\Delta^{a_2}, H_0^1(\Omega))$. So we have a double resonance situation which has not been examined in the past.

2. Mathematical background and hypotheses

The analysis of problem (1.1) uses the Sobolev space $W_0^{1,p}(\Omega)$ and the Banach space $C_0^1(\overline{\Omega}) = \{u \in C^1(\overline{\Omega}) : u|_{\partial\Omega} = 0\}$. The Poincaré inequality implies that on $W_0^{1,p}(\Omega)$ we can use the equivalent norm

$$\|u\| = \|Du\|_p \quad \text{for all } u \in W_0^{1,p}(\Omega).$$

The Banach space $C_0^1(\overline{\Omega})$ is ordered with positive (order) cone $C_+ = \{u \in C_0^1(\overline{\Omega}) : u(z) \geq 0 \text{ for all } z \in \overline{\Omega}\}$. This cone has a nonempty interior given by

$$\operatorname{int} C_+ = \{u \in C_+ : u(z) > 0 \text{ for all } z \in \Omega,\ \frac{\partial u}{\partial \mathbf{n}}|_{\partial\Omega} < 0\}$$

with $\frac{\partial u}{\partial \mathbf{n}} = (Du, \mathbf{n})_{\mathbb{R}^N}$ where $\mathbf{n}(\cdot)$ is the outward unit normal on $\partial\Omega$.

If $u : \Omega \to \mathbb{R}$ is a measurable function, then we set $u^{\pm}(z) = \max\{\pm u(z), 0\}$ for all $z \in \Omega$. Both are measurable functions and we have $u = u^+ - u^-$, $|u| = u^+ + u^-$ and if $u \in W_0^{1,p}(\Omega)$, then $u^{\pm} \in W_0^{1,p}(\Omega)$.

Let $V : W_0^{1,p}(\Omega) \to W^{-1,p'}(\Omega)$ ($\frac{1}{p} + \frac{1}{p'} = 1$) be the nonlinear operator defined by

$$\langle V(u), h\rangle = \int_\Omega [a_1(z)|Du|^{p-2} + a_2(z)|Du|^{q-2}](Du, Dh)_{\mathbb{R}^N}\, dz \quad \forall u, h \in W_0^{1,p}(\Omega).$$

This operator is continuous and strictly monotone, thus maximal monotone too and of type $(S)_+$ (see [6, p. 279]).

Let X be a Banach space and $\varphi \in C^1(X, \mathbb{R})$. We say that $\varphi(\cdot)$ satisfies the "C-condition", if the following property holds:

> Every sequence $\{u_n\}_{n\in\mathbb{N}} \subseteq X$ such that $\{\varphi(u_n)\}_{n\in\mathbb{N}} \subseteq \mathbb{R}$ is bounded, and $(1 + \|u_n\|_X)\varphi'(u_n) \to 0$ in X^*, admits a strongly convergent subsequence.

A coercive functional $\varphi \in C^1(X, \mathbb{R})$ satisfies the C-condition (see Papageorgiou-Rădulescu-Repovš [16],p. 369).

Given $\varphi \in C^1(X, \mathbb{R})$ and $c \in \mathbb{R}$, we define the sets

$$K_\varphi = \{u \in X : \varphi'(u) = 0\}, \quad \varphi^c = \{u \in X : \varphi(u) \leq c\}.$$

For a topological pair (Y_2, Y_1) with $Y_1 \subseteq Y_2 \subseteq X$ and $k \in \mathbb{N}_0$, by $H_k(Y_2, Y_1)$ we denote the k^{th}-singular homology group with integer coefficients. Given $u \in K_\varphi$ isolated, the critical groups of φ at u, are defined by

$$C_k(\varphi, u) = H_k(\varphi^c \cap U, \varphi^c \cap U \setminus \{u\}) \quad \text{for all } k \in N_0,$$

with U a neighborhood of u such that $K_\varphi \cap \varphi^c \cap U = \{u\}$.The excision property of singular homology, implies that the above definition of critical groups at u, is independent of the particular choice of the isolating neighborhood U.

Suppose that $\varphi \in C^1(X, \mathbb{R})$ satisfies the C-condition and $-\infty < \inf \varphi(K_\varphi)$. Then the critical groups of $\varphi(\cdot)$ at infinity are

$$C_k(\varphi, \infty) = H_k(X, \varphi^c) \quad \text{for all } k \in N_0, \text{ with } c < \inf \varphi(K_\varphi).$$

The Second Deformation Theorem (see [16, p. 386]) implies that the above definition is independent of the choice of the level $c < \inf \varphi(K_\varphi)$.

By $\hat{\lambda}_1^{a_1}(p)$ we denote the first eigenvalue of $(-\Delta_p^{a_1}, W_0^{1,p}(\Omega))$. We know that $\hat{\lambda}_1^{a_1}(p) > 0$ is simple, isolated and is the only eigenvalue of $(-\Delta_p^{a_1}, W_0^{1,p}(\Omega))$ with eigenfunctions of constant sign. By $\hat{u}_1(p)$ we denote the positive, L^p-normalized (that is, $\|\hat{u}_1(p)\|_p = 1$) eigenfunction corresponding to $\hat{\lambda}_1^{a_1}(p)$. If $a \in C^{0,1}(\overline{\Omega})$ (i.e. the space of all $\mathbb{R}$-valued Lipschitz functions on $\overline{\Omega}$) and $0 < \hat{c} \le \min_{\overline{\Omega}} a_1$, then the nonlinear regularity theory (see Lieberman [11]) and the nonlinear maximum principle (see Liu-Papageorgiou [13, 14]), imply that $\hat{u}_1(p) \in \operatorname{int} C_+$. We denote by $\{\hat{\lambda}_m^{a_2}\}_{m\in\mathbb{N}}$ the sequence of distinct eigenvalues of $(-\Delta^{a_2}, H_0^1(\Omega))$. We know that $\hat{\lambda}_n^{a_2}(2) \to +\infty$ as $n \to \infty$ and the sequence exhausts the set of eigenvalues of $(-\Delta^{a_2}, H_0^1(\Omega))$. In the appendix, we present in detail the main spectral properties of $(-\Delta_p^{a_1}, W_0^{1,p}(\Omega))$ and of $(-\Delta^{a_2}, H_0^1(\Omega))$.

The hypotheses on the data of (1.1) are as follows:

- (H0) Functions $a_1, a_2 \in C^{0,1}(\overline{\Omega})$, and $0 < \hat{c} \le a_1(z), a_2(z)$ for all $z \in \overline{\Omega}$.
- (H1) $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function such that $f(z,0) = 0$ for a.a. $z \in \Omega$ and
  - (i) $|f(z,x)| \le a(z)[1 + |x|^{p-1}]$ for a.a. $z \in \Omega$, all $x \in \mathbb{R}$, with $a \in L^\infty(\Omega)_+$;
  - (ii) $\limsup_{x\to\pm\infty} \frac{f(z,x)}{|x|^{p-2}x} \le \hat{\lambda}_1^{a_1}(p)$ uniformly for a.a. $z \in \Omega$;
  - (iii) if $F(z,x) = \int_0^x f(z,s)ds$, then there exists $\tau \in (2,p)$ such that
$$\lim_{x\to\pm\infty} \frac{f(z,x)x - pF(z,x)}{|x|^\tau} = +\infty \text{ uniformly for a.a. } z \in \Omega;$$
  - (iv) there exist $m \in \mathbb{N}, m \ge 2, \delta > 0$ and $\eta \in L^\infty(\Omega)$ such that
$$\begin{gathered} \eta(z) \le \hat{\lambda}_{m+1}^{a_2}(2) \quad \text{for a.a. } z \in \Omega,\ \eta \not\equiv \hat{\lambda}_{m+1}^{a_2}(2), \\ \limsup_{x\to 0} \frac{f(z,x)}{x} \le \eta(z) \quad \text{uniformly for a.a. } z \in \Omega; \\ \hat{\lambda}_1^{a_2}(2)x^2 \le f(z,x)x \text{ for a.a. } z \in \Omega,\ \text{all } |x| \le \delta. \end{gathered}$$

Remark 2.1. Hypothesis (H1)(ii) implies that we can have resonance with respect to $\hat{\lambda}_1^{a_1}(p)$ as $x \to \pm\infty$. Similarly, hypothesis $H_1(iv)$ allows for resonance to occur with respect to $\hat{\lambda}_m^{a_2}(2)(m \ge 2)$ as $x \to 0$.

We introduce the energy functional for problem (1.1), $\varphi : W_0^{1,p}(\Omega) \to \mathbb{R}$ defined by
$$\varphi(u) = \frac{1}{p}\int_\Omega a_1(z)|Du|^p\,dz + \frac{1}{2}\int_\Omega a_2(z)|Du|^2\,dz - \int_\Omega F(z,u)\,dz\,.$$
Evidently $\varphi \in C^1(W_0^{1,p}(\Omega))$.

Also we introduce the positive and negative truncations of $\varphi(\cdot)$, namely the functionals $\varphi_\pm : W_0^{1,p}(\Omega) \to \mathbb{R}$ defined by
$$\varphi_\pm(u) = \frac{1}{p}\int_\Omega a_1(z)|Du|^p\,dz + \frac{1}{2}\int_\Omega a_2(z)|Du|^2\,dz - \int_\Omega F(z,\pm u^\pm)\,dz\,.$$
Again we have $\varphi_\pm \in C^1(W_0^{1,p}(\Omega))$.

3. Three solution theorem

In this section we prove that problem (1.1) has at least three nontrivial smooth solutions. Our approach uses variational and truncation techniques and critical groups.

Proposition 3.1. *Under hypotheses* (H0), (H1)*, the functionals φ and $\varphi_\pm$ are coercive.*

Proof. We do the proof for $\varphi_+(\cdot)$, the proofs for $\varphi(\cdot), \varphi_-(\cdot)$ being similar. We argue indirectly. So, suppose that $\varphi_+(\cdot)$ is not coercive. We can find $\{u_n\}_{n\in\mathbb{N}} \subseteq W_0^{1,p}(\Omega)$ such that

$$\varphi_+(u_n) \le c_1 \quad \text{for some } c_1 > 0 \text{ and all } n \in \mathbb{N},\ \|u_n\| \to \infty. \tag{3.1}$$

If $\{u_n^+\}_{n\in\mathbb{N}} \subseteq W_0^{1,p}(\Omega)$ is bounded, then from the inequality in (3.1) and hypothesis (H1)(i) we have $\{u_n^-\}_{n\in\mathbb{N}} \subseteq W_0^{1,p}(\Omega)$ is bounded; therefore $\{u_n\}_{n\in\mathbb{N}} \subseteq W_0^{1,p}(\Omega)$ is bounded, a contradiction to (3.1). So, we can say that

$$\|u_n^+\| \to \infty. \tag{3.2}$$

Let $y_n = \frac{u_n^+}{\|u_n^+\|}, n \in \mathbb{N}$. We have $\|y_n\| = 1, y_n \ge 0$ for all $n \in \mathbb{N}$. We can assume that

$$y_n \xrightarrow{w} y \text{ in } W_0^{1,p}(\Omega), \quad y_n \to y \text{ in } L^p(\Omega), y \ge 0. \tag{3.3}$$

From the inequality in (3.1), we have

$$\frac{1}{p}\int_\Omega a_1(z)|Dy_n|^p\,dz + \frac{1}{2\|u_n^+\|^{p-2}}\int_\Omega a_2(z)|Dy_n|^2\,dz \le c_1 + \int_\Omega \frac{F(z,u_n^+)}{\|u_n^+\|^p}\,dz \tag{3.4}$$

for all $n \in \mathbb{N}$. Using hypothesis (H1)(i), we have

$$\frac{|F(z,u_n^+(z))|}{\|u_n^+\|^p} \le c_2[1 + y_n(z)^p] \quad \text{for a.a. } z \in \Omega \text{ and all } n \in \mathbb{N},\ \text{some } c_2 > 0,$$

which implies

$$\Big\{\frac{F(\cdot,u_n^+(\cdot))}{\|u_n^+\|^p}\Big\}_{n\in\mathbb{N}} \subseteq L^1(\Omega) \text{ is uniformly integrable.}$$

Then invoking the Dunford-Pettis theorem (see Papageorgiou-Winkert [18, p. 289]), we can say that at least for a subsequence,

$$\frac{F(\cdot,u_n^+(\cdot))}{\|u_n^+\|^p} \xrightarrow{w} \frac{1}{p}\hat\eta \quad \text{in } L^1(\Omega). \tag{3.5}$$

Hypothesis (H1)(ii) implies that

$$\hat\eta(z) = \vartheta(z)y(z)^p \quad \text{for a.a. } z \in \Omega, \tag{3.6}$$

with $\vartheta \in L^\infty(\Omega)$, $\vartheta(z) \le \hat\lambda_1^{a_1}(p)$ for a.a. $z \in \Omega$ (see Aizicovici-Papageorgiou-Staicu [1], proof of Proposition 16). If in (3.4) we pass to the limit as $n \to \infty$ and use (3.2) (recall $2 < p$), (3.3), (3.5), (3.6), we obtain

$$\int_\Omega a_1(z)|Dy|^p\,dz \le \int_\Omega \eta(z)y^p\,dz. \tag{3.7}$$

First assume that $\vartheta \not\equiv \hat\lambda_1^{a_1}(p)$. Using Proposition 4.1 in the appendix, we have

$$c_3\|y\|^p \le \int_\Omega a_1(z)|Dy|^p\,dz - \int_\Omega \vartheta(z)y^p\,dz \quad \text{for some } c_3 > 0. \tag{3.8}$$

From (3.7) and (3.8) it follows that $y=0$. But then from (3.4) we have

$$\int_\Omega a_1(z)|Dy_n|^p\,dz \to 0$$

which implies $y_n \to 0$ in $W_0^{1,p}(\Omega)$ (see hypotheses (H0)). This contradicts that $\|y_n\|=1$ for all $n\in\mathbb{N}$.

Next we assume that $\vartheta(z)=\hat\lambda_1^{a_1}(p)$ for a.a. $z\in\Omega$. From (3.7) and the variational characterization of $\hat\lambda_1^{a_1}(p)>0$ (see (4.2) in the appendix), we have

$$\int_\Omega a_1(z)|Dy|^p\,dz = \hat\lambda_1^{a_1}(p)\|y\|_p^p,$$

which implies $y=\beta\hat u_1(p)$ with $\beta\ge 0$ (recall that $y\ge 0$).

If $\beta=0$, then $y=0$ and as above, we have $y_n\to 0$ in $W_0^{1,p}(\Omega)$, which contradiction that $\|y_n\|=1$ for all $n\in\mathbb{N}$. Hence $y=\beta\hat u_1(p)$ with $\beta>0$ and so $y\in\operatorname{int}C_+$ (since $\hat u_1(p)\in\operatorname{int}C_+$, see hypotheses (H0)). Therefore,

$$u_n^+(z)\to+\infty \quad \text{for a.a. } z\in\Omega. \tag{3.9}$$

Hypothesis (H1)(iii) implies that given any $M>0$, we can find $\gamma>0$ such that

$$Mx^\tau \le f(z,x)x - pF(z,x) \quad \text{for a.a. } z\in\Omega, \text{ and all } x\ge\gamma. \tag{3.10}$$

We have

$$\begin{aligned}
\frac{d}{dx}\Big(\frac{F(z,x)}{x^p}\Big) &= \frac{f(z,x)x^p - px^{p-1}F(z,x)}{x^{2p}}\\
&= \frac{f(z,x)x - pF(z,x)}{x^{p+1}}\\
&\ge \frac{M}{x^{p+1-\tau}} \quad \text{for a.a. } z\in\Omega, \text{ and all } x\ge\gamma \text{ (see (3.10))};
\end{aligned}$$

therefore,

$$\frac{F(z,v)}{v^p} - \frac{F(z,x)}{x^p} \ge -\frac{M}{p-\tau}\Big[\frac{1}{v^{p-\tau}} - \frac{1}{x^{p-\tau}}\Big]$$

for a.a. $z\in\Omega$ and all $v\ge x\ge\gamma>0$.

Letting $v\to+\infty$ and using (H1)(ii), We obtain

$$\begin{aligned}
&\frac{1}{p}\hat\lambda_1^{a_1}(p) - \frac{F(z,x)}{x^p} \ge \frac{M}{p-\tau}\frac{1}{x^{p-\tau}} \quad \text{for a.a. } z\in\Omega \text{ and all } x\ge\gamma,\\
&\Rightarrow \hat\lambda_1^{a_1}(p)x^p - pF(z,x) \ge \frac{M}{p-\tau}x^\tau \quad \text{for a.a. } z\in\Omega \text{ and all } x\ge\gamma,\\
&\Rightarrow \frac{\hat\lambda_1^{a_1}(p)x^p - pF(z,x)}{x^\tau} \ge \frac{M}{p-\tau} \quad \text{for a.a. } z\in\Omega \text{ and all } x\ge\gamma.
\end{aligned}$$

Since $M>0$ is arbitrary, it follows that

$$\frac{\hat\lambda_1^{a_1}(p)x^p - pF(z,x)}{x^\tau} \to +\infty \quad \text{as } x\to\infty, \text{ uniformly for a.a. } z\in\Omega. \tag{3.11}$$

From (3.1) and (4.2) (in the appendix), we have

$$\begin{gathered}
\int_\Omega [\hat\lambda_1^{a_1}(p)(u_n^+)^p - pF(z,u_n^+)]\,dz \le pc_1 \quad \text{for all } n\in\mathbb{N},\\
\Rightarrow \int_\Omega \frac{\hat\lambda_1^{a_1}(p)(u_n^+)^p - pF(z,u_n^+)}{(u_n^+)^\tau} y_n^\tau\,dz \le \frac{pc_1}{\|u_n^+\|^\tau} \quad \text{for all } n\in\mathbb{N}.
\end{gathered} \tag{3.12}$$

Passing to the limit as $n \to \infty$ in (3.12) and using (3.2), (3.9), (3.11) and Fatou's lemma, we reach a contradiction. This proves that $\{u_n^+\}_{n\in\mathbb{N}} \subseteq W_0^{1,p}(\Omega)$ is bounded. From the first part of the proof this implies that $\{u_n\}_{n\in\mathbb{N}} \subseteq W_0^{1,p}(\Omega)$ ia bounded, contradicting (3.1). Therefore $\varphi_+(\cdot)$ is coercive. Similarly for $\varphi(\cdot)$ and $\varphi_-(\cdot)$. $\square$

Using Proposition 3.1, we can produce two constant sign smooth solutions.

Proposition 3.2. *Under hypotheses (H0), (H1), problem* (1.1) *has two constant sign solutions $u_0 \in \operatorname{int} C_+$ and $v_0 \in -\operatorname{int} C_+$, which are local minimizers of $\varphi(\cdot)$.*

Proof. From Proposition 3.1 we know that $\varphi_+(\cdot)$ is coercive. Also, using the Sobolev embedding theorem, we see that $\varphi_+(\cdot)$ is sequentially weakly lower semicontinuous. So, by the Weierstrass-Tonelli theorem, we can find $u_0 \in W_0^{1,p}(\Omega)$ such that

$$\varphi_+(u_0) = \inf[\varphi_+(u) : u \in W_0^{1,p}(\Omega)]. \tag{3.13}$$

Recall that $\hat{u}_1(2) \in \operatorname{int} C_+$ (see the appendix). Therefore, we can find $t \in (0,1)$ small such that

$$0 \le t\hat{u}_1(2)(z) \le \delta \quad \text{for all } z \in \overline{\Omega}, \tag{3.14}$$

where $\delta > 0$ is as postulated by hypothesis $H_1(iv)$. Then, using (3.14) and hypothesis (H1)(iv), we have

$$\varphi_+(t\hat{u}_1(2)) \le \frac{t^p}{p}\int_\Omega a_1(z)|D\hat{u}_1(2)|^p\,dz + \frac{t^2}{2}[\hat{\lambda}_1^{a_2}(2) - \hat{\lambda}_m^{a_2}(2)] = c_4 t^p - c_5 t^2$$

for some positive constants c_4 and $c_5 > 0$. Here we used that $\|\hat{u}_1(2)\|_2 = 1$ and that $m \ge 2$.

Since $2 < p$, choosing $t \in (0,1)$ and small, we have

$$\varphi_+(t\hat{u}_1(2)) < 0,$$

which implies $\varphi_+(u_0) < 0 = \varphi_+(0)$ (see (3.13)); thus $u_0 \ne 0$. From (3.13) we have $\varphi_+'(u_0), h\rangle = 0$ for all $h \in W_0^{1,p}(\Omega)$, which implies

$$\langle V(u_0), h\rangle = \int_\Omega f(z, u_0^+)h\,dz \quad \text{for all } h \in W_0^{1,p}(\Omega). \tag{3.15}$$

In (3.15) we choose $h = -u_0^- \in W_0^{1,p}(\Omega)$ and obtain

$$\begin{gathered}\hat{c}\|Du_0^-\|_p^p \le 0 \quad \text{(see hypotheses (H0)}\\ \Rightarrow\ u_0 \ge 0, \quad u_0 \ne 0.\end{gathered}$$

Then from (3.15) we have

$$-\Delta_p^{a_1}u_0 - \Delta^{a_2}u_0 = f(z, u_0) \quad \text{in } \Omega. \tag{3.16}$$

From Ladyzhenskaya-Uraltseva [10, p. 286], we have $u_0 \in L^\infty(\Omega)$. Then the nonlinear regularity theory of Lieberman [11] implies that $u_0 \in C_+ \setminus \{0\}$. On account of hypotheses (H1)(i),(iv), we can find $c_6 > 0$ such that $f(z,x) \ge \hat{\lambda}_m^{a_2}(2)x - c_6 x^{p-1}$ for a.a. $z \in \Omega$ and all $x \ge 0$. So, if $\hat{\vartheta} > c_6$, then

$$f(z,x) + \hat{\vartheta}x^{p-1} \ge 0 \quad \text{for a.a. } z \in \Omega \text{ and all } z \ge 0.$$

From (3.16) we have

$$-\Delta_p^{a_1}u_0 - \Delta^{a_2}u_0 + \hat{\vartheta}u_0^{p-1} \ge 0 \text{ in } \Omega,$$

which implies $u_0 \in \operatorname{int} C_+$, see [13, Lemma 1]. Note that

$$\varphi|_{C_+} = \varphi_+|_{C_+}.$$

So u_0 is a local $C_0^1(\overline{\Omega})$ minimizer of $\varphi(\cdot)$ and from [17, Proposition A3], we conclude that u_0 is a local $W_0^{1,p}(\Omega)$-minimizer of $\varphi(\cdot)$.

Similarly, working now with $\varphi_-(\cdot)$, we produce a negative solution $v_0 \in -\operatorname{int} C_+$, which is a local minimizer of $\varphi(\cdot)$. $\square$

We assume that K_φ is finite or otherwise we already have an infinity of solutions of (1.1) and so we are done. Then from Proposition 3.2 and [16, Proposition 6.2.3], we have the following result.

Corollary 3.3. *If* (H0), (H1) *hold and $u_0 \in \operatorname{int} C_+$ and $v_0 \in -\operatorname{int} C_+$ are the two constants sign solutions from Proposition 3.2, then $C_k(\varphi, u_0) = C_k(\varphi, v_0) = \delta_{k,0}\mathbb{Z}$ for all $k \in \mathbb{N}_0$.*

In what follows we denote by $E(\hat{\lambda}_1^{a_2}(2))$ the eigenspace corresponding is the eigenvalue $\hat{\lambda}_i^{a_2}(2), i \in \mathbb{N}$. We know that $E(\hat{\lambda}_i^{a_2}(2))$ is finite dimensional and $E(\hat{\lambda}_i^{a_2}(2)) \subseteq C_0^1(\overline{\Omega})$ (see the appendix).

Proposition 3.4. *If hypotheses* (H0), (H1) *hold, then $C_{d_m}(\varphi, 0) \neq 0$, where $d_m = \dim \overline{H}_m$ with $\overline{H}_m = \oplus_{i=1}^m E(\hat{\lambda}_1^{a_2}(2))$.*

Proof. We consider the C^1-functional $\psi : H_0^1(\Omega) \to \mathbb{R}$ defined by

$$\psi(u) = \frac{1}{2}\int_\Omega a_2(z)|Du|^2\,dz - \int_\Omega F(z,u)\,dz\,.$$

Let $u \in \overline{H}_m$. Since $\overline{H}_m \subseteq C(\overline{\Omega})$ is finite dimensional, all norms are equivalent and so we can find $\rho > 0$ such that

$$u \in \overline{H}_m \text{ and } \|u\| \le \rho \;\Rightarrow\; |u(z)| \le \delta \text{ for all } z \in \overline{\Omega},$$

with $\delta > 0$ as postulated by hypothesis (H1)(iv). So for $u \in \overline{H}_m$ with $\|u\| \le \rho$, we have

$$\psi(u) \le \frac{1}{2}[\|Du\|_2^2 - \hat{\lambda}_m^{a_2}(2)\|u\|_2^2] \le 0 \tag{3.17}$$

(see hypothesis (H1)(iv) and the appendix). On account of hypotheses (H1)(i),(iv), given $\varepsilon > 0$, we can find $c_\varepsilon > 0$ such that

$$F(z,x) \le \frac{1}{2}[\eta(z) + \varepsilon]x^2 + c_\varepsilon|x|^p \quad \text{for a.a. } z \in \Omega \text{ and all } x \in \mathbb{R}. \tag{3.18}$$

Then for $u \in \hat{H}_{m+1} = \overline{H}_m^\perp$, from (3.18), we have

$$\begin{aligned}\psi(u) &\ge \frac{1}{2}[\int_\Omega a_2(z)|Du|^2\,dz - \int_\Omega \eta(z)u^2\,dz - \varepsilon\|u\|_2^2] - \hat{c}_\varepsilon\|u\|^p \text{ for some } \hat{c}_\varepsilon > 0\\ &\ge \frac{1}{2}[c_7 - \frac{\varepsilon}{\hat{\lambda}_{m+1}^{a_2}}]\|u\|^2 - \hat{c}_\varepsilon\|u\|^p\end{aligned}$$

for some positive constant c_7 (see Proposition 4.2).

Choosing $\varepsilon \in (0, \hat{\lambda}_{m+1}^{a_2}(2)c_7)$, we obtain

$$\psi(u) \ge c_8\|u\|^2 - \hat{c}_\varepsilon\|u\|^p \quad \text{for some } c_8 > 0.$$

So, we can find $\rho_0 \in (0, \rho]$ such that

$$\psi(u) > 0 \quad \text{for all } u \in \hat{H}_{m+1},\ 0 < \|u\| \le \rho_0. \tag{3.19}$$

Then (3.17) and (3.19) imply that $\psi(\cdot)$ has a local linking at $u = 0$. So, invoking [16, Theorem 6.6.17], we have

$$C_{d_m}(\psi, 0) \neq 0.$$

Let $\hat{\psi} = \psi|_{W_0^{1,p}(\Omega)}$. Since $W_0^{1,p}(\Omega) \hookrightarrow H_0^1(\Omega)$ continuously and densely, using [15, Theorems 16 and 17] (see also Chang [12, p. 14]), we have

$$C_k(\hat{\psi}, 0) = C_k(\psi, 0) \quad \text{for all } k \in \mathbb{N}_0, \tag{3.20}$$

which implies $C_{d_m}(\hat{\psi}, 0) \neq 0$. Note that

$$|\varphi(u) - \hat{\psi}(u)| = \frac{1}{p}\int_\Omega a_1(z)|Du|^p\,dz \leq \frac{\|a_1\|_\infty}{p}\|u\|^p. \tag{3.21}$$

Also for all $h \in W_0^{1,p}(\Omega)$, we have

$$\begin{aligned}
|\langle \varphi'(u) - \hat{\psi}'(u), h\rangle| &= \int_\Omega a_1(z)|Du|^{p-2}(Du, Dh)_{\mathbb{R}^N}\,dz \\
&\leq \|a_1\|_\infty \int_\Omega |Du|^{p-1}|Dh|\,dz \\
&\leq \|a_1\|_\infty \|Du\|_p^{p-1}\|Dh\|_p;
\end{aligned}$$

therefore,

$$\|\varphi'(u) - \hat{\psi}'(u)\| \leq c_9\|u\|^{p-1} \quad \text{for some } c_9 > 0. \tag{3.22}$$

Recall that we assume K_φ is finite (otherwise we already have an infinity of distinct nontrivial smooth positive solutions and so we are done). Then from (3.21), (3.22) and the C^1-continuity property of critical groups (see Gasiński-Papageorgiou [6, p. 836]), we have

$$C_k(\varphi, 0) = C_k(\hat{\psi}, 0) \quad \text{for all } k \in \mathbb{N}_0,$$

which implies $C_{d_m}(\varphi, 0) \neq 0$ (see (3.20)). □

Next we will produce a third nontrivial solution which is nodal. To do this, we need some auxiliary results. Note that hypotheses (H1)(i),(iv) imply that we can find $c_{10} > 0$ such that

$$f(z, x)x \geq \hat{\lambda}_m^{a_2} x^2 - c_{10}|x|^p \text{ for a.a. } z \in \Omega \text{ and all } x \in \mathbb{R}. \tag{3.23}$$

Then (3.23) leads to the auxiliary Dirichlet problem

$$\begin{gathered}
-\Delta_p^{a_1} u(z) - \Delta^{a_2} u(z) = \hat{\lambda}_m^{a_2}(2)u(z) - c_{10}|u(z)|^{p-2}u(z) \quad \text{in } \Omega, \\
u|_{\partial\Omega} = 0.
\end{gathered} \tag{3.24}$$

Reasoning as in [13, Proposition 3], we have the following result.

Proposition 3.5. *If hypotheses* (H0) *holds and $m \geq 2$, then problem* (3.24) *has a unique positive solution $\overline{u} \in \operatorname{int} C_+$ and since the equation is odd $\overline{v} = -\overline{u} \in -\operatorname{int} C_+$ is the unique negative solution of problem* (3.24).

Using Proposition 3.5, we can produce extremal constant sign solutions for problem (1.1), that is, a smallest positive solution and a biggest negative solution.

Proposition 3.6. *If hypotheses* (H0), (H1) *hold, then problem* (1.1) *has a smallest positive solution*

$$\hat{u} \in \operatorname{int} C_+$$

and a biggest negative solution

$$\hat{v} \in -\operatorname{int} C_+.$$

Proof. Let S_+ (resp. S_-) denote the set of positive (resp. negative) solutions of (1.1). From Proposition 3.2 and its proof, we know that

$$\emptyset \neq S_+ \subseteq \operatorname{int} C_+ \quad \text{and} \quad \emptyset \neq S_- \subseteq -\operatorname{int} C_+.$$

Moreover, we know (see Filippakis-Papageorgiou [4]) that

$$S_+ \text{ is downward directed}$$

(that is, if $u_1, u_2 \in S_+$, then there exists $u \in S_+$ such that $u \leq u_1, u \leq u_2$),

$$S_- \text{ is upward directed}$$

(that is, if $v_1, v_2 \in S_-$, then there exists $v \in S_-$ such that $v_1 \leq v, v_2 \leq v$).

Next we show that

$$\overline{u} \leq u \text{ for all } u \in S_+, \quad v \leq \overline{v} \text{ for all } v \in S_-. \tag{3.25}$$

To this end, let $u \in S_+ \subseteq \operatorname{int} C_+$ and introduce the Caratheodory function $k_+ : \Omega \times \mathbb{R} \to \mathbb{R}$ defined

$$k_+(z,x) = \begin{cases} \hat{\lambda}_m^{a_2}(2)x^+ - c_{10}(x^+)^{p-1} & \text{if } x \leq u(z) \\ \hat{\lambda}_m^{a_2}(2)u(z) - c_{10}(u(z))^{p-1} & \text{if } u(z) \leq x. \end{cases} \tag{3.26}$$

We set $K_+(z,x) = \int_0^x k_+(z,s)ds$ and consider the C^1-functional $\sigma_+ : W_0^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\sigma_+(u) = \frac{1}{p}\int_\Omega a_1(z)|Du|^p\,dz + \frac{1}{2}\int_\Omega a_2(z)|Du|^2\,dz - \int_\Omega K_+(z,u)\,dz.$$

From hypotheses H_0 and (3.26), we see that $\sigma_+(\cdot)$ is coercive. Also, by the Sobolev embedding theorem $\sigma_+(\cdot)$ is sequentially weakly lower semicontinuous. So, we can find $\tilde{u} \in W_0^{1,p}(\Omega)$ such that

$$\sigma_+(\tilde{u}) = \inf[\sigma_+(u) : u \in W_0^{1,p}(\Omega)]. \tag{3.27}$$

Since $u \in \operatorname{int} C_+$, we can find $t \in (0,1)$ small such that $t\hat{u}_1(2) \leq u$ (see [16, Proposition 4.1.22]. Then using (3.26) and since $m \geq 2$ we have (by taking $t \in (0,1)$ and small, $\sigma_+(t\hat{u}_1(2)) < 0$ which implies

$$\sigma_+(\tilde{u}) < 0 = \sigma_+(0) \quad \text{(see (3.27))};$$

thus, $\tilde{u} \neq 0$. From (3.27) we have that $\langle \sigma_+', h\rangle = 0$ for all $h \in W_0^{1,p}(\Omega)$. Therefore,

$$\langle V(\tilde{u}), h\rangle = \int_\Omega k_+(z,\tilde{u})h\,dz \quad \text{for all } h \in W_0^{1,p}(\Omega). \tag{3.28}$$

In (3.28) first we use the test function $h = -\tilde{u}^- \in W_0^{1,p}(\Omega)$ and obtain $\hat{c}\|D\tilde{u}^-\|_p^p \leq 0$, hence $\tilde{u} \geq 0, \tilde{u} \neq 0$.

Next choosing $h = [\tilde{u} - u]^+ \in W_0^{1,p}(\Omega)$ in (3.28), we have

$$\langle V(\tilde{u}), (\tilde{u}-u)^+\rangle = \int_\Omega [\hat{\lambda}_m^{a_2}(2)u - c_{10}u^{p-1}](\tilde{u}-u)^+\,dz \quad \text{(see (3.26))}$$

$$\begin{aligned}&\le \int_\Omega f(z,u)(\tilde u-u)^+\,dz \quad (\text{see } (3.23))\\&=\langle V(u),(\tilde u-u)^+\rangle \quad (\text{since } u\in S_+),\end{aligned}$$

which implies $\tilde u\le u$, from the monotonicity of $V(\cdot)$). So, we have proved that

$$\tilde u\in[0,u],\tilde u\ne 0. \tag{3.29}$$

Then (3.29), (3.26), (3.28), and Proposition 3.5 imply that $\tilde u=\overline{u}\in \operatorname{int} C_+$. This in turn implies $\overline{u}\le u$ for all $u\in S_+$. Similarly, we can show that $v\le\overline{v}$ for all $v\in S_-$.

Using [8, Theorem 5.109], we can find $\{u_n\}_{n\in\mathbb{N}}\subseteq S_+$ decreasing (since S_+ is downward directed) such that

$$\inf S_+=\inf_{n\in\mathbb{N}} u_n.$$

We have

$$\begin{gathered}\langle V(u_n),h\rangle=\int_\Omega f(z,u_n)h\,dz \quad \text{for all } h\in W_0^{1,p}(\Omega) \text{ and all } n\in\mathbb{N}, \qquad (3.30)\\ \overline{u}\le u_n\le u_1 \quad \text{for all } n\in\mathbb{N} \quad (\text{see } (3.25)). \qquad (3.31)\end{gathered}$$

Choosing $h=u_n\in W_0^{1,p}(\Omega)$ in (3.30) and using (3.31) and (H1)(i), we infer that $\{u_n\}_{n\in\mathbb{N}}\subseteq W_0^{1,p}(\Omega)$ is bounded. So, we may assume that

$$u_n\xrightarrow{w}\hat u \text{ in } W_0^{1,p}(\Omega),\quad u_n\to\hat u \text{ in } L^p(\Omega),\quad \text{as } n\to\infty. \tag{3.32}$$

In (3.30) we use $h=u_n-\hat u\in W_0^{1,p}(\Omega)$, pass to the limit as $n\to\infty$ and use (3.32). We obtain $\lim_{n\to\infty}\langle V(u_n),u_n-\hat u\rangle=0$ which implies

$$u_n\to\hat u \quad \text{in } W_0^{1,p}(\Omega) \quad (\text{the } (S)_+\text{-property of } V(\cdot)). \tag{3.33}$$

Passing to the limit as $n\to\infty$ in (3.30) and using (3.33), we obtain

$$\begin{gathered}\langle V(\hat u),h\rangle=\int_\Omega f(z,\hat u)h\,dz \quad \text{for all } h\in W_0^{1,p}(\Omega),\\ \overline{u}\le\hat u \quad (\text{see } (3.31)),\end{gathered}$$

Therefore, $\hat u\in S_+\subseteq \operatorname{int} C_+$ and $\hat u=\inf S_+$.

Similarly for S_- which is upward directed and so the sequence $\{v_n\}_{n\in\mathbb{N}}\subseteq S_-$ such that $\sup S_-=\sup_{n\in\mathbb{N}} v_n$, will be increasing. □

Now we try to produce a nontrivial solution of (1.1) in the order interval

$$[\hat v,\hat u]=\{h\in W_0^{1,p}(\Omega): \hat v(z)\le h(z)\le\hat u(z) \text{ for a.a. } z\in\Omega\}.$$

On account of the extremality of $\hat u$ and $\hat v$ any such solution distinct from $\hat u$ and $\hat v$ will be nodal.

To this end, we introduce the following truncation of the reaction $f(z,\cdot)$

$$e(z,x)=\begin{cases} f(z,\hat v(z)) & \text{if } x<\hat v(z)\\ f(z,x) & \text{if } \hat v(z)\le x\le\hat u(z)\\ f(z,\hat u(z)) & \text{if } \hat u(z)<x.\end{cases} \tag{3.34}$$

Also, we consider the positive and negative truncations of $e(z,\cdot)$, namely the functions

$$e_\pm(z,x)=e(z,\pm x^\pm). \tag{3.35}$$

All three functions are Caratheodory. We set

$$E(z,x)=\int_0^x e(z,s)ds,\quad E_\pm(z,x)=\int_0^x e_\pm(z,s)ds.$$

Then we consider the C^1-functionals $w,w_\pm : W_0^{1,p}(\Omega)\to\mathbb{R}$ defined by

$$\begin{gathered}w(u)=\frac{1}{p}\int_\Omega a_1(z)|Du|^p\,dz+\frac{1}{2}\int_\Omega a_2(z)|Du|^2\,dz-\int_\Omega E(z,u)\,dz,\\ w_\pm(u)=\frac{1}{p}\int_\Omega a_1(z)|Du|^p\,dz+\frac{1}{2}\int_\Omega a_2(z)|Du|^2\,dz-\int_\Omega E_\pm(z,u)\,dz\,.\end{gathered}$$

Using (3.34) and (3.35), we can easily show that

$$K_w\subseteq[\hat v,\hat u]\cap C_0^1(\overline{\Omega}),K_{w_+}\subseteq[0,\hat u]\cap C_+,K_{w_-}\subseteq[\hat v,0]\cap(-C_+).$$

The extremality of $\hat u,\hat v$ implies that

$$K_w\subseteq[\hat v,\hat u]\cap C_0^1(\overline{\Omega}),\quad K_{w_+}=\{0,\hat u\},K_{w_-}\subseteq\{\hat v,0\}. \tag{3.36}$$

Now, we can generate the third nontrivial smooth solution of (1.1) which is nodal. By $int_{C_0^1(\overline{\Omega})}[\hat v,\hat u]$ we denote the interior in $C_0^1(\overline{\Omega})$ of $[\hat v,\hat u]\cap C_0^1(\overline{\Omega})$.

Proposition 3.7. *If* (H0), (H1) *hold, then problem* (1.1) *has a nodal solution* $y_0\in[\hat v,\hat u]\cap C_0^1(\overline{\Omega})$.

Proof. First we show that $\hat u,\hat v$ are local minimizers of $w(\cdot)$. To this end, note that $w_+(\cdot)$ is coercive (see (H0) and (3.34), (3.35)). Also, it is sequentially weakly lower semicontinuous. So, we can find $\tilde u\in W_0^{1,p}(\Omega)$ such that

$$w_+(\tilde u)=\inf[w_+(u):u\in W_0^{1,p}(\Omega)]. \tag{3.37}$$

Since $\hat u\in\operatorname{int}C_+$ as before for $t\in(0,1)$ small (at least such that $t\hat u_1(2)\le\hat u$), we have

$$\begin{gathered}w_+(\pm\hat u_1(2))<0,\\ \Rightarrow w_+(\tilde u)<0=w_+(0),\\ \Rightarrow\tilde u\neq0.\end{gathered}\tag{3.38}$$

From (3.37),(3.38) and (3.36) we infer that $\tilde u=\hat u\in\operatorname{int}C_+$. Note that

$$\begin{gathered}w|_{C_+}=w_+|_{C_+},\\ \Rightarrow\hat u\text{ is a local }C_0^1(\overline{\Omega})\text{-minimizer of }w(\cdot)\\ \Rightarrow\hat u\text{ is a local }W_0^{1,p}(\Omega)\text{-minimizer of }w(\cdot)\,;\end{gathered}$$

see Papageorgiou-Rădulescu-Zhang [17, Proposition A3].

Similarly for $\hat v\in-\operatorname{int}C_+$ using the functional $w_-(\cdot)$, we have

$$C_k(w,\hat u)=C_k(w,\hat v)=\delta_{k,0}\mathbb{Z}\quad\text{for all }k\in\mathbb{N}_0. \tag{3.39}$$

Next we show that

$$C_{d_m}(w,0)\neq0. \tag{3.40}$$

Consider the homotopy

$$h(t,u)=(1-t)\varphi(u)+tw(u)\quad\text{for all }(t,u)\in[0,1]\times W_0^{1,p}(\Omega).$$

Suppose we can find $\{(t_n,u_n)\}_{n\in\mathbb{N}}\subseteq[0,1]\times W_0^{1,p}(\Omega)$ such that

$$t_n\to t\text{ in }[0,1],\quad u_n\to0\text{ in }W_0^{1,p}(\Omega),\quad h'_u(t_n,u_n)=0\quad\text{for all }n\in\mathbb{N}. \tag{3.41}$$

From the equation in (3.41) we have

$$\langle V(u_n), h\rangle = \int_\Omega [(1-t_n)f(z,u_n) + t_n e(z,u_n)]h\,dz \tag{3.42}$$

for all $h \in W_0^{1,p}(\Omega)$ and all $n \in \mathbb{N}$,

From (3.41), (3.42), and [10, Theorem 7.1], we know that we can find $c_{11} > 0$ such that

$$u_n \in L^\infty(\Omega), \|u_n\|_\infty \le c_{11} \quad \text{for all } n \in \mathbb{N}.$$

Then the nonlinear regularity by Lieberman [11], implies that there exist $\alpha \in (0,1)$ and $c_{12} > 0$ such that

$$u_n \in C_0^{1,\alpha}(\overline{\Omega}), \quad \|u_n\|_{C_0^{1,\alpha}(\overline{\Omega})} \le c_{12}, \quad \text{for all } n \in \mathbb{N}. \tag{3.43}$$

Since $C_0^{1,\alpha}(\overline{\Omega}) \hookrightarrow C_0^1(\overline{\Omega})$ compactly, from (3.43) and (3.41) it follows that $u_n \to u$ in $C_0^1(\overline{\Omega})$. This implies

$$u_n \in \operatorname{int}_{C_0^1(\overline{\Omega})}[\hat{v},\hat{u}] \quad \text{for all } n \ge n_0$$

(recall $\hat{u} \in \operatorname{int} C_+$ and $\hat{v} \in -\operatorname{int} C_+$). Therefore, $\{u_n\}_{n\ge n_0} \subseteq K_\varphi$; see (3.34)).

But we have assumed that K_φ is finite (see the proof of Proposition 3.4. Therefore (3.41) can not be true and then the homotopy invariance property of critical groups (see [6, p. 836]) implies

$$C_k(\varphi,0) = C_k(w,0) \quad \text{for all } k \in \mathbb{N}_0, \tag{3.44}$$

Then (3.44) and Proposition 3.4 imply that (3.40) is true.

Evidently $w(\cdot)$ is coercive. Hence

$$C_k(w,\infty) = \delta_{k,0}\mathbb{Z} \quad \text{for all } k \in \mathbb{N}_0; \tag{3.45}$$

see [16, Proposition 6.2.24]. From (3.40), (3.45), and [16, Corollary 6.7.8], we know that there exists $y_0 \in K_w \subseteq [\hat{v},\hat{u}] \cap C_0^1(\overline{\Omega})$ such that

$$\begin{gathered} w(y_0) < 0 = w(0) \text{ and } C_{d_m-1}(w,y_0) \ne 0, \text{ or} \\ w(y_0) > 0 = w(0) \text{ and } C_{d_m+1}(w,y_0) \ne 0. \end{gathered} \tag{3.46}$$

Evidently $y_0 \ne 0$. Since $m \ge 2$, we have that $d_m \ge 2$. Therefore comparing (3.39) and (3.46), we conclude that $y_0 \notin \{\hat{u},\hat{v}\}$. So, finally we have $y_0 \in [\hat{v},\hat{u}] \cap C_0^1(\overline{\Omega})$ and $y_0 \notin \{0,\hat{u},\hat{v}\}$ which imply that y_0 is a nodal solution of (1.1). □

If we impose an additional condition on $f(z,\cdot)$, we can improve the conclusion of the previous proposition.

The new hypotheses on the reaction $f(z,x)$ are as follows:

- (H2) For every $\rho > 0$, there exists $\hat{\theta}_\rho > 0$ such that for a.a. $z \in \Omega$, the mapping $x \mapsto f(z,x) + \hat{\theta}_\rho x^{p-1}$ is nondecreasing on $[-\rho,\rho]$.

Proposition 3.8. *If hypotheses* (H0)–(H2) *hold, then problem* (1.1) *has a nodal solution*

$$y_0 \in \operatorname{int}_{C_0^1(\overline{\Omega})}[\hat{v},\hat{u}].$$

Proof. From Proposition 3.7, we already have a nodal solution y_0 such that

$$y_0 \in [\hat{v},\hat{u}] \cap C_0^1(\overline{\Omega}). \tag{3.47}$$

Let $\gamma(z,y) = a_1(z)|y|^{p-2}y + a_2(z)y$ for all $z \in \Omega, y \in \mathbb{R}^N$. For every $u \in W_0^{1,p}(\Omega)$ we have

$$-\operatorname{div}\gamma(z,Du) = -\Delta_p^{a_1}u - \Delta^{a_2}u.$$

Note that

$$\begin{aligned}
&\nabla_y\gamma(z,y) = a_1(z)|y|^{p-2}[id + (p-2)\frac{y\otimes y}{|y|^2}] + a_2(z)id \quad \forall z\in\Omega,; \forall y\in\mathbb{R}^N \\
&\Rightarrow (\nabla_y\gamma(z,y)\beta,\beta)_{\mathbb{R}^N} \ge \hat{c}|\beta|^2 \quad \text{for all } y,\beta\in\mathbb{R}^N.
\end{aligned}$$

Then the tangency principle of Pucci-Serrin [19, p. 35] implies that

$$y_0(z) < \hat{u}(z) \quad \text{for all } z\in\Omega. \tag{3.48}$$

Let $\rho = \max\{\|\hat{u}\|_\infty, \|\hat{v}\|_\infty\}$ and let $\hat{\theta}_\rho > 0$ be as postulated by hypothesis (H1) and (H2). Choose $\theta^*_\rho > \hat{\theta}_\rho$, we have

$$\begin{aligned}
-\Delta_p^{a_1}\hat{u} - \Delta^{a_2}\hat{u} + \theta^*_\rho\hat{u}^{p-1} &= f(z,\hat{u}) + \theta^*_\rho\hat{u}^{p-1} \\
&= f(z,\hat{u}) + \hat{\theta}_\rho\hat{u}^{p-1} + (\theta^*_\rho - \hat{\theta}_\rho)\hat{u}^{p-1} \\
&\ge f(z,y_0) + \hat{\theta}_\rho y_0^{p-1} + (\theta^*_\rho - \theta_\rho)y_0^{p-1} \\
&= f(z,y_0) + \theta^*_\rho y_0^{p-1} \\
&= -\Delta_p^{a_1}y_0 - \Delta^{a_2}y_0 + \theta^*_\rho y_0^{p-1},
\end{aligned} \tag{3.49}$$

where we used (3.47), (H1), and (H2). Since $\hat{u}\in\operatorname{int}C_+, y_0\in C_0^1(\overline{\Omega})$, from (3.48), we see that for every $K\subseteq\Omega$ compact, we have

$$0 < c_K \le \hat{u}(z) - y_0(z) \quad \text{for all } z\in K. \tag{3.50}$$

Then (3.49), (3.50) and [7, Proposition 3.2] imply that $\hat{u} - y_0 \in \operatorname{int}C_+$. Similarly we show that $y_0 - \hat{v}\in\operatorname{int}C_+$. Therefore finally we have $y_0\in\operatorname{int}_{C_0^1(\overline{\Omega})}[\hat{v},\hat{u}]$. $\square$

Concluding we can state the following multiplicity theorem for problem (1.1). We emphasize that we provide sign information for all the solutions and the three solutions are ordered.

Theorem 3.9. *(a) If* (H0), (H1) *hold, then problem* (1.1) *has at least three solutions*

$$u_0\in\operatorname{int}C_+, \quad v_0\in-\operatorname{int}C_+, \quad y_0\in[v_0,u_0]\cap C_0^1(\overline{\Omega}) \ \textit{nodal}.$$

(b) If hypotheses (H0)–(H2) *hold, then problem* (1.1) *has at least three solutions*

$$u_0\in\operatorname{int}C_+, \quad v_0\in-\operatorname{int}C_+, \quad y_0\in\operatorname{int}_{C_0^1(\overline{\Omega})}[v_0,u_0] \ \textit{nodal}.$$

4. Appendix

In this section we present some basic facts concerning the spectral properties of $(-\Delta_r^a, W_0^{1,r}(\Omega))$ and $a\in C^{0,1}(\overline{\Omega})$, $a(z)\ge\hat{c}>0$ for all $z\in\overline{\Omega}$, $1<r<\infty$. We consider the nonlinear eigenvalue problem

$$\begin{gathered}
-\Delta_r^a u(z) = \hat{\lambda}|u(z)|^{r-2}u(z) \quad \text{in } \Omega, \\
u|_{\partial\Omega} = 0.
\end{gathered} \tag{4.1}$$

We say that $\hat{\lambda}\in\mathbb{R}$ is an eigenvalue of $(-\Delta_r^a, W_0^{1,r}(\Omega))$ if problem (4.1) has a nontrivial weak solution $\hat{u}\in W_0^{1,p}(\Omega)$ known as an eigenfunction corresponding to the eigenvalue $\hat{\lambda}$.

Evidently every eigenvalue $\hat{\lambda}\ge 0$. We show that there is a smallest eigenvalue $\hat{\lambda}_1^a(r) > 0$. To see this, we minimize the Rayleigh quotient

$$R(u) = \frac{\int_\Omega a(z)|Du|^r\,dz}{\|u\|_r^r}, \quad u\in W_0^{1,r}(\Omega), u\ne 0.$$

We have

$$
\begin{aligned}
0 \le \hat{\lambda}_1^a(r) &= \inf\Big[\frac{\int_\Omega a(z)|Du|^r\,dz}{\|u\|_r^r}, \quad u \in W_0^{1,r}(\Omega), u \neq 0\Big] \\
&= \inf\Big[\int_\Omega a(z)|Du|^r\,dz,\ u \in W_0^{1,r}(\Omega),\ \|u\|_r = 1\Big],
\end{aligned} \tag{4.2}
$$

by homogeneity. The infimum in (4.2) is attained. Consider a sequence $\{u_n\}_{n\in\mathbb{N}} \subseteq W_0^{1,r}(\Omega)$ such that

$$
\int_\Omega a(z)|Du_n|^r\,dz \ \downarrow\ \hat{\lambda}_1^a(r), \|u_n\|_r = 1 \text{ for all } n \in \mathbb{N}.
$$

Therefore $\{u_n\}_{n\in\mathbb{N}} \subseteq W_0^{1,p}(\Omega)$ is bounded. So, we can assume that

$$
u_n \xrightarrow{w} \hat{u}_1 \text{ in } W_0^{1,r}(\Omega), \quad u_n \to \hat{u}_1 \text{ in } L^r(\Omega).
$$

We have that

$$
\int_\Omega a(z)|D\hat{u}_1|^r\,dz \le \liminf_{n\to\infty} \int_\Omega a(z)|Du_n|^r\,dz = \hat{\lambda}_1^a(r), \|\hat{u}_1\|_r = 1,
$$

implies

$$
\int_\Omega a(z)|D\hat{u}_1|^r\,dz = \hat{\lambda}_1^a(r) > 0, \quad \|\hat{u}_1\|_r = 1.
$$

From (4.2) and the Lagrange multiplier rule, we infer that $\hat{\lambda}_1^a(r) > 0$ is the smallest eigenvalue of $(-\Delta_r^a, W_0^{1,r}(\Omega))$. Evidently we can replace $\hat{u}_1 \in W_0^{1,p}(\Omega)$ by $|\hat{u}_1| \in W_0^{1,p}(\Omega)$. Therefore we can always assume that $\hat{u}_1 \ge 0$. The nonlinear regularity theory (see [11]) and the nonlinear maximum principle [13, 19], imply that $\hat{u}_1 \in \operatorname{int} C_+$. In fact $\hat{\lambda}_1^a(r) > 0$ is the only eigenvalue with eigenfunctions of constant sign. All other eigenvalues have eigenfunctions which are nodal (sign-changing). The proof of this fact is done along the lines of the corresponding result for $(-\Delta_r, W_0^{1,r}(\Omega))$ (see for example Gasiński-Papageorgiou [5, p. 743]).

Suppose $\hat{u}$ and $\hat{v}$ are two eigenfunctions corresponding of $\hat{\lambda}_1^a(r) > 0$. As above, we have that $\hat{u}, \hat{v} \in \operatorname{int} C_+$. Then using the nonlinear Picone's identity of Jaros [9], we have

$$
\begin{aligned}
&0 \le a(z)|D\hat{u}|^r - a(z)|D\hat{v}|^{r-2}(D\hat{v}, D(\frac{\hat{u}^r}{\hat{v}^{r-1}}))_{\mathbb{R}^N} \quad \text{for all } z \in \Omega \\
&\Rightarrow 0 \le \int_\Omega a(z)|D\hat{u}|^r\,dz - \int_\Omega -(\Delta_r^a \hat{v})\frac{\hat{u}^r}{\hat{v}^{r-1}}\,dz \\
&\quad \text{(using Green's identity, see [16, p. 34])} \\
&= \int_\Omega a(z)|D\hat{u}|^r\,dz - \hat{\lambda}_1^a(r)\|\hat{u}\|_r^r = 0,
\end{aligned}
$$

which implies $\hat{u}D\hat{v} = \hat{v}D\hat{u}$ (see Jaros [9]). In turn this implies $D(\frac{\hat{u}}{\hat{v}}) = 0$ and so $\hat{u} = \vartheta\hat{v}$ with $\vartheta > 0$. Then $\hat{\lambda}_1^a(r) > 0$ is simple. Also $\hat{\lambda}_1^a(r) > 0$ is isolated in the spectrum of $(-\Delta_r^a, W_0^{1,r}(\Omega))$. Indeed, if $\hat{\lambda}_1^a(r)$ is not isolated, we can find $\hat{\lambda}_n \downarrow \hat{\lambda}_1^a(r)$ with $\hat{\lambda}_n$ eigenvalue of $(-\Delta_r^a, W_0^{1,r}(\Omega))$ for every $n \in \mathbb{N}$. Let $\hat{u}_n \in W_0^{1,p}(\Omega)$ be an eigenfunction corresponding to $\hat{\lambda}_n$. We have

$$
-\Delta_r^a \hat{u}_n = \hat{\lambda}_n|\hat{u}_n|^{r-2}\hat{u}_n \quad \text{for all } n \in \mathbb{N}. \tag{4.3}
$$

Normalizing we may assume that $\|\hat{u}_n\|_r = 1$ for all $n \in \mathbb{N}$. Then from (4.3) it follows that $\{u_n\}_{n\in\mathbb{N}} \subseteq W_0^{1,r}(\Omega)$ is bounded. We can assume that

$$\hat{u}_n \xrightarrow{w} \hat{u}_* \text{ in } W_0^{1,r}(\Omega), \quad \hat{u}_n \to \hat{u}_* \text{ in } L^r(\Omega). \tag{4.4}$$

We have $\|\hat{u}_*\|_r = 1$. On (4.3) we act with $\hat{u}_n - \hat{u}_* \in W_0^{1,r}(\Omega)$, pass to the limit as $n \to \infty$ and use (4.4). We obtain

$$\lim_{n\to\infty} \langle A_r^a(\hat{u}_n), \hat{u}_n - \hat{u}_*\rangle = 0 \tag{4.5}$$

with $A_r^a : W_0^{1,r}(\Omega) \to W^{-1,r'}(\Omega) = W_0^{1,r}(\Omega)^* (\frac{1}{r} + \frac{1}{r'} = 1)$ being the nonlinear operator defined by

$$\langle A_r^a(u), h\rangle = \int_\Omega a(z)|Du|^{r-2}(Du, Dh)_{\mathbb{R}^N}\, dz \quad \text{for all } u, h \in W_0^{1,r}(\Omega).$$

This operator has the same properties as $V(\cdot)$. In particular, (4.5) and the $(S)_+$-property of $A_r^a(\cdot)$, imply that

$$\hat{u}_n \to \hat{u}_* \quad \text{in } W_0^{1,r}(\Omega). \tag{4.6}$$

If in (4.3) we pass to the limit as $n \to \infty$, we have

$$-\Delta_r^a \hat{u}_* = \hat{\lambda}_1^a(r)|\hat{u}_*|^{r-2}\hat{u}_* \quad \text{in } \Omega, \|\hat{u}_*\|_r = 1.$$

Then we can assume that $\hat{u}_* \in \operatorname{int} C_+$. From (4.6) and the nonlinear regularity theory of Lieberman [11], we know that there exist $\alpha \in (0,1)$ and $M > 0$ such that

$$\hat{u}_n \in C_0^{1,\alpha}(\overline{\Omega}) \text{ and } \|\hat{u}_n\|_{C_0^{1,\alpha}(\overline{\Omega})} \le M \quad \text{for all } n \in \mathbb{N}. \tag{4.7}$$

Then the compact embedding of $C_0^{1,\alpha}(\overline{\Omega})$ into $C_0^1(\overline{\Omega})$ and (4.6) imply that $\hat{u}_n \to \hat{u}_*$ in $C_0^1(\overline{\Omega})$. Since $\hat{u}_* \in \operatorname{int} C_+$, we have

$$\{u_n\}_{n\ge n_0} \subseteq C_+ \setminus \{0\},$$

which contradicts that only $\hat{\lambda}_1^a(r) > 0$ has eigenfunctions of constant sign.

Summarizing, we can state the following basic facts about the spectrum of $(-\Delta_r^a, W_0^{1,r}(\Omega))$:

- There is a smallest eigenvalue $\hat{\lambda}_1^a(r) > 0$ which has a variational characterization given by (4.2).
- $\hat{\lambda}_1^a(r)$ is simple, isolated and the corresponding eigenfunctions are of constant sign and belong in $(\operatorname{int} C_+) \cup (-\operatorname{int} C_+)$.
- If $\hat{\lambda} > \hat{\lambda}_1^a(r)$ is an eigenvalue, then $\hat{\lambda}$ has nodal eigenfunctions.

Proposition 4.1. *If $\eta \in L^\infty(\Omega)$ and $\eta(z) \le \hat{\lambda}_1^a(r)$ for a.a. $z \in \Omega, \eta \not\equiv \hat{\lambda}_1^a(r)$, then there exists $\theta > 0$ such that*

$$\theta\|u\|^r \le \int_\Omega a(z)|Du|^r\, dz - \int_\Omega \eta(z)|u|^r\, dz \quad \text{for all } u \in W_0^{1,r}(\Omega).$$

Proof. Arguing by contradiction, suppose we can find $\{u_n\}_{n\in\mathbb{N}} \subseteq W_0^{1,p}(\Omega)$ such that for all $n \in \mathbb{N}$, we have

$$\int_\Omega a(z)|Du_n|^r\, dz - \int_\Omega \eta(z)|u_n|^r\, dz < \frac{1}{n}, \|u_n\| = 1. \tag{4.8}$$

We may assume that

$$u_n \xrightarrow{w} u \text{ in } W_0^{1,r}(\Omega), \quad u_n \to u \text{ in } L^r(\Omega). \tag{4.9}$$

Passing to the limit as $n \to \infty$ in (4.8) and using (4.9), we obtain

$$\int_\Omega a(z)|Du|^r\,dz \le \int_\Omega \eta(z)|u|^r\,dz \le \hat\lambda_1^a(r)\|u\|_r^r, \tag{4.10}$$

$$\Rightarrow \int_\Omega a(z)|Du|^r\,dz = \hat\lambda_1^a(r)\|u\|_r^r \text{ (see (4.2))} \tag{4.11}$$

We claim $u \neq 0$. Otherwise we have

$$\hat c\|Du_n\|_r^r \to 0 \;\Rightarrow\; u_n \to 0 \text{ in } W_0^{1,r}(\Omega),$$

which contradicts that $\|u_n\| = 1$ for all $n \in \mathbb{N}$. From (4.11) we see that we may assume that $u \in \operatorname{int} C_+$. Then from (4.10), we have

$$\int_\Omega a(z)|Du|^r\,dz < \hat\lambda_1^a(r)\|u\|_r^r,$$

a contradiction, see (4.2). This completes the proof. □

If $r = 2$ (the linear eigenvalue problem), then the spectral theorem for compact, self-adjoint operators, provides a complete description of the spectrum $(-\Delta^a, H_0^1(\Omega))$ which consists of a sequence $\{\hat\lambda_n^a(2)\}_{n\in\mathbb{N}} \subseteq (0,\infty)$ such that $\hat\lambda_n^a(2) \to \infty$.

We denote by $E(\hat\lambda_n^a(2))$ the eigenspace corresponding to the eigenvalue $\hat\lambda_n^a(2)$. We know that $E(\hat\lambda_n^a(2))$ is finite dimensional and $E(\hat\lambda_n^a(2)) \subseteq C_0^1(\overline{\Omega})$. (standard regularity theory). Moreover, the elements of $E(\hat\lambda_n^a(2))$ have the "Unique Continuation Property" (the UCP for short), that is, if $u \in E(\hat\lambda_n^a(2))$ and $u(\cdot)$ vanishes on a set of positive measure, then $u \equiv 0$ (see de Figueiredo-Gossez [3]).

Let $\overline{H}_m = \oplus_{i=1}^m E(\hat\lambda_i^a(2)), \hat H_{m+1} = \overline{H}_m^\perp$. Then we have

$$H_0^1(\Omega) = \hat H_m \oplus \hat H_{m+1}.$$

In this case we have variational characterizations for all the eigenvalues. So, we have

$$\hat\lambda_1^a(2) = \inf\Big[\frac{\int_\Omega a(z)|Du|^2\,dz}{\|u\|_2^2} : u \in H_0^1(\Omega), u \neq 0\Big] \tag{4.12}$$

and for $m \ge 2$

$$\begin{aligned} \hat\lambda_1^a(2) &= \sup\Big[\frac{\int_\Omega a(z)|Du|^2\,dz}{\|u\|_2^2} : u \in \overline{H}_m, u \neq 0\Big] \\ &= \inf\Big[\frac{\int_\Omega a(z)|Du|^2\,dz}{\|u\|_2^2} : u \in \hat H_m, u \neq 0\Big]. \end{aligned} \tag{4.13}$$

In (4.12) and (4.13) the inf and sup are realized on the corresponding eigenspace $E(\hat\lambda_m^a(2))$.

Proposition 4.2. *If $\eta \in L^\infty(\Omega)$ and $\eta(z) \le \hat\lambda_m^a(2)$ for a.a. $z \in \Omega, \eta \not\equiv \hat\lambda_m^a(2)$, then there exists $\theta > 0$ such that*

$$\theta\|u\|^r \le \int_\Omega a(z)|Du|^2\,dz - \int_\Omega \eta(z)|u|^2\,dz \text{ for all } u \in \hat H_m.$$

Proof. If $m = 1$, then this follows from Proposition 4.1. So, assume $m \ge 2$. Arguing by contradiction, suppose we can find $\{u_n\}_{n\in\mathbb{N}} \subseteq \hat H_m$ with $\|u_n\| = 1$ for all $n \in \mathbb{N}$ such that

$$\int_\Omega a(z)|Du_n|^2\,dz - \int_\Omega \eta(z)|u_n|^2\,dz < \frac{1}{n} \quad \text{for all } n \in \mathbb{N}. \tag{4.14}$$

We can assume that

$$u_n \xrightarrow{w} u \text{ in } H_0^1(\Omega), \quad u_n \to u \text{ in } L^2(\Omega). \tag{4.15}$$

If in (4.14) we pass to the limit as $n \to \infty$ and use (4.15), we obtain

$$\int_\Omega a(z)|Du|^2\,dz \le \int_\Omega \eta(z)|u|^2\,dz \le \hat{\lambda}_m^a(2)\|u\|_2^2, \tag{4.16}$$

$$\Rightarrow \int_\Omega a(z)|Du|^2\,dz = \hat{\lambda}_m^a(2)\|u\|_2^2 \quad (\text{since } u \in \hat{H}_m, \text{ see (4.13)})$$

$$\Rightarrow u \in E(\hat{\lambda}_m^a(2)), \quad u \ne 0.$$

Then by the UCP we have $u(z) \ne 0$ for a.a. $z \in \Omega$. Using this fact in (4.16), we obtain

$$\int_\Omega a(z)|Du|^2\,dz < \hat{\lambda}_m^a(2)\|u\|_2^2$$

which contradicts (4.13). This completes the proof. □

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Zhenhai Liu
Guangxi Colleges and Universities Key Laboratory of Complex System Optimization and Big Data Processing, Yulin Normal University, Yulin 537000, China.
Guangxi Colleges and Universities Key Laboratory of Optimization Control and Engineering Calculation, Guangxi Minzu University, Nanning, Guangxi, 530006, China
Email address: `zhhliu@hotmail.com`

Nikolaos S. Papageorgiou
Department of Mathematics, National Technical University, Zografou Campus, 15780 Athens, Greece
Email address: `npapg@math.ntua.gr`