Electronic Journal of Differential Equations, Vol. 2023 (2023), No. 55, pp. 1–15.
ISSN: 1072-6691. URL: https://ejde.math.txstate.edu, https://ejde.math.unt.edu
DOI: 10.58997/ejde.2023.55

TRAVELING WAVE SOLUTIONS FOR THREE-SPECIES NONLOCAL COMPETITIVE-COOPERATIVE SYSTEMS

HONG-JIE WU, BANG-SHENG HAN, SHAO-YUE MI, LIANG-BIN SHEN

Abstract. By using a two-point boundary-value problem and a Schauder's fixed point theorem, we obtain traveling wave solutions connecting $(0,0,0)$ to an unknown positive steady state for speed $c \geq c^* = \max\{2, 2\sqrt{d_2 r_2}, 2\sqrt{d_3 r_3}\}$. Then we present some asymptotic behaviors of traveling wave solutions. In particular we show that the nonlocal effects have a great influence on the final state of traveling wave solutions at $-\infty$.

1. Introduction

We consider the three-species nonlocal competitive-cooperative system

$$
\begin{aligned}
u_t &= d_1 \Delta u + r_1 u[1 - a_1(\phi_1 * u) - b_1 v - c_1 w], \\
v_t &= d_2 \Delta v + r_2 v[1 - a_2(\phi_2 * v) + b_2 w - c_2 u], \\
w_t &= d_3 \Delta w + r_3 w[1 - a_3(\phi_3 * w) + b_3 v - c_3 u],
\end{aligned} \tag{1.1}
$$

where

$$
(\phi_i * u)(x,t) = \int_{\mathbb{R}} \phi_i(x-y) u(y,t)\, dy, \quad x \in \mathbb{R},\ t \in \mathbb{R},\ i = 1,2,3.
$$

Here the unknown functions $u(x,t)$, $v(x,t)$ and $w(x,t)$ represent the population densities of species at position x and time t, and a_i, b_i, c_i, d_i, r_i $(i = 1,2,3)$ are real constants. It is easy to see that the species u competes with the species v and w which cooperate with each other from (1.1). The positive coefficients d_i, r_i $(i = 1,2,3)$ indicate the diffusion rate and natural growth rate of u, v, w, respectively. The competition and cooperation coefficients for three species are denoted by the positive parameters a_i, b_i and c_i $(i = 1,2,3)$. To simplify the notations, we let

$$
\begin{gathered}
\frac{t}{r_1} \to t, \quad \sqrt{\frac{d_1}{r_1}} x \to x, \quad a_1 u \to u, \quad a_2 v \to v, \quad a_3 w \to w, \\
\frac{b_1}{a_2} \to b_1, \quad \frac{c_1}{a_3} \to c_1, \quad \frac{b_2}{a_3} \to b_2, \quad \frac{c_2}{a_1} \to c_2, \quad \frac{b_3}{a_2} \to b_3, \\
\frac{c_3}{a_1} \to c_3, \quad \frac{d_2}{d_1} \to d_2, \quad \frac{d_3}{d_1} \to d_3, \quad \frac{r_2}{r_1} \to r_2, \quad \frac{r_3}{r_1} \to r_3,
\end{gathered}
$$

2020 *Mathematics Subject Classification.* 35A01, 35C07, 35K55, 35K57.
Key words and phrases. Three-species system; competitive-cooperative; nonlocal effect; traveling wave solution; critical speed.

Submitted April 6, 2023. Published September 4, 2023.

then system (1.1) is converted to

$$
\begin{aligned}
u_t &= \Delta u + u[1-(\phi_1 * u) - b_1 v - c_1 w],\\
v_t &= d_2\Delta v + r_2 v[1-(\phi_2 * v) + b_2 w - c_2 u],\\
w_t &= d_3\Delta w + r_3 w[1-(\phi_3 * w) + b_3 v - c_3 u],
\end{aligned} \tag{1.2}
$$

where the bounded kernel functions $\phi_i(x)$ $(i = 1, 2, 3)$ satisfy the assumptions:

(A1) $\phi_i(x) \geq 0$ and $\int_{\mathbb{R}} \phi_i(x)dx = 1$, $i = 1, 2, 3$;

(A2) $\int_{\mathbb{R}} \phi_i(y)e^{\lambda y}\, dy < \infty$ for each $\lambda \in (0, \max\{1, \sqrt{r_2/d_2}, \sqrt{r_3/d_3}\})$.

In addition, as we shown in the paper, it suffices to assume

(A3) $0 < b_i$, $c_i < 1$ for $i = 1, 2, 3$.

We point out that if $w = 0$ in (1.2), then the system of equations is reduced to a two-species nonlocal Lotka-Volterra competitive system whose traveling waves have been discussed by Han et al. [6]. When $u = 0$, (1.2) becomes the two-species nonlocal cooperative system which have been studied by Huang and Zou [10]. In summary, two-species Lotka-Volterra systems have been extensively considered [3, 4, 5, 8, 15, 16, 18, 30]. Of course, there are studies on the traveling waves for three-species competitive systems [17, 28], but little research on cooperative systems due to the technical treatments of cooperative systems are not as convenient as competitive systems. Leung and Hou et al. [13, 14] proved that the two-species Lotka-Volterra cooperative system can be transformed into a competitive system by using variable transformation which cannot be applied to 3-dimension system [1, 2, 22]. To mitigate this technical challenge, Hung [12] proposed the following classical three-species Lotka-Volterra competitive-cooperative system for the first time

$$
\begin{aligned}
u_t &= d_1 u_{xx} + u(\lambda_1 - c_{11}u - c_{12}v + c_{13}w), \quad x \in \mathbb{R},\ t > 0,\\
v_t &= d_2 v_{xx} + v(\lambda_2 - c_{21}u - c_{22}v - c_{23}w), \quad x \in \mathbb{R},\ t > 0,\\
w_t &= d_3 w_{xx} + w(\lambda_3 + c_{31}u - c_{32}v - c_{33}w), \quad x \in \mathbb{R},\ t > 0,
\end{aligned} \tag{1.3}
$$

where competition between species u and v $(c_{12}, c_{21} > 0)$, species v and w $(c_{23}, c_{32} > 0)$, and cooperation between species u and w $(c_{13}, c_{31} > 0)$. And by transforming (1.3) into a monotonic system, they proved the existence of the traveling wave solutions for (1.3). After that, Meng and Zhang [20] obtained the asymptotic behavior and uniqueness of the traveling waves for (1.3) by using Ikehare's theorem. For more results, we can refer to [9, 21, 23, 27].

To make the model more practical in applications, nonlocal effects and time-delays have been considered [3, 6, 7, 10, 23]. Subsequently, the traveling wave solution of this model has been studied and developed, see [11, 18, 19, 25, 29]. But they are mostly concerned with the quasi-monotone case.

It is worth noting that, compared with three-species delayed Lotka-Volterra competitive-cooperative systems, there are relatively few studies on nonlocal systems. Recently, Zhang and Bao [26] introduced nonlocal effect to the diffusion term which deduced the system

$$
\begin{aligned}
\frac{\partial u}{\partial t} &= d_1(J_1 * u - u) + r_1 u(1 - a_1 u - b_1 v - c_1 w),\\
\frac{\partial v}{\partial t} &= d_2(J_2 * v - v) + r_2 v(1 - a_2 v + b_2 w - c_2 u),\\
\frac{\partial w}{\partial t} &= d_3(J_3 * w - w) + r_3 w(1 - a_3 w + b_3 v - c_3 u),
\end{aligned} \tag{1.4}
$$

where

$$J_i * z - z = \int_{\mathbb{R}} J_i(y)[z(x-y,t) - z(x,t)]\,dy, \quad i = 1,2,3,$$

and gave the existence, uniqueness and asymptotic behavior of the traveling wave solutions of (1.4) by the comparative lemma, the bilateral Laplace transform and the sliding method. We should point out that (1.4) does not destroy the comparison principle of the classical system which will not hold if nonlocal effects are introduced to the reaction term, and the method based on the comparison principle is not applicable. Therefore, inspired by [6, 24, 26], we try to (partially) solve the existence and asymptotic behavior of the traveling wave solutions of (1.2) by using the Schauder's fixed point theorem and a two-point boundary value problem.

Substituting $(u,v,w)(x,t) = (U,V,W)(\xi)$ into (1.2) and denoting $\xi = x - ct$ where $c \geq \{2, 2\sqrt{d_2 r_2}, 2\sqrt{d_3 r_3}\}$ represents the wave speed, we obtain

$$\begin{gathered} -U''(\xi) - cU'(\xi) = U(\xi)[1 - (\phi_1 * U)(\xi) - b_1 V(\xi) - c_1 W(\xi)], \\ -d_2 V''(\xi) - cV'(\xi) = r_2 V(\xi)[1 - (\phi_2 * V)(\xi) + b_2 W(\xi) - c_2 U(\xi)], \\ -d_3 W''(\xi) - cW'(\xi) = r_3 W(\xi)[1 - (\phi_3 * W)(\xi) + b_3 V(\xi) - c_3 U(\xi)]. \end{gathered} \tag{1.5}$$

Then, we have the following result.

Theorem 1.1. *Assume that* (A1)–(A3) *hold. Then, for each*

$$c > c^* = \max\{2, 2\sqrt{d_2 r_2}, 2\sqrt{d_3 r_3}\},$$

there exists a traveling wave solution $(U,V,W)(\xi)$ *satisfying* (1.5) *with the boundary conditions*

$$\begin{gathered} \liminf_{\xi \to -\infty} (U(\xi) + V(\xi) + W(\xi)) > 0, \\ \lim_{\xi \to +\infty} U(\xi) = \lim_{\xi \to +\infty} V(\xi) = \lim_{\xi \to +\infty} W(\xi) = 0. \end{gathered} \tag{1.6}$$

In particular, U, V and W are monotone decreasing on $[Z_0, +\infty)$ for some $Z_0 > 0$ (which may depend on c). Moreover, such traveling wave solution does not exist for $c < c^*$.

This article is organized as follows. In Section 2, we present the super- and sub-solutions of (1.5). The existence of traveling wave solutions connecting $(0,0)$ to an unknown positive stea dy state of (1.1) is obtained in Section 3. In Section 4, we show the proof of Theorem 1.1 .

2. Preliminaries

In this section, we first use super- and sub-solution to construct the range of traveling waves which will be used for the proof of the existence of the solution in Section 3. In the following, we construct the super- and sub-solutions of (1.5).

Supersolution. Let

$$\overline{p}_c(x) = e^{-\lambda_c x}, \quad \overline{q}_c(x) = e^{-\zeta_c x}, \quad \bar{l}_c(x) = e^{-\eta_c x}, \quad \text{for all } x \in \mathbb{R},$$

where $\lambda_c > 0$, $\zeta_c > 0$, $\eta_c > 0$ are the smaller roots of the equations

$$\lambda_c^2 - c\lambda_c + 1 = 0, \quad d_2\zeta_c^2 - c\zeta_c + r_2 = 0, \quad d_3\eta_c^2 - c\eta_c + r_3 = 0,$$

respectively. Then, it holds that

$$-\overline{p}_c'' - c\overline{p}_c' = \overline{p}_c, \quad -d_2\overline{q}_c'' - c\overline{q}_c' = r_2\overline{q}_c, \quad -d_3\bar{l}_c'' - c\bar{l}_c' = r_3\bar{l}_c. \tag{2.1}$$

Subsolution. Let

$$\underline{p}_c(x) = e^{-\lambda_c x} - Ae^{-(\lambda_c+\varepsilon)x}, \quad \underline{q}_c(x) = e^{-\zeta_c x} - Be^{-(\zeta_c+\varepsilon)x},$$
$$\underline{l}_c(x) = e^{-\eta_c x} - De^{-(\eta_c+\varepsilon)x}, \quad \text{for all } x \in \mathbb{R},$$

where small enough $\varepsilon \in (0, \min(\lambda_c, \zeta_c, \eta_c))$ satisfies

$$\kappa_c = -(\lambda_c+\varepsilon)^2 + c(\lambda_c+\varepsilon) - 1 > 0, \quad \iota_c = -d_2(\zeta_c+\varepsilon)^2 + c(\zeta_c+\varepsilon) - r_2 > 0,$$
$$\vartheta_c = -d_3(\eta_c+\varepsilon)^2 + c(\eta_c+\varepsilon) - r_3 > 0.$$

Moreover, $A, B, D > 1$ are large enough such that

$$\frac{\ln A}{\varepsilon} > \max\Big\{\frac{1}{\lambda_c-\varepsilon}\ln\frac{3Z_1^c}{A\kappa_c},\ \frac{1}{\zeta_c-\varepsilon}\ln\frac{3b_1}{A\kappa_c},\ \frac{1}{\eta_c-\varepsilon}\ln\frac{3c_1}{A\kappa_c}\Big\},$$
$$\frac{\ln B}{\varepsilon} > \max\Big\{\frac{1}{\zeta_c-\varepsilon}\ln\frac{3r_2Z_2^c}{B\iota_c},\ \frac{1}{\eta_c}\ln\frac{3r_2b_2D}{B\iota_c},\ \frac{1}{\lambda_c-\varepsilon}\ln\frac{3r_2c_2}{B\iota_c}\Big\},$$
$$\frac{\ln D}{\varepsilon} > \max\Big\{\frac{1}{\eta_c-\varepsilon}\ln\frac{3r_3Z_3^c}{D\vartheta_c},\ \frac{1}{\zeta_c}\ln\frac{3r_3b_3B}{D\vartheta_c},\ \frac{1}{\lambda_c-\varepsilon}\ln\frac{3r_3c_3}{D\vartheta_c}\Big\},$$

where

$$Z_1^c = \int_{\mathbb{R}} \phi_1(y)e^{\lambda_c y}dy, \quad Z_2^c = \int_{\mathbb{R}} \phi_2(y)e^{\zeta_c y}dy, \quad Z_3^c = \int_{\mathbb{R}} \phi_3(y)e^{\eta_c y}dy.$$

Then, for all $x > \max\big\{\frac{\ln A}{\varepsilon}, \frac{\ln B}{\varepsilon}, \frac{\ln D}{\varepsilon}\big\}$, it holds that $\underline{p}_c > 0$, $\underline{q}_c > 0$, $\underline{l}_c > 0$. So we have

$$\begin{aligned}
&-\underline{p}_c'' - c\underline{p}_c' - \underline{p}_c + \underline{p}_c(\phi_1 * \overline{p}_c) + b_1\underline{p}_c\overline{q}_c + c_1\underline{p}_c\overline{l}_c \\
&= (-\lambda_c^2 + c\lambda_c - 1)e^{-\lambda_c x} + Ae^{-(\lambda_c+\varepsilon)x}[(\lambda_c+\varepsilon)^2 - c(\lambda_c+\varepsilon) + 1] \\
&\quad + [e^{-\lambda_c x} - Ae^{-(\lambda_c+\varepsilon)x}](Z_1^c e^{-\lambda_c x} + b_1 e^{-\zeta_c x} + c_1 e^{-\eta_c x}) \\
&< -A\kappa_c e^{-(\lambda_c+\varepsilon)x} + e^{-\lambda_c x}(Z_1^c e^{-\lambda_c x} + b_1 e^{-\zeta_c x} + c_1 e^{-\eta_c x}) \\
&= e^{-(\lambda_c+\varepsilon)x}[-A\kappa_c + Z_1^c e^{-(\lambda_c-\varepsilon)x} + b_1 e^{-(\zeta_c-\varepsilon)x} + c_1 e^{-(\eta_c-\varepsilon)x}] < 0,
\end{aligned}$$

$$\begin{aligned}
&-d_2\underline{q}_c'' - c\underline{q}_c' - r_2\underline{q}_c + r_2\underline{q}_c(\phi_2 * \overline{q}_c) - b_2r_2\underline{q}_c\underline{l}_c + c_2r_2\underline{q}_c\overline{p}_c \\
&= (-d_2\zeta_c^2 + c\zeta_c - r_2)e^{-\zeta_c x} + Be^{-(\zeta_c+\varepsilon)x}[d_2(\zeta_c+\varepsilon)^2 - c(\zeta_c+\varepsilon) + r_2] \\
&\quad + r_2[e^{-\zeta_c x} - Be^{-(\zeta_c+\varepsilon)x}]\big(Z_2^c e^{-\zeta_c x} - b_2 e^{-\eta_c x} + b_2De^{-(\eta_c+\varepsilon)x} + c_2 e^{-\lambda_c x}\big) \\
&< e^{-(\zeta_c+\varepsilon)x}[-B\iota_c + r_2Z_2^c e^{-(\zeta_c-\varepsilon)x} + r_2b_2De^{-\eta_c x} + r_2c_2e^{-(\lambda_c-\varepsilon)x}] < 0,
\end{aligned}$$

and

$$\begin{aligned}
&-d_3\underline{l}_c'' - c\underline{l}_c' - r_3\underline{l}_c + r_3\underline{l}_c(\phi_3 * \overline{l}_c) - b_3r_3\underline{l}_c\underline{q}_c + c_3r_3\underline{l}_c\overline{p}_c \\
&= (-d_3\eta_c^2 + c\eta_c - r_3)e^{-\eta_c x} + De^{-(\eta_c+\varepsilon)x}[d_3(\eta_c+\varepsilon)^2 - c(\eta_c+\varepsilon) + r_3] \\
&\quad + r_3[e^{-\eta_c x} - De^{-(\eta_c+\varepsilon)x}]\big(Z_3^c e^{-\eta_c x} - b_3 e^{-\zeta_c x} + b_3Be^{-(\zeta_c+\varepsilon)x} + c_3 e^{-\lambda_c x}\big) \\
&< e^{-(\eta_c+\varepsilon)x}[-D\vartheta_c + r_3Z_3^c e^{-(\eta_c-\varepsilon)x} + r_3b_3Be^{-\zeta_c x} + r_3c_3e^{-(\lambda_c-\varepsilon)x}] < 0,
\end{aligned}$$

for all $x > \max\{\frac{\ln A}{\varepsilon}, \frac{\ln B}{\varepsilon}, \frac{\ln C}{\varepsilon}\}$. Let

$$\widetilde{p}_c(x) = \max\{0, \underline{p}_c\}, \quad \widetilde{q}_c(x) = \max\{0, \underline{q}_c\}, \quad \widetilde{l}_c(x) = \max\{0, \underline{l}_c\}, \quad x \in \mathbb{R},$$

then combining this with (2.1), we deduce that

$$\begin{aligned} -\overline{p}_c'' - c\overline{p}_c' + \overline{p}_c\big(\phi_1 * \widetilde{p}_c + b_1\widetilde{q}_c + c_1\widetilde{l}_c\big) &\geq \overline{p}_c, \\ -d_2\overline{q}_c'' - c\overline{q}_c' + r_2\big(\phi_2 * \widetilde{q}_c + c_2\widetilde{p}_c\big)\overline{q}_c &\geq r_2\overline{q}_c + r_2b_2\bar{l}_c\overline{q}_c, \\ -d_3\bar{l}_c'' - c\bar{l}_c' + r_3\big(\phi_3 * \widetilde{l}_c + c_3\widetilde{p}_c\big)\bar{l}_c &\geq r_3\bar{l}_c + r_3b_3\overline{q}_c\bar{l}_c, \end{aligned} \tag{2.2}$$

where b_i, c_i $(i = 2, 3)$ satisfy

$$\phi_2 * \widetilde{q}_c - b_2\bar{l}_c + c_2\widetilde{p}_c \geq 0 \quad \text{and} \quad \phi_3 * \widetilde{l}_c - b_3\overline{q}_c + c_3\widetilde{p}_c \geq 0.$$

In addition, we can also obtain

$$-\widetilde{p}_c'' - c\widetilde{p}_c' + \widetilde{p}_c(\phi_1 * \overline{p}_c - b_1\overline{q}_c - c_1\bar{l}_c) \leq \widetilde{p}_c, \tag{2.3}$$

for each $x \neq \ln(A)/x$,

$$-d_2\widetilde{q}_c'' - c\widetilde{q}_c' + r_2(\phi_2 * \overline{q}_c + c_2\overline{p}_c)\widetilde{q}_c \leq r_2\widetilde{q}_c + r_2b_2\widetilde{q}_c\widetilde{l}_c, \tag{2.4}$$

for each $x \neq \ln(B)/x$, and

$$-d_3\widetilde{l}_c'' - c\widetilde{l}_c' + r_3(\phi_3 * \bar{l}_c + c_3\overline{p}_c)\widetilde{l}_c \leq r_3\widetilde{l}_c + r_3b_3\widetilde{l}_c\widetilde{q}_c, \tag{2.5}$$

for each $x \neq \ln(D)/x$.

Based on the above setting, we give the existence of the traveling wave solutions of (1.5).

3. Existence of traveling wave solutions of (1.5)

The aim of this section is two-fold. Firstly, we provide a specific result assuring the existence of the solutions for the equation (1.5) in a finite interval by applying the super- and sub-solution constructed in Section 2 and a well know argument (Schauder's fixed point theorem). Secondly, by taking the limit, we derive a existence criterion of solution to (1.5) on the entire interval.

A three-point boundary value problem. For $c > \max\{2, 2\sqrt{d_2r_2}, 2\sqrt{d_3r_3}\}$, we study the following system in a finite interval $(-a, a)$:

$$\begin{gathered} -u'' - cu' = u(1 - \phi_1 * \overline{u} - b_1\overline{v} - c_1\overline{w}), \\ -d_2v'' - cv' = r_2v(1 - \phi_2 * \overline{v} + b_2\overline{w} - c_2\overline{u}), \\ -d_3w'' - cw' = r_3w(1 - \phi_3 * \overline{w} + b_3\overline{v} - c_3\overline{u}), \\ u(\pm a) = \widetilde{p}_c(\pm a),\ v(\pm a) = \widetilde{q}_c(\pm a),\ w(\pm a) = \widetilde{l}_c(\pm a), \end{gathered} \tag{3.1}$$

where $a > \max\{\frac{\ln A}{\varepsilon}, \frac{\ln B}{\varepsilon}, \frac{\ln D}{\varepsilon}\}$ and

$$\overline{u}(x) = \begin{cases} u(a), & x > a, \\ u(x), & x \in [-a, a], \\ u(-a), & x < -a, \end{cases} \qquad \overline{v}(x) = \begin{cases} v(a), & x > a, \\ v(x), & x \in [-a, a], \\ v(-a), & x < -a. \end{cases}$$

$$\overline{w}(x) = \begin{cases} w(a), & x > a, \\ w(x), & x \in [-a, a], \\ w(-a), & x < -a. \end{cases}$$

Next we first define a convex set

$$\mathcal{M}_a = \Big\{(u, v, w) \in C([-a, a], \mathbb{R}^2) : \widetilde{p}_c(x) \leq u(x) \leq \overline{p}_c, x \in (-a, a),$$

$$\widetilde{q}_c(x) \le v(x) \le \overline{q}_c, x \in (-a,a),\ \widetilde{l}_c(x) \le w(x) \le \bar{l}_c, x \in (-a,a),$$
$$u(\pm a) = \widetilde{p}_c(\pm a),\ v(\pm a) = \widetilde{q}_c(\pm a),\ w(\pm a) = \widetilde{l}_c(\pm a)\Big\},$$

to study the existence of the solution for (3.1). Then, we construct the three-point boundary value problem

$$\begin{gathered} -u'' - cu' + (\phi_1 * \overline{u}_0 + b_1\overline{v}_0 + c_1\overline{w}_0)u = u_0, \\ -d_2v'' - cv' + r_2(\phi_2 * \overline{v}_0 + c_2\overline{u}_0)v = r_2v_0 + r_2b_2w_0v_0, \\ -d_3w'' - cw' + r_3(\phi_3 * \overline{w}_0 + c_3\overline{u}_0)w = r_3w_0 + r_3b_3v_0w_0, \\ u(\pm a) = \widetilde{p}_c(\pm a), v(\pm a) = \widetilde{q}_c(\pm a), w(\pm a) = \widetilde{l}_c(\pm a), \end{gathered} \tag{3.2}$$

where $(u_0, v_0, w_0) \in \mathcal{M}_a$ and

$$\overline{u}_0(x) = \begin{cases} u_0(a), & x > a, \\ u_0(x), & x \in [-a,a], \\ u_0(-a), & x < -a, \end{cases} \quad \overline{v}_0(x) = \begin{cases} v_0(a), & x > a, \\ v_0(x), & x \in [-a,a], \\ v_0(-a), & x < -a. \end{cases}$$

$$\overline{w}_0(x) = \begin{cases} w_0(a), & x > a, \\ w_0(x), & x \in [-a,a], \\ w_0(-a), & x < -a. \end{cases}$$

Now, we define a linear operator Ψ_a which satisfies $\Psi_a(u_0, v_0, w_0) = (u, v, w)$. It is clear that the fixed point of (3.2) is a solution for (3.1). Obviously, Ψ_a is compact and continuous. Next, we prove that $\mathcal{M}_a$ is an invariant for Ψ_a. From the definition of Ψ_a and $\mathcal{M}_a$, we know that $\mathcal{M}_a \in \Psi_a(\mathcal{M}_a)$. Following, we prove $\Psi_a(\mathcal{M}_a) \in \mathcal{M}_a$. Since $(u, v, w) = (0, 0, 0)$ is a sub-solution of (3.2), we know that $u(x) > 0$, $v(x) > 0$, $w(x) > 0$ for each $x \in (-a, a)$. Given $(u_0, v_0, w_0) \in \mathcal{M}_a$ and combining with (2.2), then for $x \in (-a, a)$, it holds that

$$\begin{aligned} &- \overline{p}_c'' - c\overline{p}_c' + (\phi_1 * \overline{u}_0 + b_1\overline{v}_0 + c_1\overline{w}_0)\overline{p}_c \\ &\ge -\overline{p}_c'' - c\overline{p}_c' \\ &= \overline{p}_c \\ &\ge u_0 \\ &= -u'' - cu' + (\phi_1 * \overline{u}_0 + b_1\overline{v}_0 + c_1\overline{w}_0)u, \end{aligned}$$

$$\begin{aligned} &- d_2\overline{q}_c'' - c\overline{q}_c' + r_2(\phi_2 * \overline{v}_0 + c_2\overline{u}_0)\overline{q}_c \\ &\ge -d_2\overline{q}_c'' - c\overline{q}_c' + r_2(\phi_2 * \widetilde{q}_c + c_2\widetilde{p}_c)\overline{q}_c \\ &\ge r_2\overline{q}_c + r_2b_2\bar{l}_c\overline{q}_c \\ &\ge r_2v_0 + r_2b_2w_0v_0 \\ &= -d_2v'' - cv' + r_2(\phi_2 * \overline{v}_0 + c_2\overline{u}_0)v, \end{aligned}$$

and

$$\begin{aligned} &- d_3\bar{l}_c'' - c\bar{l}_c' + r_3(\phi_3 * \overline{w}_0 + c_3\overline{u}_0)\bar{l}_c \\ &\ge -d_3\bar{l}_c'' - c\bar{l}_c' + r_3(\phi_3 * \widetilde{l}_c + c_3\widetilde{p}_c)\bar{l}_c \\ &\ge r_3\bar{l}_c + r_3b_3\overline{q}_c\bar{l}_c \\ &\ge r_3w_0 + r_3b_3v_0w_0 \end{aligned}$$

$$= -d_3 w'' - cw' + r_3(\phi_3 * \overline{w}_0 + c_3\overline{u}_0)w.$$

In addition, we also have $u(\pm a) = \widetilde{p}_c(\pm a) \le \overline{p}_c(\pm a)$, $v(\pm a) = \widetilde{q}_c(\pm a) \le \overline{q}_c(\pm a)$, $w(\pm a) = \widetilde{l}_c(\pm a) \le \bar{l}_c(\pm a)$. Then by using the maximum principle, we obtain $u(x) \le \overline{p}_c(x)$, $v(x) \le \overline{q}_c(x)$, $w(x) \le \bar{l}_c(x)$ for each $x \in (-a, a)$. On the other hand, combining with (2.3)-(2.5), it is easy to calculate

$$\begin{aligned}
&- \widetilde{p}_c'' - c\widetilde{p}_c' + (\phi_1 * \overline{u}_0 + b_1\overline{v}_0 + c_1\overline{w}_0)\widetilde{p}_c \\
&\le -\widetilde{p}_c'' - c\widetilde{p}_c' + (\phi_1 * \overline{p}_c + b_1\overline{q}_c + c_1\bar{l}_c)\widetilde{p}_c \\
&\le \widetilde{p}_c \\
&\le u_0 \\
&= -u'' - cu' + (\phi_1 * \overline{u}_0 + b_1\overline{v}_0 + c_1\overline{w}_0)u,
\end{aligned}$$

for each $x \in (\ln(A)/\varepsilon, a)$,

$$\begin{aligned}
&- d_2\widetilde{q}_c'' - c\widetilde{q}_c' + r_2(\phi_2 * \overline{v}_0 + c_2\overline{u}_0)\widetilde{q}_c \\
&\le -d_2\widetilde{q}_c'' - c\widetilde{q}_c' + r_2(\phi_2 * \overline{q}_c + c_2\overline{p}_c)\widetilde{q}_c \\
&\le r_2\widetilde{q}_c + r_2b_2\widetilde{l}_c\widetilde{q}_c \\
&\le r_2v_0 + r_2b_2w_0v_0 \\
&= -d_2v'' - cv' + r_2(\phi_2 * \overline{v}_0 + c_2\overline{u}_0)v,
\end{aligned}$$

for each $x \in (\ln(B)/\varepsilon, a)$, and

$$\begin{aligned}
&- d_3\widetilde{l}_c'' - c\widetilde{l}_c' + r_3(\phi_3 * \overline{w}_0 + c_3\overline{u}_0)\widetilde{l}_c \\
&\le -d_3\widetilde{l}_c'' - c\widetilde{l}_c' + r_3(\phi_3 * \bar{l}_c + c_3\overline{p}_c)\widetilde{l}_c \\
&\le r_3\widetilde{l}_c + r_3b_3\widetilde{l}_c\widetilde{q}_c \\
&\le r_3w_0 + r_3b_3w_0v_0 \\
&= -d_3w'' - cw' + r_3(\phi_3 * \overline{w}_0 + c_3\overline{u}_0)w,
\end{aligned}$$

for each $x \in (\frac{\ln D}{\varepsilon}, a)$, where $u(a) = \widetilde{p}_c(a)$, $u(\frac{\ln A}{\varepsilon}) > 0 = \widetilde{p}_c(\frac{\ln A}{\varepsilon})$, $v(a) = \widetilde{q}_c(a)$, $v(\frac{\ln B}{\varepsilon}) > 0 = \widetilde{q}_c(\frac{\ln B}{\varepsilon})$ and $w(a) = \widetilde{l}_c(a)$, $w(\frac{\ln D}{\varepsilon}) > 0 = \widetilde{l}_c(\frac{\ln D}{\varepsilon})$. The maximum principle implies $u(x) \ge \widetilde{p}_c$ for each $x \in (\ln(A)/\varepsilon, a)$, $v(x) \ge \widetilde{q}_c$ for each $x \in (\ln(B)/\varepsilon, a)$ and $w(x) \ge \widetilde{l}_c$ for each $x \in (\ln(D)/\varepsilon, a)$. Then we conclude $u(x) \ge \widetilde{p}_c$, $v(x) \ge \widetilde{q}_c$, $w(x) \ge \widetilde{l}_c$ for each $x \in (-a, a)$, that is $(u, v, w) \in \mathcal{M}_a$ holds. Thus $\Psi_a(\mathcal{M}_a) \subset \mathcal{M}_a$.

Now, by using the Schauder's fixed point theorem, we can obtain that Ψ_a has a fixed point $(u_a, v_a, w_a) \in \mathcal{M}_a$ which is the solution of (3.1).

Lemma 3.1. *There exists a constant M which is independent of the number a and $c > c^*$ ($c^* = \max\{2, 2\sqrt{d_2r_2}, 2\sqrt{d_3r_3}\}$) such that each solution of problem* (3.1) *satisfies*

$$0 \le u_a \le M, \quad 0 \le v_a \le M, \quad 0 \le w_a \le M \tag{3.3}$$

for all $a > \max\left\{\frac{1}{\varepsilon}\ln\frac{A(\lambda_c+\varepsilon)}{\lambda_c}, \frac{1}{\varepsilon}\ln\frac{B(\zeta_c+\varepsilon)}{\zeta_c}, \frac{1}{\varepsilon}\ln\frac{D(\eta_c+\varepsilon)}{\eta_c}\right\}$ *and all* $x \in [-a, a]$.

Proof. Suppose that the maximum points of $u_a(x)$, $v_a(x)$ and $w_a(x)$ are x_M, x_N, $x_H \in [-a, a]$ respectively, that is

$$M_u = \max_{x\in[-a,a]} u_a(x) = u_a(x_M), \quad M_v = \max_{x\in[-a,a]} v_a(x) = v_a(x_N),$$

$$M_w = \max_{x\in[-a,a]} w_a(x) = w_a(x_H).$$

Then we have $u_a'(x_M) = 0$, $u_a''(x_M) \le 0$, $v_a''(x_N) = 0$, $v_a''(x_N) \le 0$, $w_a'(x_H) = 0$ and $w_a''(x_H) \le 0$.

Apart from this, we can also prove $x_M, x_N, x_H \in [-a, a)$. Since $u_a(a) = \widetilde{p}_c(a)$, $v_a(a) = \widetilde{q}_c(a)$ and $w_a(a) = \widetilde{l}_c(a)$, and $\widetilde{p}_c(x)$, $\widetilde{q}_c(x)$, $\widetilde{l}_c(x)$ are decreasing for $x > \max\{\frac{1}{\varepsilon}\ln\frac{A(\lambda_c+\varepsilon)}{\lambda_c}, \frac{1}{\varepsilon}\ln\frac{B(\zeta_c+\varepsilon)}{\zeta_c}, \frac{1}{\varepsilon}\ln\frac{D(\eta_c+\varepsilon)}{\eta_c}\}$, so it holds that $x_M, x_N, x_H \in [-a, a]$.

Next we prove the lemma. From the value of

$$-u_a'' - cu_a' = u_a(1 - \phi_1 * \overline{u}_a - b_1\overline{v}_a - c_1\overline{w}_a)$$

at x_M,

$$-d_2v_a'' - cv_a' = r_2v_a(1 - \phi_2 * \overline{v}_a + b_2\overline{w}_a - c_2\overline{u}_a)$$

at x_N and

$$-d_3w_a'' - cw_a' = r_3w_a(1 - \phi_3 * \overline{w}_a + b_3\overline{v}_a - c_3\overline{u}_a)$$

at x_H, we obtain

$$\begin{gathered} 1 - (\phi_1 * \overline{u}_a)(x_M) - b_1\overline{v}_a(x_M) - c_1\overline{w}_a(x_M) \ge 0, \\ 1 - (\phi_2 * \overline{v}_a)(x_N) + b_2\overline{w}_a(x_N) - c_2\overline{u}_a(x_N) \ge 0, \\ 1 - (\phi_3 * \overline{w}_a)(x_H) + b_3\overline{v}_a(x_H) - c_3\overline{u}_a(x_H) \ge 0, \end{gathered}$$

which implies

$$(\phi_1 * \overline{u}_a)(x_M) < 1, \quad (\phi_2 * \overline{v}_a)(x_N) < 1 \quad \text{and} \quad (\phi_3 * \overline{w}_a)(x_H) < 1, \tag{3.4}$$

and

$$\begin{gathered} -u_a'' - cu_a' \le u_a \le M_u, \\ -d_2v_a'' - cv_a' \le r_2v_a \le r_2M_v, \\ -d_3w_a'' - cw_a' \le r_3w_a \le r_3M_w, \end{gathered}$$

for small enough b_2 and b_3. Thus, it holds that

$$(u_a'e^{cx})' \ge -M_ue^{cx}, \quad \Big(d_2v_a'e^{\frac{c}{d_2}x}\Big)' \ge -r_2M_ve^{\frac{c}{d_2}x}, \quad \Big(d_3w_a'e^{\frac{c}{d_3}x}\Big)' \ge -r_3M_we^{\frac{c}{d_3}x}.$$

Integrating the above inequalities from x_M to $x > x_M$, x_N to $x > x_N$ and x_H to $x > x_H$, respectively, we obtain

$$\begin{gathered} u_a'(x) \ge -\frac{M_u}{c}(1 - e^{-c(x-x_M)}), \quad x \in [x_M, a), \\ v_a'(x) \ge -\frac{r_2M_v}{c}(1 - e^{-\frac{c}{d_2}(x-x_N)}), \quad x \in [x_N, a), \\ w_a'(x) \ge -\frac{r_3M_w}{c}(1 - e^{-\frac{c}{d_3}(x-x_H)}), \quad x \in [x_H, a). \end{gathered}$$

Integrating the above inequalities in the same interval again, we obtain

$$\begin{aligned} u_a(x) &\ge M_u - \frac{M_u}{c}(x - x_M) + \frac{M_u}{c}\int_{x_m}^{x} e^{-c(s-x_M)}ds \\ &= M_u\big[1 - \frac{x - x_M}{c} + \frac{1 - e^{-c(x-x_M)}}{c^2}\big] \\ &= M_u[1 - (x - x_M)^2h(c(x - x_M))] \\ &\ge M_u\big[1 - \frac{1}{2}(x - x_M)^2\big], \end{aligned}$$

$$
\begin{aligned}
v_a(x) &\geq M_v - \frac{r_2 M_v}{c}(x - x_N) + \frac{r_2 M_v}{c} e^{\frac{c}{d_2} x_N} \int_{x_N}^{x} e^{-\frac{c}{d_2} s} ds \\
&= M_v \Big[1 - \frac{r_2}{d_2}(x - x_N)^2 \Big(\frac{1}{\frac{c(x-x_N)}{d_2}} + \frac{e^{-\frac{c}{d_2}(x - x_N)}}{\frac{c^2(x-x_N)^2}{d_2^2}} - \frac{1}{\frac{c^2(x-x_N)^2}{d_2^2}} \Big) \Big] \\
&= M_v \big[1 - \frac{r_2}{d_2}(x - x_N)^2 h\big(\frac{c}{d_2}(x - x_N)\big)\big] \\
&\geq M_v \big[1 - \frac{r_2}{2d_2}(x - x_N)^2\big],
\end{aligned}
$$

and

$$
w_a(x) \geq M_w \big[1 - \frac{r_3}{2d_3}(x - x_H)^2\big],
$$

where $h(y) = \frac{e^{-y} + y - 1}{y^2} \leq 1/2$ for $y > 0$. Since $u_a(x), v_a(x), w_a(x) \in \mathcal{M}_a$, we have

$$
\begin{gathered}
u_a(a) = \widetilde{p}_c(a) \leq \overline{p}_c(a) = e^{-\lambda_c a} \leq 1, \\
v_a(a) = \widetilde{q}_c(a) \leq \overline{q}_c(a) = e^{-\zeta_c a} \leq 1, \\
w_a(a) = \widetilde{l}_c(a) \leq \bar{l}_c(a) = e^{-\eta_c a} \leq 1,
\end{gathered}
$$

which can further imply

$$
\begin{gathered}
M_u[1 - \frac{1}{2}(a - x_M)^2] \leq 1, \\
M_v[1 - \frac{r_2}{2d_2}(a - x_N)^2] \leq 1, \\
M_w[1 - \frac{r_3}{2d_3}(a - x_H)^2] \leq 1.
\end{gathered} \tag{3.5}
$$

Taking $x_0 = 1/2$, if $x_M \in (a - x_0, a)$, it follows from (3.5) that

$$
M_u \leq \big[1 - \frac{1}{2}(a - x_M)^2\big]^{-1} \leq \Big(1 - \frac{1}{2}x_0^2\Big)^{-1} \leq \frac{4}{3}.
$$

If $x_M \in [-a, a - x_0)$, then combining this with (3.4), we have

$$
\begin{aligned}
1 &\geq (\phi_1 * \overline{u}_a)(x_M) \\
&= \int_{\mathbb{R}} \phi_1(y) \overline{u}_a(x_M - y)\, dy \\
&\geq \int_{-x_0}^{0} \phi_1(y) \overline{u}_a(x_M - y)\, dy \\
&\geq M_u \int_{-x_0}^{0} \phi_1(y)\big(1 - \frac{y^2}{2}\big)\, dy
\end{aligned}
$$

From the definition of x_0, we obtain

$$
M_u \leq \Big[\int_{-x_0}^{0} \phi_1(y)(1 - \frac{y^2}{2})\, dy\Big]^{-1} \leq \frac{4}{3}\Big[\int_{-\sqrt{\frac{1}{2}}}^{0} \phi_1(y)\, dy\Big]^{-1}.
$$

From (A1), we know that

$$
\frac{4}{3}\Big(\int_{-\sqrt{\frac{1}{2}}}^{0} \phi_1(y)\, dy\Big)^{-1} \geq \frac{4}{3}.
$$

Thus, for each $x \in [-a, a)$, we have

$$M_u \le \frac{4}{3}\Big(\int_{-\sqrt{\frac{1}{2}}}^{0} \phi_1(y)\,dy\Big)^{-1}.$$

Similarly, take $y_0 = \sqrt{d_2/(2r_2)}$ and $z_0 = \sqrt{d_3/(2r_3)}$, then for each $x \in [-a, a)$, it holds

$$M_v \le \frac{4}{3}\Big(\int_{\sqrt{-\frac{d_2}{2r_2}}}^{0} \phi_2(y)\,dy\Big)^{-1} \quad \text{and} \quad M_w \le \frac{4}{3}\Big(\int_{\sqrt{-\frac{d_3}{2r_3}}}^{0} \phi_3(y)\,dy\Big)^{-1}.$$

Let

$$\begin{aligned} M = \max\Big\{&\frac{4}{3}\Big(\int_{-\sqrt{\frac{1}{2}}}^{0} \phi_1(y)\,dy\Big)^{-1}, \frac{4}{3}\Big(\int_{\sqrt{-\frac{d_2}{2r_2}}}^{0} \phi_2(y)\,dy\Big)^{-1}, \\ &\frac{4}{3}\Big(\int_{\sqrt{-\frac{d_3}{2r_3}}}^{0} \phi_3(y)\,dy\Big)^{-1}\Big\}, \end{aligned} \tag{3.6}$$

then the inequality (3.3) follows. The proof is complete □

Limit of (u_a, v_a, w_a) as $a \to +\infty$. From Lemma 3.1 and the standard elliptic estimates, we know that there exists $M_0 > 0$ such that for each $a > \max\{\frac{\ln A}{\varepsilon}, \frac{\ln B}{\varepsilon}, \frac{\ln D}{\varepsilon}\}$ and a constant $\alpha \in (0, 1)$, it holds that

$$\|u_a\|_{C^{2,\alpha}(-\frac{a}{2},\frac{a}{2})} \le M_0, \quad \|v_a\|_{C^{2,\alpha}(-\frac{a}{2},\frac{a}{2})} \le M_0, \quad \|w_a\|_{C^{2,\alpha}(-\frac{a}{2},\frac{a}{2})} \le M_0.$$

Letting $a \to +\infty$ (possibly along a subsequence), we have $u_a \to u$, $v_a \to v$ and $w_a \to w$ in $C^2_{\rm loc}(\mathbb{R})$, and $(u(x), v(x), w(x))$ satisfies

$$\begin{aligned} -u'' - cu' &= u(1 - \phi_1 * u - b_1 v - c_1 w), \quad x \in \mathbb{R}, \\ -d_2 v'' - cv' &= r_2 v(1 - \phi_2 * v + b_2 w - c_2 u), \quad x \in \mathbb{R}, \\ -d_3 w'' - cw' &= r_3 w(1 - \phi_3 * w + b_3 v - c_3 u), \quad x \in \mathbb{R}, \end{aligned}$$

and

$$\widetilde{p}_c \le u(x) \le \min\{M, \overline{p}_c\}, \quad \widetilde{q}_c \le v(x) \le \min\{M, \overline{q}_c\}. \quad \widetilde{l}_c \le w(x) \le \min\{M, \overline{l}_c\},$$

which implies

$$\lim_{x\to+\infty} u(x) = \lim_{x\to+\infty} v(x) = \lim_{x\to+\infty} w(x) = 0. \tag{3.7}$$

4. Proof of Theorem 1.1

To prove Theorem 1.1 we use the following lemmas.

Lemma 4.1. *There exists a $Z_0 > 0$ such that $u(x)$, $v(x)$ and $w(x)$ are monotonically decreasing for $x > Z_0$.*

Proof. On the contrary, suppose that $u(x)$ is not always monotonic as $x \to +\infty$. From (3.7), then there exists a sequence $x_n \to +\infty$ $(n \to +\infty)$ such that $u(x_n) \to 0$, $v(x_n) \to 0$, $w(x_n) \to 0$ $(n \to +\infty)$ and $u(x)$ achieves a local minimum at x_n, that is $u'(x_n) = 0$, $u''(x_n) \ge 0$. Since

$$-u''(x_n) - cu'(x_n) = u(x_n)(1 - (\phi_1 * u)(x_n) - b_1 v(x_n) - c_1 w(x_n)),$$

for each $n \in \mathbb{N}$, it holds that

$$(\phi_1 * u)(x_n) + b_1 v(x_n) + c_1 w(x_n) \ge 1. \tag{4.1}$$

On the other hand, from $\lim_{x\to+\infty} u(x) = 0$ and the boundedness of $u(x)$ on $C^2(\mathbb{R})$, it is easy to find that $\lim_{n\to+\infty}(\phi_1 * u)(x_n) = 0$. Then combining this with the fact that $\lim_{n\to+\infty} v(x_n) = 0$, $\lim_{n\to+\infty} w(x_n) = 0$, one can obtain a contradiction of (4.1). Therefore, $u(x)$ is always monotonically decreasing on $[Z_0, +\infty)$. In the same way, $v(x)$ and $w(x)$ are also monotonically decreasing on $[Z_0, +\infty)$. This completes the proof. □

Lemma 4.2. *There exists no traveling wave solutions of* (1.5) *for speed* $c < c^*$.

Proof. Using a contradiction argument, we suppose that there exists a traveling wave solution satisfying (1.5) and (1.6) for $c < c^*$. Take a sequence $\{z_n\}$ satisfying $z_n \to +\infty$ as $n \to +\infty$. Denote $u_n(x) = u(x + z_n)/u(z_n)$, $v_n(x) = v(x + z_n)/u(z_n)$, $w_n(x) = w(x + z_n)/w(z_n)$, then we have

$$
\begin{gathered}
-u_n''(x) - cu_n'(x) = u_n(x)(1 - (\phi_1 * \widetilde{u}_n)(x) - b_1\widetilde{v}_n(x) - c_1\widetilde{w}_n(x)), \quad x \in \mathbb{R}, \\
-d_2v_n''(x) - cv_n'(x) = r_2v_n(x)(1 - (\phi_2 * \widetilde{v}_n)(x) + b_2\widetilde{w}_n(x) - c_2\widetilde{u}_n(x)), \quad x \in \mathbb{R}, \\
-d_3w_n''(x) - cw_n'(x) = r_3w_n(x)(1 - (\phi_3 * \widetilde{w}_n)(x) + b_3\widetilde{v}_n(x) - c_3\widetilde{u}_n(x)), \quad x \in \mathbb{R},
\end{gathered}
$$

where $\widetilde{u}_n(x) = u(x + z_n)$, $\widetilde{v}_n(x) = v(x + z_n)$, $\widetilde{w}_n(x) = w(x + z_n)$. Notice that $u_n(0) = v_n(0) = w_n(0) = 1$ and $u_n(x)$, $v_n(x)$, $w_n(x)$ are monotonic decreasing on $[Z_0 - z_n, +\infty)$ for $n \in \mathbb{N}$ (where Z_0 is defined by Lemma 4.1. Since $u(x) \to 0$, $v(x) \to 0$ and $w(x) \to 0$ as $x \to +\infty$, it follows that $(\widetilde{u}_n, \widetilde{v}_n, \widetilde{w}_n) \to (0, 0, 0)$ locally uniformly at x as $n \to +\infty$. Let $(u_n, v_n, w_n) \to (\widehat{u}(x), \widehat{v}(x), \widehat{w}(x))$ in $C^2_{\rm loc}(\mathbb{R})$ as $n \to +\infty$, so we have

$$
\begin{gathered}
-\widehat{u}'' - c\widehat{u}' = \widehat{u}, \quad x \in \mathbb{R}, \\
-d_2\widehat{v}'' - c\widehat{v}' = r_2\widehat{v}, \quad x \in \mathbb{R}, \\
-d_3\widehat{w}'' - c\widehat{w}' = r_3\widehat{w}, \quad x \in \mathbb{R}.
\end{gathered} \tag{4.2}
$$

Evidently, $\widehat{v}$, $\widehat{v}$, $\widehat{w}$ are monotonically decreasing and $\widehat{u}(0) = \widehat{v}(0) = \widehat{w}(0) = 1$. In addition, it is easy to get that $\widehat{u}$, $\widehat{v}$, $\widehat{w}$ are positive. Take $\widehat{u}$ to say, if there exists a point $x_0 \in \mathbb{R}$ such that $\widehat{u}(x_0) = 0$, then from the monotonicity of the nonnegative function $\widehat{u}$, we know that for each $x \geq x_0$, $\widehat{u}(x_0) = 0$. By the uniqueness of the solutions of ordinary differential equations (4.2), we can obtain $\widehat{u}(x) = 0$ in $\mathbb{R}$, which contradicts with the fact $\widehat{u}(0) = 1$. Therefore, (4.2) admits such a solution $(\widehat{u}, \widehat{v}, \widehat{w})$ if and only if $c \geq \max\{2, 2\sqrt{d_2r_2}, 2\sqrt{d_3r_3}\}$, that is, there exists no traveling wave solutions for speed $c < \max\{2, 2\sqrt{d_2r_2}, 2\sqrt{d_3r_3}\}$. This completes the proof. □

Lemma 4.3. *Under assumptions* (A1)–(A3)*, the traveling wave* $(u(x), v(x), w(x))$ *of system* (1.5) *satisfies*

$$
\liminf_{x\to-\infty}(u(x) + v(x) + w(x)) > 0.
$$

Proof. Since $u(x)$, $v(x)$, $w(x)$ are non-negative, $\liminf_{x\to-\infty}(u(x)+v(x)+w(x)) \geq 0$. Using a contradiction argument, we suppose that $\liminf_{x\to-\infty}(u(x) + v(x) + w(x)) = 0$ which will lead to a sequence y_n satisfying $u(y_n) \to 0$, $v(y_n) \to 0$, $w(y_n) \to 0$ as $y_n \to -\infty (n \to +\infty)$. Taking $\widetilde{u}(x) = u(-x)$, $\widetilde{v}(x) = v(-x)$, $\widetilde{w}(x) = w(-x)$ and $\widetilde{c} = -c$, then $(\widetilde{u}(-y_n), \widetilde{v}(-y_n), \widetilde{w}(-y_n)) \to (0, 0, 0)$, and $(\widetilde{u}(x), \widetilde{v}(x), \widetilde{w}(x))$ satisfies

$$
\begin{gathered}
-\widetilde{u}'' - \widetilde{c}\widetilde{u}' = \widetilde{u}(1 - \phi_1 \otimes \widetilde{u} - b_1\widetilde{v} - c_1\widetilde{w}), \\
-d_2\widetilde{v}'' - \widetilde{c}\widetilde{v}' = r_2\widetilde{v}(1 - \phi_2 \otimes \widetilde{v} + b_2\widetilde{w} - c_2\widetilde{u}),
\end{gathered}
$$

$$-d_3\widetilde{w}'' - \widetilde{c}\widetilde{w}' = r_3\widetilde{w}(1 - \phi_3 \otimes \widetilde{w} + b_3\widetilde{v} - c_3\widetilde{u}),$$

where $(\phi_i \otimes z)(x) = \int_{\mathbb{R}} \phi_i(y)z(x+y)\,dy$ $(i = 1, 2, 3)$. As in the proof of Lemma 4.2, we can obtain $\widetilde{c} \geq \max\{2, 2\sqrt{d_2r_2}, 2\sqrt{d_3r_3}\}$; that is

$$c \leq \min\{-2, -2\sqrt{d_2r_2}, -2\sqrt{d_3r_3}\}$$

which contradicts Lemma 4.2. The proof is complete. □

Lemma 4.4. *Assume* (A1)–(A3) *hold If* $b_1 < \frac{1}{2M}$, $c_i < \frac{1}{2M}$ $(i = 1, 2, 3)$, *then the traveling wave* $(u(x), v(x), w(x))$ *of the system* (1.5) *satisfies*

$$\liminf_{x\to-\infty} u(x) > 0, \quad \liminf_{x\to-\infty} v(x) > 0, \quad \liminf_{x\to-\infty} w(x) > 0.$$

Proof. Since $u(x)$, $v(x)$, and $w(x)$ are nonnegative functions, we know that

$$\liminf_{x\to-\infty} u(x) \geq 0, \quad \liminf_{x\to-\infty} v(x) \geq 0, \quad \liminf_{x\to-\infty} w(x) \geq 0.$$

We will prove this lemma in two steps.

Step 1. We prove that $\liminf_{x\to-\infty} u(x) > 0$. By a contradiction argument assume that $\liminf_{x\to-\infty} u(x) = 0$. Then there must hold one of the following two cases: $u(x) \to 0$ in an oscillating or a monotonous manner as $x \to -\infty$.

Case 1.1. There exists a sequence $x_n \to -\infty$ as $n \to +\infty$, such that $u(x)$ attains local minimum at x_n and $u(x_n) \to 0$ as $n \to +\infty$. A well know argument (the Harnack inequality) shows that for each $Z > 0$ and $\delta \in (0, \frac{1-(b_1+c_1)M}{2})$, there exists a constant $N > 0$ such that $u(x) \leq \delta$ for each $n > N$ and $x \in (x_n - Z, x_n + Z)$ which can conclude that $\lim_{n\to+\infty}(\phi_1 * u)(x_n) = 0$ and

$$\begin{aligned} -u''(x_n) - cu'(x_n) &= u(x_n)[1 - (\phi_1 * u)(x_n) - b_1v(x_n) - c_1w(x_n)] \\ &\geq u(x_n)[1 - (b_1 + c_1)M - (\phi_1 * u)(x_n)] > 0, \end{aligned}$$

for large enough n. On the other hand, it is easy to obtain that

$$-u''(x_n) - cu'(x_n) \leq 0,$$

because $u(x)$ attains local minimum at x_n which implies $u'(x_n) = 0$ and $u''(x_n) \geq 0$. At this point, we reach a contradiction.

Case 1.2. $\lim_{x\to-\infty} u(x) = 0$ and there exists a large enough constant $Z > 0$ such that $u'(x) \geq 0$ for all $x < -Z$. From Lemma 4.3, we know that

$$\liminf_{x\to-\infty}(v(x) + w(x)) > 0$$

which can be divided into the following two situations.

(a) Without loss of generality, we suppose that

$$\liminf_{x\to-\infty} v(x) > 0, \liminf_{x\to-\infty} w(x) = 0$$

which admits a sequence $x_n \to -\infty (n \to +\infty)$ such that

$$\lim_{n\to+\infty} u(x_n) = 0, \quad \lim_{n\to+\infty} v(x_n) = \liminf_{x\to-\infty} v(x) = A > 0, \quad \lim_{n\to+\infty} w(x_n) = 0,$$

where A is a constant. Let $u_n(x) = u(x + x_n)/u(x_n)$, $v_n(x) = v(x + x_n)/v(x_n)$ and $w_n(x) = w(x + x_n)/w(x_n)$, then we have

$$-u_n''(x) - cu_n'(x) = u_n(x)[1 - (\phi_1 * \widehat{u}_n)(x) - b_1\widehat{v}_n(x) - c_1\widehat{w}_n(x)] \quad \text{for all } x \in \mathbb{R},$$

where $u_n'(x)$, $v_n'(x)$ and $w_n'(x)$ are defined by Lemma 4.2. Suppose that $u_n(x) \to \widetilde{u}(x)$, $v_n(x) \to \widetilde{v}(x)$, $w_n(x) \to \widetilde{w}(x)$ in $C^2_{\rm loc}(\mathbb{R})$ as $n \to +\infty$. Then by the Harnack inequality, we can conclude that $\widehat{u}_n(x) \to 0$, $\widehat{v}_n(x) \to \widehat{v}(x)$, $\widehat{w}_n(x) \to 0$ $(n \to +\infty)$. Since $u'(x) \geq 0$ for $x < -Z$, it follows that $\widetilde{u}'(x) \geq 0$ for each $x \in \mathbb{R}$. Also we that $\widetilde{u}(x)$ satisfies

$$-\widetilde{u}''(x) - c\widetilde{u}'(x) = \widetilde{u}(x)(1 - b_1\widehat{v}(x)) \quad \text{for all } x \in \mathbb{R}.$$

After integrating from 0 to $x > 0$, we obtain that

$$\begin{aligned} -\widetilde{u}'(x) + \widetilde{u}'(0) - c\widetilde{u}(x) + c\widetilde{u}(0) &= \int_0^x \widetilde{u}(y)(1 - b_1\widehat{v}(y))\,dy \\ &> (1 - b_1 M)\widetilde{u}(0)x. \end{aligned} \tag{4.3}$$

Since $\widetilde{u}(x) > 0$, $\widetilde{u}'(x) \geq 0$, $\widetilde{u}(0) = 1$, it follows that (4.3) does hold for large enough x.

(b) $\liminf_{x\to-\infty} v(x) > 0$, $\liminf_{x\to-\infty} w(x) > 0$. Processing $w(x)$ as we did for $v(x)$ in case (a), we obtain

$$-\widetilde{u}'(x) + \widetilde{u}'(0) - c\widetilde{u}(x) + c\widetilde{u}(0) > [1 - (b_1 + c_1)M]\widetilde{u}(0)x,$$

which also does not hold for large enough x. Hence,

$$\liminf_{x\to-\infty} u(x) > 0 \quad \text{for all } x \in \mathbb{R}.$$

Step 2. We prove that $\liminf_{x\to-\infty} v(x) > 0$. As in the case above, we assume that $\liminf_{x\to-\infty} v(x) = 0$ which implies that there exists a sequence $y_n \to -\infty$ as $n \to +\infty$ such that $u(y_n) \to 0 (n \to +\infty)$. We also analyze the following two cases.

Case 2.1. $v(x)$ attains local minimum at y_n. The proof is same as the Case 1.1 in Step 1, except that we take $\epsilon \in (0, \frac{1-c_2M}{2})$ here such that $v(x) < \epsilon$ for each $x \in (y_n - Z, y_n + Z)$ and $n > N$. Then we have

$$-d_2v''(y_n) - cv'(y_n) \geq r_2v(y_n)[1 - c_2M - (\phi_2 * v)(y_n)] > 0,$$

which contradicts $-d_2v''(y_n) - cv'(y_n) \leq 0$, because $v(x)$ attains local minimum at y_n.

Case 2.2. There exists large enough $Z > 0$ such that $v'(x) \geq 0$ for each $x < -Z$. As in Case 1.2 in Step 1, one can obtain

$$-d_2\widetilde{v}''(x) - c\widetilde{v}'(x) = r_2\widetilde{v}(x)(1 + b_2\widehat{w}(x) - c_2\widehat{u}(x)) \quad \text{for all } x \in \mathbb{R}. \tag{4.4}$$

From Lemma 4.3, we know that $\liminf_{x\to-\infty}(u(x) + w(x)) > 0$ which implies that $\liminf_{x\to-\infty} u(x) > 0$, and either $\liminf_{x\to-\infty} w(x) > 0$ or $\liminf_{x\to-\infty} w(x) = 0$. Then, after integrating (4.4) from 0 to $x > 0$, we obtain

$$\begin{aligned} -d_2\widetilde{v}''(x) + d_2\widetilde{v}'(0) - c\widetilde{v}(x) + c\widetilde{v}(0) &= r_2\int_0^x \widetilde{v}(y)(1 + b_2\widehat{w}(y) - c_2\widehat{u}(y))\,dy \\ &\geq r_2\int_0^x \widetilde{v}(y)(1 - c_2\widehat{u}(y))\,dy \\ &\geq r_2(1 - c_2M)\widetilde{v}(0)x. \end{aligned}$$

which does not hold for large enough x because $\widetilde{v}(x) > 0$, $\widetilde{v}'(x) \geq 0$ and $\widetilde{v}(0) = 1$. Therefore,

$$\liminf_{x\to-\infty} v(x) > 0.$$

Note that the proof of $\liminf_{x\to-\infty} v(x) > 0$ is similar to the the proof of $\liminf_{x\to-\infty} w(x) > 0$; so we omit it. This completes the proof. □

Now the proof of Theorem 1.1 follows from Lemmas 4.1, 4.2, 4.2, 4.3, and 4.4.

Acknowledgment. We are very grateful to anonymous referees for their careful reading and valuable comments. This work was supported by the Natural Science Foundation of China (12161052, 11801470, U22A20231), by the Natural Science Foundation of Sichuan Province(2022NSFSC1819), by the Central Government Funds for Guiding Local Scientific and Technological Development (2021ZYD0010), and by the Fundamental Research Funds for the Central Universities (2682022 ZTPY080).

References

[1] C.-H. Chang, C.-C. Chen, L.-C. Hung, M. Mimura, T. Ogawa; *Existence and stability of non-monotone travelling wave solutions for the diffusive Lotka-Volterra system of three competing species*, Nonlinearity 33 (2020), no. 10, 5080–5110.

[2] F.-D. Dong, W.-T. Li, J.-B. Wang; *Propagation dynamics in a three-species competition model with nonlocal anisotropic dispersal*, Nonlinear Anal. Real World Appl. 48 (2019), 232–266.

[3] J. Fang, J.-H. Wu; *Monotone traveling waves for delayed Lotka-Volterra competition systems*, Discrete Contin. Dyn. Syst. 32 (2012), no. 9, 3043–3058.

[4] J.-S. Guo, X. Liang; *The minimal speed of traveling fronts for the Lotka-Volterra competition system*, J. Dynam. Differential Equations 23 (2011), no. 2, 353–363.

[5] J.-S. Guo, C.-H. Wu; *Recent developments on wave propagation in 2-species competition systems*, Discrete Contin. Dyn. Syst. Ser. B 17 (2012), no. 8, 2713–2724.

[6] B.-S. Han, Z.-C. Wang, Z.-J. Du; *Traveling Waves for Nonlocal Lotka-Volterra Competition Systems*, Discrete Contin. Dyn. Syst. Ser. B 25 (2020), no. 5, 1959–1983.

[7] B.-S. Han, Z. Feng, W.-J. Bo; *Traveling wave phenomena of a nonlocal reaction-diffusion equation with degenerate nonlinearity.* Commun. Nonlinear Sci. Numer. Simul. 103 (2021), Paper No. 105990, 21 pp.

[8] B.-S. Han, D.-Y. Kong; *Propagation dynamics of a nonlocal reaction-diffusion system*, Discrete Contin. Dyn. Syst. 43 (2023) 2756–2780.

[9] Y.-C. Hao, G.-B. Zhang; *Stability of bistable traveling wavefronts for a nonlocal dispersal epidemic system*, Electron. J. Differential Equations 2022 (2022), no. 49, 1-21.

[10] J.-H. Huang, X.-F. Zou; *Traveling wavefronts in diffusive and cooperative Lotka-Volterra system with delays*, J. Math. Anal. Appl. 271 (2002), no. 2, 455–466.

[11] J.-H. Huang, X.-F. Zou; *Existence of traveling wavefronts of delayed reaction diffusion systems without monotonicity*, Discrete Contin. Dyn. Syst. 9 (2003), no. 4, 925–936.

[12] L.-C. Hung; *Traveling wave solutions of competitive-cooperative Lotka-Volterra systems of three species*, Nonlinear Anal. Real World Appl. 12 (2011), no. 6, 3691–3700.

[13] X.-J. Hou, A. W. Leung; *Traveling wave solutions for a competitive reaction-diffusion system and their asymptotics*, Nonlinear Anal. Real World Appl. 9 (2008), no. 5, 2196–2213.

[14] A. W. Leung, X.-J. Hou, Y. Li; *Exclusive traveling waves for competitive reaction-diffusion systems and their stabilities*, J. Math. Anal. Appl. 338 (2008), no. 2, 902–924.

[15] G.-Y. Lv, M.-X. Wang; *Traveling wave front in diffusive and competitive Lotka-Volterra system with delays*, Nonlinear Anal. Real World Appl. 11 (2010), no. 3, 1323–1329.

[16] K. Li, X. Li; *Traveling wave solutions in a delayed diffusive competition system*, Nonlinear Anal. 75 (2012), no. 9, 3705–3722.

[17] G.-C. Lu, Z.-Y. Lu; *Geometric approach for global asymptotic stability for three species competitive Gompertz models*, J. Math. Anal. Appl. 445 (2017), no. 1, 13–22.

[18] W.-T. Li, G. Lin, S. Ruan; *Existence of travelling wave solutions in delayed reaction-diffusion systems with applications to diffusion-competition systems*, Nonlinearity. 19 (2006), no. 6, 1253–1273.

[19] S.-W. Ma; *Traveling wavefronts for delayed reaction-diffusion systems via a fixed point theorem*, J. Differential Equations. 171 (2001), no. 2, 294–314.

[20] Y.-L. Meng, W.-G. Zhang; *Properties of traveling wave fronts for three species Lotka-Volterra system*, Qual. Theory Dyn. Syst. 19 (2020), no. 2, Paper No. 67, 28 pp.
[21] Z.-H. Ma, X. Wu, R. Yuan; *Nonlinear stability of traveling wavefronts for competitive-cooperative Lotka-Volterra systems of three species*, Appl. Math. Comput. 315 (2017) 331–346.
[22] Q. Liu, S. Liu, K. Y. Lam; *Stacked invasion waves in a competition-diffusion model with three species*, J. Differential Equations 271 (2021), 665–718.
[23] Y.-L. Tian, X.-Q. Zhao; *Bistable traveling waves for a competitive-cooperative system with nonlocal delays*, J. Differential Equations 264 (2018), no. 8, 5263–5299.
[24] C.-Y. Wang, L.-R. Li, Q.-Y. Zhang, R. Li; *Dynamical behaviour of a Lotka-Volterra competitive-competitive-cooperative model with feedback controls and time delays*, J. Biol. Dyn. 13 (2019), no. 1, 43–68.
[25] J.-H. Wu, X.-F. Zou; *Traveling wave fronts of reaction-diffusion systems with delay*, J. Dynam. Differential Equations 13 (2001), no. 3, 651–687.
[26] L. Zhang, X.-X. Bao; *Propagation dynamics of a three-species nonlocal competitive-cooperative system*, Nonlinear Anal. Real World Appl. 58 (2021), 103230, 17 pp.
[27] H. Zhao, S.-L. Wu; *Regular traveling waves for a reaction-diffusion equation with two nonlocal delays*, Electron. J. Differential Equations 2022 (2022), no. 82, 1-16.
[28] Y. Zhao, S.-L. Yuan, J.-L. Ma; *Survival and stationary distribution analysis of a stochastic competitive model of three species in a polluted environment*, Bull. Math. Biol. 77 (2015), no. 7, 1285–1326.
[29] X.-F. Zou, J.-H. Wu; *Existence of traveling wave fronts in delayed reaction-diffusion systems via the monotone iteration method*, Proc. Amer. Math. Soc. 125 (1997), no. 9, 2589–2598.
[30] W.-J. Zuo, D.-Q. Jiang, X.-G. Sun, T. Hayat, A. Alsaedi; *Long-time behaviors of a stochastic cooperative Lotka-Volterra system with distributed delay*, Phys. A 506 (2018), 542–559.

Hong-Jie Wu
School of Mathematics, Southwest Jiaotong University, Chengdu, Sichuan 611756, China
Email address: m17882275716@163.com

Bang-Sheng Han (corresponding author)
School of Mathematics, Southwest Jiaotong University, Chengdu, Sichuan 611756, China
Email address: hanbangsheng@swjtu.edu.cn

Shao-Yue Mi
School of Mathematics, Southwest Jiaotong University, Chengdu, Sichuan 611756, China
Email address: mishaoyue@my.swjtu.edu.cn

Liang-Bin Shen
School of Mathematics, Southwest Jiaotong University, Chengdu, Sichuan 611756, China
Email address: m18871027873@163.com