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STABILITY AND INSTABILITY OF KIRCHHOFF PLATE EQUATIONS WITH DELAY ON THE BOUNDARY CONTROL

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ABSTRACT. In this article, we consider the Kirchhoff plate equation with delay terms on the boundary control. We give instability examples of systems for some choices of delays. Finally, we prove its well-posedness, strong stability, and exponential stability under a multiplier geometric control condition.

1. Introduction

Let $\Omega \subset \mathbb{R}^2$ be a bounded open set with boundary Γ of class C^4 consisting of a clamped part $\Gamma_0 \neq \emptyset$ and a rimmed part $\Gamma_1 \neq \emptyset$ such that $\overline{\Gamma_0} \cap \overline{\Gamma_1} = \emptyset$. We consider the Kirchhoff plate equation with delay terms on the boundary controls,

$$
\begin{gathered}
\varphi_{tt}(x,t) + \Delta^2\varphi(x,t) = 0 \quad \text{in } \Omega \times (0,\infty), \\
\varphi(x,t) = \partial_\nu \varphi(x,t) = 0 \quad \text{on } \Gamma_0 \times (0,\infty), \\
\mathcal{B}_1\varphi(x,t) = -\beta_1 \partial_\nu \varphi_t(x,t) - \beta_2 \partial_\nu \varphi_t(x,t-\tau_1) \quad \text{on } \Gamma_1 \times (0,\infty), \\
\mathcal{B}_2\varphi(x,t) = \gamma_1 \varphi_t(x,t) + \gamma_2 \varphi_t(x,t-\tau_2) \quad \text{on } \Gamma_1 \times (0,\infty), \\
\varphi(x,0) = \varphi_0(x), \quad \varphi_t(x,0) = \varphi_1(x) \quad \text{in } \Omega, \\
\varphi_t(x,t) = f_0(x,t) \quad \text{on } \Gamma_1 \times (-\tau_1, 0), \\
\partial_\nu \varphi_t(x,t) = g_0(x,t) \quad \text{on } \Gamma_1 \times (-\tau_2, 0).
\end{gathered} \tag{1.1}
$$

Here and below, β_1, γ_1, τ_1, and τ_2 are positive real numbers, β_2 and γ_2 are nonzero real numbers, $\nu = (\nu_1, \nu_2)$ is the unit outward normal vector along Γ, and $\tau = (-\nu_2, \nu_1)$ is the unit tangent vector along Γ. The constant $0 < \mu < 1/2$ is the Poisson coefficient and the boundary operators $\mathcal{B}_1$ and $\mathcal{B}_2$ are defined, respectively, by

$$
\begin{gathered}
\mathcal{B}_1\varphi = \Delta\varphi + (1-\mu)\mathcal{C}_1\varphi, \\
\mathcal{B}_2\varphi = \partial_\nu \Delta\varphi + (1-\mu)\partial_\tau \mathcal{C}_2\varphi,
\end{gathered}
$$

where

$$
\begin{gathered}
\mathcal{C}_1\varphi = 2\nu_1\nu_2\varphi_{x_1x_2} - \nu_1^2\varphi_{x_2x_2} - \nu_2^2\varphi_{x_1x_1}, \\
\mathcal{C}_2\varphi = (\nu_1^2 - \nu_2^2)\varphi_{x_1x_2} - \nu_1\nu_2\left(\varphi_{x_1x_1} - \varphi_{x_2x_2}\right).
\end{gathered}
$$

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Moreover, easy computations shows that

$$\mathcal{C}_1\varphi = -\partial_\tau^2\varphi - \partial_\tau\nu_2\varphi_{x_1} + \partial_\tau\nu_1\varphi_{x_2}, \quad \mathcal{C}_2\varphi = \partial_{\nu\tau}\varphi - \partial_\tau\nu_1\varphi\varphi_{x_1} - \partial_\tau\nu_2\varphi_{x_2}. \tag{1.2}$$

To reformulate system (1.1), as in [15], we introduce the auxiliary variables

$$\begin{aligned} \eta^1(x,\rho,t) &:= \partial_\nu u_t(x, t-\rho\tau_1), \quad x \in \Gamma_1,\ \rho \in (0,1),\ t > 0, \\ \eta^2(x,\rho,t) &:= u_t(x, t-\rho\tau_2), \quad x \in \Gamma_1,\ \rho \in (0,1),\ t > 0. \end{aligned} \tag{1.3}$$

Then, system (1.1) becomes

$$\varphi_{tt} + \Delta^2\varphi = 0 \quad \text{in } \Omega \times (0,\infty), \tag{1.4}$$

$$\varphi = \partial_\nu\varphi = 0 \quad \text{on } \Gamma_0 \times (0,\infty), \tag{1.5}$$

$$\mathcal{B}_1\varphi + \beta_1\partial_\nu\varphi_t + \beta_2\eta^1(\cdot,1,t) = 0 \quad \text{on } \Gamma_1 \times (0,\infty), \tag{1.6}$$

$$\mathcal{B}_2\varphi - \gamma_1\varphi_t - \gamma_2\eta^2(\cdot,1,t) = 0 \quad \text{on } \Gamma_1 \times (0,\infty), \tag{1.7}$$

$$\tau_1\eta^1_t(\cdot,\rho,t) + \eta^1_\rho(\cdot,\rho,t) = 0 \quad \text{on } \Gamma_1 \times (0,1) \times (0,\infty), \tag{1.8}$$

$$\tau_2\eta^2_t(\cdot,\rho,t) + \eta^2_\rho(\cdot,\rho,t) = 0 \quad \text{on } \Gamma_1 \times (0,1) \times (0,\infty), \tag{1.9}$$

with the following initial conditions

$$\begin{aligned} &\varphi(\cdot,0) = \varphi_0(\cdot), \quad \varphi_t(\cdot,0) = \varphi_1(\cdot) \quad \text{in } \Omega, \\ &\eta^1(\cdot,\rho,0) = f_0(\cdot,-\rho\tau_1) \quad \text{on } \Gamma_1 \times (0,1), \\ &\eta^2(\cdot,\rho,0) = g_0(\cdot,-\rho\tau_2) \quad \text{on } \Gamma_1 \times (0,1). \end{aligned} \tag{1.10}$$

The energy of system (1.4)-(1.10) is

$$\begin{aligned} E(t) = \frac{1}{2}\Big\{ &a(\varphi,\varphi) + \int_\Omega |\varphi_t|^2 dx + \tau_1|\beta_2| \int_{\Gamma_1}\int_0^1 |\eta^1(\cdot,\rho,t)|^2\, d\rho\, d\Gamma \\ &+ \tau_2|\gamma_2| \int_{\Gamma_1}\int_0^1 |\eta^2(\cdot,\rho,t)|^2\, d\rho\, d\Gamma \Big\}, \end{aligned} \tag{1.11}$$

where the sesquilinear form $a : H^2(\Omega) \times H^2(\Omega) \mapsto \mathbb{C}$ is defined by

$$\begin{aligned} a(\varphi,\psi) = \int_\Omega \Big[&\varphi_{x_1x_1}\overline{\psi}_{x_1x_1} + \varphi_{x_2x_2}\overline{\psi}_{x_2x_2} + \mu\left(\varphi_{x_1x_1}\overline{\psi}_{x_2x_2} + \varphi_{x_2x_2}\overline{\psi}_{x_1x_1}\right) \\ &+ 2(1-\mu)\varphi_{x_1x_2}\overline{\psi}_{x_1x_2}\Big] dx. \end{aligned} \tag{1.12}$$

We first recall the following Green's formula (see [12]),

$$a(\varphi,\psi) = \int_\Omega \Delta^2\varphi\overline{\psi} dx + \int_\Gamma \left(\mathcal{B}_1\varphi\partial_\nu\overline{\psi} - \mathcal{B}_2\varphi\overline{\psi}\right) d\Gamma, \tag{1.13}$$

for all $\varphi \in H^4(\Omega)$, $\psi \in H^2(\Omega)$. For further purposes, we need a weaker version of it. As $\mathcal{D}(\overline{\Omega})$ is dense in $E(\Delta^2, L^2(\Omega)) := \{\varphi \in H^2(\Omega) : \Delta^2\varphi \in L^2(\Omega)\}$ equipped with its natural norm, we deduce that $\varphi \in E(\Delta^2, L^2(\Omega))$ (see [14, Theorem 5.6]) satisfies $\mathcal{B}_1\varphi \in H^{-1/2}(\Gamma)$ and $\mathcal{B}_2\varphi \in H^{-3/2}(\Gamma)$ with

$$\begin{aligned} a(\varphi,\psi) = \int_\Omega \Delta^2\varphi\overline{\psi} dx &+ \langle \mathcal{B}_1\varphi, \partial_\nu\psi\rangle_{H^{-1/2}(\Gamma),H^{1/2}(\Gamma)} \\ &- \langle \mathcal{B}_2\varphi, \psi\rangle_{H^{-3/2}(\Gamma),H^{3/2}(\Gamma)}, \quad \forall \psi \in H^2(\Omega). \end{aligned} \tag{1.14}$$

Similar to [1], for any regular solution $\Phi = (\varphi, \varphi_t, \eta^1, \eta^2)$ of system (1.4)-(1.10), the energy $E(t)$ satisfies the estimate

$$\frac{d}{dt}E(t) \le -(\beta_1 - |\beta_2|)\int_{\Gamma_1} |\partial_\nu \varphi_t|^2 \, d\Gamma - (\gamma_1 - |\gamma_2|)\int_{\Gamma_1} |\varphi_t|^2 \, d\Gamma. \tag{1.15}$$

Let us recall some previous works that are related to our problem. Recently, in [6], the authors considered a Kirchhoff-type parabolic problem on a geodesic ball of hyperbolic space, they derived the growth rate of the blow-up solution and the decay rate of the global solution. The stabilization of the Kirchhoff plate equation with non-linear boundary controls was addressed by Rao in [18] (in the linear case, it corresponds to the system (1.1) with $\beta_2 = \gamma_2 = 0$). He proved that the energy of solutions decays exponentially if the multiplier geometric control condition is met.

Time delays can appea in a variety of applications, including physics, chemistry, biology, and thermal phenomena, and they might depend both on the current state and on past occurrences (see [8, 11]). Since time delays frequently cause instabilities, scientists have recently become interested in controlling partial differential equations with time delays (see [3, 4, 5, 7]).

Nicaise and Pignotti [15] examined the multidimensional wave equation with boundary feedback and a delay term at the boundary, by considering the system

$$\begin{gathered}
z_{tt}(x,t) - \Delta z(x,t) = 0 \quad \text{in } \Omega \times (0,\infty), \\
z(x,t) = 0 \quad \text{on } \Gamma_D \times (0,\infty), \\
\frac{\partial z}{\partial \nu}(x,t) = -\mu_1 z_t(x,t) - \mu_2 z_t(x,t-\tau) \quad \text{on } \Gamma_N \times (0,\infty), \\
z(x,0) = z_0(x), \quad z_t(x,0) = z_1(x) \quad \text{in } \Omega, \\
z_t(x,t) = f_0(x,t) \quad \text{on } \Gamma_N \times (-\tau, 0),
\end{gathered} \tag{1.16}$$

where μ_1 and μ_2 are positive real numbers, and Ω is an open bounded domain of $\mathbb{R}^n$ with a boundary Γ of class C^2 with $\Gamma = \Gamma_D \cup \Gamma_N$, such that $\overline{\Gamma_D} \cap \overline{\Gamma_N} = \emptyset$. An exponential decay is established when $\mu_2 < \mu_1$. If this later is false, they found a sequence of delays $\{\tau_k\}_k$, $\tau_k \to 0$, for which the corresponding solutions have increased energy.

To the best of our knowledge, there are no results concerning the case of the Kirchhoff plate equation with boundary controls and time delay. We fill this gap, by examining both instability and stability of system (1.1).

The outline of this article is as follows. In section 2, we give some instability examples of system (1.1) for some particular choices of delays, when $|\beta_2| \ge \beta_1$ and $|\gamma_2| \ge \gamma_1$. In subsection 3.1 , we prove the well-posedness of our system. The subsection 3.2 is devoted to establish the strong stability of our system by following a general criteria of Arendt and Batty. Finally, in subsection 3.3, under condition (MGC), we show that system (1.1) is exponentially stable.

We complete the introduction by introducing some notation. The usual norm and semi-norm of the Sobolev space $H^s(\Omega)$ $(s > 0)$ are denoted by $\|\cdot\|_{H^s(\Omega)}$ and $|\cdot|_{H^s(\Omega)}$, respectively. By $A \lesssim B$, we mean that there exists a constant $C > 0$ independent of A and B such that $A \le CB$.

2. Instability results

In this section, we give some instability examples of system (1.1) in the cases $|\beta_2| \ge \beta_1$ and $|\gamma_2| \ge \gamma_1$. This is achieved by distinguishing between the following cases:

$$|\beta_2| = \beta_1 \text{ and } |\gamma_2| = \gamma_1, \tag{2.1}$$

$$|\beta_2| \geq \beta_1 \text{ and } |\gamma_2| \geq \gamma_1 \text{ and } |\beta_2| - \beta_1 + |\gamma_2| - \gamma_1 > 0. \tag{2.2}$$

Theorem 2.1. *If* (2.1) *or* (2.2) *hold, then there exist sequences of delays and solutions of* (1.1) *corresponding to these delays such that their standard energy is constant.*

Proof. We seek for a solution of system (1.1) of the form

$$\varphi(x,t) = e^{i\lambda t} u(x), \quad \text{with } \lambda \neq 0. \tag{2.3}$$

Inserting (2.3) in (1.1), we obtain

$$\begin{gathered} -\lambda^2 u + \Delta^2 u = 0 \quad \text{in } \Omega, \\ u = \partial_\nu u = 0 \quad \text{on } \Gamma_0, \\ \mathcal{B}_1 u = -i\lambda(\beta_1 + \beta_2 e^{-i\lambda\tau_1})\partial_\nu u \quad \text{on } \Gamma_1, \\ \mathcal{B}_2 u = i\lambda(\gamma_1 + \gamma_2 e^{-i\lambda\tau_2}) u \quad \text{on } \Gamma_1. \end{gathered} \tag{2.4}$$

Let $g \in H^2_{\Gamma_0}(\Omega)$. Multiplying the first equation in (2.4) by $\overline{g}$, then using Green's formula, we obtain

$$\begin{aligned} &-\lambda^2 \int_\Omega u\overline{g}\, dx + a(u,g) + i\lambda(\beta_1 + \beta_2 e^{-i\lambda\tau_1}) \int_{\Gamma_1} \partial_\nu u \partial_\nu \overline{g}\, d\Gamma \\ &+ i\lambda(\gamma_1 + \gamma_2 e^{-i\lambda\tau_2}) \int_{\Gamma_1} u\overline{g}\, d\Gamma = 0, \end{aligned} \tag{2.5}$$

for all $g \in H^2_{\Gamma_0}(\Omega)$. Now, since $|\beta_2| \geq \beta_1$ and $|\gamma_2| \geq \gamma_1$, then we assume that

$$\cos(\lambda\tau_1) = -\frac{\beta_1}{\beta_2} \quad \text{and} \quad \cos(\lambda\tau_2) = -\frac{\gamma_1}{\gamma_2}. \tag{2.6}$$

Thus, we choose

$$\beta_2 \sin(\lambda\tau_1) = \sqrt{\beta_2^2 - \beta_1^2} \quad \text{and} \quad \gamma_2 \sin(\lambda\tau_2) = \sqrt{\gamma_2^2 - \gamma_1^2}. \tag{2.7}$$

Inserting (2.6) and (2.7) in (2.5), we obtain

$$\begin{aligned} &-\lambda^2 \int_\Omega u\overline{g} dx + a(u,g) + \lambda\sqrt{\beta_2^2 - \beta_1^2} \int_{\Gamma_1} \partial_\nu u \partial_\nu \overline{g}\, d\Gamma \\ &+ \lambda\sqrt{\gamma_2^2 - \gamma_1^2} \int_{\Gamma_1} u\overline{g}\, d\Gamma = 0, \end{aligned} \tag{2.8}$$

for all $g \in H^2_{\Gamma_0}(\Omega)$. Now, taking $g = u$ in (2.8), we obtain

$$\begin{aligned} &-\lambda^2 \int_\Omega |u|^2 dx + a(u,u) + \lambda\sqrt{\beta_2^2 - \beta_1^2} \int_{\Gamma_1} |\partial_\nu u|^2\, d\Gamma \\ &+ \lambda\sqrt{\gamma_2^2 - \gamma_1^2} \int_{\Gamma_1} |u|^2\, d\Gamma = 0. \end{aligned} \tag{2.9}$$

Without loss of generality, we can assume that

$$\|u\|_{L^2(\Omega)} = 1. \tag{2.10}$$

Thus, from (2.9) and (2.10), we obtain

$$\lambda^2 - a(u,u) - \lambda\sqrt{\beta_2^2 - \beta_1^2} q_\nu(u) - \lambda\sqrt{\gamma_2^2 - \gamma_1^2} q(u) = 0, \tag{2.11}$$

where

$$q(u) = \int_{\Gamma_1} |u|^2 \, d\Gamma \quad \text{and} \quad q_\nu(u) = \int_{\Gamma_1} |\partial_\nu u|^2 \, d\Gamma. \tag{2.12}$$

We define

$$Z := \{z \in H^2_{\Gamma_0}(\Omega) : \|z\|_{L^2(\Omega)} = 1\}.$$

Now, we distinguish two cases.

Case 1: If (2.1), then from (2.11), we have

$$a(u, u) = \lambda^2. \tag{2.13}$$

Let us define

$$\lambda^2 := \min_{z \in Z} a(z, z). \tag{2.14}$$

Now, if u satisfies $a(u, u) = \min_{z\in Z} a(z, z)$. Then it easy to see that u is a solution of (2.5) and consequently (2.3) is a solution of (1.1). Moreover, from (2.3) and (1.11), we obtain

$$E(t) = E(0) \geq a(u, u) + \lambda^2 \int_\Omega |u|^2 dx = 2\lambda^2 > 0, \quad \forall t \geq 0.$$

Thus, the energy of (1.1) is constant and positive. Further from our assumptions

$$\cos(\lambda\tau_1) = -1, \quad \sin(\lambda\tau_1) = 0, \quad \cos(\lambda\tau_2) = -1, \quad \sin(\lambda\tau_2) = 0,$$

system (2.4) becomes

$$\begin{gathered} -\lambda^2 u + \Delta^2 u = 0 \quad \text{in } \Omega, \\ u = \partial_\nu u = 0 \quad \text{on } \Gamma_0, \\ \mathcal{B}_1 u = 0 \quad \text{on } \Gamma_1, \\ \mathcal{B}_2 u = 0 \quad \text{on } \Gamma_1. \end{gathered} \tag{2.15}$$

So, we can take a sequence $(\lambda_n)_n$ of positive real numbers defined by

$$\lambda_n^2 = \Lambda_n^2, \quad n \in \mathbb{N},$$

where Λ_n^2, $n \in \mathbb{N}$, are the eigenvalues for the bi-Laplacian operator with the boundary conditions $(2.15)_2$-$(2.15)_4$. Then, setting

$$\lambda_n\tau_1 = (2k + 1)\pi, \ k \in \mathbb{N} \quad \text{and} \quad \lambda_n\tau_2 = (2l + 1)\pi, \ l \in \mathbb{N},$$

we obtain the sequences of delays

$$\tau_{1,n,k} = \frac{(2k + 1)\pi}{\lambda_n}, \ k, n \in \mathbb{N} \quad \text{and} \quad \tau_{2,n,l} = \frac{(2l + 1)\pi}{\lambda_n}, \ l, n \in \mathbb{N},$$

which becomes arbitrarily small (or large) for suitable choices of the indices $n, k, l \in \mathbb{N}$. Therefore, we have found sets of time delays for which system (1.1) is not asymptotically stable.

Case 2: If (2.2) holds, then from (2.11), we have

$$\begin{aligned} \lambda = \frac{1}{2}\Big[&\sqrt{\beta_2^2 - \beta_1^2} q_\nu(u) + \sqrt{\gamma_2^2 - \gamma_1^2} q(u) \\ &\pm \sqrt{\Big(\sqrt{\beta_2^2 - \beta_1^2} q_\nu(u) + \sqrt{\gamma_2^2 - \gamma_1^2} q(u)\Big)^2 + 4a(u, u)}\Big]. \end{aligned} \tag{2.16}$$

Let us define

$$\lambda := \frac{1}{2}\min_{z\in Z}\Big\{\sqrt{\beta_2^2-\beta_1^2}q_\nu(z)+\sqrt{\gamma_2^2-\gamma_1^2}q(z) \\ +\sqrt{\Big(\sqrt{\beta_2^2-\beta_1^2}q_\nu(z)+\sqrt{\gamma_2^2-\gamma_1^2}q(z)\Big)^2+4a(z,z)}\Big\}. \tag{2.17}$$

Let us prove that if the minimum in the right-hand side of (2.17) is attained at u, that is

$$\begin{aligned}&\sqrt{\beta_2^2-\beta_1^2}q_\nu(u)+\sqrt{\gamma_2^2-\gamma_1^2}q(u)\\ &+\sqrt{\Big(\sqrt{\beta_2^2-\beta_1^2}q_\nu(u)+\sqrt{\gamma_2^2-\gamma_1^2}q(u)\Big)^2+4a(u,u)}\\ &:=\min_{z\in Z}\Big\{\sqrt{\beta_2^2-\beta_1^2}q_\nu(z)+\sqrt{\gamma_2^2-\gamma_1^2}q(z)\\ &\quad+\sqrt{\Big(\sqrt{\beta_2^2-\beta_1^2}q_\nu(z)+\sqrt{\gamma_2^2-\gamma_1^2}q(z)\Big)^2+4a(z,z)}\Big\},\end{aligned} \tag{2.18}$$

then u is a solution of (2.8). For this aim, take for $\varepsilon\in\mathbb{R}$ as

$$z=u+\varepsilon g \tag{2.19}$$

with $g\in H^2_{\Gamma_0}(\Omega)$ such that $\int_\Omega u\overline{g}dx=0$. Thus, we have

$$\|z\|^2_{L^2(\Omega)}=\|u\|^2_{L^2(\Omega)}+\varepsilon^2\|g\|^2_{L^2(\Omega)}=1+\varepsilon^2\|g\|^2_{L^2(\Omega)}. \tag{2.20}$$

Now, if we define

$$f(\varepsilon):=\frac{1}{1+\varepsilon^2\|g\|^2_{L^2(\Omega)}}\Big(\sqrt{\beta_2^2-\beta_1^2}q_\nu(u+\varepsilon g)+\sqrt{\gamma_2^2-\gamma_1^2}q(u+\varepsilon g)\\ +\sqrt{\Big(\sqrt{\beta_2^2-\beta_1^2}q_\nu(u+\varepsilon g)+\sqrt{\gamma_2^2-\gamma_1^2}q(u+\varepsilon g)\Big)^2+4a(u+\varepsilon g,u+\varepsilon g)}\Big);$$

thus, from (2.18), we obtain

$$f(\varepsilon)\ge f(0)=\sqrt{\beta_2^2-\beta_1^2}q_\nu(u)+\sqrt{\gamma_2^2-\gamma_1^2}q(u)\\ +\sqrt{\Big(\sqrt{\beta_2^2-\beta_1^2}q_\nu(u)+\sqrt{\gamma_2^2-\gamma_1^2}q(u)\Big)^2+4a(u,u)},$$

which gives $f'(0)=0$. Consequently, after an easy computation, we obtain

$$a(u,g)+\lambda\sqrt{\beta_2^2-\beta_1^2}\int_{\Gamma_1}\partial_\nu u\partial_\nu\overline{g}\,d\Gamma+\lambda\sqrt{\gamma_2^2-\gamma_1^2}\int_{\Gamma_1}u\overline{g}\,d\Gamma=0. \tag{2.21}$$

Since any function $\tilde g\in H^2_{\Gamma_0}(\Omega)$ can be decomposed as

$$\tilde g=\alpha u+g$$

with $\alpha\in\mathbb{R}$ and $g\in H^2_{\Gamma_0}(\Omega)$ such that $\int_\Omega u\overline{g}\,dx=0$, from (2.21) and (2.9), we obtain that u satisfies (2.8). Thus, for such $\lambda>0$,

$$\lambda\tau_1=\arccos\Big(-\frac{\beta_1}{\beta_2}\Big)+2k\pi,\ k\in\mathbb{N}\quad\text{and}\quad\lambda\tau_2=\arccos\Big(-\frac{\gamma_1}{\gamma_2}\Big)+2l\pi,\ l\in\mathbb{N},$$

define a sequences of time delays for which (1.1) is not asymptotically stable. $\square$

3. Stability results

In this section, we will prove the wellposedness, strong stability and exponential stability of system (1.4)-(1.10). For this aim, we make the following assumptions

$$\beta_1, \gamma_1 > 0, \quad \beta_2, \gamma_2 \in \mathbb{R}^*, \quad |\beta_2| < \beta_1, \quad |\gamma_2| < \gamma_1. \tag{3.1}$$

3.1. **Wellposedness of system** (1.4)-(1.10)**.** Under hypothesis (3.1), (1.15), system (1.4)-(1.10) is dissipative in the sense that its energy is non-increasing with respect to time (i.e. $E'(t) \le 0$). Let us define the Hilbert space

$$\mathbb{H} = H^2_{\Gamma_0}(\Omega) \times L^2(\Omega) \times \left(L^2(\Gamma_1 \times (0,1))\right)^2,$$

where

$$H^2_{\Gamma_0}(\Omega) = \{\mathsf{f} \in H^2(\Omega) : \mathsf{f} = \partial_\nu \mathsf{f} = 0 \text{ on } \Gamma_0\}.$$

This Hilbert space equipped with the inner product

$$\begin{aligned}(\Phi, \Phi_1)_{\mathbb{H}} = a(\varphi, \varphi_1) + \int_\Omega \psi \overline{\psi_1}\, dx + \tau_1 |\beta_2| \int_{\Gamma_1} \int_0^1 \eta^1 \overline{\eta^1_1}\, d\rho\, d\Gamma \\ + \tau_2 |\gamma_2| \int_{\Gamma_1} \int_0^1 \eta^2 \overline{\eta^2_1}\, d\rho\, d\Gamma,\end{aligned} \tag{3.2}$$

where $\Phi = (\varphi, \psi, \eta^1, \eta^2)^\top$, $\Phi^1 = (\varphi_1, \psi_1, \eta^1_1, \eta^2_1)^\top \in \mathbb{H}$. We define the linear unbounded operator $\mathbb{A} : D(\mathbb{A}) \subset \mathbb{H} \mapsto \mathbb{H}$ by

$$\begin{aligned}D(\mathbb{A}) = \Big\{ &\Phi = (\varphi, \psi, \eta^1, \eta^2)^\top \in \mathsf{D}_{\Gamma_0}(\Delta^2) \times H^2_{\Gamma_0}(\Omega) \times (L^2(\Gamma_1; H^1(0,1)))^2 \\ &: \mathcal{B}_1 \varphi = -\beta_1 \partial_\nu \psi - \beta_2 \eta^1(\cdot, 1),\ \mathcal{B}_2 \varphi = \gamma_1 \psi + \gamma_2 \eta^2(\cdot, 1), \\ &\eta^1(\cdot, 0) = \partial_\nu \psi,\ \eta^2(\cdot, 0) = \psi \text{ on } \Gamma_1 \Big\}\end{aligned}$$

where

$$\mathsf{D}_{\Gamma_0}(\Delta^2) = \left\{\varphi \in H^2_{\Gamma_0}(\Omega) : \Delta^2 \varphi \in L^2(\Omega),\ \mathcal{B}_1 \varphi \in L^2(\Gamma_1),\ \mathcal{B}_2 \varphi \in L^2(\Gamma_1)\right\}$$

and

$$\mathbb{A} \begin{pmatrix} \varphi \\ \psi \\ \eta^1 \\ \eta^2 \end{pmatrix} = \begin{pmatrix} \psi \\ -\Delta^2 \varphi \\ -\frac{1}{\tau_1} \eta^1_\rho \\ -\frac{1}{\tau_2} \eta^2_\rho \end{pmatrix}, \quad \forall \Phi = (\varphi, \psi, \eta^1, \eta^2)^\top \in D(\mathbb{A}). \tag{3.3}$$

Remark 3.1. From the fact that $2\Re(\varphi_{x_1x_1} \overline{\varphi}_{x_2x_2}) = |\varphi_{x_1x_1} + \varphi_{x_2x_2}|^2 - |\varphi_{x_1x_1}|^2 - |\varphi_{x_2x_2}|^2$, we have

$$\begin{aligned}&|\varphi_{x_1x_1}|^2 + |\varphi_{x_2x_2}|^2 + 2\mu \Re\left(\varphi_{x_1x_1} \overline{\varphi}_{x_2x_2}\right) + 2(1-\mu)|\varphi_{x_1x_2}|^2 \\ &= (1-\mu)|\varphi_{x_1x_1}|^2 + (1-\mu)|\varphi_{x_2x_2}|^2 + \mu |\varphi_{x_1x_1} + \varphi_{x_2x_2}|^2 + 2(1-\mu)|\varphi_{x_1x_2}|^2 \ge 0\,;\end{aligned} \tag{3.4}$$

consequently, from (1.12), we obtain

$$a(\varphi, \varphi) \ge (1-\mu)|\varphi|_{H^2(\Omega)}.$$

Hence the sesquilinear form a is coercive on $H^2_{\Gamma_0}(\Omega)$, since Γ_0 is non empty. On the other hand, from (1.14) (see also [18, Lemma 3.1 and Remark 3.1]), we have

$$a(\varphi, \psi) = \int_\Omega \Delta^2 \varphi \overline{\psi} dx + \int_{\Gamma_1} (\mathcal{B}_1 \varphi \partial_\nu \overline{\psi} - \mathcal{B}_2 \varphi \overline{\psi})\, d\Gamma, \tag{3.5}$$

for all $\varphi \in \mathsf{D}_{\Gamma_0}(\Delta^2)$ and all $\psi \in H^2_{\Gamma_0}(\Omega)$.

Now, if $\Phi = (\varphi, \varphi_t, \eta^1, \eta^2)^\top$ is solution of (1.4)-(1.10) and is sufficiently regular, then system (1.4)-(1.10) can be written as the first order evolution equation

$$\Phi_t = \mathbb{A}\Phi, \quad \Phi(0) = \Phi_0, \tag{3.6}$$

where $\Phi_0 = (\varphi_0, \varphi_1, f_0(\cdot, -\rho\tau_1), g_0(\cdot, -\rho\tau_2))^\top \in \mathbb{H}$.

Proposition 3.2. *Under hypothesis* (3.1)*, the unbounded linear operator* $\mathbb{A}$ *is m-dissipative in the energy space* $\mathbb{H}$.

Proof. For all $\Phi = (\varphi, \psi, \eta^1, \eta^2)^\top \in D(\mathbb{A})$, from (3.2) and (3.3), we have

$$\begin{aligned}
&\Re\left(\mathbb{A}\Phi, \Phi\right)_{\mathbb{H}} \\
&= \Re\Big\{a(\psi,\varphi) - \int_\Omega \Delta^2\varphi\overline{\psi}dx - |\beta_2|\int_{\Gamma_1}\int_0^1 \eta^1_\rho\overline{\eta^1}\,d\rho\,d\Gamma - |\gamma_2|\int_{\Gamma_1}\int_0^1 \eta^2_\rho\overline{\eta^2}\,d\rho\,d\Gamma\Big\}.
\end{aligned}$$

Using (3.5) and that $\Phi \in D(\mathbb{A})$, we obtain

$$\begin{aligned}
&\Re\left(\mathbb{A}\Phi, \Phi\right)_{\mathbb{H}} \\
&= -\beta_1\int_{\Gamma_1}|\partial_\nu\psi|^2\,d\Gamma - \Re\big\{\beta_2\int_{\Gamma_1}\eta^1(\cdot,1)\partial_\nu\overline{\psi}\,d\Gamma\big\} - \gamma_1\int_{\Gamma_1}|\psi|^2\,d\Gamma \\
&\quad - \Re\big\{\gamma_2\int_{\Gamma_1}\eta^2(\cdot,1)\overline{\psi}\,d\Gamma\big\} - \frac{|\beta_2|}{2}\int_{\Gamma_1}|\eta^1(\cdot,1)|^2\,d\Gamma + \frac{|\beta_2|}{2}\int_{\Gamma_1}|\partial_\nu\psi|^2\,d\Gamma \\
&\quad - \frac{|\gamma_2|}{2}\int_{\Gamma_1}|\eta^2(\cdot,1)|^2\,d\Gamma + \frac{|\gamma_2|}{2}\int_{\Gamma_1}|\psi|^2\,d\Gamma.
\end{aligned} \tag{3.7}$$

Now, by using Young's inequality, we obtain

$$\begin{aligned}
-\Re\big\{\beta_2\int_{\Gamma_1}\eta^1(\cdot,1)\partial_\nu\overline{\psi}\,d\Gamma\big\} &\le \frac{|\beta_2|}{2}\int_{\Gamma_1}|\eta^1(\cdot,1)|^2\,d\Gamma + \frac{|\beta_2|}{2}\int_{\Gamma_1}|\partial_\nu\psi|^2\,d\Gamma, \\
-\Re\big\{\gamma_2\int_{\Gamma_1}\eta^2(\cdot,1)\overline{\psi}\,d\Gamma\big\} &\le \frac{|\gamma_2|}{2}\int_{\Gamma_1}|\eta^2(\cdot,1)|^2\,d\Gamma + \frac{|\gamma_2|}{2}\int_{\Gamma_1}|\psi|^2\,d\Gamma.
\end{aligned}$$

Inserting the above inequalities into (3.7) and using hypothesis (3.1), we obtain

$$\Re(\mathbb{A}\Phi, \Phi)_{\mathbb{H}} \le -(\beta_1 - |\beta_2|)\int_{\Gamma_1}|\partial_\nu\psi|^2\,d\Gamma - (\gamma_1 - |\gamma_2|)\int_{\Gamma_1}|\psi|^2\,d\Gamma \le 0, \tag{3.8}$$

which implies that $\mathbb{A}$ is dissipative. Now, let us prove that $\mathbb{A}$ is maximal. For this aim, if $F = (f_1, f_2, f_3, f_4)^\top \in \mathbb{H}$, we look for $\Phi = (\varphi, \psi, \eta^1, \eta^2)^\top \in D(\mathbb{A})$ unique solution of

$$-\mathbb{A}\Phi = F. \tag{3.9}$$

Equivalently, we have the system

$$-\psi = f_1, \tag{3.10}$$

$$\Delta^2\varphi = f_2, \tag{3.11}$$

$$\frac{1}{\tau_1}\eta^1_\rho = f_3, \tag{3.12}$$

$$\frac{1}{\tau_2}\eta^2_\rho = f_4, \tag{3.13}$$

with the boundary conditions

$$\begin{gathered}\varphi = \partial_\nu \varphi = 0 \quad \text{on } \Gamma_0,\\ \mathcal{B}_1\varphi = -\beta_1 \partial_\nu \psi - \beta_2 \eta^1(\cdot,1), \quad \mathcal{B}_2\varphi = \gamma_1\psi + \gamma_2\eta^2(\cdot,1) \quad \text{on } \Gamma_1,\\ \eta^1(\cdot,0) = \partial_\nu\psi, \quad \eta^2(\cdot,0) = \psi \quad \text{on } \Gamma_1.\end{gathered} \tag{3.14}$$

From (3.10) and that $F \in \mathbb{H}$, we obtain

$$\psi = -f_1 \in H^2_{\Gamma_0}(\Omega). \tag{3.15}$$

In light of (3.12), (3.13), (3.14) and since $F \in \mathbb{H}$, we obtain

$$\eta^1_\rho \in L^2(\Gamma_1 \times (0,1)), \quad \eta^1(\cdot,\rho) = \tau_1 \int_0^\rho f_3(\cdot,s)ds + \partial_\nu \psi, \tag{3.16}$$

$$\eta^2_\rho \in L^2(\Gamma_1 \times (0,1)), \quad \eta^2(\cdot,\rho) = \tau_2 \int_0^\rho f_4(\cdot,s)ds + \psi. \tag{3.17}$$

Consequently, owing to (3.15),(3.16), (3.17), taking into account that $f_3, f_4 \in L^2(\Gamma_1 \times (0,1))$, we deduce that

$$\eta^1, \eta^2 \in L^2(\Gamma_1; H^1(0,1)).$$

It follows from (3.11), (3.14), (3.16) and (3.17) that

$$\begin{gathered}\Delta^2\varphi = f_2 \quad \text{in } \Omega,\\ \varphi = \partial_\nu\varphi = 0 \quad \text{on } \Gamma_0,\\ \mathcal{B}_1\varphi = (\beta_1+\beta_2)\partial_\nu f_1 - \tau_1\beta_2 \int_0^1 f_3(\cdot,s)ds \quad \text{on } \Gamma_1,\\ \mathcal{B}_2\varphi = -(\gamma_1+\gamma_2) f_1 + \tau_2\gamma_2 \int_0^1 f_4(\cdot,s)ds \quad \text{on } \Gamma_1.\end{gathered} \tag{3.18}$$

Let $u \in H^2_{\Gamma_0}(\Omega)$. Multiplying the first equation in (3.18) by $\overline{u}$ and integrating over Ω, then using Green's formula, we obtain

$$a(\varphi,u) = l(u), \quad \forall u \in H^2_{\Gamma_0}(\Omega), \tag{3.19}$$

where

$$\begin{aligned} l(u) = \int_\Omega f_2 \overline{u}\,dx &+ \int_{\Gamma_1} \Big((\beta_1+\beta_2)\partial_\nu f_1 - \tau_1\beta_2 \int_0^1 f_3(\cdot,s)ds\Big)\partial_\nu \overline{u}\,d\Gamma\\ &+ \int_{\Gamma_1}\Big((\gamma_1+\gamma_2)f_1 - \tau_2\gamma_2 \int_0^1 f_4(\cdot,s)ds\Big)\overline{u}\,d\Gamma. \end{aligned}$$

It is easy to see that, a is a sesquilinear, continuous and coercive form on $H^2_{\Gamma_0}(\Omega) \times H^2_{\Gamma_0}(\Omega)$ and l is an antilinear and continuous form on $H^2_{\Gamma_0}(\Omega)$. Then, thanks to Lax-Milgram theorem, (3.19) admits a unique solution $u \in H^2_{\Gamma_0}(\Omega)$. By taking the test function $\varphi \in \mathcal{D}(\Omega)$, we see that the first identity of (3.18) holds in the distributional sense, hence $\Delta^2\varphi \in L^2(\Omega)$. Going back to (3.19), and again applying Greens's formula (1.14), we find that

$$\begin{gathered}\mathcal{B}_1\varphi = (\beta_1+\beta_2)\partial_\nu f_1 - \tau_1\beta_2 \int_0^1 f_3(\cdot,s)ds \quad \text{on } \Gamma_1\,,\\ \mathcal{B}_2\varphi = -(\gamma_1+\gamma_2) f_1 + \tau_2\gamma_2 \int_0^1 f_4(\cdot,s)ds \quad \text{on } \Gamma_1\,.\end{gathered}$$

Furthermore, since $F \in \mathbb{H}$, we deduce that $\varphi \in \mathsf{D}_{\Gamma_0}(\Delta^2)$. Consequently, if we define $\Phi = (\varphi, \psi, \eta^1, \eta^2)^\top$ with $\varphi \in H^2_{\Gamma_0}(\Omega)$ the unique solution of (3.19), $\psi = -f_1$, and η^1 (resp. η^2) defined by (3.16) (resp. (3.17)), Φ belongs to $D(\mathbb{A})$ is the unique solution of (3.9). Then, $\mathbb{A}$ is an isomorphism and since $\rho(\mathbb{A})$ is open set of $\mathbb{C}$ (see [10, Theorem 6.7 (Chapter III)]), we easily obtain $R(\lambda I - \mathbb{A}) = \mathcal{H}$ for a sufficiently small $\lambda > 0$. This, together with the dissipativeness of $\mathbb{A}$, imply that $D(\mathbb{A})$ is dense in $\mathcal{H}$ and that $\mathbb{A}$ is m-dissipative in $\mathcal{H}$ (see [16, Theorems 4.5, 4.6]). □

Based on Lumer-Phillips theorem (see [16]), Proposition 3.2 implies that the operator $\mathbb{A}$ generates a C_0-semigroup of contractions $e^{t\mathbb{A}}$ in $\mathbb{H}$ which gives the well-posedness of (3.6). Then, we have the following result.

Theorem 3.3. *For all $\Phi_0 \in \mathbb{H}$, system* (3.6) *admits a unique weak solution $\Phi(t) = e^{t\mathbb{A}}\Phi_0 \in C^0(\mathbb{R}_+, \mathbb{H})$. Moreover, if $\Phi_0 \in D(\mathbb{A})$, then system* (3.6) *admits a unique strong solution $\Phi(t) = e^{t\mathbb{A}}\Phi_0 \in C^0(\mathbb{R}_+, D(\mathbb{A})) \cap C^1(\mathbb{R}_+, \mathbb{H})$.*

3.2. **Strong stability of system** (1.4)**-**(1.10)**.** The following theorem is the main result of this subsection.

Theorem 3.4. *Under hypotheses* (3.1)*, the C_0-semigroup of contraction $(e^{t\mathbb{A}})_{t\geq 0}$ is strongly stable in $\mathbb{H}$; i.e., for all $\Phi_0 \in \mathbb{H}$, the solution of* (3.6) *satisfies*

$$\lim_{t\to\infty} \|e^{t\mathbb{A}}\Phi_0\|_{\mathbb{H}} = 0.$$

According to Arendt-Batty [2], to prove Theorem 3.4, we need to prove that the operator $\mathbb{A}$ has no pure imaginary eigenvalues and $\sigma(\mathbb{A}) \cap i\mathbb{R}$ is countable. The proof of these results is not reduced to the analysis of the point spectrum of $\mathbb{A}$ on the imaginary axis since its resolvent is not compact. Hence the proof of Theorem 3.4 has been divided into the following two Lemmas.

Lemma 3.5. *For all $\lambda \in \mathbb{R}$, $i\lambda I - \mathbb{A}$ is injective i.e.,* $\ker(i\lambda I - \mathbb{A}) = \{0\}$

Proof. In accordance with Proposition 3.2, we have $0 \in \rho(\mathbb{A})$. We still need to show the result for $\lambda \in \mathbb{R}^*$. For the sake of this, suppose that $\lambda \neq 0$ and let $\Phi = (\varphi, \psi, \eta^1, \eta^2)^\top \in D(\mathbb{A})$ be such that

$$\mathbb{A}\Phi = i\lambda\Phi. \tag{3.20}$$

Equivalently, we have the system

$$\psi = i\lambda\varphi, \tag{3.21}$$

$$-\Delta^2\varphi = i\lambda\psi, \tag{3.22}$$

$$-\frac{1}{\tau_1}\eta^1_\rho = i\lambda\eta^1, \tag{3.23}$$

$$-\frac{1}{\tau_2}\eta^2_\rho = i\lambda\eta^2. \tag{3.24}$$

From (3.8), (3.20) and (3.1), we obtain

$$0 = \Re(\mathbb{A}\Phi, \Phi)_{\mathbb{H}} \leq -(\beta_1 - |\beta_2|)\int_{\Gamma_1} |\partial_\nu\psi|^2\, d\Gamma - (\gamma_1 - |\gamma_2|)\int_{\Gamma_1} |\psi|^2\, d\Gamma \leq 0.$$

Thus, we have

$$\partial_\nu\psi = \psi = 0 \quad \text{on } \Gamma_1, \tag{3.25}$$

which gives, from (3.21) and since $\lambda \neq 0$, that

$$\varphi = \partial_\nu \varphi = 0 \quad \text{on } \Gamma_1. \tag{3.26}$$

Using (3.23), (3.24), (3.25) and that $\Phi \in D(\mathbb{A})$, we obtain

$$\eta^1(\cdot, \rho) = \partial_\nu \psi e^{-i\lambda\tau_1\rho} = 0 \quad \text{on } \Gamma_1 \times (0,1), \tag{3.27}$$

$$\eta^2(\cdot, \rho) = \psi e^{-i\lambda\tau_2\rho} = 0 \quad \text{on } \Gamma_1 \times (0,1). \tag{3.28}$$

Now, from equations (3.25), (3.27), (3.28) and seeing that $\Phi \in D(\mathbb{A})$, we obtain

$$\mathcal{B}_1\varphi = \Delta\varphi + (1-\mu)\mathcal{C}_1\varphi = 0 \quad \text{on } \Gamma_1, \tag{3.29}$$

$$\mathcal{B}_2\varphi = \partial_\nu \Delta\varphi + (1-\mu)\partial_\tau \mathcal{C}_2\varphi = 0 \quad \text{on } \Gamma_1. \tag{3.30}$$

Using (3.26) and considering $\nabla\varphi = \partial_\tau\varphi\tau + \partial_\nu\varphi\nu$ on Γ_1, we obtain

$$\varphi_{x_1} = \varphi_{x_2} = 0 \quad \text{on } \Gamma_1. \tag{3.31}$$

Now, (1.2), (3.26) and (3.31), yield

$$\mathcal{C}_1\varphi = \mathcal{C}_2\varphi = 0 \quad \text{on } \Gamma_1, \tag{3.32}$$

consequently, from (3.29) and (3.30), we infer

$$\Delta\varphi = \partial_\nu \Delta\varphi = 0 \quad \text{on } \Gamma_1. \tag{3.33}$$

Inserting (3.21) in (3.22), we obtain

$$\begin{gathered} \lambda^2\varphi - \Delta^2\varphi = 0 \quad \text{in } \Omega, \\ \varphi = \partial_\nu\varphi = 0 \quad \text{on } \Gamma_0, \\ \varphi = \partial_\nu\varphi = \Delta\varphi = \partial_\nu\Delta\varphi = 0 \quad \text{on } \Gamma_1. \end{gathered} \tag{3.34}$$

Holmgren uniqueness theorem (see [13]) yields

$$\varphi = 0 \quad \text{in } \Omega. \tag{3.35}$$

Finally, from (3.21), (3.27), (3.28), and (3.35), we obtain $\Phi = 0$. $\square$

Lemma 3.6. *Under hypothesis* (3.1)*, we have* $R(i\lambda I - \mathbb{A}) = \mathbb{H}$ *for all* $\lambda \in \mathbb{R}$.

Proof. From Proposition 3.2, we have $0 \in \rho(\mathbb{A})$. We still need to show the result for $\lambda \in \mathbb{R}^\star$. For this aim, for $F = (f_1, f_2, f_3, f_4)^\top \in \mathbb{H}$, we look for $\Phi = (\varphi, \psi, \eta^1, \eta^2)^\top \in D(\mathbb{A})$ solution of

$$(i\lambda I - \mathbb{A})\Phi = F. \tag{3.36}$$

Correspondingly, we have the system

$$i\lambda\varphi - \psi = f_1, \tag{3.37}$$

$$i\lambda\psi + \Delta^2\varphi = f_2, \tag{3.38}$$

$$i\lambda\eta^1 + \frac{1}{\tau_1}\eta^1_\rho = f_3, \tag{3.39}$$

$$i\lambda\eta^2 + \frac{1}{\tau_2}\eta^2_\rho = f_4, \tag{3.40}$$

with the boundary conditions

$$\begin{gathered} \varphi = \partial_\nu\varphi = 0 \quad \text{on } \Gamma_0, \\ \mathcal{B}_1\varphi = -\beta_1\partial_\nu\psi - \beta_2\eta^1(\cdot,1), \quad \mathcal{B}_2\varphi = \gamma_1\psi + \gamma_2\eta^2(\cdot,1) \quad \text{on } \Gamma_1, \\ \eta^1(\cdot,0) = \partial_\nu\psi, \quad \eta^2(\cdot,0) = \psi \quad \text{on } \Gamma_1. \end{gathered} \tag{3.41}$$

From (3.39), (3.40) and (3.41), we deduce that

$$\eta^1(\cdot,\rho) = \partial_\nu \psi e^{-i\lambda\tau_1\rho} + \tau_1 \int_0^\rho f_3(x,s) e^{i\lambda\tau_1(s-\rho)}\,ds \quad \text{on } \Gamma_1 \times (0,1), \tag{3.42}$$

$$\eta^2(\cdot,\rho) = \psi e^{-i\lambda\tau_2\rho} + \tau_2 \int_0^\rho f_4(x,s) e^{i\lambda\tau_2(s-\rho)}\,ds \quad \text{on } \Gamma_1 \times (0,1). \tag{3.43}$$

It follows from (3.37), (3.38), (3.41), (3.42) and (3.43) that

$$\begin{gathered} -\lambda^2\varphi + \Delta^2\varphi = i\lambda f_1 + f_2 \quad \text{in } \Omega, \\ \varphi = \partial_\nu\varphi = 0 \quad \text{on } \Gamma_0, \\ \mathcal{B}_1\varphi = -C_{i\lambda}(\partial_\nu\varphi + i\lambda^{-1}\partial_\nu f_1) - F_{i\lambda} \quad \text{on } \Gamma_1, \\ \mathcal{B}_2\varphi = D_{i\lambda}(\varphi + i\lambda^{-1} f_1) + G_{i\lambda} \quad \text{on } \Gamma_1, \end{gathered} \tag{3.44}$$

where

$$\begin{gathered} C_{i\lambda} = i\lambda(\beta_1 + \beta_2 e^{-i\lambda\tau_1}), \quad F_{i\lambda} = \beta_2\tau_1 \int_0^1 f_3(x,s) e^{i\lambda\tau_1(s-1)}\,ds, \\ D_{i\lambda} = i\lambda(\gamma_1 + \gamma_2 e^{-i\lambda\tau_2}), \quad G_{i\lambda} = \gamma_2\tau_2 \int_0^1 f_4(x,s) e^{i\lambda\tau_2(s-1)}\,ds. \end{gathered}$$

Let $u \in H^2_{\Gamma_0}(\Omega)$. Multiplying the first equation in (3.44) by $\overline{u}$, integrating over Ω, then using Green's formula, we obtain

$$b(\varphi,u) = l(u), \quad \forall u \in \mathbb{V} := H^2_{\Gamma_0}(\Omega), \tag{3.45}$$

where $b(\varphi,u) = b_1(\varphi,u) + b_2(\varphi,u)$, with

$$\begin{gathered} b_1(\varphi,u) = a(\varphi,u), \\ b_2(\varphi,u) = -\lambda^2 \int_\Omega \varphi\overline{u}\,dx + C_{i\lambda}\int_{\Gamma_1} \partial_\nu\varphi\partial_\nu\overline{u}\,d\Gamma + D_{i\lambda}\int_{\Gamma_1}\varphi\overline{u}\,d\Gamma \end{gathered} \tag{3.46}$$

and

$$\begin{aligned} l(u) = &\int_\Omega (i\lambda f_1 + f_2)\overline{u}\,dx - \int_{\Gamma_1}(i\lambda^{-1}C_{i\lambda}\partial_\nu f_1 + F_{i\lambda})\partial_\nu\overline{u}\,d\Gamma \\ &- \int_{\Gamma_1}(i\lambda^{-1}D_{i\lambda} + G_{i\lambda})\overline{u}\,d\Gamma. \end{aligned} \tag{3.47}$$

Let $\mathbb{V}'$ be the dual space of $\mathbb{V}$. We define the operators $\mathbb{B} : \mathbb{V} \to \mathbb{V}'$ as $\varphi \mapsto \mathbb{B}\varphi$ and the operators $\mathbb{B}_i : \mathbb{V} \to \mathbb{V}'$ as $\varphi \mapsto \mathbb{B}_i\varphi$ for $i = 1,2$, such that

$$\begin{gathered} (\mathbb{B}\varphi)(u) = b(\varphi,u), \quad \forall u \in \mathbb{V}, \\ (\mathbb{B}_i\varphi)(u) = b_i(\varphi,u), \quad \forall u \in \mathbb{V},\ i \in \{1,2\}. \end{gathered} \tag{3.48}$$

We need to prove that the operator $\mathbb{B}$ is an isomorphism. So, we divide the proof into two steps:

Step 1. In this step, we prove that the operator $\mathbb{B}_2$ is compact. For this purpose, let us define the Hilbert space

$$H^s_{\Gamma_0}(\Omega) := \{u \in H^s(\Omega) : u = \partial_\nu u = 0 \quad \text{on } \Gamma_0\} \quad \text{with } s \in (\frac{3}{2}, 2).$$

Now, from (3.46) and a trace theorem, we obtain

$$\begin{aligned} |b_2(\varphi,u)| &\lesssim \|\varphi\|_{L^2(\Omega)}\|u\|_{H^2(\Omega)} + \|\partial_\nu\varphi\|_{L^2(\Gamma_1)}\|\partial_\nu u\|_{L^2(\Gamma_1)} + \|\varphi\|_{L^2(\Gamma_1)}\|u\|_{L^2(\Gamma_1)} \\ &\lesssim \|\varphi\|_{H^s(\Omega)}\|u\|_{H^2(\Omega)}, \end{aligned}$$

for all $s \in \left(\frac{3}{2}, 2\right)$. As $\mathbb{V}$ is compactly embedded in $H^s_{\Gamma_0}(\Omega)$ for any $s \in \left(\frac{3}{2}, 2\right)$, $\mathbb{B}_2$ is indeed a compact operator.

This compactness property and the fact that $\mathbb{B}_1$ is an isomorphism imply that the operator $\mathbb{B} = \mathbb{B}_1 + \mathbb{B}_2$ is a Fredholm operator of index zero. Now, following Fredholm alternative, we simply need to prove that the operator $\mathbb{B}$ is injective to obtain that it is an isomorphism.

Step 2. In this step, we prove that the operator $\mathbb{B}$ is injective (i.e. $\ker(\mathbb{B}) = \{0\}$). To this end, let $\phi \in \ker(\mathbb{B})$ which gives

$$b(\phi, u) = 0, \quad \forall u \in \mathbb{V}.$$

Likewise, we have

$$a(\phi, u) - \lambda^2 \int_\Omega \phi \overline{u}\, dx + C_{i\lambda} \int_{\Gamma_1} \partial_\nu \phi \partial_\nu \overline{u}\, d\Gamma + D_{i\lambda} \int_{\Gamma_1} \phi \overline{u}\, d\Gamma = 0, \quad \forall u \in \mathbb{V}.$$

Thus, we find that

$$\begin{gathered} -\lambda^2 \phi + \Delta^2 \phi = 0 \quad \text{in } \mathcal{D}'(\Omega), \\ \phi = \partial_\nu \phi = 0 \quad \text{on } \Gamma_0 \\ \mathcal{B}_1 \phi = -C_{i\lambda} \partial_\nu \phi \quad \text{on } \Gamma_1, \\ \mathcal{B}_2 \phi = D_{i\lambda} \phi \quad \text{on } \Gamma_1. \end{gathered}$$

Therefore, the vector Φ defined by

$$\Phi = (\phi, i\lambda \phi, i\lambda e^{-i\lambda \tau_1 \rho} \partial_\nu \phi, i\lambda e^{-i\lambda \tau_2 \rho} \phi)^\top$$

belongs to $D(\mathbb{A})$ and satisfies

$$i\lambda \Phi - \mathbb{A}\Phi = 0,$$

and consequently $\Phi \in \ker(i\lambda I - \mathbb{A})$. Hence Lemma 3.5 yields $\Phi = 0$ and consequently $\phi = 0$ and $\ker(\mathbb{B}) = \{0\}$.

Steps 1 and 2 guarantee that the operator $\mathbb{B}$ is isomorphism. Furthermore it is easy to see that the operator l is an antilinear and continuous form on $\mathbb{V}$. As a consequence, (3.45) admits a unique solution $\phi \in \mathbb{V}$. In (3.45), by taking test functions $u \in \mathcal{D}(\Omega)$, we see that the first identity of (3.44) holds in the distributional sense, hence $\Delta^2 \varphi \in L^2(\Omega)$. Coming back to (3.45), and again applying Green's formula (1.14), we find that

$$\begin{gathered} \mathcal{B}_1 \varphi = -C_{i\lambda}(\partial_\nu \varphi + i\lambda^{-1} \partial_\nu f_1) - F_{i\lambda} \quad \text{on } \Gamma_1, \\ \mathcal{B}_2 \varphi = D_{i\lambda}(\varphi + i\lambda^{-1} f_1) + G_{i\lambda} \quad \text{on } \Gamma_1. \end{gathered}$$

Further, since φ, $\partial_\nu \varphi$, f_1, $\partial_\nu f_1$, $\mathsf{F}_{i\lambda}$ and $\mathsf{G}_{i\lambda}$ belong to $L^2(\Gamma_1)$, we deduce that $\varphi \in \mathsf{D}_{\Gamma_0}(\Delta^2)$. As a result, if $\varphi \in \mathbb{V}$ is the unique solution of (3.45) and if we define η^1 (resp. η^2) by (3.42) (resp. (3.43)), we deduce that

$$\Phi = (\varphi, i\lambda \varphi - f_1, \eta^1, \eta^2)^\top$$

belongs to $D(\mathbb{A})$ and is the unique solution of (3.36). $\square$

Proof of Theorem 3.4. From Lemma 3.5, the operator $\mathbb{A}$ has no pure imaginary eigenvalues (i.e. $\sigma_p(\mathbb{A}) \cap i\mathbb{R} = \emptyset$). Moreover, from Lemma 3.5 and Lemma 3.6, $i\lambda I - \mathbb{A}$ is bijective for all $\lambda \in \mathbb{R}$ and since $\mathbb{A}$ is closed, we conclude, with the help of the closed graph theorem, that $i\lambda I - \mathbb{A}$ is an isomorphism for all $\lambda \in \mathbb{R}$, hence that $\sigma(\mathbb{A}) \cap i\mathbb{R} = \emptyset$. $\square$

3.3. **Exponential stability.** In this subsection, we will prove the strong stability of system (1.4)-(1.10). We start this subsection with the definition of our multiplier geometric control condition.

Definition 3.7. We say that the partition (Γ_0, Γ_1) of the boundary Γ satisfies the multiplier geometric control condition (MGC) if there exists a point $x_0 \in \mathbb{R}^2$ and a positive constant δ such that

$$h \cdot \nu \geq \delta^{-1} \quad \text{on } \Gamma_1 \quad \text{and} \quad h \cdot \nu \leq 0 \quad \text{on } \Gamma_0, \tag{3.49}$$

where $h(x) = x - x_0$.

Theorem 3.8. Under hypotheses (3.1) and (3.49), the C_0-semigroup $e^{t\mathbb{A}}$ is exponentially stable; i.e. there exists constants $M \geq 1$ and $\epsilon > 0$ independent of $\Phi_0 \in \mathbb{H}$ such that

$$\|e^{t\mathbb{A}}\Phi_0\|_{\mathbb{H}} \leq M e^{-\epsilon t}\|\Phi_0\|_{\mathbb{H}}, \quad \forall t \geq 0.$$

Since $i\mathbb{R} \subset \rho(\mathbb{A})$ (see the previous subsection), according to [9] and [17], to prove Theorem 3.8, it remains to prove that

$$\limsup_{\lambda\in\mathbb{R}, |\lambda|\to\infty} \| (i\lambda I - \mathbb{A})^{-1} \|_{\mathcal{L}(\mathbb{H})} < \infty. \tag{3.50}$$

We will prove condition (3.50) by a contradiction argument. For this purpose, suppose that (3.50) is false, then there exists $\{(\lambda_n, \Phi_n := (\varphi_n, \psi_n, \eta_n^1, \eta_n^2)^\top)\}_{n\geq 1} \subset \mathbb{R}^* \times D(\mathbb{A})$ with

$$|\lambda_n| \to \infty \text{ as } n \to \infty \quad \text{and} \quad \|\Phi_n\|_{\mathcal{H}} = 1, \ \forall n \geq 1, \tag{3.51}$$

such that

$$(i\lambda_n I - \mathbb{A})\Phi_n = F_n := (f_{1,n}, f_{2,n}, f_{3,n}, f_{4,n})^\top \to 0 \quad \text{in } \mathbb{H}, \quad \text{as } n \to \infty. \tag{3.52}$$

For simplicity, we drop the index n. Equivalently, from (3.52), we have

$$i\lambda\varphi - \psi = f_1 \to 0 \quad \text{in } H^2_{\Gamma_0}(\Omega), \tag{3.53}$$

$$i\lambda\psi + \Delta^2\varphi = f_2 \to 0 \quad \text{in } L^2(\Omega), \tag{3.54}$$

$$i\lambda\eta^1 + \frac{1}{\tau_1}\eta^1_\rho = f_3 \to 0 \quad \text{in } L^2(\Gamma_1 \times (0,1)), \tag{3.55}$$

$$i\lambda z\eta^2 + \frac{1}{\tau_2}\eta^2_\rho = f_4 \to 0 \quad \text{in } L^2(\Gamma_1 \times (0,1)). \tag{3.56}$$

Taking the inner product of (3.52) with Φ in $\mathbb{H}$ and using (3.8), we obtain

$$\begin{aligned}(\beta_1 - |\beta_2|)\int_{\Gamma_1} |\partial_\nu\psi|^2\, d\Gamma + (\gamma_1 - |\gamma_2|)\int_{\Gamma_1} |\psi|^2\, d\Gamma &\leq -\Re(\mathbb{A}\Phi, \Phi)_{\mathbb{H}} = \Re(F, \Phi)_{\mathbb{H}} \\ &\leq \|F\|_{\mathbb{H}}\|\Phi\|_{\mathbb{H}},\end{aligned}$$

From the above estimation, (3.1) and the fact that $\|F\|_{\mathbb{H}} = o(1)$ and $\|\Phi\|_{\mathbb{H}} = 1$, we obtain

$$\int_{\Gamma_1} |\partial_\nu\psi|^2\, d\Gamma = o(1) \quad \text{and} \quad \int_{\Gamma_1} |\psi|^2\, d\Gamma = o(1). \tag{3.57}$$

Lemma 3.9. *Under hypothesis* (3.1), *the solution* $\Phi = (\varphi, \psi, \eta^1, \eta^2)^\top \in D(\mathbb{A})$ *of* (3.53)-(3.56) *satisfies the following estimates*

$$\int_{\Gamma_1}\int_0^1 |\eta^1|^2\, d\rho\, d\Gamma = o(1), \quad \int_{\Gamma_1} |\eta^1(\cdot, 1)|^2\, d\Gamma = o(1), \tag{3.58}$$

$$\int_{\Gamma_1}\int_0^1 |\eta^2|^2\,d\rho\,d\Gamma = o(1), \quad \int_{\Gamma_1} |\eta^2(\cdot,1)|^2\,d\Gamma = o(1), \tag{3.59}$$

$$\int_{\Gamma_1} |\mathcal{B}_1\varphi|^2\,d\Gamma = o(1), \quad \int_{\Gamma_1} |\mathcal{B}_2\varphi|^2\,d\Gamma = o(1). \tag{3.60}$$

Proof. By (3.42), the Cauchy-Schwarz inequality, and that $\rho \in (0,1)$, we obtain

$$\begin{aligned}
\int_{\Gamma_1}\int_0^1 |\eta^1|^2\,d\rho\,d\Gamma &\le 2\int_{\Gamma_1} |\partial_\nu\psi|^2\,d\Gamma + 2\tau_1^2\int_{\Gamma_1}\int_0^1 \Big(\int_0^\rho |f_3(\cdot,s)|ds\Big)^2 d\rho\,d\Gamma \\
&\le 2\int_{\Gamma_1} |\partial_\nu\psi|^2\,d\Gamma + 2\tau_1^2\int_{\Gamma_1}\int_0^1 \rho\int_0^\rho |f_3(\cdot,s)|^2 ds\,d\rho\,d\Gamma \\
&\le 2\int_{\Gamma_1} |\partial_\nu\psi|^2\,d\Gamma + 2\tau_1^2\Big(\int_0^1 \rho\,d\rho\Big)\int_{\Gamma_1}\int_0^1 |f_3(\cdot,s)|^2 ds\,d\Gamma \\
&= 2\int_{\Gamma_1} |\partial_\nu\psi|^2\,d\Gamma + \tau_1^2\int_{\Gamma_1}\int_0^1 |f_3(\cdot,s)|^2 ds\,d\Gamma.
\end{aligned}$$

The above inequality, (3.57) and since $f_3 \to 0$ in $L^2(\Gamma_1 \times (0,1))$ lead to the first estimation in (3.58). Now, from (3.42), we deduce that

$$\eta^1(\cdot,1) = \partial_\nu\psi e^{-i\lambda\tau_1} + \tau_1\int_0^1 f_3(\cdot,s)e^{i\lambda\tau_1(s-1)}ds \quad \text{on } \Gamma_1,$$

consequently, by using Cauchy-Schwarz inequality, we infer that

$$\begin{aligned}
\int_{\Gamma_1} |\eta^1(\cdot,1)|^2\,d\Gamma &\le 2\int_{\Gamma_1} |\partial_\nu\psi|^2\,d\Gamma + 2\tau_1^2\int_{\Gamma_1}\Big(\int_0^1 |f_3(\cdot,s)|ds\Big)^2 d\Gamma \\
&\le 2\int_{\Gamma_1} |\partial_\nu\psi|^2\,d\Gamma + 2\tau_1^2\int_{\Gamma_1}\int_0^1 |f_3(\cdot,s)|^2 ds\,d\Gamma.
\end{aligned}$$

Therefore, from the above inequality, (3.57) and as $f_3 \to 0$ in $L^2(\Gamma_1 \times (0,1))$, we obtain the second estimation in (3.58). The same argument as before yields (3.59). Since $\Phi \in D(\mathbb{A})$, we have

$$\begin{gathered}
\mathcal{B}_1\varphi = -\beta_1\partial_\nu\psi - \beta_2\eta^1(\cdot,1) \quad \text{on } \Gamma_1, \\
\mathcal{B}_2\varphi = \gamma_1\psi + \gamma_2\eta^2(\cdot,1) \quad \text{on } \Gamma_1.
\end{gathered}$$

Finally, from the above equations, (3.57), (3.58), (3.59), we deduce (3.60). □

Lemma 3.10. *Under hypothesis* (3.1)*, the solution* $\Phi = (\varphi,\psi,\eta^1,\eta^2)^\top \in D(\mathbb{A})$ *of* (3.53)-(3.56) *satisfies the estimates*

$$\int_{\Gamma_1} |\partial_\nu\varphi|^2\,d\Gamma = o(\lambda^{-2}) \quad \textit{and} \quad \int_{\Gamma_1} |\varphi|^2\,d\Gamma = o(\lambda^{-2}). \tag{3.61}$$

Proof. Equation (3.53) yields

$$\begin{gathered}
i\lambda\partial_\nu\varphi = \partial_\nu\psi + \partial_\nu f_1 \quad \text{on } \Gamma_1, \\
i\lambda\varphi = \psi + f_1 \quad \text{on } \Gamma_1.
\end{gathered}$$

From the above equations, we deduce that

$$\int_{\Gamma_1} |\lambda\partial_\nu\varphi|^2\,d\Gamma \lesssim \int_{\Gamma_1} |\partial_\nu\psi|^2\,d\Gamma + \int_{\Gamma_1} |\partial_\nu f_1|^2\,d\Gamma, \tag{3.62}$$

$$\int_{\Gamma_1} |\lambda\varphi|^2 \, d\Gamma \lesssim \int_{\Gamma_1} |\psi|^2 \, d\Gamma + \int_{\Gamma_1} |f_1|^2 \, d\Gamma. \tag{3.63}$$

Using the trace theorem and that $a(f_1, f_1) = o(1)$, we obtain

$$\begin{aligned} \int_{\Gamma_1} |\partial_\nu f_1|^2 \, d\Gamma &\lesssim \|f_1\|^2_{H^2(\Omega)} \lesssim a(f_1, f_1) = o(1), \\ \int_{\Gamma_1} |f_1|^2 \, d\Gamma &\lesssim \|f_1\|^2_{H^2(\Omega)} \lesssim a(f_1, f_1) = o(1). \end{aligned}$$

Inserting the above estimations into (3.62) and (3.63), then using (3.57), we obtain the desired result. □

Lemma 3.11. *Under hypotheses* (3.1) *and* (3.49)*, the solution* $\Phi = (\varphi, \psi, \eta^1, \eta^2)^\top$ *in* $D(\mathbb{A})$ *of* (3.53)-(3.56) *satisfies the following estimates*

$$\int_\Omega |\lambda\varphi|^2 dx = o(1) \quad \textit{and} \quad a(\varphi, \varphi) = o(1). \tag{3.64}$$

Proof. Inserting (3.53) in (3.54), we obtain

$$-\lambda^2\varphi + \Delta^2\varphi = i\lambda f_1 + f_2 \quad \text{in } \Omega.$$

Multiplying the above equation by $(h \cdot \nabla\overline{\varphi})$, integrating over Ω, then taking the real part, we obtain

$$\Re\Big\{ \int_\Omega (-\lambda^2\varphi + \Delta^2\varphi)(h \cdot \nabla\overline{\varphi}) \, dx \Big\} = \Re\Big\{ \int_\Omega (i\lambda f_1 + f_2)(h \cdot \nabla\overline{\varphi}) \, dx \Big\} \tag{3.65}$$

Now, by using Green's formula and that $\varphi = 0$ on Γ_0, then using (3.61), we have

$$\begin{aligned} \Re\Big\{ -\lambda^2 \int_\Omega \varphi(h \cdot \nabla\overline{\varphi}) \, dx \Big\} &= \frac{1}{2} \int_\Omega |\lambda\varphi|^2 dx - \frac{1}{2} \int_{\Gamma_1} (h \cdot \nu) |\lambda\varphi|^2 \, d\Gamma \\ &= \frac{1}{2} \int_\Omega |\lambda\varphi|^2 dx + o(1). \end{aligned} \tag{3.66}$$

Now that $a(\varphi, \varphi) = O(1)$ and $a(f_1, f_1) = o(1)$, we obtain

$$\|\nabla\varphi\|_{L^2(\Omega)} \le \|\varphi\|_{H^2(\Omega)} \lesssim \sqrt{a(\varphi, \varphi)} = O(1), \tag{3.67}$$

$$\|f_1\|_{L^2(\Omega)}, \|\nabla f_1\|_{L^2(\Omega)} \le \|f_1\|_{H^2(\Omega)} \lesssim \sqrt{a(f_1, f_1)} = o(1). \tag{3.68}$$

Consequently, by using Green's formula, (3.61), (3.3), and $f_2 \to 0$ in $L^2(\Omega)$, we deduce that

$$\begin{aligned} &\Re\Big\{ \int_\Omega (i\lambda f_1 + f_2)(h \cdot \nabla\overline{\varphi}) \, dx \Big\} \\ &= \Re\Big\{ -i\lambda \int_\Omega (h \cdot \nabla f_1)\overline{\varphi} \, dx - 2i\lambda \int_\Omega f_1\overline{\varphi} \, dx - i\lambda \int_{\Gamma_1} f_1\overline{\varphi}(h \cdot \nu) \, d\Gamma \\ &\quad + \int_\Omega f_2(h \cdot \nabla\overline{\varphi}) \, dx \Big\} = o(1). \end{aligned}$$

Inserting (3.66) into (3.65) and using the above estimation, we acquire

$$\frac{1}{2} \int_\Omega |\lambda\varphi|^2 dx = -\Re\Big\{ \int_\Omega \Delta^2\varphi(h \cdot \nabla\overline{\varphi}) \, dx \Big\} + o(1). \tag{3.69}$$

According to [1, Lemma 5.4], for all $\varphi \in \mathsf{D}_{\Gamma_0}(\Delta^2)$, we have

$$\begin{aligned} -\Re\Big\{ \int_\Omega \Delta^2\varphi(h\cdot\nabla\overline{\varphi})\,dx \Big\} &\leq -\frac{1}{2}a(\varphi,\varphi) + \frac{\varepsilon_1 R^2}{2}\int_{\Gamma_1}|\mathcal{B}_2\varphi|^2\,d\Gamma \\ &\quad + \Big(\int_{\Gamma_1}|\mathcal{B}_1\varphi|^2\,d\Gamma\Big)^{1/2}\Big(\int_{\Gamma_1}|\partial_\nu\varphi|^2\,d\Gamma\Big)^{1/2} \\ &\quad + \frac{R^2\varepsilon_2}{2}\int_{\Gamma_1}|\mathcal{B}_1\varphi|^2\,d\Gamma, \end{aligned} \tag{3.70}$$

where $R = \|h\|_{L^\infty(\Omega)}$ and ε_1, ε_2 are positive constants. Consequently, using (3.61) and (3.60), we obtain

$$-\Re\Big\{ \int_\Omega \Delta^2\varphi(h\cdot\nabla\overline{\varphi})\,dx \Big\} \leq -\frac{1}{2}a(\varphi,\varphi) + o(1). \tag{3.71}$$

Finally, inserting (3.71) into (3.69), we obtain

$$\frac{1}{2}\int_\Omega |\lambda\varphi|^2 dx + \frac{1}{2}a(\varphi,\varphi) = o(1).$$ □

Proof of Theorem 3.8. From Lemmas 3.9 and 3.11, we deduce that $\|\Phi\|_{\mathbb{H}} = o(1)$, which contradicts (3.51). □

4. Conclusion

In this article, we considered the Kirchhoff plate equation with delay terms on the boundary control. We gave some instability examples in the cases $|\beta_2| \geq \beta_1$ and $|\gamma_2| \geq \gamma_1$. Thanks to the general criteria of Arendt and Batty, which helped to establish the strong stability of our system without using any geometric control condition. Finally, under the multiplier geometric control condition (MGC), we showed that system (1.1) is exponentially stable.

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