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GROWTH AND VALUE DISTRIBUTION OF LINEAR DIFFERENCE POLYNOMIALS GENERATED BY MEROMORPHIC SOLUTIONS OF HIGHER-ORDER LINEAR DIFFERENCE EQUATIONS

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Abstract. In this article, we investigate the relationship between growth and value distribution of meromorphic solutions for the higher-order complex linear difference equations

$$A_n(z)f(z+n)+\cdots+A_1(z)f(z+1)+A_0(z)f(z)=0 \quad \text{and} \quad =F(z),$$

and for the linear difference polynomial

$$g(z)=\alpha_n(z)f(z+n)+\cdots+\alpha_1(z)f(z+1)+\alpha_0(z)f(z)$$

generated by $f(z)$, where $A_j(z),\alpha_j(z)$ $(j=0,1,\dots,n)$, $F(z)(\not\equiv 0)$ are meromorphic functions. We improve some previous results due to Belaïdi, Chen and Zheng and others.

1. Introduction and main results

Throughout this article, we assume that the readers are familiar with the standard notations of Nevanlinna value distribution theory (see [9, 10, 17, 18]). Especially, for a meromorphic function $f(z)$ in $\mathbb{C}$, we use the notation $\rho(f)$ and $\tau(f)$ to denote the order and the type of $f(z)$ respectively, and use the notations $\lambda(f)$ and $\lambda(\frac{1}{f})$ to denote the exponent of convergence of the zeros sequence and the poles sequence of $f(z)$ respectively.

Recently, there has been an increasing interest in the study on the properties of meromorphic solutions of complex difference equations from the viewpoint of difference analogues of Nevanlinna theory (see [4, 6, 8]) and among those many good results are obtained for the case of complex linear difference equations (see [4, 5, 6, 11, 12, 14, 16, 19, 20]). For the case of complex linear differential-difference equations see [1, 2, 3, 13, 15, 21]. In particular, inspired by the results about the growth and the value distribution of differential polynomials generated by meromorphic solutions of complex linear differential equations, Latreuch and Belaïdi in [11] investigated the growth of linear difference polynomials generated by meromorphic solutions of the second order complex linear difference equation

$$f(z+2)+a(z)f(z+1)+b(z)f(z)=0, \tag{1.1}$$

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where $a(z)$ and $b(z)$ are meromorphic functions.

By denoting the shifts of a meromorphic function $f(z)$ as

$$\begin{aligned} f_0(z) &= f(z) \\ f_j(z) &= f(z+j), \quad j \in \mathbb{N}_+ , \end{aligned} \tag{1.2}$$

they stated their result as follows.

Theorem 1.1 ([11]). *Let $a(z)$ and $b(z)$ be meromorphic functions satisfying $\lambda(\frac{1}{b}) < \rho(b) < +\infty$, $\rho(a) < \rho(b)$ or $0 < \tau(a) < \tau(b) < +\infty$ if $\rho(a) = \rho(b) > 0$, and let $\alpha(z), \beta(z), \gamma(z)$ be meromorphic functions not all vanishing identically such that $\max\{\rho(\alpha), \rho(\beta), \rho(\gamma)\} < \rho(b) + 1$ and $h(z) \not\equiv 0$, where $h(z)$ is defined as follows: denote*

$$\begin{gathered} k(z) = \beta(z) - \alpha(z)a(z), \quad l(z) = \gamma(z) - \alpha(z)b(z), \\ m(z) = \gamma_1(z) - \alpha_1(z)b_1(z) - \beta_1(z)a(z) + \alpha_1(z)a(z)a_1(z), \\ n(z) = \alpha_1(z)a_1(z)b(z) - \beta_1(z)b(z), \\ h(z) = k(z)n(z) - m(z)l(z). \end{gathered}$$

If $f(z)(\not\equiv 0)$ is a meromorphic solution of (1.1), then the linear difference polynomial

$$L(f(z)) = \alpha(z)f_2(z) + \beta(z)f_1(z) + \gamma(z)f(z) \tag{1.3}$$

satisfies $\rho(L(f)) = \rho(f) \geq \rho(b) + 1$.

Note that there is a dominant coefficient $b(z)$ in Theorem 1.1. Chen and Zheng in [13] investigated a special case when $\rho(a) = \rho(b) = 1$. They considered the homogeneous complex linear difference equation

$$f_2(z) + c(z)e^{az}f_1(z) + d(z)e^{bz}f(z) = 0 \tag{1.4}$$

and the non-homogeneous complex linear difference equation

$$f_2(z) + c(z)e^{az}f_1(z) + d(z)e^{bz}f(z) = F(z), \tag{1.5}$$

and proved the following two results.

Theorem 1.2 ([3]). *Let $a(\neq 0), b(\neq a)$ be complex constants, $c(z), d(z)$ be meromorphic functions satisfying $\max\{\rho(c), \rho(d)\} < 1$, $\alpha(z), \beta(z), \gamma(z)$ be meromorphic functions not all vanishing identically such that $\max\{\rho(\alpha), \rho(\beta), \rho(\gamma)\} < 2$ and $h(z) \not\equiv 0$, where $h(z)$ is defined as follows: denote*

$$\begin{gathered} k(z) = \beta(z) - \alpha(z)c(z)e^{az}, \quad l(z) = \gamma(z) - \alpha(z)d(z)e^{bz}, \\ m(z) = \gamma_1(z) - \alpha_1(z)d_1(z)e^{bz+b} - \beta_1(z)c(z)e^{az} + \alpha_1(z)c(z)c_1(z)e^{2az+a}, \\ n(z) = \alpha_1(z)c_1(z)d(z)e^{az+bz+a} - \beta_1(z)d(z)e^{bz}, \\ h(z) = k(z)n(z) - m(z)l(z). \end{gathered}$$

Let $\varphi(z)(\not\equiv 0)$ be a meromorphic function satisfying $\rho(\varphi) < 2$ and $\psi(z) \not\equiv 0$, where

$$\psi(z) = \frac{k(z)\varphi_1(z) - m(z)\varphi(z)}{h(z)}.$$

If $f(z)(\not\equiv 0)$ is a meromorphic solution of (1.4), then the linear difference polynomial (1.3) satisfies

$$\rho(L(f) - \varphi) = \rho(L(f)) = \rho(f) \geq 2.$$

Furthermore, if $f(z)$ satisfies $\rho_2(f) < 1$, then

$$\lambda(L(f)-\varphi) = \rho(L(f)-\varphi) = \rho(L(f)) = \rho(f) \geq 2.$$

Theorem 1.3 ([3]). *Let $a, b, c(z), d(z), \alpha(z), \beta(z), \gamma(z)$ be as in Theorem 1.2. Let $\varphi(z)(\not\equiv 0), F(z)(\not\equiv 0)$ be meromorphic functions satisfying* $\max\{\rho(\varphi), \rho(F)\} < 2$ *and $\phi(z) \not\equiv 0$, where*

$$\phi(z) = \frac{k(z)[\varphi_1(z) - \alpha_1(z)F(z) - k_1(z)F(z)] - m(z)[\varphi(z) - \alpha(z)F(z)]}{h(z)}.$$

If $f(z)$ is a meromorphic solution of (1.5)*, then the linear difference polynomial* (1.3) *satisfies*

$$\rho(L(f)-\varphi) = \rho(L(f)) = \rho(f) \geq 2$$

with at most one exceptional meromorphic solution. Furthermore, if $f(z)$ satisfies $\rho_2(f) < 1$, then

$$\lambda(L(f)-\varphi) = \rho(L(f)-\varphi) = \rho(L(f)) = \rho(f) \geq 2.$$

Now, a natural question arises: would there be similar conclusions for the higher-order complex linear difference equations? So, we investigate the higher-order linear difference equation

$$A_n(z)f_n(z) + \cdots + A_1(z)f_1(z) + A_0(z)f(z) = 0 \tag{1.6}$$

and correspondingly the linear difference polynomial

$$g(z) = \alpha_n(z)f_n(z) + \cdots + \alpha_1(z)f_1(z) + \alpha_0(z)f(z) \tag{1.7}$$

generated by $f(z)$, where $A_j(z), \alpha_j(z) (j = 0, 1, \dots, n)$ are meromorphic functions.

In particular, we investigate a special case of (1.6), that is, the homogeneous linear difference equation

$$a_n(z)e^{b_n z}f_n(z) + \cdots + a_1(z)e^{b_1 z}f_1(z) + a_0(z)e^{b_0 z}f(z) = 0 \tag{1.8}$$

and correspondingly the non-homogeneous linear difference equation

$$a_n(z)e^{b_n z}f_n(z) + \cdots + a_1(z)e^{b_1 z}f_1(z) + a_0(z)e^{b_0 z}f(z) = F(z), \tag{1.9}$$

where $a_j(z)$ $(j = 0, 1, \dots, n)$, $F(z)(\not\equiv 0)$ are meromorphic functions and b_j $(j = 0, 1, \dots, n)$ are distinct complex constants.

Before stating our results, we denote

$$\xi_{0,i} = -\frac{A_i}{A_n}\alpha_n + \alpha_i, \quad 0 \leq i \leq n-1,$$

$$\xi_{j,i} = -\frac{A_i}{A_n}\xi^1_{j-1,n-1} + \xi^1_{j-1,i-1}, \quad 1 \leq j \leq n-1,\ 0 \leq i \leq n-1,$$

where

$$\xi^1_{j-1,-1}(z) \equiv 0, \quad \alpha^k_i(z) = \alpha_i(z+k),$$

$$\xi^k_{j-1,i-1}(z) = \xi_{j-1,i-1}(z+k), \quad A^k_i(z) = A_i(z+k), \quad k \in \mathbb{N}_+.$$

Then we define

$$\beta = \begin{vmatrix} \xi_{0,0} & \xi_{0,1} & \dots & \xi_{0,n-1} \\ \xi_{1,0} & \xi_{1,1} & \dots & \xi_{1,n-1} \\ & \dots & & \\ \xi_{n-1,0} & \xi_{n-1,1} & \dots & \xi_{n-1,n-1} \end{vmatrix}.$$

Also we denote

$$\eta_{0,i} = -\frac{a_i}{a_n} e^{(b_i - b_n)z} \alpha_n + \alpha_i, \quad 0 \le i \le n-1,$$

$$\eta_{j,i} = -\frac{a_i}{a_n} e^{(b_i - b_n)z} \eta^1_{j-1,n-1} + \eta^1_{j-1,i-1}, \quad 1 \le j \le n-1,\ 0 \le i \le n-1,$$

where

$$\eta^1_{j-1,-1}(z) \equiv 0, \quad a_i^k(z) = a_i(z+k),$$
$$\eta^k_{j-1,i-1}(z) = \eta_{j-1,i-1}(z+k), \quad k \in \mathbb{N}_+,$$

Then we define

$$\gamma = \begin{vmatrix} \eta_{0,0} & \eta_{0,1} & \dots & \eta_{0,n-1} \\ \eta_{1,0} & \eta_{1,1} & \dots & \eta_{1,n-1} \\ & \dots & & \\ \eta_{n-1,0} & \eta_{n-1,1} & \dots & \eta_{n-1,n-1} \end{vmatrix}.$$

Firstly, we consider the growth and the value distribution of the linear difference polynomial (1.7) generated by the meromorphic solution of (1.6), and obtain the following result.

Theorem 1.4. *Let $A_j(z)$ $(j = 0, 1, \dots, n-1)$, $A_n(z)$ $(\not\equiv 0)$ be meromorphic functions satisfying that there exists an integer $l \in \{0, 1, \dots, n\}$ such that $\lambda(\frac{1}{A_l}) < \rho(A_l) < +\infty$ and $\max\{\rho(A_j), 0 \le j \le n, j \ne l\} \le \rho(A_l)$,*

$$\sum_{\rho(A_j)=\rho(A_l),\, j\ne l} \tau(A_j) < \tau(A_l) < +\infty.$$

Let $\alpha_j(z)(j = 0, 1, \dots, n)$ be meromorphic functions not all vanishing identically such that $\beta \not\equiv 0$ and $\max\{\rho(\alpha_j), 0 \le j \le n\} < \rho(A_l) + 1$. If $f(z)(\not\equiv 0)$ is a meromorphic solution of (1.6), then the linear difference polynomial (1.7) satisfies

$$\rho(g) = \rho(f) \ge \rho(A_l) + 1.$$

Furthermore, if $f(z)$ satisfies $\rho_2(f) < 1$ and $\varphi(z)(\not\equiv 0)$ is a meromorphic function satisfying $\rho(\varphi) < \rho(A_l) + 1$ and $\omega(z) \not\equiv 0$, where

$$\omega(z) = \frac{1}{\beta} \sum_{i_0 i_1 \dots i_{n-1}} (-1)^{\tau(i_0 i_1 \dots i_{n-1})} \varphi_{i_0} \xi_{i_1,1} \xi_{i_2,2} \dots \xi_{i_{n-1},n-1},$$

then

$$\lambda(g - \varphi) = \rho(g - \varphi) = \rho(g) = \rho(f) \ge \rho(A_l) + 1.$$

As a concrete application, we consider the relationship between the growth of $f(z)$ and its forward differences $\triangle^j f$, $j \in \mathbb{N}_+$, where

$$\triangle f(z) = f(z+1) - f(z)$$
$$\triangle^j f(z) = \triangle(\triangle^{j-1} f(z)), \quad j \in \mathbb{N}_+$$

Remark 1.5. It is shown in [6, P. 106] and [7, P. 66] that for an arbitrary complex number $c(\ne 0)$, we have

$$(1 + o(1)) T(r - |c|, f(z)) \le T(r, f(z+c)) \le (1 + o(1)) T(r + |c|, f(z))$$

as $r \to +\infty$ for a general meromorphic function $f(z)$. Therefore, it is easy to obtain that

$$\rho(f(z+c)) = \rho(f), \quad \mu(f(z+c)) = \mu(f).$$

From the above remark we see that $\rho(f_j) = \rho(f)$ and $\rho(\triangle^j f) \le \rho(f)$, $j \in \mathbb{N}_+$. But the equality $\rho(\triangle^j f) = \rho(f), j \in \mathbb{N}_+$ may not hold. Next, we give some assumptions to guarantee that the equality holds.

Corollary 1.6. *Let $A_j(z)$ $(j = 0, 1, \dots, n-1)$, $A_n(z)$ $(\not\equiv 0)$ be meromorphic functions satisfying that there exists an integer $l \in \{0, 1, \dots, n\}$ such that $\lambda(\frac{1}{A_l}) < \rho(A_l) < +\infty$, $\max\{\rho(A_j), 0 \le j \le n, j \ne l\} \le \rho(A_l)$, and*

$$\sum_{\rho(A_j)=\rho(A_l),\, j\ne l} \tau(A_j) < \tau(A_l) < +\infty.$$

If $f(z)(\not\equiv 0)$ is a meromorphic solution of (1.6), then $\rho(\Delta f) = \rho(f)$. Furthermore, if

$$\beta = \sum_{j=0}^{n} A_j \cdot \sum_{j=0}^{n} (j+1) A_j^1 - \sum_{j=1}^{n} j A_j \cdot \sum_{j=0}^{n} A_j^1 \not\equiv 0,$$

then $\rho(\Delta^2 f) = \rho(f)$.

Note that there is a dominant coefficient $A_l(z)$ in Theorem 1.4 and Corollary 1.6. Next, we consider a special case of (1.6), that is (1.8), where the coefficients have the same order and admit a weaker condition on their types.

Theorem 1.7. *Let $b_j (j = 0, 1, \dots, n)$ be distinct complex constants, $a_j(z)(\not\equiv 0)$ $(j = 0, 1, \dots, n)$ be meromorphic functions satisfying $\max\{\rho(a_j), 0 \le j \le n\} < 1$, $\alpha_j(z)(j = 0, 1, \dots, n)$ be meromorphic functions not all vanishing identically such that $\gamma \not\equiv 0$ and $\max\{\rho(\alpha_j), 0 \le j \le n\} < 2$. If $f(z)(\not\equiv 0)$ is a meromorphic solution of (1.8), then the linear difference polynomial (1.7) satisfies*

$$\rho(g) = \rho(f) \ge 2.$$

Furthermore, if $f(z)$ satisfies $\rho_2(f) < 1$ and $\varphi(z)(\not\equiv 0)$ is a meromorphic function satisfying $\rho(\varphi) < 2$ and $\delta(z) \not\equiv 0$, where

$$\delta(z) = \frac{1}{\gamma} \sum_{i_0 i_1 \dots i_{n-1}} (-1)^{\tau(i_0 i_1 \dots i_{n-1})} \varphi_{i_0} \eta_{i_1,1} \eta_{i_2,2} \dots \eta_{i_{n-1},n-1},$$

then $\lambda(g - \varphi) = \rho(g - \varphi) = \rho(g) = \rho(f) \ge 2$.

Corollary 1.8. *Let $b_j (j = 0, 1, \dots, n)$ be distinct complex constants, $a_j(z)$ $(\not\equiv 0)$ $(j = 0, 1, \dots, n)$ be meromorphic functions satisfying $\max\{\rho(a_j), 0 \le j \le n\} < 1$. If $f(z)(\not\equiv 0)$ is a meromorphic solution of (1.8), then*

$$\rho(\triangle f) = \rho(f) \ge 2.$$

Furthermore, if $f(z)$ satisfies $\rho_2(f) < 1$ and $\varphi(z)(\not\equiv 0)$ be a meromorphic function satisfying $\rho(\varphi) < 2$ and

$$\delta(z) = \frac{(-1)^{n-1}}{a_n e^{b_n z} \gamma} \sum_{i=0}^{n-1} \varphi_i \sum_{j=i+1}^{n} a_j e^{b_j z} \not\equiv 0,$$

then

$$\lambda(\triangle f - \varphi) = \rho(\triangle f - \varphi) = \rho(\triangle f) = \rho(f) \ge 2.$$

Correspondingly, we consider the non-homogeneous linear difference equation (1.9) under the same condition as in Theorem 1.7 and obtain the following weaker result.

Theorem 1.9. *Let $b_j, a_j(z), \alpha_j(z) (j = 0, 1, \dots, n)$ be the same as in Theorem 1.7 and $F(z)(\not\equiv 0)$ be a meromorphic function with $\rho(F) < 1$. If $f(z)$ is a meromorphic solution of* (1.9)*, then the linear difference polynomial* (1.7) *satisfies $\rho(g) = \rho(f) \geq 2$ with at most one exceptional meromorphic solution.*

Corollary 1.10. *Let $b_j, a_j(z)(j = 0, 1, \dots, n)$ be the same as in Corollary 1.8 and $F(z)(\not\equiv 0)$ be a meromorphic function with $\rho(F) < 1$. If $f(z)$ is a meromorphic solution of* (1.9)*, then $\rho(\triangle f) = \rho(f) \geq 2$ with at most one exceptional meromorphic solution.*

Next we give some examples to illustrate our results.

Example 1.11. The function $f(z) = e^{z^2}$ with $\rho(f) = 2$ satisfies

$$e^{-4z-4} f(z + 2) - f(z) = 0.$$

Set $g(z) = -f_2(z) + f_1(z) + e^{2z} f(z)$ and $\varphi(z) = z$. Then the hypotheses of Theorem 1.4 hold. Therefore, $\lambda(g - z) = \rho(g - z) = \rho(g) = \rho(f) = 2$. And the hypotheses of Corollary 1.6 hold, so we have $\rho(\Delta^2 f) = \rho(\Delta f) = \rho(f) = 2$.

Example 1.12. The function $f(z) = e^{z^2+z+1}$ with $\rho(f) = 2$ satisfies

$$e^{-6z-12} f(z + 3) + e^{-2z-2} f(z + 1) - 2f(z) = 0.$$

Set $g(z) = -f_3(z) + e^{2z+6} f_2(z) - f_1(z) + f(z)$ and $\varphi(z) = e^z$. Then the hypotheses of Theorem 1.4 hold. Therefore, $\lambda(g - e^z) = \rho(g - e^z) = \rho(g) = \rho(f) = 2$. And the hypotheses of Corollary 1.6 hold, so we have $\rho(\Delta^2 f) = \rho(\Delta f) = \rho(f) = 2$.

Example 1.13. The function $f(z) = e^{z^2-z}$ with $\rho(f) = 2$ satisfies

$$\frac{z}{e^2} e^z f(z + 2) + e^{3z} f(z + 1) - (z + 1)e^{5z} f(z) = 0.$$

Let $g(z)$ and $\varphi(z)$ be the same as in Example 1.1. Then the hypotheses of Theorem 1.7 hold. Therefore, $\lambda(g - z) = \rho(g - z) = \rho(g) = \rho(f) = 2$. And the hypotheses of Corollary 1.8 hold, so we have $\rho(\Delta f) = \rho(f) = 2$.

Example 1.14. The function $f(z) = e^{z^2+2z}$ with $\rho(f) = 2$ satisfies

$$\frac{z}{e^{15}} e^z f(z + 3) + \frac{1}{e^8} e^{3z} f(z + 2) + \frac{1}{e^3} e^{5z} f(z + 1) - (z + 2)e^{7z} f(z) = 0.$$

Set $g(z) = f_3(z) - f_2(z) - f_1(z) + e^{2z+3} f(z)$ and $\varphi(z) = ze^z$. Then the hypotheses of Theorem 1.7 hold. Therefore, $\lambda(g - ze^z) = \rho(g - ze^z) = \rho(g) = \rho(f) = 2$. And the hypotheses of Corollary 1.8 hold, so we have $\rho(\Delta f) = \rho(f) = 2$.

2. Preparations for proofs of main results

Lemma 2.1 ([11]). *Let $A_j(z)$ $(j = 0, 1, \dots, n)$ be meromorphic functions satisfying that there exists an integer $l \in \{0, 1, \dots, n\}$ such that $\lambda(\frac{1}{A_l}) < \rho(A_l) < +\infty$ and* $\max\{\rho(A_j) : 0 \leq j \leq n, j \neq l\} \leq \rho(A_l)$,

$$\sum_{\rho(A_j)=\rho(A_l),\, j\neq l} \tau(A_j) < \tau(A_l) < +\infty.$$

If $f(z)(\not\equiv 0)$ is a meromorphic solution of (1.6)*, then $\rho(f) \geq \rho(A_l) + 1$.*

Lemma 2.2 ([19]). *Let $A_j(z) = a_j(z)e^{b_j z}$ $(j = 0, 1, \dots, n)$, where $b_j (j = 0, 1, \dots, n)$ are distinct complex constants, $a_j(z)$ $(\not\equiv 0)$ $(j = 0, 1, \dots, n)$ are meromorphic functions with $\max\{\rho(a_j), 0 \le j \le n\} < 1$, then every meromorphic solution $f(z)(\not\equiv 0)$ of (1.8) satisfies $\rho(f) \ge 2$.*

Lemma 2.3 ([19]). *Let $A_j(z)(j = 0, 1, \dots, n)$ satisfy the hypotheses of Lemma 2.2 and $F(z)(\not\equiv 0)$ be a meromorphic function with $\rho(F) < 1$, then at most one meromorphic solution $f_0(z)$ of (1.9) satisfies $1 \le \rho(f_0) \le 2$ and $\max\{\lambda(f_0), \lambda(\frac{1}{f_0})\} = \rho(f_0)$, the other solutions $f(z)$ satisfy $\rho(f) \ge 2$.*

Lemma 2.4 ([13]). *Let $c_j(j = 0, 1, \dots, n)$ be distinct complex constants, $A_j(z)$ $(j = 0, 1, \dots, n)$ and $F(z)$ $(\not\equiv 0)$ be meromorphic functions. If $f(z)$ is a meromorphic solution of*

$$A_n(z)f(z + c_n) + \dots + A_1(z)f(z + c_1) + A_0(z)f(z) = F(z)$$

and satisfies $\max\{\rho(F), \rho(A_j) : 0 \le j \le n\} < \rho(f)$ and $\rho_2(f) < 1$, then $\lambda(f) = \rho(f)$.

Lemma 2.5 ([18]). *Let $f(z)$ and $g(z)$ be non-constant meromorphic functions of orders $\rho(f)$ and $\rho(g)$ respectively. Then we have*

$$\rho(f + g) \le \max\{\rho(f), \rho(g)\},$$
$$\rho(fg) \le \max\{\rho(f), \rho(g)\}.$$

Furthermore, if $\rho(f) > \rho(g)$, then $\rho(f + g) = \rho(fg) = \rho(f)$.

3. Proofs of main results

Proof of Theorem 1.4. By Lemma 2.1, we have $\rho(f) \ge \rho(A_l) + 1$. By substituting

$$f_n = -\frac{1}{A_n}(A_{n-1}f_{n-1} + \dots + A_1 f_1 + A_0 f_0)$$

into (1.7), we obtain

$$\begin{aligned} g_0 &= -\frac{\alpha_n}{A_n}(A_{n-1}f_{n-1} + \dots + A_1 f_1 + A_0 f_0) + \alpha_{n-1}f_{n-1} + \dots + \alpha_0 f_0 \\ &= (-\frac{A_{n-1}}{A_n}\alpha_n + \alpha_{n-1})f_{n-1} + \dots + (-\frac{A_1}{A_n}\alpha_n + \alpha_1)f_1 + (-\frac{A_0}{A_n}\alpha_n + \alpha_0)f_0 \\ &= \xi_{0,n-1}f_{n-1} + \dots + \xi_{0,1}f_1 + \xi_{0,0}f_0. \end{aligned}$$

Then

$$\begin{aligned} g_1 &= \xi^1_{0,n-1}f_n + \dots + \xi^1_{0,1}f_2 + \xi^1_{0,0}f_1 \\ &= -\frac{\xi^1_{0,n-1}}{A_n}(A_{n-1}f_{n-1} + \dots + A_1 f_1 + A_0 f_0) + \xi^1_{0,n-2}f_{n-1} + \dots + \xi^1_{0,0}f_1 \\ &= (-\frac{A_{n-1}}{A_n}\xi^1_{0,n-1} + \xi^1_{0,n-2})f_{n-1} + \dots + (-\frac{A_1}{A_n}\xi^1_{0,n-1} + \xi^1_{0,0})f_1 + (-\frac{A_0}{A_n}\xi^1_{0,n-1})f_0 \\ &= \xi_{1,n-1}f_{n-1} + \dots + \xi_{1,1}f_1 + \xi_{1,0}f_0. \end{aligned}$$

By repeating the above argument $n - 2$ times, we obtain

$$\begin{gathered} \xi_{0,0}f_0 + \xi_{0,1}f_1 + \dots + \xi_{0,n-1}f_{n-1} = g_0 \\ \xi_{1,0}f_0 + \xi_{1,1}f_1 + \dots + \xi_{1,n-1}f_{n-1} = g_1 \\ \dots \\ \xi_{n-1,0}f_0 + \xi_{n-1,1}f_1 + \dots + \xi_{n-1,n-1}f_{n-1} = g_{n-1} \end{gathered} \tag{3.1}$$

Then by Cramer's Rule and the assumption that $\beta \not\equiv 0$, the system of equations (3.1) has a unique solution $f(z)$, and

$$f = \frac{1}{\beta}\begin{vmatrix} g_0 & \xi_{0,1} & \dots & \xi_{0,n-1} \\ g_1 & \xi_{1,1} & \dots & \xi_{1,n-1} \\ \vdots & \vdots & & \vdots \\ g_{n-1} & \xi_{n-1,1} & \dots & \xi_{n-1,n-1} \end{vmatrix}. \tag{3.2}$$

On the one hand, by Lemma 2.5, Remark 1.5 and the assumption $\max\{\rho(\alpha_j), 0 \le j \le n\} < \rho(A_l) + 1 \le \rho(f)$, it is clear from (1.7) that $\rho(g) \le \rho(f)$. On the other hand, by Lemma 2.5, Remark 1.5 and the assumptions $\max\{\rho(A_j), 0 \le j \le n, j \ne l\} \le \rho(A_l), \max\{\rho(\alpha_j), 0 \le j \le n\} < \rho(A_l) + 1$, it is clear from (3.2) that $\rho(f) \le \rho(g)$. Hence,

$$\rho(g) = \rho(f) \ge \rho(A_l) + 1.$$

Set $G(z) = g(z) - \varphi(z)$. By Lemma 2.5 and the assumption $\rho(\varphi) < \rho(A_l) + 1$, we have

$$\rho(G) = \rho(g - \varphi) = \rho(g) = \rho(f) \ge \rho(A_l) + 1.$$

By substituting $g(z) = G(z) + \varphi(z)$ into (3.2), we obtain

$$\begin{aligned} f &= \frac{1}{\beta}\begin{vmatrix} G_0 + \varphi_0 & \xi_{0,1} & \dots & \xi_{0,n-1} \\ G_1 + \varphi_1 & \xi_{1,1} & \dots & \xi_{1,n-1} \\ \vdots & \vdots & & \vdots \\ G_{n-1} + \varphi_{n-1} & \xi_{n-1,1} & \dots & \xi_{n-1,n-1} \end{vmatrix} \\ &= \frac{1}{\beta}\sum_{i_0 i_1 \dots i_{n-1}} (-1)^{\tau(i_0 i_1 \dots i_{n-1})} (G+\varphi)_{i_0} \xi_{i_1,1}\xi_{i_2,2}\dots\xi_{i_{n-1},n-1} \\ &= \frac{1}{\beta}\sum_{i_0 i_1 \dots i_{n-1}} (-1)^{\tau(i_0 i_1 \dots i_{n-1})} G_{i_0} \xi_{i_1,1}\xi_{i_2,2}\dots\xi_{i_{n-1},n-1} \\ &\quad + \frac{1}{\beta}\sum_{i_0 i_1 \dots i_{n-1}} (-1)^{\tau(i_0 i_1 \dots i_{n-1})} \varphi_{i_0} \xi_{i_1,1}\xi_{i_2,2}\dots\xi_{i_{n-1},n-1} \\ &= \frac{1}{\beta}\sum_{i_0 i_1 \dots i_{n-1}} (-1)^{\tau(i_0 i_1 \dots i_{n-1})} G_{i_0} \xi_{i_1,1}\xi_{i_2,2}\dots\xi_{i_{n-1},n-1} + \omega, \end{aligned} \tag{3.3}$$

where $\{i_0, i_1, \dots, i_{n-1}\} = \{0, 1, \dots, n-1\}$. Then by substituting (3.3) into (1.6), we obtain

$$\begin{aligned} &A_n(\frac{1}{\beta_n}\sum_{i_0 i_1 \dots i_{n-1}} (-1)^{\tau(i_0 i_1 \dots i_{n-1})} G_{i_0+n} \xi^n_{i_1,1}\xi^n_{i_2,2}\dots\xi^n_{i_{n-1},n-1} + \omega_n) \\ &+ A_{n-1}(\frac{1}{\beta_{n-1}}\sum_{i_0 i_1 \dots i_{n-1}} (-1)^{\tau(i_0 i_1 \dots i_{n-1})} G_{i_0+n-1} \xi^{n-1}_{i_1,1}\xi^{n-1}_{i_2,2}\dots\xi^{n-1}_{i_{n-1},n-1} + \omega_{n-1}) \\ &+ \dots + A_0(\frac{1}{\beta}\sum_{i_0 i_1 \dots i_{n-1}} (-1)^{\tau(i_0 i_1 \dots i_{n-1})} G_{i_0} \xi_{i_1,1}\xi_{i_2,2}\dots\xi_{i_{n-1},n-1} + \omega) = 0; \end{aligned}$$

that is,

$$\frac{A_n}{\beta_n}\sum_{n-1, i_1 \dots i_{n-1}} (-1)^{\tau(n-1, i_1 \dots i_{n-1})} \xi^n_{i_1,1}\xi^n_{i_2,2}\dots\xi^n_{i_{n-1},n-1} G_{2n-1}$$

$$
\begin{aligned}
&+ (\frac{A_n}{\beta_n} \sum_{n-2, i_1 \dots i_{n-1}} (-1)^{\tau(n-2, i_1 \dots i_{n-1})} \xi^n_{i_1,1} \xi^n_{i_2,2} \dots \xi^n_{i_{n-1},n-1} \\
&+ \frac{A_{n-1}}{\beta_{n-1}} \sum_{n-1, i_1 \dots i_{n-1}} (-1)^{\tau(n-1, i_1 \dots i_{n-1})} \xi^{n-1}_{i_1,1} \xi^{n-1}_{i_2,2} \dots \xi^{n-1}_{i_{n-1},n-1}) G_{2n-2} \\
&+ \dots + \frac{A_0}{\beta} \sum_{0, i_1 \dots i_{n-1}} (-1)^{\tau(0, i_1 \dots i_{n-1})} \xi_{i_1,1} \xi_{i_2,2} \dots \xi_{i_{n-1},n-1} G_0 \\
&= -(A_n \omega_n + A_{n-1} \omega_{n-1} + \dots + A_0 \omega).
\end{aligned}
$$

By Lemma 2.5, Remark 1.5 and the assumption $\max\{\rho(\varphi), \rho(A_j), \rho(\alpha_j), 0 \le j \le n\} < \rho(A_l)+1$, we have $\rho(\omega) < \rho(A_l)+1$. Then, by Lemma 2.1 and the assumption $\omega \not\equiv 0$, ω is not a solution of (1.6), that is,

$$
A_n \omega_n + A_{n-1} \omega_{n-1} + \dots + A_0 \omega \not\equiv 0.
$$

Hence, by Lemma 2.4, we have $\lambda(G) = \rho(G)$, that is,

$$
\lambda(g - \varphi) = \rho(g - \varphi) = \rho(g) = \rho(f) \ge \rho(A_l) + 1.
$$

□

Proof of Corollary 1.6. Set $g(z) = \triangle f(z) = f_1(z) - f(z)$. Then by (1.7), we have $\alpha_0 = -1$, $\alpha_1 = 1$, $\alpha_2 = \alpha_3 = \dots = \alpha_n = 0$. So,

$$
\begin{aligned}
\beta &= \begin{vmatrix}
-1 & 1 & 0 & \dots & 0 & 0 \\
0 & -1 & 1 & \dots & 0 & 0 \\
\vdots & \vdots & \vdots & & \vdots & \vdots \\
0 & 0 & 0 & \dots & 1 & 0 \\
0 & 0 & 0 & \dots & -1 & 1 \\
-\frac{A_0}{A_n} & -\frac{A_1}{A_n} & -\frac{A_2}{A_n} & \dots & -\frac{A_{n-2}}{A_n} & -\frac{A_{n-1}}{A_n} - 1
\end{vmatrix} \\
&= -\frac{1}{A_n} \begin{vmatrix}
0 & 1 & 0 & \dots & 0 & 0 \\
0 & 0 & 1 & \dots & 0 & 0 \\
\vdots & \vdots & \vdots & & \vdots & \vdots \\
0 & 0 & 0 & \dots & 1 & 0 \\
0 & 0 & 0 & \dots & 0 & 1 \\
\sum_{j=0}^n A_j & \sum_{j=1}^n A_j & \sum_{j=2}^n A_j & \dots & \sum_{j=n-2}^n A_j & A_{n-1} + A_n
\end{vmatrix} \\
&= \frac{(-1)^n}{A_n} \sum_{j=0}^n A_j.
\end{aligned} \tag{3.4}
$$

Since $\max\{\rho(A_j),\, 0 \le j \le n,\, j \ne l\} \le \rho(A_l) < +\infty$,

$$
\sum_{\rho(A_j) = \rho(A_l),\, j \ne l} \tau(A_j) < \tau(A_l) < +\infty,
$$

by (3.4) and Lemma 2.5, we have $\rho(\beta) = \rho(A_l) > 0$, which means that $\beta \not\equiv 0$. Then by Theorem 1.4, we have $\rho(\Delta f) = \rho(g) = \rho(f)$.

Now, set $g(z) = \triangle^2 f(z) = f_2(z) - 2f_1(z) + f(z)$. Then by (1.7), we have $\alpha_0 = 1$, $\alpha_1 = -2$, $\alpha_2 = 1$, $\alpha_3 = \dots = \alpha_n = 0$. So,

$$\beta = \begin{vmatrix} 1 & -2 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \dots & -2 & 1 \\ -\frac{A_0}{A_n} & -\frac{A_1}{A_n} & \dots & -\frac{A_{n-2}}{A_n}+1 & -\frac{A_{n-1}}{A_n}-2 \\ \beta_{n,1} & \beta_{n,2} & \dots & \beta_{n,n-1} & \beta_{n,n} \end{vmatrix}$$

where the bottom row of the above matrix has values

$$\beta_{n,1} = \frac{A_0}{A_n}(\frac{A^1_{n-1}}{A^1_n}+2), \quad \beta_{n,2} = \frac{A_1}{A_n}(\frac{A^1_{n-1}}{A^1_n}+2) - \frac{A^1_0}{A^1_n}\,,$$

$$\beta_{n,n-1} = \frac{A_{n-2}}{A_n}(\frac{A^1_{n-1}}{A^1_n}+2) - \frac{A^1_{n-3}}{A^1_n}, \quad \beta_{n,n} = \frac{A_{n-1}}{A_n}(\frac{A^1_{n-1}}{A^1_n}+2) - \frac{A^1_{n-2}}{A^1_n}+1\,.$$

The above determinant is equal to

$$\begin{aligned} &\begin{vmatrix} 0 & 0 & 1 & \dots & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 1 \\ -\frac{1}{A_n}\sum_{j=0}^n A_j & -\frac{1}{A_n}\sum_{j=1}^n jA_j & * & \dots & * & * \\ \hat{\beta}_{n,1} & \hat{\beta}_{n,2} & * & \dots & * & * \end{vmatrix} \\ &= \frac{1}{A_nA^1_n}\Big(\sum_{j=0}^n A_j \cdot \sum_{j=0}^n (j+1)A^1_j - \sum_{j=1}^n jA_j \cdot \sum_{j=0}^n A^1_j\Big), \end{aligned}$$

where

$$\hat{\beta}_{n,1} = \Big(\frac{A^1_{n-1}+2A^1_n}{A_nA^1_n}\Big)\sum_{j=0}^n A_j - \frac{1}{A^1_n}\sum_{j=0}^n A^1_j\,,$$

$$\hat{\beta}_{n,2} = \Big(\frac{A^1_{n-1}+2A^1_n}{A_nA^1_n}\Big)\sum_{j=1}^n jA_j - \frac{1}{A^1_n}\sum_{j=0}^n (j+1)A^1_j\,.$$

Since $\beta \not\equiv 0$, by Theorem 1.4, we have $\rho(\Delta^2 f) = \rho(g) = \rho(f)$. □

Proof of Theorem 1.7. By Lemma 2.2, we have $\rho(f) \ge 2$. By substituting

$$f_n = -\frac{1}{a_ne^{b_nz}}(a_{n-1}e^{b_{n-1}z}f_{n-1} + \dots + a_1e^{b_1z}f_1 + a_0e^{b_0z}f_0)$$

into (1.7), we obtain

$$\begin{aligned} g_0 &= -\frac{\alpha_n}{a_ne^{b_nz}}(a_{n-1}e^{b_{n-1}z}f_{n-1} + \dots + a_0e^{b_0z}f_0) + \alpha_{n-1}f_{n-1} + \dots + \alpha_0f_0 \\ &= (-\frac{a_{n-1}}{a_n}e^{(b_{n-1}-b_n)z}\alpha_n + \alpha_{n-1})f_{n-1} + \dots + (-\frac{a_1}{a_n}e^{(b_1-b_n)z}\alpha_n + \alpha_1)f_1 \\ &\quad + (-\frac{a_0}{a_n}e^{(b_0-b_n)z}\alpha_n + \alpha_0)f_0 \\ &= \eta_{0,n-1}f_{n-1} + \dots + \eta_{0,1}f_1 + \eta_{0,0}f_0. \end{aligned}$$

Then

$$\begin{aligned}
g_1 &= \eta^1_{0,n-1} f_n + \dots + \eta^1_{0,1} f_2 + \eta^1_{0,0} f_1 \\
&= -\frac{\eta^1_{0,n-1}}{a_n e^{b_n z}} (a_{n-1} e^{b_{n-1} z} f_{n-1} + \dots + a_1 e^{b_1 z} f_1 + a_0 e^{b_0 z} f_0) + \eta^1_{0,n-2} f_{n-1} \\
&\quad + \dots + \eta^1_{0,0} f_1 \\
&= (-\frac{a_{n-1}}{a_n} e^{(b_{n-1}-b_n)z} \eta^1_{0,n-1} + \eta^1_{0,n-2}) f_{n-1} + \dots \\
&\quad + (-\frac{a_1}{a_n} e^{(b_1-b_n)z} \eta^1_{0,n-1} + \eta^1_{0,0}) f_1 + (-\frac{a_0}{a_n} e^{(b_0-b_n)z} \eta^1_{0,n-1}) f_0 \\
&= \eta_{1,n-1} f_{n-1} + \dots + \eta_{1,1} f_1 + \eta_{1,0} f_0.
\end{aligned}$$

By repeating the above argument $n-2$ times, we obtain

$$\begin{gathered}
\eta_{0,0} f_0 + \eta_{0,1} f_1 + \dots + \eta_{0,n-1} f_{n-1} = g_0 \\
\eta_{1,0} f_0 + \eta_{1,1} f_1 + \dots + \eta_{1,n-1} f_{n-1} = g_1 \\
\dots \\
\eta_{n-1,0} f_0 + \eta_{n-1,1} f_1 + \dots + \eta_{n-1,n-1} f_{n-1} = g_{n-1}
\end{gathered} \tag{3.5}$$

Then by Cramer's Rule and the assumption $\gamma \not\equiv 0$, the equation system (3.5) has a unique solution $f(z)$, and

$$f = \frac{1}{\gamma} \begin{vmatrix} g_0 & \eta_{0,1} & \dots & \eta_{0,n-1} \\ g_1 & \eta_{1,1} & \dots & \eta_{1,n-1} \\ \vdots & \vdots & & \vdots \\ g_{n-1} & \eta_{n-1,1} & \dots & \eta_{n-1,n-1} \end{vmatrix}. \tag{3.6}$$

On the one hand, by Lemma 2.5, Remark 1.5 and the assumption $\max\{\rho(\alpha_j), 0 \le j \le n\} < 2 \le \rho(f)$, it is clear from (1.7) that $\rho(g) \le \rho(f)$. On the other hand, by Lemma 2.5, Remark 1.5 and the assumptions $\max\{\rho(a_j), 0 \le j \le n\} < 1, \max\{\rho(\alpha_j), 0 \le j \le n\} < 2$, it is clear from (3.6) that $\rho(f) \le \rho(g)$. Hence,

$$\rho(g) = \rho(f) \ge 2.$$

Set $G(z) = g(z) - \varphi(z)$. By Lemma 2.5 and the assumption $\rho(\varphi) < 2$, we have

$$\rho(G) = \rho(g - \varphi) = \rho(g) = \rho(f) \ge 2.$$

By substituting $g(z) = G(z) + \varphi(z)$ into (3.6), we obtain

$$
\begin{aligned}
f &= \frac{1}{\gamma}\begin{vmatrix} G_0+\varphi_0 & \eta_{0,1} & \dots & \eta_{0,n-1} \\ G_1+\varphi_1 & \eta_{1,1} & \dots & \eta_{1,n-1} \\ \vdots & \vdots & & \vdots \\ G_{n-1}+\varphi_{n-1} & \eta_{n-1,1} & \dots & \eta_{n-1,n-1} \end{vmatrix} \\
&= \frac{1}{\gamma}\sum_{i_0 i_1\dots i_{n-1}} (-1)^{\tau(i_0 i_1\dots i_{n-1})}(G+\varphi)_{i_0}\eta_{i_1,1}\eta_{i_2,2}\dots\eta_{i_{n-1},n-1} \\
&= \frac{1}{\gamma}\sum_{i_0 i_1\dots i_{n-1}} (-1)^{\tau(i_0 i_1\dots i_{n-1})}G_{i_0}\eta_{i_1,1}\eta_{i_2,2}\dots\eta_{i_{n-1},n-1} \\
&\quad + \frac{1}{\gamma}\sum_{i_0 i_1\dots i_{n-1}} (-1)^{\tau(i_0 i_1\dots i_{n-1})}\varphi_{i_0}\eta_{i_1,1}\eta_{i_2,2}\dots\eta_{i_{n-1},n-1} \\
&= \frac{1}{\gamma}\sum_{i_0 i_1\dots i_{n-1}} (-1)^{\tau(i_0 i_1\dots i_{n-1})}G_{i_0}\eta_{i_1,1}\eta_{i_2,2}\dots\eta_{i_{n-1},n-1} + \delta,
\end{aligned}
\tag{3.7}
$$

where $\{i_0, i_1, \dots, i_{n-1}\} = \{0, 1, \dots, n-1\}$. Then by substituting (3.7) into (1.8), we obtain

$$
\begin{aligned}
&a_n e^{b_n z}\Big(\frac{1}{\gamma_n}\sum_{i_0 i_1\dots i_{n-1}} (-1)^{\tau(i_0 i_1\dots i_{n-1})}G_{i_0+n}\eta^n_{i_1,1}\eta^n_{i_2,2}\dots\eta^n_{i_{n-1},n-1} + \delta_n\Big) \\
&+ a_{n-1}e^{b_{n-1}z}\Big(\frac{1}{\gamma_{n-1}}\sum_{i_0 i_1\dots i_{n-1}} (-1)^{\tau(i_0 i_1\dots i_{n-1})}G_{i_0+n-1}\eta^{n-1}_{i_1,1}\eta^{n-1}_{i_2,2}\dots\eta^{n-1}_{i_{n-1},n-1} + \delta_{n-1}\Big) \\
&+ \dots + a_0 e^{b_0 z}\Big(\frac{1}{\gamma}\sum_{i_0 i_1\dots i_{n-1}} (-1)^{\tau(i_0 i_1\dots i_{n-1})}G_{i_0}\eta_{i_1,1}\eta_{i_2,2}\dots\eta_{i_{n-1},n-1} + \delta\Big) = 0,
\end{aligned}
$$

that is,

$$
\begin{aligned}
&\frac{a_n e^{b_n z}}{\gamma_n}\sum_{n-1,i_1\dots i_{n-1}} (-1)^{\tau(n-1,i_1\dots i_{n-1})}\eta^n_{i_1,1}\eta^n_{i_2,2}\dots\eta^n_{i_{n-1},n-1}G_{2n-1} \\
&+ \Big(\frac{a_n e^{b_n z}}{\gamma_n}\sum_{n-2,i_1\dots i_{n-1}} (-1)^{\tau(n-2,i_1\dots i_{n-1})}\eta^n_{i_1,1}\eta^n_{i_2,2}\dots\eta^n_{i_{n-1},n-1} \\
&+ \frac{a_{n-1}e^{b_{n-1}z}}{\gamma_{n-1}}\sum_{n-1,i_1\dots i_{n-1}} (-1)^{\tau(n-1,i_1\dots i_{n-1})}\eta^{n-1}_{i_1,1}\eta^{n-1}_{i_2,2}\dots\eta^{n-1}_{i_{n-1},n-1}\Big)G_{2n-2} \\
&+ \dots + \frac{a_0 e^{b_0 z}}{\gamma}\sum_{0,i_1\dots i_{n-1}} (-1)^{\tau(0,i_1\dots i_{n-1})}\eta_{i_1,1}\eta_{i_2,2}\dots\eta_{i_{n-1},n-1}G_0 \\
&= -(a_n e^{b_n z}\delta_n + a_{n-1}e^{b_{n-1}z}\delta_{n-1} + \dots + a_0 e^{b_0 z}\delta).
\end{aligned}
$$

By Lemma 2.5, Remark 1.5 and the assumption $\max\{\rho(\varphi), \rho(a_j), \rho(\alpha_j), 0 \le j \le n\} < 2$, we have $\rho(\delta) < 2$. Then by Lemma 2.2 and the assumption $\delta \not\equiv 0$, δ is not a solution of (1.8), that is,

$$
a_n e^{b_n z}\delta_n + a_{n-1}e^{b_{n-1}z}\delta_{n-1} + \dots + a_0 e^{b_0 z}\delta \not\equiv 0.
$$

Hence, by Lemma 2.4, we have $\lambda(G) = \rho(G)$, that is,

$$
\lambda(g - \varphi) = \rho(g - \varphi) = \rho(g) = \rho(f) \ge 2.
$$

□

Proof of Corollary 1.8. Set $g(z) = \triangle f(z) = f_1(z) - f(z)$. Then by (1.7), we have $\alpha_0 = -1$, $\alpha_1 = 1$, $\alpha_2 = \alpha_3 = \dots = \alpha_n = 0$. So,

$$\begin{aligned}\gamma &= \begin{vmatrix} -1 & 1 & 0 & \dots & 0 & 0 \\ 0 & -1 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & 0 \\ 0 & 0 & 0 & \dots & -1 & 1 \\ -\frac{a_0e^{b_0z}}{a_ne^{b_nz}} & -\frac{a_1e^{b_1z}}{a_ne^{b_nz}} & -\frac{a_2e^{b_2z}}{a_ne^{b_nz}} & \dots & -\frac{a_{n-2}e^{b_{n-2}z}}{a_ne^{b_nz}} & -\frac{a_{n-1}e^{b_{n-1}z}}{a_ne^{b_nz}} - 1 \end{vmatrix} \\ &= -\frac{1}{a_ne^{b_nz}} \begin{vmatrix} 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & 0 \\ 0 & 0 & 0 & \dots & 0 & 1 \\ \sum_{j=0}^n a_je^{b_jz} & \sum_{j=1}^n a_je^{b_jz} & * & \dots & * & \sum_{j=n-1}^n a_je^{b_jz} \end{vmatrix} \\ &= \frac{(-1)^n}{a_ne^{b_nz}} \sum_{j=0}^n a_je^{b_jz}. \end{aligned} \tag{3.8}$$

By the assumptions that $b_j (j = 0, 1, \dots, n)$ are distinct complex constants and $a_j(z) \not\equiv 0 (j = 0, 1, \dots, n)$, we have $\sum_{j=0}^n a_je^{b_jz} \not\equiv 0$, consequently $\gamma \not\equiv 0$. Then by Theorem 1.7, we have $\rho(\triangle f) = \rho(f) \geq 2$. Since

$$\begin{aligned}\delta &= \frac{1}{\gamma} \begin{vmatrix} \varphi_0 & 1 & 0 & \dots & 0 & 0 \\ \varphi_1 & -1 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ \varphi_{n-3} & 0 & 0 & \dots & 1 & 0 \\ \varphi_{n-2} & 0 & 0 & \dots & -1 & 1 \\ \varphi_{n-1} & -\frac{a_1e^{b_1z}}{a_ne^{b_nz}} & -\frac{a_2e^{b_2z}}{a_ne^{b_nz}} & \dots & -\frac{a_{n-2}e^{b_{n-2}z}}{a_ne^{b_nz}} & -\frac{a_{n-1}e^{b_{n-1}z}}{a_ne^{b_nz}} - 1 \end{vmatrix} \\ &= \frac{1}{\gamma} \begin{vmatrix} \varphi_0 & 1 & 0 & \dots & 0 & 0 \\ \varphi_1 & 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ \varphi_{n-3} & 0 & 0 & \dots & 1 & 0 \\ \varphi_{n-2} & 0 & 0 & \dots & 0 & 1 \\ \varphi_{n-1} & -\frac{\sum_{j=1}^n a_je^{b_jz}}{a_ne^{b_nz}} & -\frac{\sum_{j=2}^n a_je^{b_jz}}{a_ne^{b_nz}} & \dots & -\frac{\sum_{j=n-2}^n a_je^{b_jz}}{a_ne^{b_nz}} & -\frac{\sum_{j=n-1}^n a_je^{b_jz}}{a_ne^{b_nz}} \end{vmatrix} \\ &= \frac{(-1)^{n-1}}{a_ne^{b_nz}\gamma} \sum_{i=0}^{n-1} \varphi_i \sum_{j=i+1}^n a_je^{b_jz} \not\equiv 0, \end{aligned}$$

by Theorem 1.7, we have

$$\lambda(\triangle f - \varphi) = \rho(\triangle f - \varphi) = \rho(\triangle f) = \rho(f) \geq 2.$$

□

Proof of Theorem 1.9. By Lemma 2.3, we have $\rho(f) \geq 2$ with at most one exceptional meromorphic solution. Next, we assume that $\rho(f) \geq 2$. By substituting

$$f_n = \frac{1}{a_n e^{b_n z}}(F - a_{n-1}e^{b_{n-1}z}f_{n-1} - \dots - a_1 e^{b_1 z}f_1 - a_0 e^{b_0 z}f_0)$$

into (1.7), we obtain

$$\begin{aligned}
g_0 &= \frac{\alpha_n}{a_n e^{b_n z}}(F - a_{n-1}e^{b_{n-1}z}f_{n-1} - \dots - a_0 e^{b_0 z}f_0) + \alpha_{n-1}f_{n-1} + \dots + \alpha_0 f_0 \\
&= \frac{\alpha_n}{a_n e^{b_n z}}F + (-\frac{a_{n-1}}{a_n}e^{(b_{n-1}-b_n)z}\alpha_n + \alpha_{n-1})f_{n-1} + \dots \\
&\quad + (-\frac{a_1}{a_n}e^{(b_1-b_n)z}\alpha_n + \alpha_1)f_1 + (-\frac{a_0}{a_n}e^{(b_0-b_n)z}\alpha_n + \alpha_0)f_0 \\
&= \frac{\alpha_n}{a_n e^{b_n z}}F + \eta_{0,n-1}f_{n-1} + \dots + \eta_{0,1}f_1 + \eta_{0,0}f_0.
\end{aligned}$$

Then

$$\begin{aligned}
&g_1 - \frac{\alpha_n^1}{a_n^1 e^{b_n(z+1)}}F_1 \\
&= \eta_{0,n-1}^1 f_n + \dots + \eta_{0,1}^1 f_2 + \eta_{0,0}^1 f_1 \\
&= \frac{\eta_{0,n-1}^1}{a_n e^{b_n z}}(F - a_{n-1}e^{b_{n-1}z}f_{n-1} - \dots - a_1 e^{b_1 z}f_1 - a_0 e^{b_0 z}f_0) + \eta_{0,n-2}^1 f_{n-1} \\
&\quad + \dots + \eta_{0,0}^1 f_1 \\
&= \frac{\eta_{0,n-1}^1}{a_n e^{b_n z}}F + (-\frac{a_{n-1}}{a_n}e^{(b_{n-1}-b_n)z}\eta_{0,n-1}^1 + \eta_{0,n-2}^1)f_{n-1} \\
&\quad + \dots + (-\frac{a_1}{a_n}e^{(b_1-b_n)z}\eta_{0,n-1}^1 + \eta_{0,0}^1)f_1 + (-\frac{a_0}{a_n}e^{(b_0-b_n)z}\eta_{0,n-1}^1)f_0 \\
&= \frac{\eta_{0,n-1}^1}{a_n e^{b_n z}}F + \eta_{1,n-1}f_{n-1} + \dots + \eta_{1,1}f_1 + \eta_{1,0}f_0.
\end{aligned}$$

By repeating the above argument $n-2$ times, we obtain

$$\begin{gathered}
\eta_{0,0}f_0 + \eta_{0,1}f_1 + \dots + \eta_{0,n-1}f_{n-1} = g_0 - \frac{\alpha_n}{a_n e^{b_n z}}F, \\
\eta_{1,0}f_0 + \eta_{1,1}f_1 + \dots + \eta_{1,n-1}f_{n-1} = g_1 - \frac{\alpha_n^1}{a_n^1 e^{b_n(z+1)}}F_1 - \frac{\eta_{0,n-1}^1}{a_n e^{b_n z}}F, \\
\dots, \\
\eta_{n-1,0}f_0 + \eta_{n-1,1}f_1 + \dots + \eta_{n-1,n-1}f_{n-1} \\
= g_{n-1} - \frac{\alpha_n^{n-1}}{a_n^{n-1}e^{b_n(z+n-1)}}F_{n-1} - \sum_{t=0}^{n-2}\frac{\eta_{n-2-t,n-1}^{t+1}}{a_n^t e^{b_n(z+t)}}F_t\,.
\end{gathered} \tag{3.9}$$

Then by Cramer's Rule and the assumption $\gamma \not\equiv 0$, the equation system (3.9) has a unique solution $f(z)$, and

$$f = \frac{1}{\gamma}\begin{vmatrix} g_0 - \frac{\alpha_n}{a_n e^{b_n z}} F & \eta_{0,1} & \cdots & \eta_{0,n-1} \\ g_1 - \frac{\alpha_n^1}{a_n^1 e^{b_n(z+1)}} F_1 - \frac{\eta_{0,n-1}^1}{a_n e^{b_n z}} F & \eta_{1,1} & \cdots & \eta_{1,n-1} \\ \vdots & \vdots & & \vdots \\ g_{n-1} - \frac{\alpha_n^{n-1}}{a_n^{n-1} e^{b_n(z+n-1)}} F_{n-1} - \sum_{t=0}^{n-2} \frac{\eta_{n-2-t,n-1}^{t+1}}{a_n^t e^{b_n(z+t)}} F_t & \eta_{n-1,1} & \cdots & \eta_{n-1,n-1} \end{vmatrix}. \tag{3.10}$$

On the one hand, by Lemma 2.5, Remark 1.5 and the assumption $\max\{\rho(\alpha_j), 0 \le j \le n\} < 2 \le \rho(f)$, it is clear from (1.7) that $\rho(g) \le \rho(f)$. On the other hand, by Lemma 2.5, Remark 1.5 and the assumptions $\max\{\rho(a_j), 0 \le j \le n\} < 1, \max\{\rho(\alpha_j), 0 \le j \le n\} < 2, \rho(F) < 1$, it is clear from (3.10) that $\rho(f) \le \rho(g)$. Hence, $\rho(g) = \rho(f) \ge 2$. □

Proof of Corollary 1.10. We have

$$\gamma = \frac{(-1)^n}{a_n e^{b_n z}} \sum_{j=0}^{n} a_j e^{b_j z} \not\equiv 0$$

as in (3.8). So, by Theorem 1.9, we have $\rho(\triangle f) = \rho(f) \ge 2$ with at most one exceptional meromorphic solution. □

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