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A KAM THEOREM FOR DEGENERATE INFINITE-DIMENSIONAL REVERSIBLE SYSTEMS

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Abstract. In this article, we establish a Kolmogorov-Arnold-Moser (KAM) theorem for degenerate infinite-dimensional reversible systems under a non-degenerate condition of Rüssmann type. This theorem broadens the scope of applicability of degenerate KAM theory, previously confined to Hamiltonian systems, by incorporating infinite-dimensional reversible systems. Using this theorem, we obtain the existence and linear stability of quasi-periodic solutions for a class of non-Hamiltonian but reversible beam equations with non-linearities in derivatives.

1. Introduction

In Kolmogorov-Arnold-Moser (KAM) theory, the non-degeneracy condition typically refers to the regularity assumption of tangent frequencies with respect to external parameters. This condition plays a vital role in ensuring the persistence of KAM tori under small perturbations. Degenerate KAM theory for finite-dimensional Hamiltonian systems has been widely developed since the works of Arnold [1] and Pjartly [15]. A complete geometric definition of non-degeneracy condition was given by Rüssmann [18], which is not only applicable to maximal dimensional KAM tori but also to lower dimensional elliptic tori. See also other important works by Brjuno [5], Cheng and Sun [6] and Xu, Qiu, and You [22].

KAM theory for degenerate infinite-dimensional Hamiltonian systems was initially established by Xu, Qiu, and You [21]. They proved a KAM theorem under an analytic Rüssmann degeneracy assumption. Subsequently, Bambusi, Berti, and Magistrelli [3] introduced a different Rüssmann degeneracy condition from that in [21] and established a new KAM theorem for a class of nonlinear wave equations. Building upon the non-degeneracy condition in [3], Baldi, Berti, Haus and Montalto derived a new degenerate KAM theorem particularly applicable to gravitational wave equations. The proof is based on the Nash-Moser iteration, KAM reduction and pseudo-differential calculus. Furthermore, Gao and Liu[8] recently developed a KAM theorem for degenerate infinite-dimensional Hamiltonian systems with double normal frequencies, successfully applying it to nonlinear wave equations and nonlinear Schrödinger equations with periodic boundary conditions.

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On the other hand, KAM theory for infinite-dimensional Hamiltonian systems has also been extended to infinite-dimensional reversible systems [2, 4, 7, 9, 12, 11, 13, 14, 24]. However, to the best of our knowledge, these results are all based on classical non-degeneracy conditions, and there is still no works on KAM theory for degenerate infinite-dimensional reversible systems. In contrast, some progress has been made in the KAM theory for finite-dimensional reversible systems under Rüssmann degeneracy conditions [19, 20, 23].

The primary motivation of this article is to bridge the gap in the literature concerning the degenerate KAM theory for infinite-dimensional reversible systems. The study aims to generalize the existing theorems and to provide new insights into the applicability of degenerate KAM theory in a broader range of systems, particularly in infinite-dimensional reversible settings. By doing so, we hope to advance the understanding of the existence and stability properties of quasi-periodic motions in more complex systems that were previously not well-covered by the classical non-degeneracy conditions.

In this article, we establish a KAM theorem for degenerate infinite-dimensional reversible systems, which is, to the best of our knowledge, a new result in the field. Our method combines and extends the techniques from previous works on degenerate KAM theory for infinite-dimensional Hamiltonian systems and finite-dimensional reversible systems, allowing us to study new systems by the combination of degeneracy and reversibility in an infinite-dimensional setting. As an application, we use our KAM theorem to prove the existence and linear stability of quasi-periodic solutions for a class of non-Hamiltonian but reversible beam equations with derivative non-linearities.

We consider a family of vector fields with normal form

$$\begin{aligned} N &= \omega(\xi)\frac{\partial}{\partial\theta} + \mathrm{i}\Omega(\xi)z\frac{\partial}{\partial z} - \mathrm{i}\Omega(\xi)\bar{z}\frac{\partial}{\partial\bar{z}} \\ &= \sum_{j=1}^{n}\omega_j(\xi)\frac{\partial}{\partial\theta_j} + \sum_{j\geq 1}\Big(\mathrm{i}\Omega_j(\xi)z_j\frac{\partial}{\partial z_j} - \mathrm{i}\Omega_j(\xi)\bar{z}_j\frac{\partial}{\partial\bar{z}_j}\Big), \end{aligned}$$

on the phase space $\mathscr{P}^{a,p} = \mathbb{T}^n \times \mathbb{R}^n \times l^{a,p} \times l^{a,p} \ni (\theta, I, z, \bar{z})$, where $\mathbb{T}^n (1 \leq n < +\infty)$ is the standard $n-$torus, $l^{a,p}$ is the Hilbert space of all complex sequence $z = (z_1, z_2, \dots,)$ with the norm defined by

$$\|z\|_{a,p}^2 = \sum_{j\geq 1}|z_j|j^{2p}e^{2aj} < +\infty, \quad a \geq 0, p \geq 0.$$

The parameter $\xi \in \Pi \subset \mathbb{R}^n$, where Π is a bounded, connected closed domain in $\mathbb{R}^n$. Note that N is reversible with respect to the involution map $S : (\theta, I, z, \bar{z}) \to (-\theta, I, \bar{z}, z)$:

$$N \circ S = -DS \cdot N,$$

where DS is the tangent map of S.

The motion equations of S-reversible vector fields N are

$$\begin{aligned} \dot{\theta} &= \omega(\xi), \\ \dot{I} &= 0, \\ \dot{z} &= \mathrm{i}\Omega(\xi)z, \\ \dot{\bar{z}} &= -\mathrm{i}\Omega(\xi)\bar{z}. \end{aligned} \tag{1.1}$$

Obviously, for each $\xi \in \Pi$, $\mathscr{T}^n = \mathbb{T}^n \times \{0,0,0\} \subset \mathscr{P}^{a,p}$ is an invariant torus of the reversible systems (1.1) with the frequencies $\omega(\xi) = (\omega_1(\xi), \dots, \omega_n(\xi))$.

Now we consider the small perturbations of N : $X = N + P$, where the perturbations P are S-reversible. To vector fields X, the associated reversible equations are

$$\begin{aligned} \dot\theta &= \omega(\xi) + P^{(\theta)}, \\ \dot I &= P^{(I)}, \\ \dot z &= \mathrm{i}\Omega(\xi) z + P^{(z)}, \\ \dot{\bar z} &= -\mathrm{i}\Omega(\xi)\bar z + P^{(\bar z)}. \end{aligned} \tag{1.2}$$

The definition of $P^{(\theta)}, P^{(I)}, P^{(z)}, P^{(\bar z)}$ will be given below. We are concerned with the persistence of invariant tori for reversible systems (1.2). For this purpose, we first give some notations and assumptions.

Let $C^{\mathcal{N},1}(\Pi)$ be the $\mathcal{N}$-order Lipschitz continuously differentiable function space. If Π is a closed set, the derivatives of function on Π are understood in the sense of Whitney. Consequently, the space $C^{\mathcal{N},1}(\Pi)$ is also understood in the sense of Whitney, where the integer $\mathcal{N}$ will be decided below.

We use the following assumptions.

(A1) (Non-degeneracy condition) Suppose for all $\xi \in \Pi$,

$$\operatorname{rank}\{\frac{\partial \omega}{\partial \xi}\} = \mathtt{r},$$

$$\operatorname{rank}\{\frac{\partial^\beta \omega}{\partial \xi^\beta} : \forall \beta, 1 \le |\beta| \le n - \mathtt{r} + 1\} = n,$$

where $\frac{\partial \omega}{\partial \xi}$ is a function vector group of all 1-order partial derivatives of ω, and $\frac{\partial^\beta \omega}{\partial \xi^\beta} = \big(\frac{\partial^\beta \omega_1}{\partial \xi^\beta}, \dots, \frac{\partial^\beta \omega_n}{\partial \xi^\beta}\big)$. Moreover, for some $\mathcal{N} \ge n - \mathtt{r} + 1, \omega \in \{C^{\mathcal{N},1}(\Pi)\}^n$ with $\|\omega\|_{C^{\mathcal{N},1}(\Pi)} \triangleq \max_{1\le j\le n} \|\omega_j\|_{C^{\mathcal{N},1}(\Pi)} \le M_1$.

(A2) (Spectral asymptotic) There exist $d \ge 1$ and $\delta < d - 1$ such that

$$\Omega_j(\xi) = bj^d + b'j^{d'} + \dots + O(j^\delta), b > 0, d' < d,$$

where the dots stand for finite lower order terms of j and $bj^d + b'j^{d'} + \dots$ are independent of the parameter ξ. Moreover, Ω_j satisfies

$$\|\Omega_j - bj^d - b'j^{d'} - \dots\|_{C^{\mathcal{N},1}(\Pi)} \le M_2 j^\delta, \quad \forall j \ge 1.$$

(A3) (Regularity of perturbations) For all $\xi \in \Pi$, the S-reversible perturbation vector field $P : \mathscr{P}^{a,p} \to \mathscr{P}^{a,\bar p}$ is real analytic with $p \le \bar p$ and $p - \bar p < d - 1$. Without loss of generality, suppose $p - \bar p < \delta$.

We denote by $D(s,r)$ a complex neighborhood of $\mathbb{T}^n$ in $\mathscr{P}^{a,p}$:

$$D(s,r) = \{(\theta, I, z, \bar z) \in \mathscr{P}^{a,p} : |\operatorname{Im}\theta| < s, |I| < r, \|z\|_{a,p} < r, \|\bar z\|_{a,p} < r\},$$

where $|\operatorname{Im}\theta|$ is the imaginary part of θ and $|\cdot|$ is the sup-norm for n-dimensional vector, $r > 0$ is a radius of neighborhood (Note that it is different from that in assumption (A1.). Let $\|\cdot\|^* = \|\cdot\|_{C^{\mathcal{N},1}(\Pi)}$.

Now we consider the vector field

$$P(v) = (P^{(\theta)}(v), P^{(I)}(v), P^{(z)}(v), P^{(\bar z)}(v)),$$

where $v = (\theta, I, z, \bar{z}) \in D(s, r)$. It is convenient to rewrite it as a partial differential operator

$$P(v) = P^{(\theta)}(v)\frac{\partial}{\partial\theta} + P^{(I)}(v)\frac{\partial}{\partial I} + P^{(z)}(v)\frac{\partial}{\partial z} + P^{(\bar{z})}(v)\frac{\partial}{\partial\bar{z}},$$

where

$$P^{(\theta)}(v)\frac{\partial}{\partial\theta} = \sum_{j=1}^{n} P^{(\theta_j)}(v)\frac{\partial}{\partial\theta_j},$$

and similarly for $P^{(I)}(v)\frac{\partial}{\partial I}, P^{(z)}(v)\frac{\partial}{\partial z}, P^{(\bar{z})}(v)\frac{\partial}{\partial\bar{z}}$.

Suppose the vector field $W(\xi) = (W^{(\theta)}(\xi), W^{(I)}(\xi), W^{(z)}(\xi), W^{(\bar{z})}(\xi)) \in \mathscr{P}^{a,\bar{p}}$, with the weighted norm

$$\|W\|_r^* = \|W\|_{\bar{p},r} = |W^{(\theta)}|^* + \frac{1}{r}|W^{(I)}|^* + \frac{1}{r}\|W^{(z)}\|_{a,\bar{p}}^* + \frac{1}{r}\|W^{(\bar{z})}\|_{a,\bar{p}}^*,$$

where $|W^{(\theta)}|^* = |(|W_1^{(\theta)}|^*, \dots, |W_n^{(\theta)}|^*)|$, $\|W^{(z)}\|_{a,\bar{p}}^* = \|(\|W_1^{(z)}\|^*, \|W_2^{(z)}\|^*, \dots)\|_{a,\bar{p}}^*$. For a function $f(v,\xi)$ on $D \times \Pi$, we define its norm by

$$\|f\|_D^* = \sup_{v\in D} \|f(v,\cdot)\|^*.$$

Similarly we define the norm of vector field W as

$$\|W\|_{r,D}^* = \|W\|_{\bar{p},r,D} = |W^{(\theta)}|_D^* + \frac{1}{r}|W^{(I)}|_D^* + \frac{1}{r}\|W^{(z)}\|_{a,\bar{p},D}^* + \frac{1}{r}\|W^{(\bar{z})}\|_{a,\bar{p},D}^*.$$

For $k \in \mathbb{Z}^n$ and $l \in \mathbb{Z}^{+\infty}$, we denote $|k| = \sum_{j=1}^{n} |k_j|, |l| = \sum_{j=1}^{\infty} |l_j|$, $|l|_\delta = \sum_{j=1}^{\infty} |l_j|j^\delta$, and $[l]_d = \{1, |\sum_{j=1}^{\infty} l_j j^d|\}$. We denote $A_k = (1 + |k|)^\tau$, where $\tau \le \tau_0$ with

$$\tau_0 = \begin{cases} (n + \frac{2}{d-1})(n - r + 1), & \text{if } d > 1, \\ (n + 1 + \frac{\kappa+n-r+1}{\kappa})(n - r + 1), & \text{if } d = 1, \end{cases}$$

where $\kappa = \min\{d, d - 1 - \delta, d - d'\}$.

Theorem 1.1. *Suppose the S-reversible vector field $X = N + P$ satisfies* (A1)–(A3). *Then there exist K and L such that for k, l with $0 \ne |k| \le K, |l| \le 2$ and $|l|_k \le L$, the inequality*

$$|\langle k, \omega(\xi)\rangle + \langle l, \Omega(\xi)\rangle| \ge \alpha_0 > 0$$

holds for all $\forall\xi \in \Pi$. Then for sufficiently small $\alpha > 0 (\alpha > \alpha_0)$, there exists sufficiently small $\epsilon_0 = \epsilon_0\ (\alpha) > 0$ such that if $\epsilon = \|P\|_{r,D(s,r)}^* < \epsilon_0$, then there exists a nonempty Cantorian subset Π_* of Π and embeddings $\Phi_*(\cdot,\cdot) : \mathbb{T}^n \times \Pi_* \to \mathscr{P}^{a,\bar{p}}$ satisfying

$$\|\Phi_* - \Phi_0\|_r^* \le c\epsilon,$$

where Φ_0 is the trivial embedding $\mathbb{T}^n \times \Pi_* \to \mathscr{T}^n = \mathbb{T}^n \times \{0,0,0\} \subset \mathscr{P}^{a,\bar{p}}$, and for all $\xi \in \Pi_*, \Phi_*(\mathbb{T}^n, \xi)$ is a real analytic embedding of a rotational torus for reversible systems (1.2) *at ξ with its frequencies $\omega_*(\xi)$ satisfying $|\omega_* - \omega|^* \le c\epsilon$. Moreover, the Lebesgue measure* $\operatorname{meas}(\Pi - \Pi_*) \le c(\operatorname{diam} Pi)^{n-1}\alpha^\mu$, *where*

$$\mu = \begin{cases} \frac{1}{n-r+1}, & \text{if } d > 1, \\ \frac{\kappa}{(\kappa+n-r+1)(n-r+1)}, & \text{if } d = 1. \end{cases}$$

The Cantorian subset Π_* in Theorem1.1 arises from the small divisor conditions encountered during the KAM step,

$$|\langle k,\omega(\xi)\rangle+\langle l,\Omega(\xi)\rangle|\geq\alpha\frac{[l]_d}{A_k},\forall k\in\mathbb{Z}^n,\ l\in\mathbb{Z}^{+\infty},\ |l|\leq 2,|k|+|l|\neq 0. \tag{1.3}$$

In other words, the KAM iteration can only be performed for those parameters that satisfy the inequality (1.3). Hence, we need to exclude certain points where this inequality does not hold. The subsequent Theorem 1.2 ensures that the set Π is not empty if α is sufficiently small.

Theorem 1.2. *Let*

$$R_{k,l}(\alpha)=\big\{\xi:|\langle k,\omega(\xi)\rangle+\langle l,\Omega(\xi)\rangle|<\alpha\frac{[l]_d}{A_k}\big\},$$

where $k\in\mathbb{Z}^n,l\in\mathbb{Z}^{+\infty}$, $|l|\leq 2,|k|+|l|\neq 0$. *If* (A1) *and* (A2) *hold, then there exist* K *and* L *such that for sufficiently small* $\alpha>0$,

$$\operatorname{meas}\big(\cup_{|k|\geq K or |l|_k\geq L}R_{k,l}(\alpha)\big)\leq c(\operatorname{diam}(Pi)^{n-1}\alpha^{\mu},$$

where μ *is defined in Theorem 1.1.*

The proof of the above theorem follows a similar approach as in [21] and is omitted here.

In the subsequent sections we focus on the proof and application of Theorem 1.1. In Section 2, we derive and solve the homological equation. Section 3 is dedicated to constructing one step of the iterative scheme. The complete iteration sequences, forming the basis of the proof of Theorem 1.1, are presented in Section 4. Finally, in Section 5, we apply Theorem 1.1 to a class of non-Hamiltonian but reversible beam equations.

2. Homological equation

At each step of the KAM iteration, we obtain the S-invariant change of variables Φ through the time 1-map $\Phi_F^t|_{t=1}$ of the flow generated by the S-invariant vector field F. The vector field F and the correction $\hat{N}$ to the normal form N are solutions of the homological equation

$$[F,N]+\hat{N}=R, \tag{2.1}$$

where the symbol $[\cdot,\cdot]$ represents the Lie bracket of vector fields, and the vector field R is the truncation of the Taylor polynomial of the reversible vector field P,

$$\begin{aligned}R=R^{\theta}\frac{\partial}{\partial\theta}&+\big(R^{I}(\theta)+R^{II}(\theta)I+R^{Iz}(\theta)z+R^{I\bar{z}}(\theta)\bar{z}\big)\frac{\partial}{\partial I}\\&+\big(R^{z}(\theta)+R^{zI}(\theta)I+R^{zz}(\theta)z+R^{z\bar{z}}(\theta)\bar{z}\big)\frac{\partial}{\partial z}\\&+\big(R^{\bar{z}}(\theta)+R^{\bar{z}I}(\theta)I+R^{\bar{z}z}(\theta)z+R^{\bar{z}\bar{z}}(\theta)\bar{z}\big)\frac{\partial}{\partial\bar{z}}.\end{aligned} \tag{2.2}$$

The reversiblity of P implies that

$$\begin{gathered}R^{\theta}(\theta)=R^{\theta}(-\theta),\quad R^{I}(\theta)=-R^{I}(-\theta),\quad R^{z}(\theta)=-R^{\bar{z}}(-\theta),\\R^{II}(\theta)=-R^{II}(-\theta),\quad R^{Iz}(\theta)=-R^{I\bar{z}}(-\theta),\quad R^{zI}(\theta)=-R^{\bar{z}I}(-\theta),\\R^{zz}(\theta)=-R^{\bar{z}\bar{z}}(-\theta),\quad R^{z\bar{z}}(\theta)=-R^{\bar{z}z}(-\theta).\end{gathered} \tag{2.3}$$

The normal form of R is defined as

$$\begin{aligned}[R] &= [R^{\theta}]\frac{\partial}{\partial\theta} + [\operatorname{diag} R^{zz}]z\frac{\partial}{\partial z} + [\operatorname{diag} R^{\bar z\bar z}]\bar z\frac{\partial}{\partial \bar z} \\ &= \sum_{b=1}^{n}[R^{\theta_b}]\frac{\partial}{\partial\theta_b} + \sum_{j\geq 1}[R^{z_j z_j}]z_j\frac{\partial}{\partial z_j} + \sum_{j\geq 1}[R^{\bar z_j\bar z_j}]\bar z_j\frac{\partial}{\partial \bar z_j}.\end{aligned}$$

We choose the normal correction $\hat N = [R]$. Thus (2.1) is a homological equation of F. Below we solve this homological equation and estimate the generating vector field F.

Lemma 2.1. *Suppose that uniformly on $\Pi_+ \subset \Pi$,*

$$|\langle k, \omega(\xi)\rangle + \langle l, \Omega(\xi)\rangle| \geq \alpha\frac{[l]_d}{A_k}, \tag{2.4}$$

for all $k\in\mathbb{Z}^n$, $l\in\mathbb{Z}^{+\infty}$ with $|l|\leq 2, |k|+|l|\neq 0$. Then the homological equation (2.1) has solution F and $\hat N$ that are normalized by $[F]=0, [\hat N]=\hat N$. Moreover, they satisfy $F\circ S = DS\cdot F$ and

$$\|\hat N\|^*_{r,D(s,r)} \leqslant \|R\|^*_{r,D(s,r)}, \quad \|F\|^*_{r,D(s-\sigma,r)} \leqslant \frac{cM}{\alpha^{\mathcal{N}+2}\sigma^b}\|R\|^*_{r,D(s,r)},$$

where $b=(\mathcal{N}+2)\tau+\mathcal{N}+1+\frac{n}{2}$, $M=(M_1+M_2)^{\mathcal{N}+2}$, $\|\cdot\|^* = \|\cdot\|_{C^{\mathcal{N},1}(\Pi_+)}$.

Proof. The homological equation (2.1) splits into:

$$\partial_\omega F_i^{\theta} = R_i^{\theta} - [R_i^{\theta}], \quad i=1,\dots,n; \tag{2.5}$$

$$\partial_\omega F_i^{I} = R_i^{I}, \quad i=1,\dots,n; \tag{2.6}$$

$$\partial_\omega F_{ij}^{II} = R_{ij}^{II}, \quad i,j=1,\dots,n; \tag{2.7}$$

$$\partial_\omega F_{ij}^{Iz} + \mathrm{i}\Omega_j F_{ij}^{Iz} = R_{ij}^{Iz}, \quad i=1,\dots,n, \quad j\geq 1; \tag{2.8}$$

$$\partial_\omega F_{ij}^{I\bar z} - \mathrm{i}\Omega_j F_{ij}^{I\bar z} = R_{ij}^{I\bar z}, \quad i=1,\dots,n, \quad j\geq 1; \tag{2.9}$$

$$\partial_\omega F_i^{z} - \mathrm{i}\Omega_i F_i^{z} = R_i^{z}, \quad i\geq 1; \tag{2.10}$$

$$\partial_\omega F_{ij}^{zI} - \mathrm{i}\Omega_i F_{ij}^{zI} = R_{ij}^{zI}, \quad i\geq 1, \quad j=1,\dots,n; \tag{2.11}$$

$$\partial_\omega F_{ij}^{zz} + \mathrm{i}\Omega_j F_{ij}^{zz} - \mathrm{i}\Omega_i F_{ij}^{zz} = R_{ij}^{zz} - [R_{ij}^{zz}]\delta_{ij}, \quad i,j\geq 1; \tag{2.12}$$

$$\partial_\omega F_{ij}^{z\bar z} - \mathrm{i}\Omega_i F_{ij}^{z\bar z} - \mathrm{i}\Omega_j F_{ij}^{z\bar z} = R_{ij}^{z\bar z}, \quad i,j\geq 1; \tag{2.13}$$

$$\partial_\omega F_i^{\bar z} + \mathrm{i}\Omega_i F_i^{\bar z} = R_i^{\bar z}, \quad i\geq 1; \tag{2.14}$$

$$\partial_\omega F_{ij}^{\bar zI} + \mathrm{i}\Omega_i F_{ij}^{\bar zI} = R_{ij}^{\bar zI}, \quad i\geq 1, \quad j=1,\dots,n; \tag{2.15}$$

$$\partial_\omega F_{ij}^{\bar zz} + \mathrm{i}\Omega_j F_{ij}^{\bar zz} + \mathrm{i}\Omega_i F_{ij}^{\bar zz} = R_{ij}^{\bar zz}, \quad i,j\geq 1; \tag{2.16}$$

$$\partial_\omega F_{ij}^{\bar z\bar z} - \mathrm{i}\Omega_i F_{ij}^{\bar z\bar z} + \mathrm{i}\Omega_j F_{ij}^{\bar z\bar z} = R_{ij}^{\bar z\bar z} - [R_{ij}^{\bar z\bar z}]\delta_{ij}, \quad i,j\geq 1. \tag{2.17}$$

Below we only consider the homological equations (2.10) and (2.12), the other equations can be analyzed similarly. Firstly, we consider the equation (2.10). Note that

$$\|R^z\|^*_{a,p,D(s,r)} \leq r\|R\|^*_{r,D(s,r)}.$$

Let

$$F_i^z = \sum_{k\in\mathbb{Z}^n} F_i^z(k)e^{\mathrm{i}\langle k,\theta\rangle}, \quad R_i^z = \sum_{k\in\mathbb{Z}^n} R_i^z(k)e^{\mathrm{i}\langle k,\theta\rangle},$$

where $\{F_i^z(k) \mid k \in \mathbb{Z}^n\}$ are the Fourier coefficients of F_i^z. The vector R^z is an analytic mapping from $D(s,r)$ into $l^{a,\bar{p}}$, its Fourier coefficients $R^z(k)$ satisfy the l^2-estimates (ref. [16]):

$$\sum_k \left(\|R^z(k)\|_{a,\bar{p}}^*\right)^2 \mathrm{e}^{2|k|s} \leqslant 2^n(\|R^z\|_{a,\bar{p},D(s)}^*)^2.$$

By Lemma 6.8 and the non-resonant condition (2.4), for $\xi \in \Pi_+$, we have $\bar{\omega}(\xi) = (\omega(\xi), \Omega_i(\xi)) \in O_+$, and $\frac{1}{\langle k,\omega(\xi)\rangle + \Omega_j(\xi)} = G_{\bar{k}}[\bar{\omega}(\xi)]$, where $\bar{k} = (k,1) \in \mathbb{Z}^{n+1}$. By deriving function $G_{\bar{k}}[\bar{\omega}(\xi)]$, we have

$$\|G_{\bar{k}}[\bar{\omega}(\xi)]\|^* \leqslant \frac{cA_k^{\mathcal{N}+2}|k|^{\mathcal{N}+1}M}{\alpha^{\mathcal{N}+2}}.$$

By Lemma 6.9, we have

$$\begin{aligned}
\|F_i^z(k)\|^* &\leqslant \frac{cA_k^{\mathcal{N}+2}|k|^{\mathcal{N}+1}M}{\alpha^{\mathcal{N}+2}}\|R_i^z(k)\|^*,\\
\|F^z(k)\|^* &\leqslant \frac{cA_k^{\mathcal{N}+2}|k|^{\mathcal{N}+1}M}{\alpha^{\mathcal{N}+2}}\|R^z(k)\|^*.
\end{aligned}$$

Thus

$$\|F^z\|_{a,\bar{p},D(s-\sigma)}^* \le \frac{cMB_\sigma}{\alpha^{\mathcal{N}+2}}\|R^z\|_{a,\bar{p},D(s,r)}^*,$$

where

$$B_\sigma = \Big(\sum_k A_k^{2(\mathcal{N}+2)}|k|^{2(\mathcal{N}+1)}\mathrm{e}^{-2|k|\sigma}\Big)^{1/2} \leqslant \frac{c}{\sigma^b}.$$

Then

$$\frac{1}{r}\|F^z\|_{a,\bar{p},D(s-\sigma)}^* \leqslant \frac{cM}{\alpha^{\mathcal{N}+2}\sigma^b}\|R\|_{r,D(s,r)}^*.$$

We now consider the homological equation (2.12) given by

$$\partial_\omega F_{ij}^{zz} + \mathrm{i}\Omega_j F_{ij}^{zz} - \mathrm{i}\Omega_i F_{ij}^{zz} = R_{ij}^{zz} - [R_{ij}^{zz}]\delta_{ij}, \quad i,j \ge 1.$$

Since $R^{zz} = \frac{\partial}{\partial z}R^{(z)}|_{(z,\bar{z},I)=0}$, by the Cauchy estimate, we have

$$|||R^{zz}|||_{D(s)} \le \frac{1}{r}\|R^{(z)}\|_{a,\bar{p},D(s,r)}^* \le \|R\|_{r,D(s,r)}^*,$$

where $|||\cdot|||$ denotes the operator norm from $l^{a,p} \to l^{a,\bar{p}}$.

Note that the operator R^{zz} is equivalent to the operator $\widetilde{R} = \big(v_i R_{ij}^{zz} w_j\big)$ from $l^2 \to l^2$ and $|||\widetilde{R}|||_{2,D(s)} = |||R^{zz}|||_{D(s)}$, where v_i and w_j are some weights, $|||\cdot|||_2$ is the operator norm from $l^2 \to l^2$. Thus we need only consider the norm of l^2-operator $\widetilde{R}$.

Let $\widetilde{R} = \sum_k \widetilde{R}(k)\mathrm{e}^{\mathrm{i}\langle k,\theta\rangle}$, $\quad \widetilde{F} = \sum_k \widetilde{F}(k)\mathrm{e}^{\mathrm{i}\langle k,\theta\rangle}$, we have

$$\mathrm{i}\widetilde{F}_{ij}(k) = \frac{\widetilde{R}_{ij}(k)}{\langle k,\omega\rangle + \Omega_j - \Omega_i}, |k| + |i-j| \neq 0,$$

where $\{\widetilde{F}_{ij}(k) : k \in \mathbb{Z}^n\}$ are the Fourier coefficients of the $(i,j)-$components of the $\widetilde{F}$. By Lemma 6.8 and taking $\bar{\omega} = (\omega, \Omega_i - \Omega_j) \in O_+$, by the same way as the above and combining $2|i^d - j^d| \geqslant |i-j|\,|i^{d-1} + j^{d-1}|$, we know

$$\|\frac{1}{\langle k,\omega\rangle + \Omega_i - \Omega_j}\|^* \leqslant \frac{cA_k^{\mathcal{N}+2}|k|^{\mathcal{N}+1}M}{\alpha^{\mathcal{N}+2}|i-j|}, i \neq j.$$

Again by Lemma 6.9,

$$\|\widetilde{F}_{ij}(k)\|^* \le \frac{cA_k^{\mathcal{N}+2}|k|^{\mathcal{N}+1}M}{\alpha^{\mathcal{N}+2}|i-j|}\|\widetilde{R}_{ij}(k)\|^*, i \neq j.$$

Hence, by Lemma 6.6, $|||\widetilde{F}(k)|||_2 \leqslant \frac{cA_k^{\mathcal{N}+2}|k|^{\mathcal{N}+1}M}{\alpha^{\mathcal{N}+2}}|||\widetilde{R}(k)|||_2$. Summing up for k as the above, we have

$$|||\widetilde{F}|||_{2,D(s-\alpha)} \leqslant \frac{cB_\sigma M}{\alpha^{\mathcal{N}+2}}|||\widetilde{R}|||_{2,D(s)}.$$

Going back to the original operator norm we have

$$\frac{1}{r}\|F^{zz}z\|^*_{a,\bar{p},D(s-\sigma,r)} \leqslant \frac{cM}{\alpha^{\mathcal{N}+2}\sigma^b}\|R\|^*_{r,D(s,r)}.$$

The proof is complete. □

3. KAM step

At the ν-th step of the iteration scheme, we are given an S-reversible vector field $X_\nu = N_\nu + P_\nu$, where $N_\nu = \omega_\nu\frac{\partial}{\partial\theta} + \mathrm{i}\Omega_\nu z\frac{\partial}{\partial z} - \mathrm{i}\Omega_\nu\bar{z}\frac{\partial}{\partial\bar{z}}$ is a normal S-reversible vector field, and P_ν is an S-reversible the perturbation vector field.

To simplify notation, we drop the index "ν″ and write "+″ for "$\nu+1$″. Thus, $P = P_\nu$, $P_+ = P_{\nu+1}$, and so on.

To proceed with the next step of the iteration, we assume that the perturbation vector field is sufficiently small, and we can choose $0 \le \eta \le \frac{1}{8}$ such that

$$|P|^*_{r,D(s,r)} \le \frac{\alpha\sigma^{b+1}\eta}{c_0}, \tag{3.1}$$

where c_0 is some suitably large constant depending only on n and τ.

3.1. **Approximation estimates.** We approximate P by its Taylor polynomial R of the form (2.2). This approximation extends to the corresponding components $P^{(\theta)}$, $P^{(I)}$, $P^{(z)}$, and $P^{(\bar{z})}$ of the vector field P. Since P is analytic, the components $R^{(\theta)}$, $R^{(I)}$, $R^{(z)}$, $R^{(\bar{z})}$, and their remainders are given by certain Cauchy integrals. We then obtain the estimates

$$\|R\|^*_{r,D(s,r)} \le c\|P\|^*_{r,D(s,r)}\,, \tag{3.2}$$

$$\|P-R\|^*_{\eta r,D(s,4\eta r)} \le c\eta\|P\|^*_{r,D(s,r)}\,. \tag{3.3}$$

3.2. **Solution of the homological equation.** By Lemma 2.1 and the above estimates, we obtain

$$\|N\|^*_{r,D(s,r)} \le c\|P\|^*_{r,D(s,r)}\,, \tag{3.4}$$

$$\|F\|^*_{r,D(s-\sigma,r)} \leqslant \frac{cM}{\alpha^{\mathcal{N}+2}\sigma^b}\|P\|^*_{r,D(s,r)}, \tag{3.5}$$

$$\|\mathscr{D}F\|^*_{r,r,D(s-2\sigma,\frac{r}{2})} \leqslant \frac{cM}{\alpha^{\mathcal{N}+2}\sigma^{b+1}}\|P\|^*_{r,D(s,r)}. \tag{3.6}$$

where $\mathscr{D}F$ is the differential of F with respect to $(\theta, I, z, \bar{z})$, and the operator norm $\|\cdot\|_{\bar{r},r}$ is defined by $\|L\|_{\bar{r},r} = \sup_{w\neq 0}\frac{\|Lw\|_{\bar{p},\bar{r}}}{\|w\|_{p,r}}$.

3.3. **Coordinate transformation.** From Lemma 2.1, F is defined on $D(s-\sigma,r)\times\Pi_+$. By Lemma 6.11 and Cauchy's inequality, we have th4e following lemma.

Lemma 3.1. *If $\|F\|^*_{r,D(s-\sigma,r)}\le\sigma$, then for $\xi\in\Pi_+$, the flow $\phi^t_F(\cdot,\xi)$ exists on $D(s-2\sigma,\frac{r}{2})$ for $|t|\le 1$ and it maps $D(s-\sigma,\frac{r}{2})$ into $D(s-\sigma,r)$. Moreover, for $|t|\le 1$,*

$$\|\phi^t_F-\mathrm{id}\|^*_{r,D(s-2\sigma,r/2)},\quad \sigma\|\mathscr{D}\phi^t_F-\mathrm{Id}\|^*_{r,r,D(s-3\sigma,r/4)}\leqslant c\|F\|^*_{r,D(s-\sigma,r)},\tag{3.7}$$

where id *and* Id *are identity mapping and unit operator, respectively.*

We choose $E=\frac{\|P\|^*_{r,D(s,r)}}{\alpha^{N+2}\sigma^{b+1}}$, by Lemmas 2.1 and 3.1, it follows that for $|t|\le 1$,

$$\|\Phi^t_F-\mathrm{id}\|^*_{r,D(s-2\sigma,r/2)},\quad \sigma\|\mathscr{D}\Phi^t_F-\mathrm{Id}\|^*_{r,r,D(s-3\sigma,r/4)}\leqslant cME.\tag{3.8}$$

Lemma 3.2. *If the S-reversible vector field $W(\cdot,\xi)$ on $V=D(s-4\sigma,2\eta r)$ depends on the parameter $\xi\in\Pi_+$ with $\|W\|^*_{r,V}<+\infty$, F is an S-invariant vector field, and $\Phi=\Phi^t_F:U=D(s-5\sigma,\eta r)\to V$, then $\Phi^*W=(\mathscr{D}\Phi)^{-1}W\circ\Phi$ is also an S-reversible vector field and if E is small, we have $\|\Phi^*W\|^*_{\eta r,U}\le c\|W\|^*_{\eta r,V}$.*

Proof. See [11] for the verification of the reversibility of Φ^*W. Since $\Phi^*W=(\mathscr{D}\Phi)^{-1}W\circ\Phi$, we have

$$\|\Phi^*W\|^*_{\eta r,U}\le\|(\mathscr{D}\Phi)^{-1}\|^*_{\eta r,\eta r,U}\|W\circ\Phi\|^*_{\eta r,U},$$

By (3.8) and $\eta^2=E$,

$$\frac{1}{\sigma}\|\Phi-id\|^*_{\eta r,U},\quad \|\mathscr{D}\Phi-Id\|^*_{\eta r,\eta r,U}\le 1,$$

then by Lemma 6.10 it follows that $\|W\circ\Phi\|^*_{\eta r,U}\le\|W\|^*_{\eta r,v}$. Again we have

$$\|(\mathscr{D}\Phi)^{-1}\|^*_{\eta r,\eta r,U}\le 1+\|(\mathscr{D}\Phi)^{-1}-id\|^*_{\eta r,\eta r,U}\le 2,$$

Hence $\|\Phi^*W\|^*_{\eta r,U}\le c\|W\|^*_{\eta r,v}$. □

3.4. **New vector field.** The map $\Phi=\Phi^t_F|_{t=1}$ defined above transforms X into $X_+=\Phi^*X=N_++P_+$ on $D(s-5\sigma,\eta r)$, where $N_+=N+\hat{N}$,

$$P_+=(\Phi^1_F)^*(P-R)+\int_0^1(\Phi^t_F)^*[R(t),F]dt,$$

with $R(t)=(1-t)\hat{N}+tR$. By Lemma 3.2, we have

$$\|(\Phi^1_F)^*(P-R)\|^*_{\eta r,D(s-5\sigma,\eta r)}\le\|P-R\|^*_{\eta r,D(s-4\sigma,2\eta r)},$$
$$\|(\Phi^1_F)^*[R(t),F]\|^*_{\eta r,D(s-5\sigma,\eta r)}\le\|[R(t),F]\|^*_{\eta r,D(s-4\sigma,2\eta r)}.$$

In (3.3), we had already estimated $\|P-R\|^*_{\eta r,D(s,4\eta r)}$, so it remains to consider the Lie bracket $[R(t),F]$.

Using the Cauchy estimate we obtain

$$\begin{aligned}\|[R(t),F]\|^*_{\eta r,D(s-4s,2\eta r)}&\leqslant\frac{c}{\eta}\|\mathscr{D}R(t)\cdot F+\mathscr{D}F\cdot R(t)\|^*_{r,D(s-4\sigma,2\eta r)}\\&\leqslant\frac{c}{\eta\sigma}\|R(t)\|^*_{r,D(s,r)}\cdot\|F\|^*_{r,D(s-\sigma,r)}.\end{aligned}$$

Then we arrive at the estimate

$$\|P_+\|^*_{\eta r,D(s-5\sigma,\eta r)}\le c\eta\|P\|^*_{r,D(s,r)}+cM\eta\|P\|^*_{r,D(s,r)}\le cM\eta\|P\|^*_{r,D(s,r)}.\tag{3.9}$$

3.5. **New normal form.** We consider

$$N_+ = N + \hat{N} = w_+ \frac{\partial}{\partial\theta} + \mathrm{i}\Omega_+ z \frac{\partial}{\partial z} - \mathrm{i}\Omega_+ \bar{z} \frac{\partial}{\partial \bar{z}}$$

with $\|\hat{N}\|^*_{r,D(s,r)} \le \|P\|^*_{r,D(s,r)}$, where $\omega_+ = \omega + \hat{\omega}$, $\Omega_+ = \Omega + \hat{\Omega}$. This implies $|\hat{\omega}|^*_D \le \|P\|^*_{r,D(s,r)}, \|\hat{\Omega}\|^*_{-\delta,D(s,r)} \le \|P\|^*_{r,D(s,r)}$, where $\|\hat{\Omega}\|^*_{-\delta} = \sup_{j\ge1} j^{-\delta}\|\hat{\Omega}_j\|^*$. With $-\delta \le \bar{p} - p$, we have

$$|\hat{\omega}|^* + \|\hat{\Omega}\|^*_{-\delta,D(s,r)} \le \|P\|^*_{r,D(s,r)}. \tag{3.10}$$

For $|k| \le K$, we observe that $|l|_\delta \le |l|_{d-1} \le [l]_d$, hence

$$|\langle k, \hat{\omega}\rangle + \langle l, \hat{\Omega}\rangle \le |k| \cdot |\hat{\omega}| + |l|_\delta \cdot |\hat{\Omega}|_{-\delta} \le c|k|A_k \frac{[l]_d}{A_k}\|P\|^*_r \le c\hat{\alpha}\frac{[l]_d}{A_k}.$$

with $\hat{\alpha} \ge c|k|A_k\|P\|^*_r$. Using the bound for the old divisors, the new ones then satisfy

$$|\langle k, \omega_+(\xi)\rangle + \langle l, \Omega_+(\xi)\rangle| \ge \alpha_+ \frac{[l]_d}{A_k}, \quad |k| \le K, \tag{3.11}$$

on Π with $\alpha_+ = \alpha - \hat{\alpha}$.

4. Iteration and convergence

In Section 3, we present a detailed process of one cycle of the KAM iteration. Now, we choose a sequence of relevant parameters that allow us to perform infinitely many iterations of the KAM step. For $\nu \ge 0$, define

$$\alpha_\nu = \frac{\alpha_0}{\nu^{\frac{2}{\nu}}}, \quad \sigma_{\nu+1} = \frac{\sigma_\nu}{2},$$

$$\epsilon_{\nu+1} = \frac{c_1 M_\nu \epsilon_\nu^{\frac{3}{2}}}{(\alpha_\nu^{\mathcal{N}+2}\sigma_\nu^{b+1})^{1/2}}, \quad M_\nu = (M_1 + M_2 + c_1(\epsilon_1 + \dots + \epsilon_{\nu-1}))^{\mathcal{N}+1},$$

$$\lambda_\nu = \frac{\alpha_\nu}{M_\nu}, \quad \eta_\nu^2 = \frac{\epsilon_\nu}{\alpha_\nu^{\mathcal{N}+2}\sigma_\nu^{b+1}}, \quad E_\nu = \frac{\epsilon_\nu}{\alpha_\nu^{\mathcal{N}+2}\sigma_\nu^{b+1}}.$$

Furthermore, $s_{\nu+1} = s_\nu - 5\sigma_\nu$, $r_{\nu+1} = \eta_v r_\nu$, and $D_\nu = D(s_\nu, r_\nu)$. As initial value fix $\sigma_0 = \frac{s_0}{40} \le \frac{1}{4}$ so that $s_0 > s_1 > \dots \ge \frac{s_0}{2}$, and assume that

$$\epsilon_0 \le \gamma_0 \alpha_0^{\mathcal{N}+2}\sigma_0^{a+1}, \quad \gamma_0 \le (c_1 + c_2)^{-2},$$

where c_1 is twice as large as the constants that appear in Section 3, $c_2 \triangleq c \cdot \hat{M} \ge c_1 M_\nu$, $\hat{M} = (M_1 + M_2 + c_1)^{\mathcal{N}+1}$. Let $A_k = (1 + |k|)^\tau$, $(1 + k_0)^{\tau+1} = \frac{1}{c_1\gamma_0}$ and $1 + K_\nu = (1 + K_0) \cdot 2^\nu$.

Lemma 4.1 (Iterative lemma)**.** *Assume the reversible vector field $X_\nu = N_\nu + P_\nu$ is regular on $D_\nu \times \Pi_\nu$, where N_ν is a normal form with coefficients satisfying $|\omega_\nu|^* + \|\Omega_\nu\|^*_{-\delta} \le M_\nu$,*

$$|\langle k, \omega_\nu(\xi)\rangle + \langle l, \Omega_\nu(\xi)\rangle| \ge \alpha_\nu \frac{[l]_d}{A_k}, \quad 0 \ne |k| \le K, |l| \le 2, |l|_k \le L,$$

on Π_ν. P_ν satisfies

$$\|P_\nu\|^*_{r_\nu,D_\nu} \le \epsilon_\nu.$$

Then there exists a family of real analytic, S-invariant transformations $\Phi_{\nu+1} : D_{\nu+1} \times \Pi_{\nu+1} \to D_\nu$, and a closed subset

$$\Pi_{\nu+1} = \Pi_\nu \setminus \cup_{|k|\ge K} R^{\nu+1}_{k,l}(\alpha_{\nu+1}),$$

where

$$R_{k,l}^{\nu+1}(\alpha_{\nu+1}) = \{\xi \in \Pi_\nu | |\langle k, \omega_{\nu+1}\rangle + \langle l, \Omega_\nu\rangle| < \alpha_{\nu+1}\frac{[l]_d}{A_{k_\nu}}\}$$

such that for $X_{\nu+1} = (\Phi_{\nu+1})^* X_\nu = N_{\nu+1} + P_{\nu+1}$, *the estimates*

$$|\omega_{\nu+1} - \omega_\nu|^* \le c\epsilon_\nu, \quad \|\Omega_{\nu+1} - \Omega_\nu\|_{-\delta}^* \le c\epsilon_\nu.$$

hold and the same assumptions as above are satisfied with '$\nu+1$' in place of 'ν'.

Proof. By induction one verifies that $\epsilon_\nu \le \gamma_0 \alpha_\nu^{\mathcal{N}+2}\sigma_\nu^{b+1}/2^{2b+2+2v+\frac{4(\mathcal{N}+2)}{\nu}}$ for all $\nu \ge 0$. With the definition of η_ν this implies $\epsilon_\nu \le \alpha_\nu^{\mathcal{N}+2}\sigma_\nu^{b+1}\eta_\nu/c_1$. So the smallness condition (3.1) of the KAM step is satisfied, and there exists an S-invariant transformation $\Phi_{\nu+1} : D_{\nu+1} \times \Pi_{\nu+1} \to D_\nu$ taking X_ν into $X_{\nu+1} = N_{\nu+1} + P_{\nu+1}$. The new error satisfies the estimate

$$\|P_{\nu+1}\|_{r_{\nu+1}, D_{\nu+1}}^* \le cM_\nu \eta_\nu \epsilon_\nu = \frac{cM_\nu \epsilon_\nu^{\frac{3}{2}}}{(\alpha_\nu^{\mathcal{N}+2}\sigma_\nu^{b+1})^{1/2}} = \epsilon_{\nu+1}.$$

One verifies that $c_1\epsilon_\nu \le c_1\gamma_0\alpha_\nu^{\mathcal{N}+2}\sigma_\nu^{b+1} \le \alpha_\nu - \alpha_{\nu-1}/K_\nu \cdot A_{K_\nu}$, hence

$$ckA_{k_\nu}\|P_\nu\|_{r_\nu}^* \le \alpha_\nu - \alpha_{\nu-1}.$$

So by (3.11) the small divisor estimates hold for the new frequencies with parameter $\alpha_{\nu+1}$ up to $|k| \le K_\nu$. Removing from Π_ν the union of resonance zones $R_{k,l}^{\nu+1}(\alpha_{\nu+1})$ for $|k| \ge K_\nu$ we obtain the parameter domain $\Pi_{\nu+1} \subset \Pi_\nu$ with the desired properties. □

With (3.7), (3.8) and (3.10), we also obtain the following estimates.

Lemma 4.2. *For* $\nu \ge 0$,

$$\frac{1}{\sigma_\nu}\|\Phi_\nu - id\|_{r_\nu, D_{\nu+1}}^*, \|\mathscr{D}\Phi - Id\|_{r_\nu, r_\nu, D_{\nu+1}}^* \le \frac{c\epsilon_\nu}{\alpha_\nu^{\mathcal{N}+2}\sigma_\nu^{b+1}}, \tag{4.1}$$

$$|\omega_{\nu+1} - \omega_\nu|^*, \quad \|\Omega_{\nu+1} - \Omega_\nu\|_{-\delta}^* \le c_1\epsilon_\nu, \tag{4.2}$$

$$|\omega_\nu - \omega|^*, \quad \|\Omega_\nu - \Omega\|_{-\delta}^* \le c_1(\epsilon_1 + \dots + \epsilon_\nu). \tag{4.3}$$

Proof of Theorem 1.1. Suppose the assumption of Theorem 1.1 are satisfied. To apply the Lemma 4.1 with $\nu = 0$, set $N_0 = N$, $P_0 = P$, $s_0 = s$, $r_0 = r$, $E_0 = E$, $\alpha_0 = \alpha$, and $\gamma = \gamma_0\alpha_0^{\mathcal{N}+1}$. The smallness condition is satisfied because

$$\|P_0\|_{r_0, D(s_0, r_0)}^* = \|P\|_{r, D(s,r)}^* \le \gamma\alpha = \gamma_0\sigma_0^{a+1}\alpha_0^{\mathcal{N}+1} = \epsilon_0.$$

The small divisor conditions are satisfied by setting $\Pi_0 = \Pi \setminus \cup_{k,l} R_{k,l}^0(\alpha_0)$. Then the Iterative Lemma can be applied, and we can obtain a decreasing sequence of domains $D_\nu \times \Pi_\nu$ and S-invariant transformations $\{\Phi^\nu = \Phi_1 \circ \dots \circ \Phi_\nu\}$ for $\nu \ge 1$, such that $(\Phi^\nu)^* X = N_\nu + P_\nu$. Moreover, the estimates in Lemma 4.2 hold.

To prove the convergence of $\{\Phi^\nu\}$, we note that the operator norm $\|\cdot\|_{r,s}$ satisfies $\|A \cdot B\|_{r,s} \le \|A\|_{r,r} \cdot \|B\|_{s,s}$ for $r \ge s$. We thus obtain

$$\|\Phi^{\nu+1} - \Phi^\nu\|_{r_1, D_{\nu+1}}^* \le \|\mathscr{D}\Phi^\nu\|_{r_1, r_\nu, D_\nu}^* \cdot \|\Phi_{\nu+1} - id\|_{r_\nu, D_{\nu+1}}^*, \tag{4.4}$$

and

$$\begin{aligned}\|\mathscr{D}\Phi^\nu\|_{r_1, r_\nu, D_{\nu+1}}^* &\le \|\mathscr{D}\Phi_1\|_{r_1, r_1, D_2}^* \cdots \|\mathscr{D}\Phi_\nu\|_{r_\nu, r_\nu, D_{\nu+1}}^* \\ &\le \prod_{j=1}^{\nu}\|\mathscr{D}\Phi_j\|_{r_j, r_j, D_{j+1}}^* \le \prod_{j=1}^{\nu}(1 + 2^{-j}) < +\infty,\end{aligned}$$

for all $\nu \geq 0$. Then we have

$$\|\Phi^{\nu+1} - \Phi^{\nu}\|^*_{r_1, D_{\nu+1}} \leq c\|\Phi_{\nu+1} - id\|^*_{r_\nu, D_{\nu+1}},$$

So $\{\Phi^\nu\}$ is convergent on $D_* \times \Pi_* = \bigcap_{\nu \geq 1} D_\nu \times \Pi_\nu$.

Let $\lim_{\nu\to\infty} \Phi^\nu = \Phi_*$ and $(\Phi_*)^* X = N_* + P_*$, where $N_* = w_* \frac{\partial}{\partial\theta} + \mathrm{i}\Omega_* z \frac{\partial}{\partial z} - \mathrm{i}\Omega_* \bar{z} \frac{\partial}{\partial \bar{z}}$. Since $\omega_* = \lim_{\nu\to\infty} \omega_\nu$ by (4.3) it follows that $|\omega_* - \omega|^* \leq c\epsilon$. Since $\|P_\nu\|^*_{r_\nu, D_\nu} \leq \epsilon_\nu$ and $\lim_{\nu\to\infty} \|P_\nu - P_*\|^*_{r_\nu, D_\nu} = 0$, it follows that $P_* = 0$ on $D_* \times \Pi_*$. By (4.4) it follows that $\|\Phi_* - id\|^* \leq c\epsilon$.

Now we estimate the measure of Π_*. By Theorem 1.2,

$$\mathrm{meas}(\Pi - \Pi_\nu) \leq c(\mathrm{diam}(Pi)^{n-1} \frac{\alpha^\nu}{\nu^2},$$

so $\mathrm{meas}(\Pi - \Pi_*) \leq \sum_{\nu=1}^{\infty} \mathrm{meas}(\Pi - \Pi_\nu) \leq c \cdot (\mathrm{diam}(Pi)^{n-1} \alpha^\nu$. This implies that if α is sufficiently small, Π_* is nonempty. □

5. Application to the beam equation

5.1. **Reversible seting of beam equation.** We consider the beam equation

$$u_{tt} + u_{xxxx} + \bar{m}u + uu_t^2 = 0, \quad \bar{m} > 0,\ x \in [0, \pi], \tag{5.1}$$

with Navier boundary condition

$$\begin{gathered} u(t, 0) = 0 = u(t, \pi), \\ u_{xx}(t, 0) = 0 = u_{xx}(t, \pi). \end{gathered} \tag{5.2}$$

We denote the operator D as $Du = (\partial_{xxxx} + \bar{m})^{1/2} u$ and introduce $v = u_t$, then (5.1) takes the form

$$\begin{gathered} u_t = v, \\ v_t = -D^2 u - uu_t^2. \end{gathered} \tag{5.3}$$

We set

$$\begin{gathered} w = \frac{1}{\sqrt{2}}(Du - \mathrm{i}v), \\ \bar{w} = \frac{1}{\sqrt{2}}(Du + \mathrm{i}v). \end{gathered} \tag{5.4}$$

We then obtain

$$\begin{gathered} w_t = \mathrm{i}Dw - \frac{\mathrm{i}}{4} D^{-1}(w + \bar{w}) \cdot (w - \bar{w})^2, \\ \bar{w}_t = -\mathrm{i}Dw + \frac{\mathrm{i}}{4} D^{-1}(w + \bar{w}) \cdot (w - \bar{w})^2. \end{gathered} \tag{5.5}$$

The eigenvalues of the operator D are $\lambda_j = \sqrt{j^4 + \bar{m}}$, $j \geq 1$, and the corresponding eigenfunctions are $\phi_j(x) = \sqrt{\frac{2}{\pi}} \sin(jx)$, which form a complete orthonormal basis in $L^2[0, \pi]$. Let $w = \sum_{j\geq 1} q_j \phi_j(x)$ and $\bar{w} = \sum_{j\geq 1} q_j \phi_j(x)$. System (5.5) is then transformed into the lattice form

$$\begin{gathered} \dot{q}_j = \mathrm{i}\lambda_j q_j + Q^{(q_j)}, \\ \dot{\bar{q}}_j = -\mathrm{i}\lambda_j \bar{q}_j + Q^{(\bar{q}_j)}, \end{gathered} \tag{5.6}$$

where

$$Q^{(q_j^\sigma)} = \sum_{l,m,n\geq 1} \sum_{\alpha,\beta,\gamma \in \{+,-\}} Q^{(q_j^\sigma)}_{\alpha\beta\gamma, lmn} q_l^\alpha q_m^\beta q_n^\gamma, \tag{5.7}$$

$$Q^{(q_j^\sigma)}_{\alpha\beta\gamma,lmn} = -\frac{\mathrm{i}\sigma}{4}\lambda_l^{-1} a_{lmnj}, \quad \sigma \in \{+,-\}, \tag{5.8}$$

$$a_{lmnj} = \int_0^\pi \phi_l(x)\phi_m(x)\phi_n(x)\phi_j(x)dx = \begin{cases} \text{constant}, & \pm l \pm m \pm n = j, \\ 0, & \text{otherwise}. \end{cases} \tag{5.9}$$

In particular,

$$a_{lljj} = \begin{cases} \frac{3}{2\pi}, & \text{if } l = j, \\ \frac{1}{\pi}, & \text{if } l \neq j. \end{cases}$$

The system (5.6) can be reinterpreted as a reversible context with respect to the involution $S_0(q,\bar{q}) = (\bar{q}, q)$. To this system we associate the S_0-reversible vector field

$$\bar{X}(q,\bar{q}) = \Lambda(q,\bar{q}) + Q(q,\bar{q}) \tag{5.10}$$

where

$$\Lambda = \sum_{\sigma=\pm} \Lambda^{(q^\sigma)}(q,\bar{q})\frac{\partial}{\partial q^\sigma}, \quad Q = \sum_{\sigma=\pm} Q^{(q^\sigma)}(q,\bar{q})\frac{\partial}{\partial q^\sigma}$$

with

$$\Lambda^{(q^\sigma)}(q,\bar{q}) = (\Lambda^{(q_j^\sigma)})_{j\geq 1} = (-\sigma \mathrm{i}\lambda_j q_j^\sigma)_{j\geq 1},$$
$$Q^{(q^\sigma)}(q,\bar{q}) = (Q^{(q_j^\sigma)}(q,\bar{q}))_{j\geq 1}.$$

Now we establish the regularity of the nonlinear reversible vector field Q. To this end, let l_b^2 and L^2 be the Hilbert spaces of all bi-infinite, square summable sequences with complex coefficients and all square-integrable complex valued functions on $[-\pi,\pi]$, respectively. Define the inverse discrete Fourier transform $\mathcal{F} : l_b^2 \to L^2$ by

$$\mathcal{F}q = \frac{1}{\sqrt{2\pi}}\sum_{j\in\mathbb{Z}} q_j e^{ijx}, \quad q = (q_j)_{j\in\mathbb{Z}} \in l_b^2.$$

This defines an isometry between the two spaces.

For $a \geq 0$ and $p \geq 0$, define the subspaces $l_b^{a,p} \subset l_b^2$ consisting of all bi-infinite sequences $q = (q_j)_{j\in\mathbb{Z}}$ with norm

$$\|q\|_{a,p}^2 = |q_0|^2 + \sum_{j\neq 0} |q_j|^2 |j|^{2p} e^{2|j|a}.$$

These subspaces induce subspaces $W^{a,p} \subset L^2$ under $\mathcal{F}$, normed by $\|\mathcal{F}q\|_{a,p} = \|q\|_{a,p}$.

Lemma 5.1. *For $a \geq 0, p \geq 0$, the vector field Q is an analytic map from a neighborhood of the origin of $l_2^{a,p} \times l_2^{a,p}$ into $l_2^{a,p} \times l_2^{a,p}$, with*

$$|||Q|||_{a,p} = O(\|q\|_{a,p}^3),$$

where the norm $|||Q|||_{a,p} = \|Q^{(q)}\|_{a,p} + \|Q^{(\bar{q})}\|_{a,p}$.

Proof. Let $q \in l_2^{a,p}$. Considered as the functions on $[-\pi,\pi]$, w and $\bar{w}$ are in $W^{a,p}$ with $\|w\|_{a,p} \leq \|q\|_{a,p}, \|\bar{w}\|_{a,p} \leq \|\bar{q}\|_{a,p}$. $D^{-1}w$ and $D^{-1}\bar{w}$ are in $W^{a,p+2}$ with $\|D^{-1}w\|_{a,p+2} \leq \|q\|_{a,p}, \|D^{-1}\bar{w}\|_{a,p+2} \leq \|\bar{q}\|_{a,p}$, for $\bar{m} > 0$. We denote

$$f(w,\bar{w}) = -\frac{\mathrm{i}}{4}D^{-1}(w+\bar{w})\cdot(w-\bar{w})^2.$$

By the algebraic property and the analyticity of $f(w,\bar{w})$, the function $f(w,\bar{w})$ also belongs to $W^{a,p}$ with

$$\begin{aligned}
\|f(w,\bar{w})\|_{a,p} &= \| -\frac{\mathrm{i}}{4} D^{-1}(w+\bar{w})\cdot(w-\bar{w})^2\|_{a,p} \\
&\le c(\|D^{-1}w\cdot w^2\|_{a,p} + 2\|D^{-1}w\cdot w\bar{w}\|_{a,p} + \|D^{-1}w\cdot \bar{w}^2\|_{a,p} \\
&\quad + \|D^{-1}\bar{w}\cdot w^2\|_{a,p} + 2\|D^{-1}\bar{w}\cdot w\bar{w}\|_{a,p} + \|D^{-1}\bar{w}\cdot \bar{w}^2\|_{a,p}) \\
&\le c(\|D^{-1}w\|_{a,p+2}\|w\|_{a,p}\|w\|_{a,p} + 2\|D^{-1}w\|_{a,p+2}\|w\|_{a,p}\|\bar{w}\|_{a,p} \\
&\quad + \|D^{-1}w\|_{a,p+2}\|\bar{w}\|_{a,p}\|\bar{w}\|_{a,p} + \|D^{-1}\bar{w}\|_{a,p+2}\|w\|_{a,p}\|w\|_{a,p} \\
&\quad + 2\|D^{-1}\bar{w}\|_{a,p+2}\|w\|_{a,p}\|\bar{w}\|_{a,p} + \|D^{-1}\bar{w}\|_{a,p+2}\|\bar{w}\|_{a,p}\|\bar{w}\|_{a,p}) \\
&\le c\|q\|^3_{a,p},
\end{aligned}$$

in a sufficiently small neighborhood of the origin. By $Q^{(q_j)} = \int_0^\pi f(w,\bar{w})\phi_j(x)dx$ the components of the Fourier sine coefficients of $f(w,\bar{w})$. Therefore $Q^{(q)}$ belongs to $l_2^{a,p}$ with

$$\|Q^{(q)}\|_{a,p} = \|f(w,\bar{w})\|_{a,p} \le c\|q\|^3_{a,p},$$

i.e., $|||Q|||_{a,p} = O(\|q\|^3_{a,p})$. □

5.2. **Birkhoff normal form.** In this section, to apply the KAM theorem, we transform the reversible vector field $\bar{X}$ into a Birkhoff normal form. Before proving the proposition, we give the following lemma.

Lemma 5.2 ([10, Lemma 4.1])**.** *If $l,m,n,j \in \mathbb{Z}_0$ satisfy $\alpha l + \beta m + \gamma n - \sigma j = 0$ and*

$$((l,\alpha),(m,\beta),(n,\gamma),(j,-\sigma)) \ne ((p,+),(p,-),(q,+),(q,-)),$$

then

$$|\alpha\lambda_l + \beta\lambda_m + \gamma\lambda_n - \sigma\lambda_j| \ge \frac{1}{\sqrt{(h^4+\bar{m})^3}},$$

with $h = min\{l,m,n,j\}$.

Proposition 5.3. *There exists a real analytic, S_0-invariant transformation Γ in a neighborhood of the origin of $l_2^{a,p}\times l_2^{a,p}$, which transforms the S_0-reversible vector field $\bar{X}$ into*

$$\Gamma^*\bar{X} = (D\Gamma)^{-1}\bar{X}\circ\Gamma = \Lambda + \bar{Q} + \hat{Q} + K$$

such that the corresponding fields $\bar{Q},\hat{Q}$ and K are analytic maps from a neighborhood of the origin of $l_2^{a,p}\times l_2^{a,p}$ into $l_2^{a,p}\times l_2^{a,p}$ where

$$|||K|||_{a,p} = O(\|q\|^5_{a,p}).$$

Proof. We denote $A_{\bar{a},j} = \{(l,m,n)\in\mathbb{N}^3 : min\{l,m,n\}\le \bar{a}\}$, $B_{\bar{a},\sigma j} = \{(l,m,n)\in A_{\bar{a},j} : ((l,\alpha),(m,\beta),(n,\gamma)) = ((p,+),(p,-),(q,\sigma))$. We write $Q = \bar{Q}+\tilde{Q}+\hat{Q}$ where

$$\bar{Q} = \sum_{\sigma=\pm,j}\ \sum_{(l,m,m)\in B_{\bar{a},\sigma j},\alpha,\beta,\gamma} Q^{(q_j^\sigma)}_{\alpha\beta\gamma,lmn} q_l^\alpha q_m^\beta q_n^\gamma \frac{\partial}{\partial q_j^\sigma},$$

$$\tilde{Q} = \sum_{\sigma=\pm,j}\ \sum_{(l,m,m)\in A_{\bar{a},j}\setminus B_{\bar{a},\sigma j},\alpha,\beta,\gamma} Q^{(q_j^\sigma)}_{\alpha\beta\gamma,lmn} q_l^\alpha q_m^\beta q_n^\gamma \frac{\partial}{\partial q_j^\sigma},$$

$$\hat{Q}=\sum_{\sigma=\pm,j}\sum_{(l,m,m)\in A^c_{\bar{a},j},\alpha,\beta,\gamma}Q^{(q_j^\sigma)}_{\alpha\beta\gamma,lmn}q_l^\alpha q_m^\beta q_n^\gamma\frac{\partial}{\partial q_j^\sigma}.$$

Then the reversible vector field $\bar{X}$ becomes

$$\bar{X}=\Lambda+\bar{Q}+\tilde{Q}+\hat{Q}. \tag{5.11}$$

Let $\Gamma=\phi_F^1$ be the time−1-map of the flow generated by the vector field

$$F=\sum_{\sigma=\pm}\sum_{l,m,m,j\geq 1,\alpha,\beta,\gamma}F^{(q_j^\sigma)}_{\alpha\beta\gamma,lmn}q_l^\alpha q_m^\beta q_n^\gamma\frac{\partial}{\partial q_j^\sigma}$$

with coefficients

$$F^{(q_j^\sigma)}_{\alpha\beta\gamma,lmn}=\begin{cases}\frac{\mathrm{i}Q^{(q_j^\sigma)}_{\alpha\beta\gamma,lmn}}{\alpha\lambda_l+\beta\lambda_m+\gamma\lambda_n-\sigma\lambda_j}, & \text{if } (l,m,n,\alpha,\beta,\gamma)\in A_{\bar{a},j}\setminus B_{\bar{a},\sigma j},\\ 0, & \text{otherwise.}\end{cases}$$

Note that F is well-defined because of Lemma 5.2.

By the Taylor series expansion of $(\phi_F^t)^*X$ at $t=0$, we have

$$\begin{aligned}\Gamma^*X&=(\phi_F^1)^*X\\&=\Lambda+[\Lambda,F]+Q+\int_0^1(1-t)\left(\phi_F^t\right)^*[[\Lambda,F],F]\,dt+\int_0^1\left(\phi_F^t\right)^*[Q,F]\,dt.\end{aligned}$$

The last line constitutes the higher order term K.

By a direct calculation, we have

$$[\Lambda,q_l^\alpha q_m^\beta q_n^\gamma\frac{\partial}{\partial q_j^\sigma}]=\mathrm{i}(\alpha\lambda_l+\beta\lambda_m+\gamma\lambda_n-\sigma\lambda_j)q_l^\alpha q_m^\beta q_n^\gamma\frac{\partial}{\partial q_j^\sigma}.$$

It follows that

$$\begin{aligned}&[\Lambda,F]+Q\\&=\sum_{\sigma=\pm}\sum_{l,m,n,j\geq 1,\alpha,\beta,\gamma}[\sigma\mathrm{i}(\alpha\lambda_l+\beta\lambda_m+\gamma\lambda_n-\sigma\lambda_j)F^{(q_j^\sigma)}_{\alpha\beta\gamma,lmn}+Q^{(q_j^\sigma)}_{\alpha\beta\gamma,lmn}]q_l^\alpha q_m^\beta q_n^\gamma\frac{\partial}{\partial q_j^\sigma}\\&=\bar{Q}+\hat{Q}.\end{aligned}$$

Hence formally we have $\Gamma^*\bar{X}=\Lambda+\bar{Q}+\hat{Q}+K$ as claimed.

Now we investigate the analyticity of Γ. We first prove that F is an analytic vector field on $l_2^{a,p}\times l_2^{a,p}$ of the order three at the origin. Indeed, from Lemma 5.2 we have

$$\begin{aligned}|F^{(q_j)}|&=|\sum_{l,m,n\geq 1}\sum_{\alpha,\beta,\gamma}F^{(q_j)}_{\alpha\beta\gamma,lmn}q_l^\alpha q_m^\beta q_n^\gamma|\\&\leq\sum_{\pm l\pm m\pm n=j}\sum_{\alpha,\beta,\gamma}|F^{(q_j)}_{\alpha\beta\gamma,lmn}||q_l^\alpha||q_m^\beta||q_n^\gamma|\\&\leq c_1\sum_{\pm l\pm m\pm n=j}\sum_{\alpha,\beta,\gamma}|q_l^\alpha||q_m^\beta||q_n^\gamma|.\end{aligned}$$

Let $w_{\sigma j}=(|q_j^\sigma|), j\geq 1,\sigma=\pm$, we have

$$\sum_{\pm l\pm m\pm n=j}\sum_{\alpha,\beta,\gamma}|q_l^\alpha||q_m^\beta||q_n^\gamma|=\sum_{\pm l\pm m\pm n=j}\sum_{\alpha,\beta,\gamma}|w_{\alpha l}||w_{\beta m}||w_{\gamma n}|$$

$$= \sum_{\pm l \pm m \pm n = j, l,m,n \neq 0} |w_l||w_m||w_n|.$$

Let $\tilde{w_j} = |w_j| + |w_{-j}|, j \neq 0$, and for $\forall j, \tilde{w_j} \geq |w_j|$. Then we prove that

$$\sum_{\pm l \pm m \pm n = j, l,m,n \neq 0} |w_l||w_m||w_n| \leq c_2(\tilde{w} * \tilde{w} * \tilde{w})_j.$$

In fact, for each combination of $\pm l \pm m \pm n = j$, for instance, considering the sum

$$\sum_{l-m-n=j} |w_l||w_m||w_n|.$$

Let $l' = l$, $m' = -m$, $n' = n$. Then

$$\begin{gathered}
|w_l| = |w_{l'}| \leq \tilde{w}_{l'}, \\
|w_m| = |w_{-m'}| \leq \tilde{w}_{m'}, \\
|w_n| = |w_{-n'}| \leq \tilde{w}_{n'}, \\
\sum_{l-m-n=j} |w_l||w_m||w_n| = \sum_{l'+m'+n'=j} |w_{l'}||w_{-m'}||w_{-n'}| \leq \sum_{l'+m'+n'=j} \tilde{w}_{l'}\tilde{w}_{m'}\tilde{w}_{n'}.
\end{gathered}$$

By the definition of discrete convolution,

$$\sum_{l'+m'+n'=j} \tilde{w}_{l'}\tilde{w}_{m'}\tilde{w}_{n'} = (\tilde{w} * \tilde{w} * \tilde{w})_j.$$

Therefore, we obtain

$$\begin{aligned}
\|F^{(q)}\|_{a,p} &= \sqrt{\sum_{j\geq 1} e^{2aj} j^{2p} |F^{(q_j)}|^2} \\
&\leq \sqrt{\sum_{j\neq 0} e^{2aj} j^{2p} c_1^2 c_2^2 |(\tilde{w} * \tilde{w} * \tilde{w})_j|^2} \\
&\leq c_1 c_2 \|\tilde{w} * \tilde{w} * \tilde{w}\|_{a,p} \\
&\leq c_1 c_2 \|\tilde{w}\|_{a,p}^3 \\
&\leq c\|q\|_{a,p}^3,
\end{aligned}$$

where c depends on h and $\bar{m}$. The analyticity of F follows from the analyticity of each component function and its local boundedness. The estimate for K can get from $\|F^{(q)}\|_{a,p} = O(\|q\|_{a,p}^3)$ and $\|Q^{(q)}\|_{a,p} = O(\|q\|_{a,p}^3)$.

By straightforward calculations, one can verify that $F \circ S_0 = DS_0 \cdot F$. So Γ is S_0-invariant. This completes the proof. □

5.3. **Applying the KAM Theorem.** Given an index set $J = \{j_1 \leq j_2 \leq \dots \leq j_n\} \subseteq \mathbb{N}$, and denote $\mathbb{N}_1 = \mathbb{N} \backslash J$. We introduce the action-angle coordinates (θ, I) and normal coordinates $(z, \bar{z})$ by the transformation Ψ:

$$\begin{gathered}
q_{j_b}^{\pm} = \sqrt{I_b + \xi_b} e^{\pm \mathrm{i}\theta_b}, \quad b = 1, 2, \dots, n, \\
q_j^{\pm} = z_j^{\pm}, \quad j \in \mathbb{N}_1,
\end{gathered}$$

depending on parameters $\xi = (\xi_1, \dots, \xi_n) \in \mathbb{R}_+^n$. We obtain a new vector field

$$X = \Psi^*(\Gamma^* \bar{X}) = N + P \tag{5.12}$$

where

$$N = \omega(\xi)\frac{\partial}{\partial\theta} + \mathrm{i}\Omega(\xi)z\frac{\partial}{\partial z} - \mathrm{i}\Omega(\xi)\bar{z}\frac{\partial}{\partial\bar{z}}, \tag{5.13}$$

$$P = \sum_{w\in\{\theta,I,z,\bar{z}\}} P^w(\theta,I,z,\bar{z};\xi)\frac{\partial}{\partial w} \tag{5.14}$$

with

$$\omega_b = \lambda_{j_b} - \frac{1}{4}\sum_{k=1}^n \lambda_{j_k}^{-1} a_{j_kj_kj_bj_b}\xi_k, \quad \Omega_j = \lambda_j - \frac{1}{4}\sum_{k=1}^n \lambda_{j_k}^{-1} a_{j_kj_kjj}\xi_k \tag{5.15}$$

$$\begin{aligned} P^{(\theta_b)} &= -\frac{1}{4}\sum_{k=1}^n \lambda_{j_k}^{-1} a_{j_kj_kj_bj_b} I_k - \frac{1}{4}\sum_{l\in\mathbb{N}_1}\lambda_l^{-1} a_{llj_bj_b}|z_l|^2 \\ &\quad + \frac{(\hat{Q}^{(q_{j_b})} + K^{(q_{j_b})})\circ\Psi}{2\mathrm{i}q_{j_b}} - \frac{(\hat{Q}^{(\bar{q}_{j_b})} + K^{(\bar{q}_{j_b})})\circ\Psi}{2\mathrm{i}\bar{q}_{j_b}}, \end{aligned} \tag{5.16}$$

$$P^{(I_b)} = (\hat{Q}^{(q_{j_b})} + K^{(q_{j_b})})\circ\Psi\bar{q}_{j_b} + (\hat{Q}^{(\bar{q}_{j_b})} + K^{(\bar{q}_{j_b})})\circ\Psi q_{j_b}, \tag{5.17}$$

$$\begin{aligned} P^{(z_j^\sigma)} &= -\frac{\sigma\mathrm{i}}{4}\Big(\sum_{k=1}^n \lambda_{j_k}^{-1} a_{j_kj_kjj} I_k z_j^\sigma + \sum_{l\in\mathbb{N}_1}\lambda_l^{-1} a_{lljj}|z_l|^2 z_j^\sigma\Big) \\ &\quad + (\hat{Q}^{(q_j^\sigma)} + K^{(q_j^\sigma)})\circ\Psi. \end{aligned} \tag{5.18}$$

Lemma 5.4. *The vector field X is reversible with respect to the involution*

$$S(\theta,I,z,\bar{z}) = (-\theta,I,\bar{z},z).$$

The proof of the above lemma follows a similar approach as in [11, Lemma 6.4] and is omitted here.

Verification of Assumptions (A1)–(A3). The frequency vector ω can be rewritten as

$$\omega(\xi) = \hat{\alpha} + A\xi$$

where

$$\hat{\alpha} = \begin{pmatrix} \lambda_{j_1} \\ \lambda_{j_2} \\ \vdots \\ \lambda_{j_n} \end{pmatrix},$$

and

$$A = (-\frac{1}{4})\begin{pmatrix} \lambda_{j_1}^{-1}a_{j_1j_1j_1j_1} & \lambda_{j_2}^{-1}a_{j_2j_2j_1j_1} & \lambda_{j_3}^{-1}a_{j_3j_3j_1j_1} & \cdots & \lambda_{j_n}^{-1}a_{j_nj_nj_1j_1} \\ \lambda_{j_1}^{-1}a_{j_1j_1j_2j_2} & \lambda_{j_2}^{-1}a_{j_2j_2j_2j_2} & \lambda_{j_3}^{-1}a_{j_3j_3j_2j_2} & \cdots & \lambda_{j_n}^{-1}a_{j_nj_nj_2j_2} \\ \lambda_{j_1}^{-1}a_{j_1j_1j_3j_3} & \lambda_{j_2}^{-1}a_{j_2j_2j_3j_3} & \lambda_{j_3}^{-1}a_{j_3j_3j_3j_3} & \cdots & \lambda_{j_n}^{-1}a_{j_nj_nj_3j_3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \lambda_{j_1}^{-1}a_{j_1j_1j_nj_n} & \lambda_{j_2}^{-1}a_{j_2j_2j_nj_n} & \lambda_{j_3}^{-1}a_{j_3j_3j_nj_n} & \cdots & \lambda_{j_n}^{-1}a_{j_nj_nj_nj_n} \end{pmatrix}.$$

As we can find that

$$|\frac{\partial\omega}{\partial\xi}| = |A| = (-\frac{1}{4})^n\lambda_{j_1}^{-1}\dots\lambda_{j_n}^{-1}(\frac{3}{2\pi} + n - 1)(\frac{1}{2\pi})^{n-1} \neq 0.$$

It is easy to see $\mathrm{rank}\{\frac{\partial\omega}{\partial\xi}\} = \mathtt{n}$ and there exist a positive constant M_1 such that $\|\omega\|_{C^{\mathcal{N},1}(\Pi)} \le M_1$. So the Assumption (A1) is satisfied.

From Taylor's formula we know that

$$\Omega_j = \sqrt{j^4 + \bar{m}} - \frac{1}{4\pi}\Big(\frac{\xi_1}{\lambda_{j_1}} + \cdots + \frac{\xi_n}{\lambda_{j_n}}\Big) = j^2 + O(j^{-2}),$$

and there exist a positive constant M_2 such that

$$\|\Omega_j - j^2\|_{C^{\mathcal{N},1}(\Pi)} \le M_2 j^{-2}, \quad \forall j \ge 1.$$

So Assumption (A2) is satisfied. From Proposition 5.3 and (5.12) we know that the perturbation P is analytic, thus Assumption (A3) holds. And we have verified all the assumption of Theorem 1.1.

6. Appendix

Definition 6.1 (Infinite-dimensional reversible system)**.** Suppose S is an involution map: $S(\theta, I, z, \bar{z}) = (-\theta, I, \bar{z}, z)$ on $\mathscr{P}^{a,p}$. We say that an infinite-dimensional system

$$\begin{pmatrix} \dot{\theta} \\ \dot{I} \\ \dot{z} \\ \dot{\bar{z}} \end{pmatrix} = X(\theta, I, z, \bar{z})$$

is called reversible with respect to S (or S-reversible), if $X \circ S = -DS \cdot X$. i.e.,

$$X(S(\theta, I, z, \bar{z})) = -DS(\theta, I, z, \bar{z}) \cdot X(\theta, I, z, \bar{z}), \quad \forall (\theta, I, z, \bar{z}) \in \mathscr{P}^{a,p},$$

where $DS(\theta, I, z, \bar{z})$ is the tangent map of S.

Definition 6.2. A transformation Φ is called invariant with respect to above involution S (or S-invariant) if $\Phi \circ S = S \circ \Phi$.

Definition 6.3. Suppose S is an involution map: $S^2 = id$. A vector field X is called invariant with respect to S (or S-invariant), if $DS \cdot X = X \circ S$.

Lemma 6.4. *If X is S-reversible, Y is S-invariant and the transformation Φ is S-invariant, then $[X, Y]$ and $\Phi^* X$ are both S-reversible. In particular, the flow ϕ_Y^t of Y are S-invariant, thus $(\phi_Y^t)^* X$ is S-reversible.*

Lemma 6.5 ([17])**.** *The convolution $w * v$ of two complex sequences w, v in $l_2^{a,p}$ is defined as $(w * v)_j = \sum_m w_{j-m} v_m$. If $a \ge 0, p > \frac{1}{2}$, then $l_2^{a,p}$ is a Hilbert algebra with respect to the convolution of sequences, and*

$$\|w * v\|_p \le c \|w\|_p \|v\|_p,$$

the constant c depends only on p.

Lemma 6.6 ([16, Lemma A.1])**.** *If $A = (A_{ij})$ is a bounded linear operator on l^2, then $B = (B_{ij})$ with*

$$B_{ij} = \frac{|A_{ij}|}{|i-j|}, \quad i \ne j,$$

and $B_{ii} = 0$ is a bounded linear operator on l^2, and $\|B\| \le \frac{\Pi}{\sqrt{3}} \|A\|$.

Lemma 6.7 (Cauchy's estimate,[16, Lemma A.3])**.** *Let E and F be two complex Banach spaces with the norms $\|\cdot\|_E$ and $\|\cdot\|_F$. The first-order derivatives $d_v G$ of G at v is a linear mapping from E to F, whose operator norm is defined by*

$$\|\mathrm{d}_v G\| = \sup_{u \ne 0} \frac{\|\mathrm{d}_v G(u)\|_F}{\|u\|_E}.$$

If G is analytic on B_r, the open ball of radius r around v in E, then

$$\|\mathrm{d}_v G\|_{F,E} \leqslant \frac{1}{r} \sup_{u\in B_r} \|G(u)\|_F.$$

Lemma 6.8 ([6, Lemma A.4]). *Let $\bar{k} = (k, 1)$, with $k \in \mathbb{Z}^n, \bar{\omega} = (\omega, \Omega)$ with $\omega \in \mathbb{R}^n$ and $\Omega \in \mathbb{R}^1$, and*

$$O_+ = \big\{\bar{\omega} | \langle \bar{k}, \bar{\omega}\rangle| \geqslant \frac{\alpha}{|k|^\tau}, \forall k = (k, 1), k \in \mathbb{Z}^n\big\}.$$

Let $G_{\bar{k}} = \frac{1}{\langle \bar{k},\bar{\omega}\rangle}$, then $G_{\bar{k}}$ is infinitively differentiable on O_+ with respect to $\bar{\omega}$ in the sense of Whitney and the lth derivatives of $G_k(\bar{\omega})$ is

$$\frac{\partial^l}{\partial \bar{\omega}^l} G_k(\bar{\omega}) = (-1)^{|l|} l! k^l [G_{\bar{k}}(\bar{\omega})]^{|l|+1}.$$

Lemma 6.9 ([21, Lemma A.5]). *If f_1 and f_2 belong to $C^{\mathcal{N},1}(\Pi)$, then $f_1 \cdot f_2$ belongs to $C^{\mathcal{N},1}(\Pi)$ and*

$$\|f_1 \cdot f_2\|_{C^{\mathcal{N},1}(\Pi)} \leqslant \|f_1\|_{C^{\mathcal{N},1}(\Pi)} \cdot \|f_2\|_{C^{\mathcal{N},1}(m)}.$$

Lemma 6.10 ([21, Lemma A.6]). *Let $f(\Phi, \xi)$ be analytic in Φ on $D \subset \mathscr{P}^{a,p}$ and belong to $C^{\mathcal{N},1}(\Pi)$ in ξ, where D is an open set of $\mathscr{P}^{a,p}$. Let $\|\cdot\|$ be the norm of $\mathscr{P}^{a,p}$. Denote by $D_{-\varphi} = \{\Phi \in D, \mathrm{dist}(\Phi, \partial D) > \rho\}$, where ∂D is the boundary of D. If $\Phi : \varphi \in D_0 \subset D \to \Phi(\varphi; \xi)\epsilon D_{-\phi}$, for all $\xi \in \Pi$, and $\sup_{\varphi\in D_0} \|\Phi(\varphi; \cdot) - \mathrm{id}\|^* \leqslant \rho$, then for $\forall \varphi \in D_0$,*

$$\|f[\Phi(\varphi; \cdot), \cdot]\|^* \leqslant c \sup_{\Phi\in D} \|f(\Phi, \cdot)\|^*,$$

where $\|\cdot\|^*$ denotes the norm of $C^{\mathcal{N},1}(\Pi)$ and c only depends on n and $\mathcal{N}$.

Lemma 6.11 ([21, Lemma A.7]). *Let D be an open domain in a complex Banach space E with the norm $\|\cdot\|$. $X : (\Phi, \xi) \in D \times \Pi \to E$ is a parameter dependent vector field on D. Suppose X to be analytic in Φ on D and belong to $C^{\mathcal{N},1}(\Pi)$ in ξ. If $\sup_{\phi\in D} \|X(\Phi; \cdot)\|^* \leqslant \rho$, then for each $\xi \in \Pi$, its flow $\Phi^t(\cdot; \xi)$ exists on $D_{-\rho}$ for $|t| \leqslant 1$ and maps $D_{-2\rho}$ into $D_{-\rho}$, where $D_{-\rho}$ and $\|\cdot\|^*$ are defined as those in Lemma6.10. Moreover, on $D_{-2\rho}$*

$$\|\Phi(\cdot; \cdot) - \mathrm{id}\|^* \leqslant c \sup_{\Phi\in D} \|X(\Phi; \cdot)\|^*.$$

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