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EXISTENCE OF TWO INFINITE FAMILIES OF SOLUTIONS FOR SINGULAR SUPERLINEAR EQUATIONS ON EXTERIOR DOMAINS

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ABSTRACT. In this article we study radial solutions of $\Delta u + K(|x|)f(u) = 0$ in the exterior of the ball of radius $R > 0$ in $\mathbb{R}^N$ with $N > 2$ where f grows superlinearly at infinity and is singular at 0 with $f(u) \sim \frac{1}{|u|^{q-1}u}$ and $0 < q < 1$ for small u. We assume $K(|x|) \sim |x|^{-\alpha}$ for large $|x|$ and establish existence of two infinite families of sign-changing solutions when $N + q(N-2) < \alpha < 2(N-1)$.

1. INTRODUCTION

In this article we are interested in radial solutions of

$$\Delta u + K(|x|)f(u) = 0 \quad \text{on } \mathbb{R}^N \backslash B_R, \quad u = 0 \quad \text{on } \partial B_R, \quad u \to 0 \text{ as } |x| \to \infty, \tag{1.1}$$

when $N > 2$ and where B_R is the ball of radius $R > 0$ centered at the origin.

Assuming $u(x) = u(|x|) = u(r)$ the above problem becomes

$$u'' + \frac{N-1}{r}u' + K(r)f(u) = 0 \quad \text{for } R < r < \infty, \tag{1.2}$$

$$u(R) = 0, \quad \lim_{r\to\infty} u(r) = 0. \tag{1.3}$$

Numerous papers have proved existence of positive solutions of these equations with various nonlinearities $f(u)$ and for various functions $K(|x|) \sim |x|^{-\alpha}$ with $\alpha > 0$. See for example [1, 4, 5, 7, 11, 12, 13].

Here we prove existence of two infinite families of solutions including sign-changing solutions for this equation. We have also proved the existence of sign-changing solutions in other recent papers [2, 3, 9, 10].

We use the following assumptions:

(H1) $f : \mathbb{R}\backslash\{0\} \to \mathbb{R}$ is odd, locally Lipschitz, and

$$f(u) = |u|^{p-1}u + g(u) \text{ with } p > 1$$

for large $|u|$ and $\lim_{u\to\infty} \frac{|g(u)|}{|u|^p} = 0$.

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(H2) There exists a locally Lipschitz $g_1 : \mathbb{R} \to \mathbb{R}$ such that

$$f(u) = \frac{1}{|u|^{q-1}u} + g_1(u) \text{ with } 0 < q < 1 \text{ for small } |u| \text{ and } \; g_1(0) = 0.$$

(H3) $f > 0$ on $(0, \infty)$.

Let $F(u) = \int_0^u f(t)\,dt$. Since f is odd then F is even. Also, since $0 < q < 1$ (by (H2)) it follows that f is integrable at 0 and therefore F is continuous with $F(0) = 0$. Also since $f > 0$ on $(0, \infty)$ it follows that $F(u) > 0$ for $u > 0$. Since $F(u)$ is even then $F(u) > 0$ for $u \neq 0$.

We also assume $K(r) > 0$ and $K'(r)$ are continuous on $[R, \infty)$. In addition, we assume that

(H4) there exist α_1, α_2 and positive K_1, K_2, K_3 such that

$$\frac{K_1}{r^{\alpha_1}} \le K \le \frac{K_2}{r^{\alpha_2}} \quad \text{and} \quad \frac{r|K'|}{K} \le K_3 \quad \text{on } [R, \infty), \tag{1.4}$$

where $N + q(N-2) < \alpha_2 \le \alpha_1 < 2(N-1)$.

In this article we prove the following result.

Theorem 1.1. *Let $N > 2$ and assume* (H1)–(H4). *If $R > 0$, then there exist two infinite families $u_n^{\pm}$ of solutions to* (1.2)-(1.3). *If $R > 0$ is sufficiently large then there are 2 solutions, $u_n^{\pm}$, with n interior zeros on (R, ∞) for all positive integers n and there is 1 positive solution. If $R > 0$ is sufficiently small then there is an $n_0 \ge 0$ such that there are 2 solutions with n zeros on (R, ∞) for all $n > n_0$ and there is one solution with n_0 zeros on (R, ∞).*

We remark that the solutions of (1.2)-(1.3) have continuous second derivatives except at points where $u(r_0) = 0$ because $\lim_{u \to 0} |f(u)| = \infty$. Solutions, however, do turn out to be $C^1[R, \infty)$. In addition, we will see in Lemma 2.1 that if $a > 0$ then $u(r)$ and $u'(r)$ cannot both be zero at any $r \in [R, \infty)$. In particular, if $u(z) = 0$ then $u'(z) \neq 0$ and so by (H2) it follows that $r^{N-1}Kf(u)$ is integrable at z. Therefore, by a $C^1[R, \infty)$ solution of (1.2)-(1.3) we mean $u \in C^1[R, \infty)$ such that $r^{N-1}u' + \int_R^r t^{N-1}Kf(u)\,dt = R^{N-1}u'(R)$ for $r \ge R$, $u(R) = 0$, and $\lim_{r \to \infty} u(r) = 0$.

2. Preliminaries

Let $R > 0$. We begin our analysis of (1.2)-(1.3) by first making the change of variables $u(r) = v(r^{2-N}) = v(t)$ and obtaining

$$v''(t) + h(t)f(v(t)) = 0,$$

where

$$0 < h(t) = \frac{t^{\frac{2(N-1)}{2-N}} K(t^{\frac{1}{2-N}})}{(N-2)^2}.$$

Henceforth we denote $R_1 = R^{2-N}$.

We now attempt to solve the initial value problem

$$v_a'' + h(t)f(v_a) = 0 \quad \text{for } 0 < t < R_1, \tag{2.1}$$

$$v_a(0) = 0, \quad v_a'(0) = a > 0 \tag{2.2}$$

and then try to find values of a so that

$$v_a(R_1) = 0. \tag{2.3}$$

Let

$$\tilde{\alpha}_1 = \frac{2(N-1)-\alpha_1}{N-2}, \quad \tilde{\alpha}_2 = \frac{2(N-1)-\alpha_2}{N-2}.$$

It follows from (H4) and the definition of h that there exist positive h_1, h_2, h_3 such that

$$0 < h_1 t^{-\tilde{\alpha}_1} \le h(t) \le h_2 t^{-\tilde{\alpha}_2} \quad \text{and} \quad \frac{t|h'|}{h} \le h_3, \tag{2.4}$$

where $0 < \tilde{\alpha}_1 \le \tilde{\alpha}_2 < 1-q$.

First we prove existence of a solution to (2.1)-(2.2) on $[0, \epsilon_0]$ for some $\epsilon_0 > 0$. To do this we reformulate (2.1)-(2.2) as an appropriate integral equation. Let us suppose first that v_a is a solution (2.1)-(2.2). Integrating on $(0, t)$ gives:

$$v_a' + \int_0^t h(x) f(v_a(x))\, dx = a \quad \text{for } a > 0. \tag{2.5}$$

Integrating on $(0, t)$ gives

$$v_a + \int_0^t \int_0^s h(x) f(v_a(x))\, dx\, ds = at \quad \text{for } a > 0. \tag{2.6}$$

A bit of care needs to be taken here because we first need to know that the integral in (2.5) is defined. To see this notice that if v_a is a solution of (2.1)-(2.2) then for sufficiently small $t > 0$ we have $\frac{a}{2} t \le v_a \le at$. In addition, it follows from (H1) and (H2) that there is a constant $f_1 > 0$ such that $f(v_a) \le f_1(v_a^{-q} + v_a^p)$ and therefore by (2.4) we have

$$\begin{aligned} 0 < h(t) f(v_a) &\le f_1 h_2 \Big(\frac{t^{-\tilde{\alpha}_2}}{v_a^q} + t^{-\tilde{\alpha}_2} v_a^p \Big) \\ &\le f_1 h_2 \Big(\frac{t^{-\tilde{\alpha}_2}}{(\frac{a}{2})^q t^q} + t^{-\tilde{\alpha}_2 + p} a^p \Big) \\ &= f_1 h_2 \Big(\frac{2^q}{a^q} t^{-\tilde{\alpha}_2 - q} + t^{-\tilde{\alpha}_2 + p} a^p \Big). \end{aligned} \tag{2.7}$$

From (2.4) we have $1 - \tilde{\alpha}_2 - q > 0$ and $1 - \tilde{\alpha}_2 + p > 0$ so it follows from (2.7) that $h(t) f(v_a)$ is integrable near $t = 0$. Thus the integral in (2.5) is defined and is a continuous function. It then follows that (2.6) is also defined.

Now using (H2) we see that (2.6) is equivalent to

$$v_a + \int_0^t \int_0^s h(x) \Big(\frac{1}{v_a^q(x)} + g_1(v_a) \Big)\, dx\, ds = at. \tag{2.8}$$

Next let $v_a = tw$ in (2.8) which gives

$$w = a - \frac{1}{t} \int_0^t \int_0^s h(x) \Big(\frac{1}{x^q w^q(x)} + g_1(xw) \Big)\, dx\, ds. \tag{2.9}$$

We now define

$$S_\epsilon = \{ w \in C[0, \epsilon] : w(0) = a > 0, \text{ and } |w - a| \le \frac{a}{2} \text{ for all } t \in [0, \epsilon] \}.$$

Here $C[0, \epsilon]$ is the set of real-valued continuous functions on $[0, \epsilon]$ with the supremum norm $\| \cdot \|$. We define $T : S_\epsilon \to C[0, \epsilon]$ by $Tw(0) = a$ and

$$Tw = a - \frac{1}{t} \int_0^t \int_0^s h(x) \Big(\frac{1}{x^q w^q(x)} + g_1(xw) \Big)\, dx\, ds \quad \text{for } t > 0.$$

As mentioned in (2.4) and (2.7) it follows that $0 < \frac{h(x)}{x^q} \le h_2 x^{-\tilde{\alpha}_2 - q}$ and $\tilde{\alpha}_2 + q < 1$. Hence $x^{-\tilde{\alpha}_2 - q}$ is integrable on $(0, \epsilon)$. Then it is straightforward to show T maps S_ϵ into S_ϵ if $\epsilon > 0$ is sufficiently small. Next let L be the Lipschitz constant for the function g_1 defined in (H2) and suppose $w_1, w_2 \in S$. Using the mean value theorem and the fact that $\frac{a}{2} \le w_i \le a$ for $i = 1, 2$ on $[0, \epsilon]$ we see that

$$
\begin{aligned}
|Tw_1 - Tw_2| &\le \frac{1}{t}\int_0^t \int_0^s \Big(qh_2\big(\frac{2}{a}\big)^{q+1} x^{-\tilde{\alpha}_2 - q} + Lx^{1-\tilde{\alpha}_2}\Big)|w_1 - w_2|\,dx\,ds \\
&\le \|w_1 - w_2\|\Big(\frac{qh_2}{(1-\tilde{\alpha}_2 - q)(2 - \tilde{\alpha}_2 - q)}\big(\frac{2}{a}\big)^{q+1} t^{1-\tilde{\alpha}_2 - q} \qquad (2.10)\\
&\quad + \frac{L}{(2-\tilde{\alpha}_2)(3-\tilde{\alpha}_2)} t^{2-\tilde{\alpha}_2}\Big).
\end{aligned}
$$

Since the term in parentheses in (2.10) goes to 0 as $t \to 0^+$, it follows that there exists $\epsilon_0 > 0$ and a c with $0 < c < 1$ so that

$$\|Tw_1 - Tw_2\| \le c\|w_1 - w_2\| \quad \text{for all } w_i \in S_{\epsilon_0}.$$

Thus T is a contraction and so by the contraction mapping principle T has a unique fixed point [8]. Therefore, we obtain a unique solution of (2.6) on $[0, \epsilon_0]$. It then follows that the integral term in (2.6) is differentiable which implies that v_a is differentiable and satisfies (2.5).

Next we let

$$E_a = \frac{v_a'^2}{2h} + F(v_a). \qquad (2.11)$$

Recall from the comments after (H3) that $F(v_a) \ge 0$. Therefore from (2.1) and (2.4) it follows that

$$|E_a'| = \big| -\frac{h'}{2h^2} v_a'^2\big| \le \big|\frac{th'}{h}\big|\frac{v_a'^2}{2th} \le \frac{h_3 E_a}{t}. \qquad (2.12)$$

Thus $\big(\frac{E_a}{t^{h_3}}\big)' \le 0$ for $t > 0$ and therefore integrating on $(\epsilon_0/2, t)$ (with the ϵ_0 in the proof of existence) gives

$$\frac{v_a'^2}{2h} + F(v_a) = E_a(t) \le C_1 t^{h_3} \le C_1 R_1^{h_3},$$

where $C_1 = E_a(\epsilon_0/2).(\epsilon_0/2)^{h_3}$.

Thus v_a and v_a' are uniformly bounded on a largest interval of the form $[\epsilon_0/2, T] \subset [\epsilon_0/2, R_1]$. It then follows from this that v_a and v_a' are defined and continuous on all of $[0, R_1]$. In addition, it also follows from this that the v_a vary continuously with respect to a.

Lemma 2.1. *Assume* (H1)–(H4) *and let* v_a *solve* (2.1)-(2.2) *with* $a > 0$. *Then* $|v_a| + |v_a'| > 0$ *on* $[0, R_1]$.

Proof. First since $v_a(0) = 0$ and $v_a'(0) = a > 0$ it follows that v_a and v_a' cannot both be zero at any $t \in [0, \epsilon]$ for some $\epsilon > 0$. Suppose now that there is a $t_0 \in (0, R_1]$ such that $v_a(t_0) = v_a'(t_0) = 0$. Thus $E_a(t_0) = 0$ and then from (2.12) it follows that $(E_a t^{h_3})' \ge 0$ on (t, t_0). Integrating this on (t, t_0) yields $E_a \le 0$ on (t, t_0). Since $E_a \ge 0$ it follows then that $E_a \equiv 0$ on $[0, t_0]$ and thus $v_a = v_a' = 0$ on $[0, t_0]$. This however contradicts that $v_a'(0) = a > 0$. Thus the lemma follows. ☐

Lemma 2.2. *Assume* (H1)–(H4) *and let* v_a *solve* (2.1)-(2.2) *with* $a > 0$. *Then* v_a *only has a finite number of zeros on* $[0, R_1]$.

Proof. First since $v_a(0) = 0$ and $v_a'(0) = a > 0$ it follows that $v_a > 0$ on $(0, \epsilon)$ for some $\epsilon > 0$. Now suppose $v_a(z_k) = 0$ for $z_k \in [\epsilon/2, R_1]$ with $z_1 < z_2 < \cdots \leq R_1$. Then there exists z^* with $\epsilon/2 < z^* \leq R_1$ such that $z_k \to z^* \in [\epsilon/2, R_1]$ and $v_a(z^*) = 0$. In addition, it follows from Lemma 2.1 that $v_a'(z_k) \neq 0$ and thus there exist local extrema, M_k, with $z_k < M_k < z_{k+1}$ and $v_a'(M_k) = 0$. Thus we see $M_k \to z^*$ and $v_a'(z^*) = 0$. But this along with $v_a(z^*) = 0$ contradicts Lemma 2.1. Thus v_a has only a finite number of zeros on $[0, R_1]$. □

Lemma 2.3. *Assume* (H1)–(H4) *and let v_a solve* (2.1)-(2.2). *Suppose $a > 0$ is sufficiently small. Then v_a has a local maximum, $M_{1,a}$, and a zero, $z_{1,a}$, on $(0, R_1)$. In addition, $z_{1,a} \to 0$, $v_a'(z_{1,a}) \to 0$, and $v_a(M_{1,a}) \to 0$ as $a \to 0^+$. More generally, if $a > 0$ is sufficiently small and $k \geq 1$ then v_a has k zeros, $z_{i,a}$, and k local extrema, $M_{i,a}$, with $0 < M_{1,a} < z_{1,a} < M_{2,a} < z_{2,a} < \cdots < M_{k,a} < z_{k,a}$ on $(0, R_1)$. In addition, $\lim_{a\to 0^+} z_{i,a} = 0, \lim_{a\to 0^+} v_a'(z_{i,a}) = 0$, and $\lim_{a\to 0^+} |v_a(M_{i,a})| = 0$ for $1 \leq i \leq k$.*

Proof. From (2.6) we have

$$v_a + \int_0^t \int_0^s h(x) f(v_a(x))\, dx\, ds = at. \tag{2.13}$$

Suppose now that $v_a > 0$ on $(0, R_1)$. Then from (H2) and (H3) there is a constant $f_2 > 0$ such that $f(v_a) \geq f_2 v_a^{-q}$. In addition, from (2.4) we see that $h(t) \geq h_1 t^{-\tilde{\alpha}_1}$ and $1 - \tilde{\alpha}_1 - q > 0$. Substituting into (2.13) gives

$$\int_0^t \int_0^s h(x) f(v_a(x))\, dx\, ds \geq f_2 h_1 \int_0^t \int_0^s x^{-\tilde{\alpha}_1} v_a^{-q}(x)\, dx\, ds. \tag{2.14}$$

Also, it follows from (2.1) and (H3) that when $v_a > 0$ we have $v_a'' < 0$ and so integrating this inequality twice on $(0, t)$ gives

$$0 < v_a < at. \tag{2.15}$$

Substituting this into (2.14) gives

$$\begin{aligned} f_2 h_1 \int_0^t \int_0^s x^{-\tilde{\alpha}_1} v_a^{-q}\, dx\, ds &\geq \frac{f_2 h_1}{a^q} \int_0^t \int_0^s x^{-\tilde{\alpha}_1 - q}\, dx\, ds \\ &= \frac{f_2 h_1 t^{2-\tilde{\alpha}_1 - q}}{a^q (1 - \tilde{\alpha}_1 - q)(2 - \tilde{\alpha}_1 - q)}. \end{aligned} \tag{2.16}$$

Substituting this expression into (2.13)-(2.14) gives

$$0 < v_a \leq at - \frac{f_2 h_1 t^{2-\tilde{\alpha}_1 - q}}{a^q (1 - \tilde{\alpha}_1 - q)(2 - \tilde{\alpha}_1 - q)}. \tag{2.17}$$

However, the right-hand side of (2.17) is zero when

$$t = \Big(\frac{a^{q+1}(1 - \tilde{\alpha}_1 - q)(2 - \tilde{\alpha}_1 - q)}{f_2 h_1} \Big)^{\frac{1}{2-\tilde{\alpha}_1 - q}}$$

and notice that this value of t is less than or equal to R_1 if $a > 0$ is sufficiently small. Thus (2.17) yields a contradiction and therefore v_a has a first zero, $z_{1,a}$, and $0 < z_{1,a} < R_1$ if $a > 0$ is sufficiently small. In addition, the above argument shows that

$$0 < z_{1,a} \leq \Big(\frac{a^{q+1}(1 - \tilde{\alpha}_1 - q)(2 - \tilde{\alpha}_1 - q)}{f_2 h_1} \Big)^{\frac{1}{2-\tilde{\alpha}_1 - q}} \to 0 \quad \text{as } a \to 0^+. \tag{2.18}$$

Thus

$$\lim_{a\to 0^+} z_{1,a} = 0. \tag{2.19}$$

Next we examine the following identity which is straightforward to establish by differentiation and (2.1),

$$\frac{1}{2}v_a'^2 + h(t)F(v_a) + \int_0^t (-h'(s))F(v_a)\,ds = \frac{1}{2}a^2. \tag{2.20}$$

Evaluating at $z_{1,a}$ gives

$$\frac{1}{2}v_a'^2(z_{1,a}) = \frac{1}{2}a^2 + \int_0^{z_{1,a}} h'(s)F(v_a)\,ds. \tag{2.21}$$

Since $F(t) = \int_0^t f(s)\,ds$ it follows from (H1) and (H2) that there is a constant $f_3 > 0$ such that

$$F(v_a) \le f_3(v_a^{1-q} + v_a^{p+1}) \quad \text{when } v_a > 0. \tag{2.22}$$

Also from (2.4) we have

$$\frac{t|h'|}{h} \le h_3 \quad \text{and so} \quad |h'| \le h_2h_3t^{-1-\tilde\alpha_2}. \tag{2.23}$$

Substituting this into the right-hand side of (2.21) and using (2.15), (2.22) gives

$$\begin{aligned}\int_0^{z_{1,a}} h'(s)F(v_a)\,ds &\le \int_0^{z_{1,a}} f_3h_2h_3t^{-1-\tilde\alpha_2}(a^{1-q}t^{1-q} + a^{p+1}t^{p+1})\,dt \\ &= f_3h_2h_3\Big(\frac{a^{1-q}z_{1,a}^{1-\tilde\alpha_2-q}}{1-\tilde\alpha_2-q} + \frac{a^{p+1}z_{1,a}^{1-\tilde\alpha_2+p}}{1-\tilde\alpha_2+p}\Big) \\ &\le f_3h_2h_3a^{1-q}R_1^{1-\tilde\alpha_2-q}\Big(\frac{1}{1-\tilde\alpha_2-q} + \frac{a^{p+q}R_1^{p+q}}{1-\tilde\alpha_2+p}\Big).\end{aligned} \tag{2.24}$$

Thus substituting (2.22) and (2.24) into (2.21) gives

$$\frac{1}{2}v_a'^2(z_{1,a}) \le \frac{1}{2}a^2 + f_3h_2h_3a^{1-q}R_1^{1-\tilde\alpha_2-q}\Big(\frac{1}{1-\tilde\alpha_2-q} + \frac{a^{p+q}R_1^{p+q}}{1-\tilde\alpha_2+p}\Big) \to 0 \tag{2.25}$$

as $a \to 0^+$. Therefore,

$$\lim_{a\to 0^+} v_a'(z_{1,a}) = 0. \tag{2.26}$$

Next since $v_a(0) = v_a(z_{1,a}) = 0$ and $v_a'(0) = a > 0$ it follows that there is a local maximum, $M_{1,a}$, with $0 < M_{1,a} < z_{1,a}$. Evaluating (2.20) at $M_{1,a}$ gives

$$h(M_{1,a})F(v_a(M_{1,a})) = \frac{1}{2}a^2 + \int_0^{M_{1,a}} h'(t)F(v_a)\,dt. \tag{2.27}$$

Estimating as in (2.24)-(2.24) but now on $[0, M_{1,a}]$ (instead of $[0, z_{1,a}]$) we again obtain

$$\int_0^{M_{1,a}} h'(t)F(v_a)\,dt \le f_3h_2h_3a^{1-q}R_1^{1-\tilde\alpha_2-q}\Big(\frac{1}{1-\tilde\alpha_2-q} + \frac{a^{p+q}R_1^{p+q}}{1-\tilde\alpha_2+p}\Big). \tag{2.28}$$

Then from (2.27)-(2.28) and (2.4) we obtain

$$F(v_a(M_{1,a})) \le \frac{f_3h_2h_3a^{1-q}R_1^{1-\tilde\alpha_2+\tilde\alpha_1-q}}{h_1}\Big(\frac{1}{1-\tilde\alpha_2-q} + \frac{a^{p+q}R_1^{p+q}}{1-\tilde\alpha_2+p}\Big) \to 0 \tag{2.29}$$

as $a \to 0^+$. Therefore,

$$\lim_{a\to 0^+} v_a(M_{1,a}) = 0. \tag{2.30}$$

In a similar way we can show v_a has as many zeros as desired by choosing $a > 0$ sufficiently small and we can also similarly establish the analogs of (2.19), (2.26), and (2.30). This completes the proof of the lemma. □

Lemma 2.4. *Assume* (H1)–(H4) *and let* v_a *solve* (2.1)-(2.2). *If* $a > 0$ *is sufficiently large then* v_a *has a local maximum,* $M_{1,a}$*, on* $(0, R_1)$.

Proof. Suppose not and so suppose v_a is increasing on $(0, R_1)$ for all sufficiently large $a > 0$. Then $v_a > 0$ on $(0, R_1)$ and so it follows from (2.1) that $v_a'' < 0$ on $(0, R_1)$.

We now claim that $v_a(t_0) \to \infty$ as $a \to \infty$ for any fixed t_0 with $0 < t_0 \le R_1$. So suppose not. Thus suppose $0 < v_a \le C_2$ on $(0, t_0]$ where C_2 is independent of a. Using (2.15) and (2.22) we see that

$$\begin{aligned} F(v_a) &\le f_3(v_a^{1-q} + v_a^{p+1}) = f_3 v_a^{1-q}(1 + v_a^{p+q}) \\ &\le f_3 v_a^{1-q}(1 + C_2^{p+q}) = f_3 C_3 v_a^{1-q} \end{aligned} \tag{2.31}$$

where $C_3 = 1 + C_2^{p+q}$.

Then using (2.15) in (2.31) we obtain

$$F(v_a) \le f_3 C_3 v_a^{1-q} \le f_3 C_3 a^{1-q} t^{1-q}. \tag{2.32}$$

Substituting this into (2.20) and using (2.4) we then have $h(t) \le h_2 t^{-\tilde{\alpha}_2}$ and $|h'| \le h_2 h_3 t^{-\tilde{\alpha}_2 - 1}$. This gives

$$\begin{aligned} h(t)F(v_a) + \int_0^t (-h'(s))F(v_a)\,ds &\le f_3 h_2 C_3 \Big(1 + \frac{h_3}{1 - \tilde{\alpha}_2 - q}\Big) a^{1-q} t^{1-\tilde{\alpha}_2 - q} \\ &= C_4 a^{1-q} t^{1-\tilde{\alpha}_2 - q} \\ &\le C_4 a^{1-q} t_0^{1-\tilde{\alpha}_2 - q} \end{aligned} \tag{2.33}$$

where $C_4 = f_3 h_2 C_3 \left(1 + \frac{h_3}{1-\tilde{\alpha}_2 - q}\right)$. Therefore from (2.20) and (2.33) we see that

$$\frac{1}{2} v_a'^2 \ge \frac{1}{2} a^2 - C_4 t_0^{1-\tilde{\alpha}-q} a^{1-q} \ge \frac{1}{2} a^2 - C_4 R_1^{1-\tilde{\alpha}-q} a^{1-q} \ge \frac{1}{8} a^2$$

for a sufficiently large. Thus $v_a' \ge a/2$ for a sufficiently large, and integrating this on $(0, t_0)$ gives

$$C_2 \ge v_a(t_0) \ge \frac{a}{2} t_0 \to \infty \quad \text{as } a \to \infty.$$

Hence we obtain a contradiction. Thus it follows that if v_a is increasing on $[0, R_1]$ then $v_a(t_0) \to \infty$ as $a \to \infty$ for every t_0 with $0 < t_0 \le R_1$.

Next it follows that if v_a is increasing on $[0, R_1]$ then since f is superlinear (by (H1)) then

$$\frac{h(t) f(v_a)}{v_a} \to \infty$$

uniformly on $[t_0, R_1]$ for any $t_0 > 0$ as $a \to \infty$. Therefore assuming v_a is increasing on $[0, R_1]$ we see that

$$I_a = \inf_{[t_0, R_1]} \frac{h(t) f(v_a)}{v_a} \to \infty \quad \text{as } a \to \infty. \tag{2.34}$$

Next we rewrite (2.1) as

$$v_a'' + \Big(\frac{h(t) f(v_a)}{v_a}\Big) v_a = 0. \tag{2.35}$$

Assuming v_a is increasing on $[0, R_1]$, we let y solve

$$y'' + I_a y = 0 \tag{2.36}$$

with $y(t_0) = v_a(t_0)$ and $y'(t_0) = v_a'(t_0)$. Thus

$$y = v_a(t_0)\cos(\sqrt{I_a}(t - t_0)) + \frac{v_a'(t_0)}{\sqrt{I_a}}\sin(\sqrt{I_a}(t - t_0))$$

and so it follows that y is $2\pi/\sqrt{I_a}$-periodic. Thus y must have a local maximum on $[t_0, t_0 + \frac{2\pi}{\sqrt{I_a}}]$. In addition, it follows from (2.34) that $[t_0, t_0 + \frac{2\pi}{\sqrt{I_a}}] \subset [t_0, R_1]$ if a is sufficiently large. We will now show that v_a must have a local maximum on $[t_0, t_0 + \frac{2\pi}{\sqrt{I_a}}] \subset [t_0, R_1]$ if a is sufficiently large. This is essentially the Sturm Comparison Theorem [6] but we write out the details because they are brief.

Let $a > 0$ be sufficiently large so that y has a local maximum $M < R_1$ and that $y' > 0$ on $[t_0, M]$. Multiplying (2.35) by y, (2.36) by v_a, and subtracting gives

$$(yv_a' - y'v_a)' + \Big(\frac{h(t)f(v_a)}{v_a} - I_a\Big)yv_a = 0. \tag{2.37}$$

Integrating this on $[t_0, M]$ and using $y'(M) = 0$, $y(t_0) = v_a(t_0)$, and $y'(t_0) = v_a'(t_0)$ gives

$$y(M)v_a'(M) + \int_{t_0}^{M} \Big(\frac{h(t)f(v_a)}{v_a} - I_a\Big)yv_a\,dt = 0. \tag{2.38}$$

On $[t_0, M]$ we have $y > 0$, $v_a > 0$. In addition, the term in parentheses in (2.38) is nonnegative. Thus we see $y(M)v_a'(M) \le 0$ and therefore $v_a'(M) \le 0$ since $y(M) > 0$. Now if $v_a'(M) < 0$ then since $v_a'(t_0) > 0$ it follows that v_a has a local maximum, $M_{1,a}$, with $t_0 < M_{1,a} < M$. On the other hand, if $v_a'(M) = 0$ then from (2.1) it follows that $v_a''(M) < 0$ and therefore M is a local maximum for v_a and we set $M_{1,a} = M$. Therefore in both cases we see that v_a has a local maximum, $M_{1,a}$, with $0 < M_{1,a} < R_1$ and $v_a' > 0$ on $[0, M_{1,a})$ if $a > 0$ is sufficiently large. ☐

Lemma 2.5. *Assume* (H1)–(H4) *and let v_a solve* (2.1)-(2.2). *Suppose $a > 0$ is sufficiently large so that v_a has a smallest local maximum $M_{1,a}$ with $v_a' > 0$ on $[0, M_{1,a})$ and $M_{1,a} < R_1$. Then* $\lim_{a\to\infty} v_a(M_{1,a}) = \infty$ *and* $\lim_{a\to\infty} M_{1,a} = 0$.

Proof. We first show that $v_a(M_{1,a}) \to \infty$ as $a \to \infty$. So suppose not. Mimicking the proof of Lemma 2.4, suppose there is a $C_5 > 0$ such that $v_a(M_{1,a}) \le C_5$. Then using (2.31)-(2.32) and evaluating (2.20) and (2.33) at $t = M_{1,a}$ gives

$$\begin{aligned}
\frac{1}{2}a^2 &= h(M_{1,a})F(v_a(M_{1,a})) + \int_0^{M_{1,a}} (-h'(s))F(v_a)\,ds \\
&\le f_3h_2C_5\Big(1 + \frac{h_3}{1 - \tilde{\alpha}_2 - q}\Big)a^{1-q}t^{1-\tilde{\alpha}_2-q} \\
&= C_6a^{1-q}M_{1,a}^{1-\tilde{\alpha}_2-q} \\
&\le C_6a^{1-q}R_1^{1-\tilde{\alpha}_2-q}
\end{aligned} \tag{2.39}$$

where $C_6 = f_3h_2C_5(1 + \frac{h_3}{1-\tilde{\alpha}_2-q})$. Thus

$$\frac{1}{2}a^{1+q} \le C_6R_1^{1-\tilde{\alpha}_2-q}. \tag{2.40}$$

However, the left-hand side of (2.40) goes to infinity as $a \to \infty$ but the right-hand side stays finite. Hence we obtain a contradiction and therefore we must have

$$\lim_{a\to\infty} v_a(M_{1,a}) = \infty. \tag{2.41}$$

Next we show $M_{1,a} \to 0$ as $a \to \infty$. By (H1) it follows that

$$f(v_a) \geq f_4 v_a^p \text{ when } v_a > 0 \text{ for some constant } f_4 > 0. \tag{2.42}$$

We integrate (2.1) on $(t, M_{1,a})$ and estimate using the fact that v_a is increasing on $(t, M_{1,a})$ to obtain:

$$v_a' = \int_t^{M_{1,a}} h(s) f(v_a)\, ds \geq f_4 v_a^p \int_t^{M_{1,a}} h(s)\, ds. \tag{2.43}$$

Dividing by v_a^p, recalling $p > 1$, and integrating on $(\frac{M_{1,a}}{2}, M_{1,a})$ gives

$$\frac{v_a^{1-p}(\frac{M_{1,a}}{2})}{p-1} \geq \frac{v_a^{1-p}(\frac{M_{1,a}}{2}) - v_a^{1-p}(M_{1,a})}{p-1} \geq f_3 \int_{\frac{M_{1,a}}{2}}^{M_{1,a}} \int_s^{M_{1,a}} h(s)\, ds. \tag{2.44}$$

Since $v_a'' < 0$ it follows that v_a is concave and thus $v_a(\lambda x + (1-\lambda)y) \geq \lambda v_a(x) + (1-\lambda)v_a(y)$ for $0 \leq \lambda \leq 1$. In particular, for $x = v_a(M_{1,a})$, $y = 0$, and $\lambda = \frac{1}{2}$ we obtain $v_a(\frac{M_{1,a}}{2}) \geq \frac{v_a(M_{1,a})}{2}$. Then it follows from this and (2.41) that $v_a(\frac{M_{1,a}}{2}) \to \infty$ as $a \to \infty$. Since $p > 1$ it follows then that the left-hand side of (2.44) goes to 0 as $a \to \infty$ and thus we must have

$$\lim_{a\to\infty} M_{1,a} = 0. \tag{2.45}$$

This completes the proof. □

Lemma 2.6. *Assume* (H1)–(H4) *and let v_a solve* (2.1)-(2.2)*. Suppose $a > 0$ is sufficiently large. Then v_a has a zero, $z_{1,a}$, with $M_{1,a} < z_{1,a} < R_1$. In addition, $v_a > 0$ and $v_a' < 0$ on $(M_{1,a}, z_{1,a})$. Further $\lim_{a\to\infty} z_{1,a} = 0$, $\lim_{a\to\infty} v_a(M_{1,a}) = \infty$, and $\lim_{a\to\infty} v_a'(z_{1,a}) = -\infty$. More generally, if a is sufficiently large and $k \geq 1$ then v_a has k zeros, $z_{i,a}$, and k local extrema, $M_{i,a}$, with $0 < M_{1,a} < z_{1,a} < M_{2,a} < z_{2,a} < \cdots < M_{k,a} < z_{k,a}$ on $(0, R_1)$. In addition, $\lim_{a\to\infty} z_{i,a} = 0$, $\lim_{a\to\infty} |v_a'(z_{i,a})| = \infty$, and $\lim_{a\to\infty} |v_a(M_{i,a})| = \infty$ for $1 \leq i \leq k$.*

Proof. It follows from Lemma 2.5 that

$$\lim_{a\to\infty} v_a(M_{1,a}) = \infty. \tag{2.46}$$

Assume now that $v_a > 0$ on $(M_{1,a}, R_1)$. Then using (2.42) and integrating on $(M_{1,a}, t)$ we obtain

$$-v_a' \geq f_4 v_a^p \int_{M_{1,a}}^t h(s)\, ds.$$

Dividing by v_a^p, integrating on $(M_{1,a}, t)$, and using (2.4) gives

$$\begin{aligned} v_a^{1-p} &\geq v_a^{1-p} - v_a^{1-p}(M_{1,a}) \\ &\geq (p-1) f_4 \int_{M_{1,a}}^t \int_{M_{1,a}}^s h(x)\, dx\, ds \\ &= \frac{(p-1) f_4 R_1^{-\tilde{\alpha}_1}}{2} (t - M_{1,a})^2. \end{aligned} \tag{2.47}$$

Evaluating (2.47) at $t = \frac{R_1+M_{1,a}}{2}$ we see

$$v_a^{1-p}\Big(\frac{R_1+M_{1,a}}{2}\Big) \geq \frac{(p-1)f_4 R_1^{-\tilde{\alpha}_1}}{2}\Big(\frac{R_1-M_{1,a}}{2}\Big)^2$$

and therefore

$$v_a^{p-1}\Big(\frac{R_1+M_{1,a}}{2}\Big) \leq \frac{8R_1^{\tilde{\alpha}_1}}{(p-1)f_4(R_1-M_{1,a})^2}. \tag{2.48}$$

By (2.45) we see then for large a that

$$v_a\Big(\frac{R_1+M_{1,a}}{2}\Big) \leq \Big(\frac{32R_1^{\tilde{\alpha}_1-2}}{(p-1)f_4}\Big)^{\frac{1}{p-1}}. \tag{2.49}$$

Using that $v_a'' < 0$ when $v_a > 0$ and the mean value theorem we see there is a c_a with $M_{1,a} < c_a < \frac{R_1+M_{1,a}}{2}$ such that

$$\begin{aligned} v_a(M_{1,a}) - v_a\Big(\frac{R_1+M_{1,a}}{2}\Big) &= -v_a'(c_a)\Big(\frac{R_1-M_{1,a}}{2}\Big) \\ &\leq -v_a'\Big(\frac{R_1+M_{1,a}}{2}\Big)\Big(\frac{R_1}{2}\Big). \end{aligned} \tag{2.50}$$

Since $v_a' > 0$ on $(0, M_{1,a})$ it follows from (2.41) and (2.49) that the left-hand side of (2.50) goes to infinity as $a \to \infty$. And then from (2.45) and (2.50) it follows that

$$v_a'\Big(\frac{R_1+M_{1,a}}{2}\Big) \to -\infty \quad \text{as } a \to \infty. \tag{2.51}$$

Since $v_a'' < 0$ when $v_a > 0$ it follows that v_a' is decreasing when $v_a > 0$ so:

$$v_a' < v_a'\Big(\frac{R_1+M_{1,a}}{2}\Big) \text{ for } t > \frac{R_1+M_{1,a}}{2}.$$

Integrating this on $(\frac{R_1+M_{1,a}}{2}, R_1)$ gives

$$v_a(R_1) < v_a\Big(\frac{R_1+M_{1,a}}{2}\Big) + v_a'\Big(\frac{R_1+M_{1,a}}{2}\Big)\Big(\frac{R_1-M_{1,a}}{2}\Big). \tag{2.52}$$

It follows from (2.49) that the first term on the right-hand side (2.52) is bounded. Then from (2.45) we have $M_{1,a} \to 0$ as $a \to \infty$ and this along with (2.51) implies that the right-hand side of (2.52) becomes negative while the left-hand side stays positive. Thus we obtain a contradiction and therefore there exists $z_{1,a}$ with $M_{1,a} < z_{1,a} < R_1$ such that $v_a(z_{1,a}) = 0$ and $v_a > 0$ on $(M_{1,a}, z_{1,a})$.

From the mean value theorem and that $v_a'' < 0$ when $v_a > 0$ it follows that there is a d_a such that $M_{1,a} < d_a < z_{1,a}$ and

$$v_a(M_{1,a}) = |v_a(z_{1,a}) - v_a(M_{1,a})| = |v_a'(d_a)||z_{1,a} - M_{1,a}| \leq |v_a'(d_a)|R_1 \leq |v_a'(z_{1,a})|R_1$$

and since the left-hand side goes to infinity by (2.46) it then follows from the above inequality that

$$\lim_{a\to\infty} v_a'(z_{1,a}) = -\infty. \tag{2.53}$$

Next it follows from evaluating (2.47) at $\frac{M_{1,a}+z_{1,a}}{2}$ that we obtain

$$v^{1-p}\Big(\frac{M_{1,a}+z_{1,a}}{2}\Big) \geq \frac{(p-1)f_4 R_1^{-\tilde{\alpha}_1}}{2}\Big(\frac{M_{1,a}-z_{1,a}}{2}\Big)^2. \tag{2.54}$$

Since $v_a'' < 0$ when $v_a > 0$ it follows that v_a is concave. Then it follows from this and (2.46) that $v_a(\frac{M_{1,a}+z_{1,a}}{2}) \geq \frac{v_a(M_{1,a})}{2} + \frac{v_a(z_{1,a})}{2} = \frac{v_a(M_{1,a})}{2} \to \infty$. Thus we see the left-hand side of (2.54) goes to 0 as $a \to \infty$ and therefore $z_{1,a} - M_{1,a} \to 0$. Since $M_{1,a} \to 0$ by Lemma 2.5 we see then that

$$\lim_{a\to\infty} z_{1,a} = 0. \tag{2.55}$$

In a similar way we can show that v_a as many zeros as desired on $(0, R_1)$ by choosing $a > 0$ sufficiently large, and we can obtain the analogs of (2.46), (2.53), and (2.55). This completes the proof. □

Lemma 2.7. *Assume* (H1)–(H4) *and let* v_a *solve* (2.1)-(2.2) *with* $a > 0$*. If* R_1 *is sufficiently small then there are values of* $a > 0$ *such that* $v_a > 0$ *on* $(0, R_1)$*. Also, if* R_1 *is sufficiently large then* v_a *has at least one zero on* $(0, R_1)$ *for all* $a > 0$*. Similarly, if* $R_1 > 0$ *is sufficiently large then* v_a *has at least* k *zeros on* $(0, R_1)$ *for all* $a > 0$*.*

Proof. We prove the second part first. It follows from (H1)–(H3) that there is a constant $f_5 > 0$ such that $\frac{f(v)}{v} \ge f_5$ for all $v \neq 0$. In addition, we know from (2.4) that $h(t) \ge h_1 t^{-\tilde{\alpha}_1} \ge h_1 R_1^{-\tilde{\alpha}_1}$. Thus $\frac{h(t)f(v_a)}{v_a} \ge \frac{f_5}{R_1^{\tilde{\alpha}_1}}$.

Next we consider

$$\begin{gathered} w'' + \Big(\frac{f_5 h_1}{R_1^{\tilde{\alpha}_1}}\Big) w = 0, \\ w(0) = 0, w'(0) = a. \end{gathered}$$

Thus:

$$w = c \sin\Big(\sqrt{\frac{f_5 h_1}{R_1^{\tilde{\alpha}_1}}}\, x\Big)$$

for some $c > 0$, and so w has a zero on $[0, \sqrt{\frac{R_1^{\tilde{\alpha}_1}}{f_5 h_1}}\,\pi]$. It follows then from the Sturm Comparison Theorem [6] that v_a has at least one zero on $[0, R_1]$ if $\sqrt{\frac{R_1^{\tilde{\alpha}_1}}{f_5 h_1}}\,\pi < R_1$. That is, if

$$R_1 > (\frac{\pi^2}{f_5 h_1})^{\frac{1}{2-\tilde{\alpha}_1}} = \Big(\frac{\pi^2}{f_5 h_1}\Big)^{\frac{N-2}{\alpha_1 - 2}}.$$

Similarly, v_a has at least k zeros on $[0, R_1]$ if

$$R_1 > \Big(\frac{k^2\pi^2}{f_5 h_1}\Big)^{\frac{1}{2-\tilde{\alpha}_1}} = \Big(\frac{k^2\pi^2}{f_5 h_1}\Big)^{\frac{N-2}{\alpha_1 - 2}}.$$

Next we show that if R_1 is sufficiently small then there is a value of $a > 0$ such that $v_a > 0$ on $(0, R_1)$. First since $f(v_a) > 0$ for $v_a > 0$ by (H3) there is a constant $f_6 > 0$ such that $f(v_a) \ge f_6 > 0$ for $v_a > 0$. Thus it follows from this and (2.4) that $h(t)f(v_a) \ge f_6 h_1 t^{-\tilde{\alpha}_1}$. Suppose now that v_a has a zero, z_a, on $(0, R_1)$. Then there is an M_a with $0 < M_a < z_a$ such that v_a has a local maximum at M_a. Substituting $t = M_a$ into (2.5) then gives

$$\frac{f_6 h_1 M_a^{1-\tilde{\alpha}_1}}{1-\tilde{\alpha}_1} \le \int_0^{M_a} f_6 h_1 t^{-\tilde{\alpha}_1}\, dt \le \int_0^{M_a} h(t) f(v_a)\, dt = a.$$

It follows from this that

$$\lim_{a\to 0^+} M_a = 0. \tag{2.56}$$

Returning to (2.20) and evaluating at M_a we see that

$$\frac{1}{2}a^2 = h(M_a)F(v_a(M_a)) + \int_0^{M_a} (-h'(t))F(v_a)\,dt. \tag{2.57}$$

Then using (2.15), (2.22), and (2.4) we see that

$$\begin{aligned}\int_0^{M_a} (-h'(t))F(v_a)\,dt &\le f_3h_2h_3 \int_0^{M_a} t^{-\tilde{\alpha}_2-1}(a^{1-q}t^{1-q} + a^{p+1}t^{p+1})\,dt \\ &= f_3h_2h_3a^{1-q}\Big(\frac{R_1^{1-\tilde{\alpha}_2-q}}{1-\tilde{\alpha}_2-q} + \frac{a^{p+q}R_1^{1-\tilde{\alpha}_2+p}}{1-\tilde{\alpha}_2+p}\Big).\end{aligned} \tag{2.58}$$

Similarly,

$$h(M_a)F(v_a(M_a)) \le f_3h_2a^{1-q}\big(R_1^{1-\tilde{\alpha}_2-q} + a^{p+q}R_1^{1-\tilde{\alpha}_2+p}\big). \tag{2.59}$$

Now substituting (2.58)-(2.59) into (2.57) gives

$$\frac{1}{2}a^2 \le f_3h_2a^{1-q}\big(C_7R_1^{1-\tilde{\alpha}_2-q} + a^{p+q}C_8R_1^{1-\tilde{\alpha}_2+p}\big), \tag{2.60}$$

where $C_7 = (1 + \frac{h_3}{1-\tilde{\alpha}_2-q})$ and $C_8 = (1 + \frac{h_3}{1-\tilde{\alpha}_2+p})$. Select $a = 1$ and we see (2.60) becomes

$$1 \le 2f_3h_2\left(C_7R_1^{1-\tilde{\alpha}_2-q} + C_8R_1^{1-\tilde{\alpha}_2+p}\right) \tag{2.61}$$

Now if R_1 is sufficiently small we see that this violates (2.61). Thus if R_1 is sufficiently small and if $a = 1$ then $v_a > 0$ on $(0, R_1)$. This completes the proof. □

3. Proof of Theorem 1.1

We saw from Lemma 2.2 that v_a has a finite number of zeros on $(0, R_1)$ for $a > 0$. Thus there exists an $a > 0$ such that v_a has the *least* number of zeros on $(0, R_1)$ among all $a > 0$. We denote the number of zeros of this particular v_a as $n_0 \ge 0$. (There may be more than one choice of a such that v_a has n_0 zeros on $(0, R_1)$ but choose one such a). Now let

$$S_{n_0} = \{a > 0 : v_a \text{ solves (2.1)-(2.2) and has exactly } n_0 \text{ zeros on } (0, R_1)\}.$$

From the above comments it follows that S_{n_0} is nonempty and from Lemma 2.6 it follows that S_{n_0} is bounded above.

Next let $a_{n_0} = \sup S_{n_0}$. We now prove that $v_{a_{n_0}}$ has exactly n_0 zeros on $(0, R_1)$ and $v_{a_{n_0}}(R_1) = 0$. From the definition of n_0 it follows that $v_{a_{n_0}}$ has at least n_0 zeros on $(0, R_1)$. Now if $v_{a_{n_0}}$ has an $(n_0 + 1)$st zero on $(0, R_1)$ then by continuity with respect to initial conditions then so does v_a for a close to a_{n_0} and $a < a_{n_0}$ but if $a < a_{n_0}$ then v_a has only n_0 zeros. Thus $v_{a_{n_0}}$ has exactly n_0 zeros on $(0, R_1)$. Now suppose $v_{a_{n_0}}(R_1) \neq 0$. Without loss of generality suppose that $v_{a_{n_0}}(R_1) > 0$. Now if a is close to a_{n_0} and $a > a_{n_0}$ then by continuity with respect to initial conditions and the fact that if $v_a(z) = 0$ then $v_a'(z) \neq 0$ it follows that $v_a(R_1) > 0$ and also v_a has n_0 zeros on $(0, R_1)$. But since $a > a_{n_0}$ then v_a has at least $n_0 + 1$ zeros on $(0, R_1)$ and so we obtain a contradiction. Thus it must be the case that $v_{a_{n_0}}(R_1) = 0$ and thus we obtain a solution of (2.1)-(2.2). Then by Lemma 2.1 it follows that $v_{a_{n_0}}'(R_1) \neq 0$ so let us assume without loss of generality that $v_{a_{n_0}}'(R_1) < 0$.

In a similar way we now define

$$S_{n_0+1} = \{a > 0 : v_a \text{ solves (2.1)-(2.2) and has exactly } n_0 + 1 \text{ zeros on } (0, R_1)\}.$$

It follows from Lemma 2.6 that S_{n_0+1} is bounded from above. For $a > a_{n_0}$ and a sufficiently close to a_{n_0} it follows again by continuity with respect to initial conditions that v_a has an (n_0+1)st zero $z_{n_0+1} < R_1$ and z_{n_0+1} is close to R_1. In addition, since $v'_{a_{n_0}}(R_1) < 0$ it follows that $v'_a(z_{n_0+1}) < 0$. Thus v_a has exactly n_0+1 zeros on $(0, R_1)$ for $a > a_{n_0}$ and a sufficiently close to a_{n_0}. Therefore S_{n_0+1} is nonempty.

Similarly we define $a_{n_0+1} = \sup S_{n_0+1}$ and we can similarly show that $v_{a_{n_0+1}}$ has exactly n_0+1 zeros on $(0, R_1)$ and $v_{a_{n_0+1}}(R_1) = 0$.

Continuing in this way we see that we can find an infinite number of solutions, v_{a_n}, where v_{a_n} has exactly n zeros on $(0, R_1)$ and $v_{a_n}(R_1) = 0$ for each $n \geq n_0$. Thus we have found one infinite family of solutions of (2.1)-(2.2).

Next we let

$$b_{n_0} = \inf S_{n_0}.$$

By the above comments S_{n_0} is nonempty and by definition S_{n_0} is bounded below. Then $b_{n_0} \leq a_{n_0}$ and by a similar argument we can show that $v_{b_{n_0}}$ has exactly n_0 zeros on $(0, R_1)$ and $v_{a_{n_0}}(R_1) = 0$. Now it may be the case that $a_{n_0} = b_{n_0}$ so there may be only one solution with n_0 zeros. Next we let

$$b_{n_0+1} = \inf S_{n_0+1}.$$

Then we have $b_{n_0+1} < b_{n_0} \leq a_{n_0} < a_{n_0+1}$ and we can show $v_{b_{n_0+1}}$ has exactly n_0+1 zeros on $(0, R_1)$ and $v_{b_{n_0+1}}(R_1) = 0$. Since $b_{n_0+1} < a_{n_0+1}$ it follows that we have two solutions, $v_{a_{n_0}}$ and $v_{b_{n_0}}$, with n_0+1 zeros on $(0, R_1)$. Continuing in this way we see that if $n > n_0$ we can find a second infinite family of solutions of (2.1)-(2.2), v_{b_n}, where v_{b_n} has exactly n zeros on $(0, R_1)$ and $v_{b_n}(R_1) = 0$.

Finally, we let $u_n^+(t) = v_{a_n}(t^{\frac{1}{2-N}})$ and $u_n^-(t) = v_{b_n}(t^{\frac{1}{2-N}})$ for all $n \geq n_0$. This completes the proof of Theorem 1.1. □

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