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# A BIHARMONIC EQUATION WITH DISCONTINUOUS NONLINEARITIES

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Abstract. We study the biharmonic equation with discontinuous nonlinearity and homogeneous Dirichlet type boundary conditions

$$\begin{aligned} \Delta^2 u &= H(u-a)q(u) \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega, \\ \frac{\partial u}{\partial n} &= 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{1}$$

where $\Delta$ is the Laplace operator, $a > 0$, $H$ denotes the Heaviside function, $q$ is a continuous function, and $\Omega$ is a bounded domain in $\mathbb{R}^N$ with $N \geq 3$.

Adapting the method introduced by Ambrosetti and Badiale (The Dual Variational Principle), which is a modification of Clarke and Ekeland's Dual Action Principle, we prove the existence of nontrivial solutions to (1). This method provides a differentiable functional whose critical points yield solutions to (1) despite the discontinuity of $H(s-a)q(s)$ at $s = a$.

Considering $\Omega$ of class $\mathcal{C}^{4,\gamma}$ for some $\gamma \in (0,1)$, and the function $q$ constrained under certain conditions, we show the existence of two non-trivial solutions. Furthermore, we prove that the free boundary set $\Omega_a = \{x \in \Omega : u(x) = a\}$ has measure zero when $u$ is a minimizer of the action functional.

## 1. Introduction

The main objective of this work is to study the existence of solutions to the PDE

$$\begin{aligned} \Delta^2 u &= H(u-a)q(u) \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega, \\ \frac{\partial u}{\partial n} &= 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{1.1}$$

where $\Delta$ is the Laplace operator, $a > 0$, $H$ denotes the Heaviside function, $q \in \mathcal{C}(\mathbb{R})$, and $\Omega$ is a domain of $\mathbb{R}^N$ with $N \geq 3$.

The action functional associated with (1.1) is given by

$$J(u) = \int_\Omega \big((\Delta u)^2 - Q(u)\big)\, d\mathbf{x} \quad \forall u \in H_0^2(\Omega), \tag{1.2}$$

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where $Q(t) := \int_0^t H(s-a)q(s)\,d\boldsymbol{s}$, and $H_0^2(\Omega)$ denotes de Sobolev space of square integrable functions having square integrable first and second order partial derivatives and vanishing in $\partial\Omega$ together with its first order partial derivatives. Since $H$ is not continuous at $s=a$, $Q$ need not be differentiable at $s=a$, and, therefore, $J$ need not be differentiable. We bypass this difficulty using the Dual Variational Principle introduced by Ambrosetti and Badiale (1989) which yields a differentiable functional even when $Q$ is not continuous.

## 2. Preliminaries

Throughout this article we assume that $q$ is a continuous function and that

$$q(s) \geq 0 \text{ for all } s \geq 0,\ q \text{ is non-decreasing;} \tag{2.1}$$

$$q(s) \leq \alpha|s| + c_0, \text{ with } 0 < \alpha < \mu_1 \text{ and } c_0 \text{ a constant,} \tag{2.2}$$

where $\mu_1$ is the first eigenvalue of the biharmonic operator with homogeneous Dirichlet boundary conditions.

Let us consider the multivalued function $\hat{q}$ defined by

$$\hat{q}(s) := \begin{cases} q(s) & \text{if } s > a, \\ [0, q(a)] & \text{if } s = a, \\ 0 & \text{if } s < a. \end{cases}$$

**Definition 2.1.** A function $u : \Omega \to \mathbb{R}$ is called a *multi valued solution* of the PDE (1) if $u \in H_0^2(\Omega) \cap H^4(\Omega)$ and $u$ satisfies

$$\Delta^2 u \in \hat{q}(u), \quad \text{a.e. in } \Omega.$$

**Definition 2.2.** Let $u$ a solution of (1). The set

$$\Omega_a = \{x \in \Omega : u(x) = a\}$$

is called the *free boundary.*

Letting $p(s) = H(s-a)q(s)$, we rewrite (1) as

$$\begin{gathered} \Delta^2 u = p(u) \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega, \\ \frac{\partial u}{\partial n} = 0 \quad \text{on } \partial\Omega. \end{gathered} \tag{2.3}$$

**Definition 2.3.** A function $u : \Omega \to \mathbb{R}$ is called a *solution* to the PDE (2.3) if $u \in H_0^2(\Omega) \cap H^4(\Omega)$ and $u$ satisfies

$$\Delta^2 u = p(u) \quad \text{a.e. in } \Omega.$$

Let us define $p_m(s) := p(s) + ms$. Note that, for $m > 0$, the function $p_m$ is strictly increasing and (2.3) is equivalent to

$$\begin{gathered} \Delta^2 u + mu = p_m(u) \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega, \\ \frac{\partial u}{\partial n} = 0 \quad \text{on } \partial\Omega. \end{gathered} \tag{2.4}$$

Let us consider the multivalued function $\hat{p}$ defined by

$$\hat{p}(s) := \begin{cases} p_m(s) & \text{if } s \neq a, \\ [ma, ma + q(a)] & \text{if } s = a, \end{cases}$$

where $b = q(a)$.

Let $p^*$ denote the generalized inverse of $\hat{p}$ given by

$$p^*(w) = s \iff w \in \hat{p}(s).$$

**Remark 2.4.** The function $p^*$ is a continuous though $\hat{p}$ is a multivalued function, and

$$p^*(w) = a \iff ma \le w \le p_m(a) = ma + q(a).$$

Defining $P^*(w) := \int_0^w p^*(s)\,d\boldsymbol{s}$, we see that $P^* \in \mathcal{C}^1(\mathbb{R})$. Also, from (2.2),

$$\frac{w}{m+\alpha} - \frac{c_0 + q(a)}{m} \le p^*(w) \le \frac{w}{m} \quad \text{for all } w \in \mathbb{R}. \tag{2.5}$$

From the above inequalities we obtain

$$P^*(w) \ge \frac{1}{2}\frac{w^2}{m+\alpha} - \frac{c_0 + q(a)}{m}|w| \quad \text{for all } w \in \mathbb{R}, \tag{2.6}$$

$$P^*(w) \le \frac{w^2}{2m} \quad \text{for all } w \in \mathbb{R}. \tag{2.7}$$

Assuming that $\Omega$ of class $\mathcal{C}^2$, for every $w \in L^2(\Omega)$ the problem

$$\begin{gathered} (\Delta^2 + m)v = w \quad \text{in } \Omega, \\ v = 0 \quad \text{on } \partial\Omega, \\ \frac{\partial v}{\partial n} = 0 \quad \text{on } \partial\Omega \end{gathered}$$

has a unique weak solution $v \in H_0^2(\Omega) \cap H^4(\Omega)$. Defining $v = G(w)$, elliptic regularity theory implies that $G$ is a continuous linear operator from $L^2(\Omega)$ into $H_0^2(\Omega) \cap H^4(\Omega))$. Moreover,

$$\int_\Omega w(x)G(w)(x)dx \le \frac{1}{m+\mu_1}\int_\Omega w^2(x)dx. \tag{2.8}$$

Next we define $f : L^2(\Omega) \to \mathbb{R}$ by

$$f(w) := \int_\Omega \Big(P^*(w) - \frac{1}{2}wG(w)\Big)\,d\mathbf{x}.$$

Since $P^*$ is a differentiable function, $f \in \mathcal{C}^1(L^2(\Omega))$.

## 3. Main results

**Lemma 3.1.** *If $w \in L^2(\Omega)$ is a critical point of $f$, then $u := G(w)$ is a solution to (2.3) in the sense that $u \in H_0^2(\Omega) \cap H^4(\Omega)$ and $\Delta^2 u = p(u)$ a.e. in $\Omega$.*

*Proof.* Let $w \in L^2(\Omega)$ be such that $f'(w) = 0$, then $p^*(w) = G(w)$ a.e. in $\Omega$. Hence $u := G(w) \in H_0^2(\Omega) \cap H^4(\Omega)$ and satisfies $(\Delta^2 + m)u = w$. This implies that $p^*(w) = u$ a.e. in $\Omega$, and from the definition of $p^*$ we obtain that $w \in \hat{p}(u)$, and hence

$$\Delta^2 u + mu \in \hat{p}(u) \quad \text{a.e. in } \Omega.$$

For $x \in \Omega \setminus \Omega_a$, i.e., when $u(x) \ne a$ we have $\hat{p}(u(x)) = mu(x) + p(u(x))$ and then $\Delta^2 u(x) = p(u(x))$ a.e. $x \in \Omega \setminus \Omega_a$.

Since $u$ is constant a.e. in $\Omega_a$, $\Delta^2 u = 0$ a.e. in $\Omega_a$. Therefore,

$$\Delta^2 u + p_m(u(x)) = mu(x) + H(0)q(a) = ma \quad \text{a.e. in } \Omega.$$

Thus $\Delta^2 u = p(u)$ a.e. in $\Omega_a$. These show that $u$ is a solution of (2.3). $\square$

Next we apply the *direct method of the calculus of variations* to prove the existence of a solution (2.3).

**Theorem 3.2** (First existence theorem)**.** *There exists $w_0 \in L^2(\Omega)$ such that*

$$f(w_0) = \min_{w\in L^2(\Omega)} f(w).$$

*Fixing $u_0 := G(w_0)$, where $u_0$ is a solution of (2.3), the set*

$$\Omega_a = \{x \in \Omega : u_0(x) = a\}$$

*has zero measure.*

*Proof.* For $w \in L^2(\Omega)$, from (2.8) and (2.6),

$$f(w) \geq \frac{1}{2}\Big[\frac{1}{m+\alpha} - \frac{1}{m+\mu_1}\Big]\|w\|^2_{L^2(\Omega)} - C\|w\|_{L^2(\Omega)}. \tag{3.1}$$

The hypothesis $0 < \alpha < \mu_1$ and the inequality (3.1) implies

$$\lim_{\|u\|_{L^2(\Omega)}\to+\infty} f(u) = +\infty. \tag{3.2}$$

That is, $f$ is coercive. Let $\hat{m} = \inf_{w\in L^2(\Omega)} f(w)$. From the coercivity of $f$, we have $\hat{m} > -\infty$. This and the compactness of $G$ imply that $f$ attains its global minimum at some $w_0$. Let $u_0 = G(w_0)$ be a solution of (2.3).

Let $\chi$ denote the characteristic function of $\Omega_a$. This results in

$$\begin{aligned}\frac{d}{d\varepsilon} f(w_0+\varepsilon\chi) &= \int_\Omega (p^*(w_0+\varepsilon\chi) - \varepsilon G(\chi) - G(w_0))\chi\, d\mathbf{x}\\ &= \int_{\Omega_a} p^*(w_0+\varepsilon\chi)\, d\mathbf{x} - \varepsilon\int_\Omega \chi G(\chi)\, d\mathbf{x} - \int_{\Omega_a} u_0\, d\mathbf{x}\end{aligned}$$

for every $\varepsilon \in \mathbb{R}$. From $G(w_0) = u_0$ and $\Delta^2 u_0 = 0$ a.e. in $\Omega_a$, it follows that $w_0 = ma$ a.e. in $\Omega_a$. Hence, taking $0 < \varepsilon < b$, one finds that

$$ma \leq w_0 + \varepsilon\chi \leq ma + b = ma + q(a)$$

a.e. in $\Omega_a$. Then $p^*(w_0(x) + \varepsilon\chi(x)) = a$ a.e. in $\Omega_a$ and

$$\int_{\Omega_a} p^*(w_0+\varepsilon\chi)\, d\mathbf{x} = \int_{\Omega_a} a\, d\mathbf{x} = a|\Omega_a| = \int_{\Omega_a} u_0\, d\mathbf{x}.$$

Since $\chi \in L^2(\Omega)$ by the definition of $G$ there exists $z \in H^2_0(\Omega) \cap H^4(\Omega)$ such that $z = G(\chi)$, it follows that

$$(G(\chi) \mid \chi) = \int_\Omega (z\Delta^2 z + mz^2)\, d\mathbf{x}.$$

The above equalities imply

$$\frac{d}{d\varepsilon} f(w_0+\varepsilon\chi) = -\varepsilon\Big(\int_\Omega (\Delta z)^2\, d\mathbf{x} + m\|z\|^2_{L^2(\Omega)}\Big).$$

If $|\Omega_a| > 0$, it follows that

$$\frac{d}{d\varepsilon} f(w_0+\varepsilon\chi) < 0$$

a contradiction, because $w_0$ is the global minimum of $f$. □

We note that the last arguments of the proof are valid for any local minimum of $f$. The next lemma and Lemma 3.5 prove that the graph $f$ satisfies the geometric hypotheses of the Mountain-Pass theorem.

**Lemma 3.3.** *For each $a > 0$ and $m > 0$, there exists $\epsilon > 0$ and $\gamma > 0$ such that if $\|u\|_{L^2(\Omega)} \le \epsilon$ then $f(u) \ge \gamma \|u\|^2_{L^2(\Omega)}$. Hence $f$ attains a strict local minimum at $u = 0$.*

*Proof.* Let $\alpha_1 \in (\alpha, \mu_1)$. Since $p^*(s) = ms$ for all $s \in (-\infty, a]$, $P^*(s) = \frac{s^2}{2m}$ for any $s \in (-\infty, ma]$. Also, from (2.2), there exists $c_1 \ge ma$ such that

$$P^*(s) \ge \frac{1}{2(m+\alpha_1)} s^2 \quad \text{for } s \ge c_1. \tag{3.3}$$

For $v \in L^2(\Omega)\setminus\{0\}$, let $W = \{x \in \Omega; ma \le v(x) \le c_1\}$, $v_1 = \chi_{\Omega\setminus W} v$ and $v_2 = \chi_W v$, where $\chi_S$ denotes the characteristic function of the set $S$. Thus,

$$\int_\Omega P^*(v_1)dx \ge \frac{1}{2(m+\alpha_1)} \int_\Omega v_1^2(x)dx. \tag{3.4}$$

Letting $|W|$ denote the Lebesgue measure of the set $W$, we have

$$|W| \le \frac{\|v_2\|^2_{L^2(\Omega)}}{m^2a^2} = \frac{\|v_2\|^2_{L^2(W)}}{m^2a^2}. \tag{3.5}$$

Since $p^*(ma) = a$, for $s \in [ma, c_1]$ we have $P^*(s) \ge \frac{a}{2c_1}s^2$. Therefore

$$\frac{a}{2c_1}\int_W v_2^2(x)dx \le \int_W P^*(v_2(x))dx \le \frac{c_1^2}{2m}|W| \le \frac{c_1^2}{2m^3a^2}\int_W v_2^2(x)dx. \tag{3.6}$$

From the definition of $\mu_1$, we have $\int_\Omega G(v_1)v_1dx \le \frac{1}{m+\mu_1}\int_\Omega v_1^2dx$. By regularity properties of elliptic operators, there exist $p > 2$ and $K > 0$ such that

$$\|G(u)\|_{L^p(\Omega)} \le K(p)\|u\|_{L^2(\Omega)} \quad \text{for all } u \in L^2(\Omega). \tag{3.7}$$

Hence, for $i = 1, 2$, see (3.5),

$$\begin{aligned}
\int_\Omega v_2(x)G(v_i(x))dx &= \int_W v_2(x)G(v_i(x))dx \\
&\le \|v_2\|_{L^2(\Omega)}\Big(\int_W (G(v_i))^2(x)dx\Big)^{1/2} \\
&\le \|v_2\|_{L^2(\Omega)}\Big(\int_W (G(v_i))^p(x)dx\Big)^{1/p}|W|^{(p-2)/2p} \\
&\le K(p)\|v_2\|_{L^2(\Omega)}\|v_i\|_{L^2(\Omega)}|W|^{(p-2)/2p} \\
&\le \frac{K(p)}{(ma)^{(p-2)/p}}\|v_2\|^{2(p-1)/p}_{L^2(\Omega)}\|v_i\|_{L^2(\Omega)}.
\end{aligned} \tag{3.8}$$

Therefore,

$$\begin{aligned}
&\int_\Omega v(x)G(v(x))dx \\
&= \int_\Omega (v_1G(v_1) + v_2G(v_1) + v_1G(v_2) + v_2G(v_2))dx \\
&\le \frac{1}{m+\mu_1}\|v_1\|^2_{L^2(\Omega)} + \int_\Omega (2v_2G(v_1) + v_2G(v_2))dx \\
&= \frac{1}{m+\mu_1}\|v_1\|^2_{L^2(\Omega)} + \int_W (2v_2G(v_1) + v_2G(v_2))dx \\
&\le \frac{1}{m+\mu_1}\|v_1\|^2_{L^2(\Omega)} \\
&\quad + \frac{K(p)}{(ma)^{(p-2)/p}}\|v_2\|^{2(p-1)/p}_{L^2(\Omega)}\left(2\|v_1\|_{L^2(\Omega)} + \|v_2\|_{L^2(\Omega)}\right) \\
&\le \frac{1}{m+\mu_1}\|v_1\|^2_{L^2(\Omega)} + C\|v_2\|^{2(p-1)/p}_{L^2(\Omega)}\left(\|v_1\|_{L^2(\Omega)} + \|v_2\|_{L^2(\Omega)}\right),
\end{aligned} \tag{3.9}$$

with $C > 0$ independent of $v$. Combining (3.4), (3.6), and (3.9), we have

$$\begin{aligned}
f(v) &= \int_\Omega \left[P^*(v(x)) - \frac{1}{2}v(x)G(v(x))\right]dx \\
&\ge \frac{1}{2(m+\alpha_1)}\|v_1\|^2_{L^2(\Omega)} + \frac{a}{2c_1}\|v_2\|^2_{L^2(\Omega)} - \frac{1}{2(m+\mu_1)}\|v_1\|^2_{L^2(\Omega)} \\
&\quad - C\|v_2\|^{2(p-1)/p}_{L^2(\Omega)}\left(\|v_1\|_{L^2(\Omega)} + \|v_2\|_{L^2(\Omega)}\right) \\
&\ge \frac{\mu_1-\alpha_1}{4(m+\alpha_1)(m+\mu_1)}\|v_1\|^2_{L^2(\Omega)} + \frac{a}{2c_1}\|v_2\|^2_{L^2(\Omega)} \\
&\quad - C\|v_2\|^{2(p-1)/p}_{L^2(\Omega)}\left(\|v_1\|_{L^2(\Omega)} + \|v_2\|_{L^2(\Omega)}\right) \\
&\ge \gamma_1\|v\|^2_{L^2(\Omega)} - 2C\|v\|^{1+2(p-1)/p}_{L^2(\Omega)} \\
&\ge \gamma_1\|v\|^2_{L^2(\Omega)}\Big(1 - \frac{2C}{\gamma_1}\|v\|^{(3p-2)/p}_{L^2(\Omega)}\Big),
\end{aligned} \tag{3.10}$$

where

$$\gamma_1 = \min\{\frac{\mu_1-\alpha_1}{4(m+\alpha_1)(m+\mu_1)}, \frac{a}{2c_1}\}.$$

Since $p > 2$, $(3p-2)/p > 0$. Hence taking $\epsilon = (\gamma_1/(4C))^{p/(3p-2)}$ and $\gamma = \gamma_1/2$, the lemma is proven. $\square$

The next lemmas show that, under suitable conditions on $\Omega$ and an appropriate relationship between $a$ and $q(a)$, $f$ possesses a pair of non-trivial critical points: a negative global minimum and a positive Mountain-Pass critical point.

**Definition 3.4.** Let $U$ be a domain in $\mathbb{R}^N$, $k \in \mathbb{N}$, $\gamma \in [0,1)$, and $\varepsilon > 0$. We say that $U$ is $\varepsilon$-close in $\mathcal{C}^{k,\gamma}$-sense to the unit ball $B$ if there exists a surjective mapping $g \in \mathcal{C}^{k,\gamma}(\overline{B};\overline{U})$ such that

$$\|g - Id\|_{\mathcal{C}^{k,\gamma}(\overline{B};\overline{U})} \le \varepsilon.$$

In 2020 Grunau and Sweers[13] show that there is $\varepsilon_N > 0$ such that if $\Omega$ is $\varepsilon$-close in $\mathcal{C}^{4,\gamma}$-sense to the unitary ball $B$ with $\varepsilon < \varepsilon_N$, then the first eigenfunction $\varphi_1$ for the first eigenvalue $\mu_1$ of

$$\begin{gathered}\Delta^2\varphi=\mu\varphi \quad \text{in } \Omega,\\ \varphi=0 \quad \text{on } \partial\Omega,\\ \frac{\partial\varphi}{\partial n}=0 \quad \text{on } \partial\Omega\end{gathered}$$

is unique (up to normalization), and $\varphi_1>0$ in $\Omega$.

**Lemma 3.5.** *Let $\Omega$ be $\varepsilon$-close in $\mathcal{C}^{k,\gamma}$-sense to the unit ball $B$. If*

$$\frac{q(a)}{a}=\frac{b}{a}>2\mu_1\frac{\|\varphi_1\|_{L^1(\Omega)}}{\|\varphi_1\|^2_{L^2(\Omega)}}, \tag{3.11}$$

*then $f(b\varphi_1)<0$.*

*Proof.* Since $0<b\varphi_1(x)\le b$ and $p^*(w)\le a$, for $0\le w\le b$, it follows that

$$\begin{aligned} f(b\varphi_1)&=\int_\Omega P^*(b\varphi_1)\,d\mathbf{x}-\frac12 b^2\int_\Omega G(\varphi_1)\varphi_1\,d\mathbf{x}\\ &\le ba\|\varphi_1\|_{L^1(\Omega)}-\frac{b^2}{2\mu_1}\|\varphi_1\|^2_{L^2(\Omega)}. \end{aligned}$$

This and (3.11) imply $f(b\varphi_1)<0$. □

Finally, we prove that $f$ satisfies a weak form of (PS) condition.

**Lemma 3.6.** *Let $\{w_k\}_{k\in\mathbb{N}}$ in $L^2(\Omega)$ be such that $\{f'(w_k)\}_{k\in\mathbb{N}}$ converges to $0$ and $\{f(w_k)\}_{k\in\mathbb{N}}$ converges to a real number $c$, then there exists $w\in L^2(\Omega)$ with $f(w)=c$, $f'(w)=0$, and $w_k\rightharpoonup w$.*

*Proof.* The coercivity of the functional $f$ implies, up to subsequences, the existence of $w\in L^2(\Omega)$ such that $w_n\rightharpoonup w$ in $L^2(\Omega)$. From $f'(w_k)\to 0$ and the compactness of $G$, it follows that $G(w_n)\to v:=G(w)$, strongly in $L^2(\Omega)$, and a.e. in $\Omega$. Let $\Gamma=\{x\in\Omega: v(x)=a\}$ and $\Omega_1=\Omega\setminus\Gamma$.

Let us begin studying the convergence in $\Omega_1$. Since $p\in\mathcal{C}(\mathbb{R}\backslash\{a\})$ and $p^*(w_k)\to v$ a.e. in $\Omega$, hence $w_k\to p(v)$ a.e. in $\Omega_1$. Clearly, $|w|\le C_1|p^*(w)|+C_2$; this and the convergence of $\{p^*(w_k)\}_{k\in\mathbb{N}}$ in $L^2(\Omega)$ imply that there exists $h\in L^2(\Omega)$ such that $|w_k|\le h$ for every $k\in\mathbb{N}$. Applying the Lebesgue dominated convergence theorem: $w_k\to p(v)$ a.e. in $L^2(\Omega_1)$. From the uniqueness of the weak limit, one infers that $w=p(v)$ in $L^2(\Omega_1)$. Since $p^*$ is asymptotically linear, it follows that

$$p^*(w_k)\to p^*(w) \text{ in } L^2(\Omega_1), \quad \text{and} \quad \int_{\Omega_1}P^*(w_k)\,d\mathbf{x}\to\int_{\Omega_1}P^*(w)\,d\mathbf{x}. \tag{3.12}$$

On the other hand, for a.e. $x\in\Gamma$, one has $w(x)=mv(x)=ma$ and hence $p^*(w(x))=p^*(ma)=a=v(x)$. This jointly with (3.12) imply $p^*(w)=v$, which in turn$f'(w)v=0$, hence $f'(w)=0$. In a similar way, from (3.12) and the definition of $P^*(s)$ for $s\in[ma,ma+b]$, one finds that

$$\int_\Omega P^*(w_k)\,d\mathbf{x}\to\int_\Omega P^*(w)\,d\mathbf{x}.$$

Letting $c=\int_\Omega[P^*(w)-\frac12 wG(w)]\,d\mathbf{x}$ it follows that $f(w)=c$, which completes the proof. □

**Theorem 3.7.** *Assume that the domain $\Omega$ is $\varepsilon$-close in $\mathcal{C}^{k,\gamma}$-sense to the unit ball $B$. Suppose that* (2.1), (2.2), *and* (3.11) *hold. Then the problem* (2.3) *has two distinct solutions $u_0 \neq u_1$, and one of these solutions, obtained through the minimizer, has a free boundary set of measure zero.*

*Proof.* Let $w_0$ be the global minimum of $f$ given by Theorem 3.2. By Lemma 3.5, $f(w_0) < 0$. Hence $w_0 \neq 0$ and $u_0 = G(w_0)$ is a non-trivial solution of (2.3) and the free boundary $\Omega_a(u_0) = \{x \in \Omega : u_0(x) = a\}$ has zero measure.

Taking $\rho = \epsilon/2 > 0$ and $\beta = \gamma\epsilon/2 > 0$ in Lemma 3.3 we see that $f(u) \geq \beta > 0$ for $\|u\|_{L^2(\Omega)} = \rho > 0$. This Lemmas 3.5, and 3.6 allow us to apply the Mountain-Pass Theorem (see [5]), yielding a second non-trivial critical point $w_1$, with $f(w_1) \geq \beta > 0$. Hence $u_1 = G(w_1) \neq 0$ is a second non-trivial solution of (2.3). Since $f(w_0) < 0 < f(w_1)$, $w_0 \neq w_1$ and as a consequence $u_0 \neq u_1$.

Finally, the zero measure of $\Omega_a(u_0)$ follows from the fact that $u_0$ minimizes $f$ over all functions with zero measure on the set $\Omega_a(u_0)$, as proven in Theorem 3.2. However, it is possible for the free boundary of $u_1$ to have positive measure.

Therefore, by Lemma 3.1, problem (2.3) has two different solutions $u_0 \neq u_1$, with the free boundary of $u_0$ having zero measure. □

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