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UNIFORM ATTRACTORS OF NON-AUTONOMOUS SUSPENSION BRIDGE EQUATIONS WITH MEMORY

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Abstract. In this article, we investigate the long-time dynamical behavior of non-autonomous suspension bridge equations with memory and free boundary conditions. We first establish the well-posedness of the system by means of the maximal monotone operator theory. Secondly, the existence of uniformly bounded absorbing set is obtained. Finally, asymptotic compactness of the process is verified, and then the existence of uniform attractors is proved for non-autonomous suspension bridge equations with memory term.

1. Introduction

In this article, we focus on the long-time dynamical behavior of solutions for the following non-autonomous suspension bridge equations with memory in $\Omega = (0,\pi)\times(-l,l)\subset\mathbb{R}^2$,

$$\begin{gathered} u_{tt}+\alpha\Delta^2u+\beta u_t-\int_0^\infty \mu(s)\Delta^2u(t-s)ds+f(u(x,y,t))=g(x,y,t), \\ (x,y)\in\Omega,\ t\geq\tau,\ \tau\in\mathbb{R}, \end{gathered} \tag{1.1}$$

with the boundary conditions

$$\begin{gathered} u(0,y,t)=u_{xx}(0,y,t)=u(\pi,y,t)=u_{xx}(\pi,y,t)=0, \quad y\in(-l,l),\ t\geq\tau, \\ u_{yy}(x,\pm l,t)+\sigma u_{xx}(x,\pm l,t)=0, \quad x\in(0,\pi), t\geq\tau, \\ u_{yyy}(x,\pm l,t)+(2-\sigma)u_{xxy}(x,\pm l,t)=0, \quad x\in(0,\pi), t\geq\tau, \end{gathered} \tag{1.2}$$

and initial conditions

$$u(x,y,t)=u_0^\tau(x,y), \quad u_t(x,y,t)=v_0^\tau(x,y), \quad (x,y)\in\Omega,\ t\geq\tau,\ \tau\in\mathbb{R}, \tag{1.3}$$

where α, β are positive constant, β is the damping coefficient, $0<\sigma<\frac{1}{2}$ is the Poisson ratio, f is the nonlinear term, g is the external force. Since we have in mind a long narrow rectangle, that is $l\ll\pi$, it is reasonable to assume that the forcing term g does not depend on y, see [9] for details. So, we now assume that $g(x,t)=g(x,y,t)$ and $g\in L^2_{\rm loc}(\mathbb{R}^+;L^2(\Omega))$. The assumptions on $\mu(s)$, $f(u)$ will be given in details in the next section.

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We know that the earlier suspension bridge equations are derived from the mathematical model of a one-dimensional hinged beam suspended by hangers, which describes the deflection of the roadbed in the vertical plane, see [15, 17]. As a new problem in the field of nonlinear analysis in 1990, Lazer and McKenna [16] introduced the following one-dimensional suspension bridge equation

$$\begin{gathered} u_{tt} + EIu_{xxxx} + \delta u_t + ku^+ = W(x) + \varepsilon f(x,t), \quad (x,t) \in (0,L) \times \mathbb{R}^+, \\ u(0,t) = u(L,t) = u_{xx}(0,t) = u_{xx}(L,t) = 0, \quad t \geq 0. \end{gathered} \tag{1.4}$$

In 1998, Ahmed and Harbi [1] made a rigorous mathematical analysis for the coupled suspension bridge equations, they studied the dynamical behavior of system under the different conditions, which are clamped, hinged and mixed boundary condition (one end clamped and the other one hinged), respectively, and gave the relevant numerical simulation and physical interpretation.

A series of important works have investigated around the existence of a global attractor for suspension bridge equations, see for example [3, 9, 14, 17, 18, 19, 20, 24, 25, 27, 28, 29, 30] and the references therein. Ma and Zhong [18] first obtained the global attractor of the weak solution for coupled suspension bridge equations in 2005, and they further studied the existence of strong solution and strong global attractor for beam-string coupling system in [30]. Bochicchio, Giorgi and Vuk [3] proved the existence and regularity of the global attractor with finite fractal dimension for the extensible suspension bridge equation. Park and Kang [25] studied existence of global attractor for suspension bridge equation with nonlinear damping in 2011. Recently, Wang and Ma surveyed the long-term behavior of solutions for the suspension bridge equation with either time delay or state delay, see [28, 29].

When $\mu = 0$ in (1.1), Ferrero and Gazzola [9] introduced the following model of suspension bridges

$$u_{tt}(x,y,t) + \Delta^2 u(x,y,t) + \alpha u_t(x,y,t) + f(x,y,u) = g(x,y,t),$$

for $(x,y) \in \Omega$ and $t > 0$. The above model regards the suspension bridges as a rectangular plate of length π with the same boundary value conditions as (1.2). They obtained the well-posedness of the system and analyzed several other boundary value problems. For further details on mathematical models for suspension bridge, we refer the reader to the new book [11] published by Gazzola.

More much work related to the above-mentioned rectangular plate models for suspension bridge can be found in [2, 4, 5, 10, 12, 13, 22, 23, 27] and reference therein. For example, Messaoudi et al. [23] considered the suspension bridge problem with memory under the above-mentioned boundary conditions and initial data in 2016, and established the well-posedness of the system and the existence of global attractors. Al-Gwaiz et al. [2] studied the bending and stretching energy about the rectangular plate model proposed in [9]. Berchio et al. [5] investigated the structural instability of nonlinear plate modeling suspension bridges. In 2019, Wang and Ma [27] paid attention to the following nonlinear plate modeling suspension bridges with time delay in $\Omega = (0,\pi) \times (-l,l)$ under the same conditions (1.2) as in [9],

$$\begin{gathered} \partial_{tt} u + \Delta^2 u + \gamma_1 \partial_t u + \gamma_2 \partial_t u(x,y,t-h) + f(u(x,y,t)) = g(x,y,t), \\ (x,y) \in \Omega,\ t \geq \tau,\ \tau \in \mathbb{R}, \end{gathered} \tag{1.5}$$

where $\gamma_1 > 0$ is the damped coefficient, $\gamma_2 \in \mathbb{R}$. $\partial_t u(x,y,t-h)$ is the delay term, $h > 0$ represents the time delay. The existence of uniform attractors was achieved for (1.5) and (1.2).

To the best of our knowledge, we do not find any results of non-autonomous suspension bridge equations with history memory, so we focus on the long-time dynamical behavior of problem (1.1)-(1.3). For this purpose, as in [8], we shall add a new variable η^t to the system, which corresponds to the relative displacement history, that is,

$$\eta^t = \eta^t(x,y,s) = u(x,y,t) - u(x,y,t-s), \quad (x,y) \in \Omega,\ s \in \mathbb{R}^+,\ t \geq \tau, \tag{1.6}$$

then differentiating with respect to t, it is easy to see that

$$\eta^t_t(x,y,s) = -\eta^t_s(x,y,s) + u_t(x,y,t), \quad (x,y) \in \Omega,\ s \in \mathbb{R}^+,\ t \geq \tau. \tag{1.7}$$

Thus, taking $\alpha - \int_0^\infty \mu(s)ds = 1$, problem (1.1)-(1.3) is equivalent to

$$\begin{gathered} u_{tt} + \Delta^2 u + \beta u_t + \int_0^\infty \mu(s)\Delta^2\eta^t(s)ds + f(u(x,y,t)) = g(x,t), \\ (x,y) \in \Omega,\ t \geq \tau,\ \tau \in \mathbb{R}, \\ \eta^t_t(x,y,s) = -\eta^t_s(x,y,s) + u_t(x,y,t), \quad (x,y) \in \Omega,\ s \in \mathbb{R}^+,\ t \geq \tau, \end{gathered} \tag{1.8}$$

with boundary conditions

$$\begin{gathered} u(0,y,t) = u_{xx}(0,y,t) = u(\pi,y,t) = u_{xx}(\pi,y,t) = 0, \\ y \in (-l,l), t \geq \tau, \\ u_{yy}(x,\pm l,t) + \sigma u_{xx}(x,\pm l,t) = 0, \quad x \in (0,\pi), t \geq \tau, \\ u_{yyy}(x,\pm l,t) + (2-\sigma)u_{xxy}(x,\pm l,t) = 0, \quad x \in (0,\pi), t \geq \tau, \end{gathered} \tag{1.9}$$

$$\begin{gathered} \eta^t(0,y,s) = \eta^t_{xx}(0,y,s) = \eta^t(\pi,y,s) = \eta^t_{xx}(\pi,y,s) = 0, \\ y \in (-l,l),\ s \in \mathbb{R}^+, \\ \eta^t_{yy}(x,\pm l,s) + \sigma\eta^t_{xx}(x,\pm l,s) = 0, \quad x \in (0,\pi),\ s \in \mathbb{R}^+, \\ \eta^t_{yyy}(x,\pm l,s) + (2-\sigma)\eta^t_{xxy}(x,\pm l,s) = 0, \quad x \in (0,\pi),\ s \in \mathbb{R}^+, \end{gathered} \tag{1.10}$$

and initial conditions

$$\begin{gathered} u(x,y,\tau) = u_0^\tau(x,y), \quad (x,y) \in \Omega,\ \tau \in \mathbb{R}, \\ u_t(x,y,\tau) = v_0^\tau(x,y), \quad (x,y) \in \Omega,\ \tau \in \mathbb{R}, \\ \eta^\tau(x,y,s) = \eta_0^\tau(x,y,s), \quad (x,y) \in \Omega,\ s \in \mathbb{R}^+, \\ \eta^t(x,y,0) = 0, \quad (x,y) \in \Omega. \end{gathered} \tag{1.11}$$

We denote

$$z(t) = (u(t), u_t(t), \eta^t(s)), \quad z_0 = (u_0^\tau, v_0^\tau, \eta_0^\tau).$$

The rest of this article is organized as follows. In Section 2, we present some basic concepts and abstract conclusion. After that we establish the well-posedness of the system by means of the maximal monotone operator theory, and further obtain the existence of the uniformly bounded absorbing set and asymptotical compactness; ultimately, the existence of uniform attractors to (1.8)-(1.11) is proved in Section 3.

All C throughout the paper represent real positive numbers, each C is not exactly the same in the same line, and $C(\cdot)$ denotes a positive constant depending on the quantities in parentheses.

2. Preliminaries

In this section, we will give some preliminaries on the existence and uniqueness of solutions to our problem (1.8)-(1.11), and recall some definitions and results concerning the existence of uniform attractors. Firstly, let us introduce the phase space as in [9]

$$H^2_*(\Omega) = \{w \in H^2(\Omega) : w(0,y) = w(\pi,y) = 0,\, \forall y \in (-l,l)\},$$

equipped with the inner product and norm

$$(u,v)_{H^2_*} = \int_\Omega [\Delta u \Delta v + (1-\sigma)(2u_{xy}v_{xy} - u_{xx}v_{yy} - u_{yy}v_{xx})]\,dx\,dy,$$
$$\|u\|_{H^2_*} = \Big[\int_\Omega [(\Delta u)^2 + 2(1-\sigma)(u_{xy}^2 - u_{xx}u_{yy})]\,dx\,dy\Big]^{1/2}.$$

It has been proven that $\|\cdot\|_{H^2_*}$ is a norm on H^2_* which is equivalent to the usual $H^2(\Omega)$-norm in [9, Lemma 4.1]. Moreover, H^2_* is a Hilbert space endowed with the scalar product $(\cdot,\cdot)_{H^2_*}$.

For the new variable η^t, we introduce the weighted L^2-space

$$\mathcal{M} = L^2_\mu(\mathbb{R}^+; H^2_*(\Omega)) = \Big\{\xi : \mathbb{R}^+ \to H^2_*(\Omega) : \int_0^\infty \mu(s)\|\xi(s)\|^2_{H^2_*} ds < \infty\Big\},$$

which is a Hilbert space endowed with inner product and norm

$$(\xi,\zeta)_\mathcal{M} = \int_0^\infty \mu(s)(\xi(s),\zeta(s))_{H^2_*} ds, \quad \|\xi\|^2_\mathcal{M} = \int_0^\infty \mu(s)\|\xi(s)\|^2_{H^2_*} ds,$$

respectively. Now, the phase space is defined as

$$\mathcal{H} = H^2_*(\Omega) \times L^2(\Omega) \times \mathcal{M},$$

equipped with the inner product and norm

$$(U,V)_\mathcal{H} = (u,\tilde{u})_{H^2_*(\Omega)} + (v,\tilde{v}) + (w,\tilde{w})_\mathcal{M},$$
$$\|U\|^2_\mathcal{H} = (U,U)_\mathcal{H} = \|u\|^2_{H^2_*(\Omega)} + \|v\|^2 + \|w\|^2_\mathcal{M},$$

respectively. where $\|\cdot\| = \|\cdot\|_{L^2(\Omega)}$, and

$$U = (u,v,w)^T, \quad V = (\tilde{u},\tilde{v},\tilde{w})^T \in \mathcal{H}.$$

Next, we assume that the memory and nonlinear term satisfy the following conditions:

(H1) The memory kernel $\mu(\cdot) \in C^1(\mathbb{R}^+) \cap L^1(\mathbb{R}^+)$ and satisfies

$$\mu(s) \ge 0, \quad \mu'(s) \le 0, \quad \forall s \in \mathbb{R}^+, \tag{2.1}$$
$$\int_0^\infty \mu(s)ds = k_0 > 0, \quad \forall s \in \mathbb{R}^+, \tag{2.2}$$
$$\mu'(s) + k_1\mu(s) \le 0, \quad \text{for some } k_1 > 0,\ \forall s \in \mathbb{R}^+. \tag{2.3}$$

(H2) The nonlinear function $f \in C^1(\mathbb{R})$ and satisfies

$$|f(s_1) - f(s_2)| \le C(|s_1|^p + |s_2|^p)|s_1 - s_2|, \quad \forall s_1,s_2 \in \mathbb{R},\ p > 0, \tag{2.4}$$
$$-c \le F(s) \le sf(s), \quad \forall s \in \mathbb{R}, \tag{2.5}$$

where $F(s) = \int_0^s f(\nu)d\nu$.

Lemma 2.1 ([9]). *Let $u \in H_*^2(\Omega)$ and suppose that $1 \le p < +\infty$. Then there exists a positive constant $c_* = c_*(\Omega, p) > 0$ such that*

$$\|u\|_{L^p(\Omega)} \le c_* \|u\|_{H_*^2(\Omega)}. \tag{2.6}$$

To obtain the existence of uniform attractors corresponding to (1.8)-(1.11), we also need the following definitions and abstract results.

Definition 2.2. Let E be a metric space, Σ be a parameter set, and $\sigma \in \Sigma$ be a time symbol. A family of two-parameter operators $\{U_\sigma(t,\tau)\} = \{U_\sigma(t,\tau) | t, \tau \in \mathbb{R}, t \ge \tau\}$ is called a process acting on E, if

(i) $U_\sigma(t,s)U_\sigma(s,\tau) = U_\sigma(t,\tau)$ for all $t \ge s \ge \tau$ and $\tau \in \mathbb{R}$;
(ii) $U_\sigma(\tau,\tau) = I$ for all $\tau \in \mathbb{R}$.

Let $\{T(h) | h \ge 0\}$ be the translation semigroup on Σ. We say that a family of processes $\{U_\sigma(t,\tau)\}_{\sigma\in\Sigma}$ satisfies the translation identity if

$$T(h)\Sigma = \Sigma, \tag{2.7}$$

$$U_\sigma(t+h, \tau+h) = U_{T(h)}(t,\tau), \quad \forall \sigma \in \Sigma,\ t \ge \tau,\ \tau \in \mathbb{R},\ h \ge 0. \tag{2.8}$$

Definition 2.3 ([10, 7]). Let E be a Banach space, and $\mathcal{B}$ be a bounded subset of E and Σ be a symbol space. We call a function $\phi(\cdot,\cdot;\cdot,\cdot)$ defined on $(E\times E)\times(\Sigma\times\Sigma)$ be a contractive function on $\mathcal{B}\times\mathcal{B}$, if for any sequence $\{x_n\}_{n=1}^\infty \subset \mathcal{B}$ and $\{\sigma_n\}_{n=1}^\infty \subset \Sigma$, there are subsequence $\{x_{n_k}\}_{k=1}^\infty \subset \{x_n\}_{n=1}^\infty$ and $\{\sigma_{n_k}\}_{k=1}^\infty \subset \{\sigma_n\}_{n=1}^\infty$ such that

$$\lim_{k\to\infty}\lim_{l\to\infty} \phi(x_{n_k}, x_{n_l}; \sigma_{n_k}, \sigma_{n_l}) = 0.$$

We denote the set of all contractive functions on $\mathcal{B}\times\mathcal{B}\times\Sigma\times\Sigma$ by $\mathfrak{C}(\mathcal{B},\mathcal{B};\Sigma,\Sigma)$.

Theorem 2.4 ([26]). *Let $\{U_\sigma(t,\tau)\}_{\sigma\in\Sigma}$ be a family of processes satisfying the translation identity* (2.8) *on a Banach space E and having a bounded uniformly (w.r.t.$\sigma \in \Sigma$) absorbing set $\mathcal{B}_0 \subset E$. Moreover, assume that for any $\varepsilon > 0$ there exist $T = T(\varepsilon)$ and $\phi_T \in \mathfrak{C}(\mathcal{B}_0,\mathcal{B}_0;\Sigma,\Sigma)$ such that*

$$\|U_{\sigma_1}(T,\tau)x - U_{\sigma_2}(T,\tau)y\| \le \varepsilon + \phi_T(x,y;\sigma_1,\sigma_2), \quad \forall x,y \in \mathcal{B}_0,\ \forall \sigma_1,\sigma_2 \in \Sigma. \tag{2.9}$$

Then $\{U_\sigma(t,\tau)\}_{\sigma\in\Sigma}$ is uniformly (with respect to $\sigma \in \Sigma$) asymptotically compact in E.

Theorem 2.5 ([10, 26]). *Let E be a complete metric space, $\{U_\sigma(t,\tau)\}_{\sigma\in\Sigma}$ be a family of processes satisfying the translation identity* (2.8) *on E. Then $\{U_\sigma(t,\tau)\}_{\sigma\in\Sigma}$ has a compactly uniform (with respect to $\sigma \in \Sigma$) attractor $\mathcal{A}_\Sigma$ in E if and only if*

(i) *$\{U_\sigma(t,\tau)\}_{\sigma\in\Sigma}$ has a bounded uniformly (with respect to $\sigma \in \Sigma$) absorbing set $\mathcal{B}_0 \subset E$;*
(ii) *$\{U_\sigma(t,\tau)\}_{\sigma\in\Sigma}$ is uniformly (with respect to $\sigma \in \Sigma$) asymptotically compact in E.*

Let X be a Banach space with space. Then $L^p_{\rm loc}(\mathbb{R}^+;X)$ denotes all functions with spatial values in Banach space X and time variable locally p-power integrable in the Bochner sense; that is, the norm $\int_{t_1}^{t_2} \|\cdot\|_X^p ds < \infty$ for any time interval $[t_1,t_2] \subset \mathbb{R}^+$. Moreover, the space $L^2_b(\mathbb{R}^+;X)$ denotes all translation bounded functions in $L^2_{\rm loc}(\mathbb{R}^+;X)$ satisfying

$$\|\sigma\|^2_{L^2_b(\mathbb{R}^+;X)} = \sup_{t\in\mathbb{R}^+} \int_t^{t+1} \|\sigma(s)\|_X^2 ds < +\infty, \quad \forall \sigma \in L^2_b(\mathbb{R}^+;X)$$

Now, we define the symbol space so as to obtain the asymptotic behavior of the solutions to problem (1.8)-(1.11). For an arbitrary function $g_0 \in L^\infty(\mathbb{R}^+; L^2(\Omega)) \cap W_b^{1,r}(\mathbb{R}^+; L^r(\Omega))(r > 1)$, then we define the symbol space $\mathcal{H}(g_0)$ as

$$\mathcal{H}(g_0) = [g_0(x, t + r) | r \in \mathbb{R}^+]_{L^{2,w}_{\rm loc}(\mathbb{R}^+; L^2(\Omega))},$$

where $L^{2,w}_{\rm loc}(\mathbb{R}^+; L^2(\Omega))$ denotes the space $L^{2,w}_{\rm loc}(\mathbb{R}^+; L^2(\Omega))$ endowed with local weak convergence topology, and [ ] denotes the closure of a set in a topological space $L^{2,w}_{\rm loc}(\mathbb{R}^+; L^2(\Omega))$. Thus, for any $g \in \mathcal{H}(g_0)$, (1.8)-(1.11) with g_0 instead of g possesses a corresponding process $\{U_{g_0}(t, \tau)\}$ acting on $\mathcal{H}$. The translation semigroup $\{T(r) | r \geq 0\}$ satisfies (2.7) and (2.8), namely,

$$\begin{gathered} T(r)\mathcal{H}(g_0) = \mathcal{H}(g_0), \\ U_g(t + r, \tau + r) = U_{T(r)g(t)}(t, \tau), \end{gathered}$$

for all $g \in \mathcal{H}(g_0)$, $t \geq \tau$, $\tau \in \mathbb{R}$, $r \geq 0$.

Proposition 2.6 ([7]). *Let E be reflexive separable Banach space. Then the following statements hold:*

- (i) $\|g\|_{L_b^2(\mathbb{R}^+;E)} \leq \|g_0\|_{L_b^2(\mathbb{R}^+;E)}$, *for all* $g \in \mathcal{H}(g_0)$;
- (ii) *the translation group $T(t)$ is weakly continuous on $\mathcal{H}(g_0)$;*
- (iii) $T(r)\mathcal{H}(g_0) = \mathcal{H}(g_0)$, *for all* $r \in \mathbb{R}$.

Proposition 2.7 ([26]). *Let $\sigma \in L^\infty(\mathbb{R}^+; L^2(\Omega)) \cap W_b^{1,r}(\mathbb{R}^+; L^r(\Omega))(r > 1)$. Then there exists $M > 0$ such that*

$$\sup_{t \in \mathbb{R}^+} \|\sigma(x, t + s)\|_{L^2(\Omega)} \leq M, \quad \forall s \in \mathbb{R}^+.$$

Proposition 2.8 ([26]). *Let $\sigma \in L^\infty(\mathbb{R}^+; L^2(\Omega)) \cap W_b^{1,r}(\mathbb{R}^+; L^r(\Omega))(r > 1)$, $s_i \in \mathbb{R}(i = 1, 2, \cdots)$, $\{u_n(t) | t \geq 0, n = 1, 2, \cdots\}$ be bounded in $H^2(\Omega) \cap H_0^1(\Omega)$, and $\{u_{n_t}(t) | n = 1, 2, \cdots\}$ be bounded for any $T_1 > 0$ in $L^\infty(0, T_1; L^2(\Omega))$. Then there exist subsequence $\{u_{n_k}\}_{k=1}^\infty \subset \{u_n\}_{n=1}^\infty$ and $\{s_{n_k}\}_{k=1}^\infty \subset \{s_n\}_{n=1}^\infty$, such that*

$$\lim_{k \to \infty} \lim_{l \to \infty} \int_0^T \int_s^t \int_\Omega (\sigma(x, \tau + s_{n_k}) - \sigma(x, \tau + s_{n_l})) \partial_t (u_{n_k} - u_{n_l})(\tau) dx d\tau ds = 0.$$

3. Well-posedness and uniformly bounded absorbing set

In this section, we will establish the well-posedness of problem (1.8)-(1.11). To achieve this, we set $U = (u, v, \eta^t)^T$, where $v = u_t$, initial data $U_\tau = (u_0^\tau, v_0^\tau, \eta_0^\tau)^T$. Then problem (1.8)-(1.11) is transformed into

$$\begin{gathered} U_t + AU = F(U), \\ U(\tau) = U_\tau, \end{gathered} \tag{3.1}$$

where

$$AU = \begin{pmatrix} -v \\ \Delta^2 u + \beta v + \int_0^\infty \mu(s) \Delta^2 \eta^t(s) ds \\ \eta_s^t(s) - v \end{pmatrix}, \quad F(U) = \begin{pmatrix} 0 \\ -f(u) + g(x, t) \\ 0 \end{pmatrix},$$

$$U_\tau = \begin{pmatrix} u_0^\tau \\ v_0^\tau \\ \eta_0^\tau \end{pmatrix},$$

and the domain of A is

$$D(A) = \big\{(u, v, \eta^t) \in \mathcal{H} : u \in H^4(\Omega),\ v \in H^2_*(\Omega),\ \eta^t \in L^2_\mu(\mathbb{R}^+, H^4(\Omega)),\\ \text{and (1.9)-(1.10) hold}\big\}.$$

To obtain the well-posedness of problem (1.8)-(1.11), we first need to prove the following statement.

Lemma 3.1. *The operator $A : D(A) \subset \mathcal{H} \to \mathcal{H}$ is maximal monotone.*

Proof. Letting $U = (u, v, \eta^t)^T$ we have

$$\begin{aligned}(AU, U)_{\mathcal{H}} &= (-v, u)_{H^2_*(\Omega)} + (\Delta^2 u, v) + (\beta v, v) + (\eta^t, v)_{\mathcal{M}} + (\eta^t_s, \eta^t)_{\mathcal{M}} + (-v, \eta^t)_{\mathcal{M}}\\ &= \beta\|v\|^2 + (\eta^t_s, \eta^t)_{\mathcal{M}},\end{aligned}$$

From (H1), we infer that

$$\begin{aligned}(\eta^t_s, \eta^t)_{\mathcal{M}} &= \int_0^\infty \mu(s)(\eta^t_s(s), \eta^t(s))_{H^2_*} ds\\ &= -\frac{1}{2}\int_0^\infty \mu'(s)\|\eta^t(s)\|_{H^2_*} ds\\ &\geq \frac{k_1}{2}\|\eta^t\|_{\mathcal{M}}.\end{aligned} \tag{3.2}$$

Using (3.2), we arrive at

$$(AU, U)_{\mathcal{H}} \geq \beta\|v\|^2 + \frac{k_1}{2}\|\eta^t\|_{\mathcal{M}} \geq 0, \tag{3.3}$$

thus, A is monotone. Next, we prove that A is maximal, so we need to prove that $\mathcal{R}(I + A) = \mathcal{H}$. We prove that there exists $\widetilde{U} = (\widetilde{u}, \widetilde{v}, \widetilde{\eta}^t)^T \in \mathcal{H}$ such that

$$U + AU = \widetilde{U} \tag{3.4}$$

has a solution $U = (u, v, \eta^t)^T \in D(A)$. Equation (3.4) can be written

$$\begin{gathered}u - v = \widetilde{u},\\ v + \Delta^2 u + \beta v + \int_0^\infty \mu(s)\Delta^2\eta^t(s) ds = \widetilde{v},\\ \eta^t + \eta^t_s - v = \widetilde{\eta}^t.\end{gathered} \tag{3.5}$$

Inserting $(3.5)_1$ into $(3.5)_2$, we obtain

$$\begin{gathered}u + \Delta^2 u + \beta v + \int_0^\infty \mu(s)\Delta^2\eta^t(s) ds = \widetilde{u} + \widetilde{v},\\ \eta^t + \eta^t_s - v = \widetilde{\eta}^t,\end{gathered} \tag{3.6}$$

then, for any $\mathcal{U} = (u, v, \eta^t) \in V = H^4(\Omega) \times H^2_*(\Omega) \times L^2_\mu(\mathbb{R}^+, H^4(\Omega))$, problem (3.6) is equivalent to

$$\mathcal{L}_1(\mathcal{U}, \overline{\mathcal{U}}) = \mathcal{L}_2(\overline{\mathcal{U}}), \quad \forall \overline{\mathcal{U}} = (\overline{u}, \overline{v}, \overline{\eta}^t) \in V,$$

where $\mathcal{L}_1 : V \times V \to \mathbb{R}$ is the bilinear operator, $\mathcal{L}_2 : V \to \mathbb{R}$ is the linear operator with the following forms, respectively,

$$\begin{aligned}\mathcal{L}_1(\mathcal{U}, \overline{\mathcal{U}}) = (u, \overline{u}) + (\Delta^2 u, \overline{u})_{H^2_*} + (\beta v, \overline{v}) + \int_0^\infty \mu(s)(\eta, \overline{u})_{H^2_*} ds\\ + (\eta^t, \overline{\eta}^t)_{\mathcal{M}} + (\eta^t_s, \overline{\eta}^t)_{\mathcal{M}} - (v, \overline{v}),\end{aligned}$$

$$\mathcal{L}_2(\overline{\mathcal{U}}) = ((\widetilde{u} + \widetilde{v}), \overline{u})_{H_*^2} + (\widetilde{\eta}^t, \overline{\eta}^t)_{\mathcal{M}}.$$

Obviously, $\mathcal{L}_1$ is a bilinear and continuous from on $V \times V$, $\mathcal{L}_2$ is a linear and continuous from on V. Moreover, for some $C_1 > 0$, we have

$$\mathcal{L}_1(\mathcal{U}, \mathcal{U}) \geq C_1 \|\mathcal{U}\|_V^2.$$

Furthermore, there exist $C_2, C_3 > 0$ such that

$$\begin{aligned} |\mathcal{L}_1(\mathcal{U}, \overline{\mathcal{U}})| &\leq \|u\|\|\overline{u}\| + \|u\|_{H_*^2}\|\overline{u}\|_{H_*^2} + \beta\|v\|\|\overline{v}\| + \|\eta\|_{\mathcal{M}}\|\overline{u}\|_{H_*^2} \\ &\quad + \|\eta^t\|_{\mathcal{M}}\|\overline{\eta}^t\|_{\mathcal{M}} + \|\eta_s^t\|_{\mathcal{M}}\|\overline{\eta}^t\|_{\mathcal{M}} + \|v\|\|\overline{v}\| \\ &\leq C_2\|\mathcal{U}\|_V\|\overline{\mathcal{U}}\|_V, \\ |\mathcal{L}_2(\overline{\mathcal{U}})| &\leq \|\widetilde{u} + \widetilde{v}\|_{H_*^2}\|\overline{u}\|_{H_*^2} + \|\widetilde{\eta}^t\|_{\mathcal{M}}\|\overline{\eta}^t\|_{\mathcal{M}} \leq C_3\|\overline{\mathcal{U}}\|_V. \end{aligned}$$

By the Lax-Milgram theorem, equation (3.6) admits an unique (weak) solution $\mathcal{U} \in V$. In addition, from (3.5)-(3.6), we deduce that

$$\begin{gathered} v = u_t = u - \widetilde{u} \in H_*^2(\Omega), \\ \Delta^2 u = \widetilde{u} + \widetilde{v} - u - \beta(\widetilde{u} - u) - \int_0^\infty \mu(s)\Delta^2\eta^t(s)ds \in L^2(\Omega), \\ \eta^t - \widetilde{\eta}^t = -\eta_s^t(s) + u_t(t) \in L_\mu^2(\mathbb{R}^+, H^4(\Omega)). \end{gathered}$$

Then $(u, v, \eta^t) \in D(A)$. Hence, $\mathcal{R}(I + A) = \mathcal{H}$, which completes the proof. □

Theorem 3.2. *Assume* (H1) *and* (H2) *and* $U_\tau \in \mathcal{H}$. *Then problem* (3.1) *has a unique global solution* $U = (u, u_t, \eta^t) \in C([\tau, +\infty]; \mathcal{H})$.

Proof. From Lemma 3.1, we know that the operator A is monotone and maximal, and F obviously satisfies locally Lipschitz from (2.4). Therefore, by the Hille-Yosida theorem, we obtain the existence of a unique weak local solution for (1.8)-(1.11); that is,

$$U = (u, u_t, \eta^t) \in C([\tau, T_{max}], \mathcal{H}), \quad \text{for all } T_{max} > 0.$$

Next, we prove that the solution is global, namely, $T_{max} = \infty$. For this purpose, we need to prove that $\|U(t)\|_{\mathcal{H}}$ is uniformly bounded with respect to time. For simplicity, from now on we set $d\varpi = dx\,dy$. Multiplying the first equation of (1.8) by u_t and integrating over Ω, we have

$$\frac{d}{dt}\Big(\frac{1}{2}\|u\|_{H_*^2}^2 + \frac{1}{2}\|u_t\|^2 + \int_\Omega F(u)d\varpi\Big) + \beta\|u_t\|^2 + (\eta^t, u_t)_{\mathcal{M}} = (g(t), u_t), \tag{3.7}$$

multiplying the second equation of (1.8) by η^t and integrating over $\mathcal{M}$, we obtain

$$\frac{1}{2}\frac{d}{dt}\|\eta^t\|_{\mathcal{M}}^2 + (\eta_s^t, \eta^t)_{\mathcal{M}} = (\eta^t, u_t)_{\mathcal{M}}. \tag{3.8}$$

Then, by (3.2) and (3.7)-(3.8), we obtain

$$\frac{d}{dt}E(t) = -\beta\|u_t\|^2 - (\eta_s^t, \eta^t)_{\mathcal{M}} + (g(t), u_t), \tag{3.9}$$

where

$$E(t) = \frac{1}{2}\|u\|_{H_*^2}^2 + \frac{1}{2}\|u_t\|^2 + \frac{1}{2}\|\eta^t\|_{\mathcal{M}}^2 + \int_\Omega F(u)d\varpi. \tag{3.10}$$

By Hölder's inequality and Young's inequality, for $0 < \xi \leq 2\beta$, we have

$$(g(t), u_t) = \int_\Omega g(t)u_t d\varpi \leq \|g(t)\|\|u_t\| \leq \frac{1}{2\xi}\|g(t)\|^2 + \frac{\xi}{2}\|u_t\|^2. \tag{3.11}$$

Using (3.9)-(3.11), we deduce that

$$\frac{d}{dt}E(t) \le -(\beta - \frac{\xi}{2})\|u_t\|^2 - \frac{k_1}{2}\int_0^\infty \mu(s)\|\eta^t(s)\|_{H_*^2} ds + \frac{1}{2\xi}\|g(t)\|^2, \tag{3.12}$$

integrating (3.12) over (τ, t), it is easy to see that

$$E(t) \le E(\tau) + \frac{1}{2\xi}\int_\tau^t \|g(s)\|^2 ds. \tag{3.13}$$

By Proposition 2.6, we know that $\|g\|^2_{L_b^2(\mathbb{R}_\tau;L^2(\Omega))} \le \|g_0\|^2_{L_b^2(\mathbb{R}_\tau;L^2(\Omega))}$. Then

$$E(t) \le E(\tau) + \frac{1}{2\xi}\|g_0\|^2_{L_b^2(\mathbb{R}_\tau;L^2(\Omega))}, \tag{3.14}$$

and by (2.5), we obtain

$$E(t) \ge \frac{1}{2}\|u\|^2_{H_*^2} + \frac{1}{2}\|u_t\|^2 + \frac{1}{2}\|\eta^t\|^2_{\mathcal{M}} - c|\Omega|. \tag{3.15}$$

Thus, for each $t \ge \tau$, we obtain

$$E(t) \ge C_4\big\|(u(t), u_t(t), \eta^t(t))\big\|^2_{\mathcal{H}} - C_5. \tag{3.16}$$

This and (3.14) imply

$$\big\|(u(t), u_t(t), \eta^t(t))\big\|^2_{\mathcal{H}} \le \frac{1}{C_4}\Big(E(\tau) + \frac{1}{2\xi}\|g_0\|^2_{L_b^2(\mathbb{R}_\tau;L^2(\Omega))} + C_5\Big) \le C_6 \tag{3.17}$$

for all $t \ge \tau$, which completes the proof. □

Remark 3.3. From Theorem 3.2, we deduce that problem (1.8)-(1.11) generates a family of processes $\{U_g(t,\tau)\}, g \in \mathcal{H}(g_0)$ in the space $\mathcal{H}$. Then we define a family of two-parameter operators $U_g(t,\tau) : \mathcal{H} \to \mathcal{H}$ given by

$$U_g(t,\tau)(u_0^\tau, v_0^\tau, \eta_0^\tau) = (u(t), u_t(t), \eta^t(s)). \tag{3.18}$$

where $(u(t), u_t(t), \eta^t(s))$ is the unique global solution of (1.8)-(1.11) corresponding to initial data $(u_0^\tau, v_0^\tau, \eta_0^\tau)$, and $\{U_g(t,\tau)\},\ g \in \mathcal{H}(g_0)$ satisfies Definition 2.2. Moreover, for all initial data $z_0^\tau = (u_0^\tau, v_0^\tau, \eta_0^\tau)$ and $z_1^\tau = (u_1^\tau, v_1^\tau, \eta_1^\tau)$, we let $z_\tau = z_0^\tau - z_1^\tau$. Then there exists a positive constant C depending on z_0^τ and z_1^τ, such that

$$\|U_g(t,\tau)z_0^\tau - U_g(t,\tau)z_1^\tau\|_{\mathcal{H}} \le e^{CT}(\|z_\tau\|^2_{\mathcal{H}} + \|g_1(t) - g_2(t)\|^2_{L_b^2(\mathbb{R}_\tau;L^2(\Omega))}), \tag{3.19}$$

for $\tau \le t \le T$. This shows that solutions of (1.8)-(1.11) depend continuously on the initial data.

Next, we prove the existence of a uniformly absorbing set in $\mathcal{H}$. We need to introduce a Lyapunov functional

$$L(t) = PE(t) + Q\Phi(t), \tag{3.20}$$

where P, Q are positive constants, which will be defined later, and $\Phi(t) = (u_t, u)$.

Lemma 3.4. *Let Q be small enough and P be large enough. Then there exist θ_1 and $\theta_2 > 0$ such that*

$$\theta_1\|(u(t), u_t(t), \eta^t(t))\|^2_{\mathcal{H}} - c_1 \le L(t) \le \theta_2\|(u(t), u_t(t), \eta^t(t))\|^2_{\mathcal{H}} + Pc_2. \tag{3.21}$$

Proof. Firstly, we prove the left inequality of (3.21). Choosing Q small enough and then P large enough such that $\frac{P-Q}{2}>0$, $\frac{P-Qc_*}{2}>0$. By (3.15), we obtain

$$\begin{aligned} L(t) &\geq \frac{P}{2}\|u\|^2_{H^2_*}+\frac{P}{2}\|u_t\|^2+\frac{P}{2}\|\eta^t\|^2_{\mathcal{M}}-cP|\Omega|-\frac{Qc_*^2}{2}\|u\|^2_{H^2_*}-\frac{Q}{2}\|u_t\|^2 \\ &\geq \Big(\frac{P-Qc_*^2}{2}\Big)\|u\|^2_{H^2_*}+\Big(\frac{P-Q}{2}\Big)\|u_t\|^2+\frac{P}{2}\|\eta^t\|^2_{\mathcal{M}}-cP|\Omega| \\ &\geq \theta_1\|(u(t),u_t(t),\eta^t(t))\|^2_{\mathcal{H}}-c_1. \end{aligned} \tag{3.22}$$

where $\theta_1=\min\{\frac{P-Qc_*^2}{2},\frac{P-Q}{2},\frac{P}{2}\}$, $c_1=cP|\Omega|$. Moreover, by (2.4), (2.6), (3.17), and $H^2(\Omega)\hookrightarrow L^p(\Omega)$, $(1\leq p\leq\infty)$, we arrive at

$$\begin{aligned} \int_\Omega F(u(t))d\varpi &\leq \int_\Omega |u||f(u)|d\varpi \\ &\leq \int_\Omega |u||f(u)-f(0)|d\varpi+\int_\Omega |u||f(0)|d\varpi \\ &\leq C\int_\Omega |u|^2|u|^p d\varpi+\frac{1}{2}\int_\Omega |u|^2 d\varpi+\frac{1}{2}|\Omega||f(0)|^2 \\ &\leq c_*^2C\big(\|u\|^p_{L^\infty(\Omega)}+1\big)\|u\|^2_{\mathcal{H}^2_*(\Omega)}+c_2. \end{aligned} \tag{3.23}$$

Applying Young's inequality, Sobolev's embedding Theorem and (3.23), we deduce that

$$\begin{aligned} L(t) &\leq \frac{P}{2}\|u\|^2_{H^2_*}+\frac{P}{2}\|u_t\|^2+\frac{P}{2}\|\eta^t\|^2_{\mathcal{M}}+P\int_\Omega F(u(t))d\varpi+\frac{Qc_*^2}{2}\|u\|^2_{H^2_*}+\frac{Q}{2}\|u_t\|^2 \\ &\leq \Big(\frac{P+Qc_*^2}{2}\Big)\|u\|^2_{H^2_*}+\Big(\frac{P+Q}{2}\Big)\|u_t\|^2+\frac{P}{2}\|\eta^t\|^2_{\mathcal{M}}+P\int_\Omega |u||f(u(t))|d\varpi \\ &\leq \Big(\frac{P+c_*^2[Q+2PC(\|u\|^p_{L^\infty(\Omega)}+1)]}{2}\Big)\|u\|^2_{H^2_*} \\ &\quad+\Big(\frac{P+Q}{2}\Big)\|u_t\|^2+\frac{P}{2}\|\eta^t\|^2_{\mathcal{M}}+Pc_2 \\ &\leq \theta_2\|(u(t),u_t(t),\eta^t(t))\|^2_{\mathcal{H}}+Pc_2. \end{aligned}$$

This completes the proof. □

Lemma 3.5. *The function $\Phi(t)=(u_t,u)$ satisfies*

$$\begin{aligned} \Phi'(t) &\leq \Big(1+\frac{\beta}{2\zeta}\Big)\|u_t\|^2+\Big((\beta\zeta+2\zeta)\frac{c_*^2}{2}-1\Big)\|u\|^2_{\mathcal{H}^2_*} \\ &\quad+\frac{1}{2\zeta}\|\eta^t\|^2_{\mathcal{M}}+\frac{1}{2\zeta}\|g(t)\|^2+c|\Omega|. \end{aligned} \tag{3.24}$$

Proof. By $(1.8)_1$, we have

$$\begin{aligned} \Phi'(t) &= (u_{tt},u)+\|u_t\|^2 \\ &= \|u_t\|^2-\|u\|^2_{\mathcal{H}^2_*(\Omega)}-\beta(u_t,u)-(\eta^t,u)_{\mathcal{M}}-(f(u),u)+(g(t),u). \end{aligned} \tag{3.25}$$

Using Young's inequality, Hölder's inequality, and (2.6), for each $\zeta>0$, we have

$$-\beta(u_t,u)\leq\frac{\beta}{2\zeta}\|u_t\|^2+\frac{\beta\zeta}{2}\|u\|^2\leq\frac{\beta}{2\zeta}\|u_t\|^2+\frac{c_*^2\beta\zeta}{2}\|u\|^2_{\mathcal{H}^2_*}, \tag{3.26}$$

$$-(\eta^t,u)_{\mathcal{M}} \le \frac{\zeta}{2}\|u\|^2 + \frac{1}{2\zeta}\|\eta^t\|^2_{\mathcal{M}} \le \frac{c_*^2\zeta}{2}\|u\|^2_{\mathcal{H}^2_*} + \frac{1}{2\zeta}\|\eta^t\|^2_{\mathcal{M}}, \tag{3.27}$$

$$(g(t),u) \le \frac{1}{2\zeta}\|g(t)\|^2 + \frac{\zeta}{2}\|u\|^2 \le \frac{1}{2\zeta}\|g(t)\|^2 + \frac{c_*^2\zeta}{2}\|u\|^2_{\mathcal{H}^2_*}. \tag{3.28}$$

From (2.5), it is easy to see that

$$-(f(u),u) \le c|\Omega|. \tag{3.29}$$

Inserting (3.26)-(3.29) into (3.25), we arrive at (3.24), which completes the proof. □

Theorem 3.6. *Under the assumption of Theorem 3.2, a family of processes $\{U_g(t,\tau)\}$, $g \in \mathcal{H}(g_0)$ corresponding to (1.8)-(1.11) possesses a bounded uniformly (with respect to $g \in \mathcal{H}(g_0)$) absorbing set B in $\mathcal{H}$.*

Proof. From (3.12), (3.20) and (3.24), for $P, Q > 0$ it follows that

$$\begin{aligned} L'(t) &= PE'(t) + Q\Phi'(t) \\ &\le -Q\Big(1-(\beta\zeta+2\zeta)\frac{c_*^2}{2}\Big)\|u\|^2_{\mathcal{H}^2_*} - \Big((\beta-\frac{\xi}{2})P - (1+\frac{\beta}{2\zeta})Q\Big)\|u_t\|^2 \\ &\quad - \Big(\frac{k_1P}{2} - \frac{Q}{2\zeta}\Big)\|\eta^t\|^2_{\mathcal{M}} + \frac{P\zeta+Q\xi}{2\xi\zeta}\|g(t)\|^2 + cQ|\Omega|. \end{aligned} \tag{3.30}$$

where $\zeta, \xi > 0$. Choosing first ζ, ξ small enough such that

$$1-(\beta\zeta+2\zeta)\frac{c_*^2}{2} > 0, \quad \beta - \frac{\xi}{2} > 0,$$

after that, choosing again Q small enough, and then P large enough such that

$$\frac{k_1P}{2} - \frac{Q}{2\zeta} > 0, \quad (\beta-\frac{\xi}{2})P - (1+\frac{\beta}{2\zeta})Q > 0.$$

Thus, there exist positive constants φ_1, φ_2, φ_3, φ_4, and φ_5 such that

$$\frac{d}{dt}L(t) \le -\varphi_1\|u\|^2_{\mathcal{H}^2_*} - \varphi_2\|u_t\|^2 - \varphi_3\|\eta^t\|^2_{\mathcal{M}} + \varphi_4\|g(t)\|^2 + \varphi_5, \tag{3.31}$$

choosing $\varphi = \min\{\varphi_1, \varphi_2, \varphi_3\}$ yields

$$\frac{d}{dt}L(t) \le -\varphi\Big(\|u\|^2_{\mathcal{H}^2_*} + \|u_t\|^2 + \|\eta^t\|^2_{\mathcal{M}}\Big) + \varphi_4\|g(t)\|^2 + \varphi_5. \tag{3.32}$$

By Lemma 3.4, we claim that

$$\frac{d}{dt}L(t) + \varrho L(t) \le c_3\|g(t)\|^2 + c_4, \tag{3.33}$$

where $\varrho = \varphi/\theta_2$. Using Gronwall Lemma for (3.33), leads to

$$\begin{aligned} L(t) &\le L(\tau)e^{-\varrho(t-\tau)} + c_3\int_\tau^t e^{-\varrho(t-s)}\|g(s)\|^2 ds + c_4\int_\tau^t e^{-\varrho(t-s)} ds \\ &\le L(\tau)e^{-\varrho(t-\tau)} + \frac{c_3}{1-e^{-\varrho}}\sup_{t\ge\tau}\int_t^{t+1}\|g(s)\|^2 ds + c_6 \\ &\le L(\tau)e^{-\varrho(t-\tau)} + \frac{c_5}{1-e^{-\varrho}}\|g_0\|_{L^2_b(\mathbb{R}_\tau;L^2(\Omega))} + c_6. \end{aligned} \tag{3.34}$$

If for any bounded set $B \subseteq \mathcal{H}$, and the initial data $(u_0^\tau, v_0^\tau, \eta_0^\tau) \in B$, there exists a constant $C_B > 0$ such that $L(\tau) \leq C_B$, then we deduce from (3.34) that

$$L(t) \leq C_B e^{-\varrho(t-\tau)} + \frac{c_5}{1-e^{-\varrho}} \|g_0\|_{L_b^2(\mathbb{R}_\tau; L^2(\Omega))} + c_6, \tag{3.35}$$

for any $t \geq t_0$. It follows that

$$\|(u, u_t, \eta^t)\|_{\mathcal{H}}^2 \leq \frac{1}{\theta_1}(L(t) + c_1) = R^2.$$

This means that a family of processes $\{U_g(t,\tau)\}$ generated by (1.8)-(1.11) has an uniformly absorbing ball $B = B(0,R) = \{(u, u_t, \eta^t) \in \mathcal{H} : \|(u, u_t, \eta^t)\|_{\mathcal{H}}^2 \leq R^2\} \subseteq \mathcal{H}$ for any $g \in \mathcal{H}(g_0)$, which completes the proof. □

Next, we prove the asymptotic compactness of a family of processes $\{U_g(t,\tau)\}$, $g \in \mathcal{H}(g_0)$ associated with (1.8)-(1.11) in $\mathcal{H}$. Our main results are as follows.

Theorem 3.7. *Assume that* (H1) *and* (H2) *hold and* $g \in \mathcal{H}(g_0)$. *Then a family of processes* $\{U_g(t,\tau)\}$, $g \in \mathcal{H}(g_0)$ *corresponding to* (1.8)-(1.11) *is uniformly (with respect to* $g \in \mathcal{H}(g_0)$*) asymptotically compact in* $\mathcal{H}$.

Proof. Let $z^1 = (u^1, u_t^1, \eta^1)$ and $z^2 = (u^2, u_t^2, \eta^2)$ be two solutions of (1.8)-(1.11) with the initial data $z_0^1 = (u_0^{1\tau}, v_0^{1\tau}, \eta_0^{1\tau})$, $z_0^2 = (u_0^{2\tau}, v_0^{2\tau}, \eta_0^{2\tau})$ and the symbols g^1, g^2, respectively.

Set $z = z^1 - z^2 = (u, u_t, \eta^t)$, the initial data $z_0 = z_0^1 - z_0^2 = (u_0^\tau, v_0^\tau, \eta_0^\tau)$. Then (u, u_t, η^t) satisfies the equations

$$\begin{gathered} u_{tt} + \Delta^2 u + \beta u_t + \int_0^\infty \mu(s)\Delta^2\eta^t(s)ds + f(u^1) - f(u^2) = g^1(t) - g^2(t), \\ \eta_t^t(x,y,s) = -\eta_s^t(x,y,s) + u_t(x,y,t), \end{gathered} \tag{3.36}$$

with boundary conditions (1.9)-(1.10). We denote

$$\widetilde{E}(t) = \frac{1}{2}\|u\|_{H_*^2}^2 + \frac{1}{2}\|u_t\|^2 + \frac{1}{2}\|\eta^t\|_{\mathcal{M}}^2, \tag{3.37}$$

$$\widetilde{L}(t) = P_1\widetilde{E}(t) + Q_1\widetilde{\Phi}(t), \tag{3.38}$$

where $\widetilde{\Phi}(t) = (u_t, u)$. Obviously, $\widetilde{E}(t)$ and $\widetilde{L}(t)$ are equivalent. Then there exist two positive constants γ_1 and γ_2 depending on P_1, Q_1 such that

$$\gamma_1\widetilde{E}(t) \leq \widetilde{L}(t) \leq \gamma_2\widetilde{E}(t), \tag{3.39}$$

where $P_1 > 0$ large enough and $Q_1 > 0$ small enough.

First, multiplying $(3.36)_1$ by u_t and integrating over Ω, multiplying $(3.36)_2$ by η^t and integrating over $\mathcal{M}$, then adding them, we have

$$\begin{aligned} \frac{d}{dt}\widetilde{E}(t) &\leq -\beta\|u_t\|^2 - \frac{k_1}{2}\|\eta^t\|_{\mathcal{M}}^2 + \int_\Omega (f(u^2) - f(u^1))u_t d\varpi \\ &\quad + \int_\Omega (g^1(t) - g^2(t))u_t d\varpi, \end{aligned} \tag{3.40}$$

according to the proof of Lemma 3.5, there exists $\zeta > 0$ such that

$$\begin{aligned} \widetilde{\Phi}'(t) &\leq (1 + \frac{\beta}{2\zeta})\|u_t\|^2 + \Big((\beta\zeta + \zeta)\frac{c_*^2}{2} - 1\Big)\|u\|_{\mathcal{H}_*^2}^2 \\ &\quad + \frac{1}{2\zeta}\|\eta^t\|_{\mathcal{M}}^2 + \int_\Omega (f(u^2) - f(u^1))u d\varpi + \int_\Omega (g^1(t) - g^2(t))u d\varpi, \end{aligned} \tag{3.41}$$

combining with (3.40) and (3.41), it is easy to see that

$$\begin{aligned}\widetilde{L}'(t) &= P_1\widetilde{E}'(t) + Q_1\widetilde{\Phi}'(t)\\ &\leq -Q_1\Big(1-(\beta\zeta+\zeta)\frac{c_*^2}{2}\Big)\|u\|_{\mathcal{H}_*^2}^2 - \Big(P_1\beta - (1+\frac{\beta}{2\zeta})Q_1\Big)\|u_t\|^2\\ &\quad - \Big(\frac{k_1P_1}{2} - \frac{Q_1}{2\zeta}\Big)\|\eta^t\|_{\mathcal{M}}^2 + P_1\int_\Omega (f(u^2)-f(u^1))u_t d\varpi\\ &\quad + Q_1\int_\Omega (f(u^2)-f(u^1))ud\varpi + P_1\int_\Omega (g^1(t)-g^2(t))u_t d\varpi\\ &\quad + Q_1\int_\Omega (g^1(t)-g^2(t))ud\varpi.\end{aligned} \tag{3.42}$$

First of all, taking ζ small enough such that

$$1-(\beta\zeta+\zeta)\frac{c_*^2}{2} > 0.$$

After that, choosing Q_1 small enough and then P_1 large enough, such that

$$\frac{k_1P_1}{2} - \frac{Q_1}{2\zeta} > 0, \quad P_1\beta - \big(1+\frac{\beta}{2\zeta}\big)Q_1 > 0.$$

Thus, there exists $\psi > 0$ such that

$$\begin{aligned}\frac{d}{dt}\widetilde{L}(t) &\leq -\psi\widetilde{E}'(t) + P_1\int_\Omega (f(u^2)-f(u^1))u_t d\varpi + Q_1\int_\Omega (f(u^2)-f(u^1))ud\varpi\\ &\quad + P_1\int_\Omega (g^1(t)-g^2(t))u_t d\varpi + Q_1\int_\Omega (g^1(t)-g^2(t))ud\varpi,\end{aligned}$$

thanks to (3.39), we have

$$\begin{aligned}\frac{d}{dt}\widetilde{L}(t) + \chi\widetilde{L}(t) &\leq P_1\int_\Omega (f(u^2)-f(u^1))u_t d\varpi + Q_1\int_\Omega (f(u^2)-f(u^1))ud\varpi\\ &\quad + P_1\int_\Omega (g^1(t)-g^2(t))u_t d\varpi + Q_1\int_\Omega (g^1(t)-g^2(t))ud\varpi,\end{aligned} \tag{3.43}$$

where $\chi = \frac{\psi}{\gamma_2}$. Integrating (3.43) over $[\tau, t]$, we conclude that

$$\begin{aligned}\widetilde{L}(t) &\leq \widetilde{L}(\tau)e^{-\chi(t-\tau)} + P_1\int_\tau^t\int_\Omega e^{-\chi(t-s)}(f(u^2)-f(u^1))u_t d\varpi ds\\ &\quad + Q_1\int_\tau^t\int_\Omega e^{-\chi(t-s)}(f(u^2)-f(u^1))ud\varpi ds\\ &\quad + P_1\int_\tau^t\int_\Omega e^{-\chi(t-s)}(g^1(s)-g^2(s))u_t d\varpi ds\\ &\quad + Q_1\int_\tau^t\int_\Omega e^{-\chi(t-s)}(g^1(s)-g^2(s))ud\varpi ds.\end{aligned} \tag{3.44}$$

For each $\epsilon > 0$, there exists $T > \tau$, such that $\widetilde{L}(\tau)e^{-\chi(t-\tau)} \leq \epsilon$ for $t \geq T$. Then, by (3.39) and (3.44), we deduce that

$$\begin{aligned}\widetilde{E}(t) &\leq \epsilon + P_1 \int_\tau^t \int_\Omega e^{-\chi(t-s)}(f(u^2) - f(u^1))u_t d\varpi ds \\ &\quad + Q_1 \int_\tau^t \int_\Omega e^{-\chi(t-s)}(f(u^2) - f(u^1))u d\varpi ds \\ &\quad + P_1 \int_\tau^t \int_\Omega e^{-\chi(t-s)}(g^1(s) - g^2(s))u_t d\varpi ds \\ &\quad + Q_1 \int_\tau^t \int_\Omega e^{-\chi(t-s)}(g^1(s) - g^2(s))u d\varpi ds \\ &:= \epsilon + \phi_T\Big((u_0^{1\tau}, v_0^{1\tau}, \eta_0^{1\tau}), (u_0^{2\tau}, v_0^{2\tau}, \eta_0^{2\tau}); g^1, g^2\Big).\end{aligned} \tag{3.45}$$

Now, we prove $\phi_T(\cdot,\cdot;\cdot,\cdot) \in \mathfrak{C}(\mathcal{B},\mathcal{B};\Sigma,\Sigma)$ for every fixed $T > \tau$. By Theorem 3.6, we know that

$$\cup_{g\in\mathcal{H}(g_0)} \cup_{t\in[\tau,T]} U_g(t,\tau)B$$

is bounded in $\mathcal{H}$. Let the sequence $(u_{0n}^\tau, v_{0n}^\tau, \eta_{0n}^\tau) \in B$, $g^n \in \mathcal{H}(g_0), n = 1, 2, \cdots$. Because B is bounded, the corresponding sequence of solutions (u_n, v_n, η_n^t) associated with the system (1.8)-(1.11) is uniformly bounded in $\mathcal{H}$. Without loss of generality, we assume that

(i) $u^m \to u$ weak star in $L^\infty(\tau, T; H_*^2(\Omega))$,
(ii) $u_t^m \to u_t$ weak star in $L^\infty(\tau, T; L^2(\Omega))$,
(iii) $u^m \to u$ in $L^2(\tau, T; L^2(\Omega))$,
(iv) $u^m(\tau) \to u(\tau), u^m(T) \to u(T)$ in $L^k(\Omega)$, $k < \infty$.

Then, applying Proposition 2.7 and (iii), it follows that

$$\lim_{n\to\infty}\lim_{m\to\infty} \int_\tau^t \int_\Omega (g^n(x,s) - g^m(x,s))(u^n(s) - u^m(s)) d\varpi ds = 0, \tag{3.46}$$

from Proposition 2.8, we have

$$\lim_{n\to\infty}\lim_{m\to\infty} \int_\tau^t \int_\Omega (g^n(x,s) - g^m(x,s))(u_t^n(s) - u_t^m(s)) d\varpi ds = 0. \tag{3.47}$$

On the other hand, since $f(u^m) \to f(u)$ weak star in $L^2(\tau, T; \mathcal{H})$, and exploiting (ii)-(iii), it follows that

$$\lim_{n\to\infty}\lim_{m\to\infty} \int_\tau^t \int_\Omega (f(u^n(r)) - f(u^m(r)))(u_t^n(r) - u_t^m(r)) d\varpi dr = 0, \tag{3.48}$$

$$\lim_{n\to\infty}\lim_{m\to\infty} \int_\tau^t \int_\Omega (f(u^n(r)) - f(u^m(r)))(u^n(r) - u^m(r)) d\varpi dr = 0. \tag{3.49}$$

Therefore, from (3.46) and (3.49), we deduce that $\phi_T \in \mathfrak{C}(\mathcal{B},\mathcal{B};\Sigma,\Sigma)$, which completes the proo. □

Finally, by Theorems 3.6 and 3.7, we conclude the main result of this article.

Theorem 3.8. *Assume that* (H1) *and* (H2) *hold and* $g \in \mathcal{H}(g_0)$. *Then a family of processes* $\{U_g(t,\tau)\}$, $g \in \mathcal{H}(g_0)$ *corresponding to* (1.8)-(1.11) *has a compactly uniform* (*w.r.t.*$g \in \mathcal{H}(g_0)$) *attractor* $\mathcal{A}_\Sigma$ *in* $\mathcal{H}$.

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