

MAXIMAL REGULARITY FOR FRACTIONAL DIFFERENCE EQUATIONS OF ORDER $2 < \alpha < 3$ ON UMD SPACES

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ABSTRACT. In this article, we study the ℓ^p -maximal regularity for the fractional difference equation

$$\Delta^\alpha u(n) = Tu(n) + f(n), \quad (n \in \mathbb{N}_0).$$

We introduce the notion of α -resolvent sequence of bounded linear operators defined by the parameters T and α , which gives an explicit representation of the solution. Using Blunck's operator-valued Fourier multipliers theorems on $\ell^p(\mathbb{Z}; X)$, we give a characterization of the ℓ^p -maximal regularity for $1 < p < \infty$ and X is a UMD space.

1. INTRODUCTION

In this article, we study the unique representation of solutions and the ℓ^p -maximal regularity for the fractional difference equation

$$\Delta^\alpha u(n) = Tu(n) + f(n), \quad (n \in \mathbb{N}_0) \tag{1.1}$$

with the initial conditions $u(0) = u(1) = u(2) = 0$, where T is a bounded linear operator defined on a Banach space X , $f : \mathbb{N}_0 \rightarrow X$ is an X -valued sequence, $2 < \alpha < 3$ and $1 < p < \infty$. Here we denote by \mathbb{N}_0 the set of non-negative integers, the discrete fractional operator Δ^α corresponds to sampling, by means of the Poisson distribution, of the Riemann-Liouville fractional derivative [17] (see the precise definition in the second section).

The fractional difference equation (1.1) is the counterparts of fractional differential equations in discrete time which arises as models for several biological and physical applications. Much literature have been devoted to such problems [2, 6]. For instance, Blunck studied the ℓ^p -maximal regularity for (1.1) with the initial condition $u(0) = 0$ when $\alpha = 1$, and T substitutes for $T - I$. He established the following result: when the underlying Banach space X is a UMD space and $1 < p < \infty$, equation (1.1) has the ℓ^p -maximal regularity if and only if $\{z : |z| = 1, z \neq 1\} \subset \rho(T)$ and the set

$$\{(z - 1)(z - T)^{-1} : |z| = 1, z \neq 1\}$$

is Rademacher bounded (R-bounded) [7].

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Later, Lizama further considered the ℓ^p -maximal regularity for (1.1) with the initial condition $u(0) = 0$ when $0 < \alpha \leq 1$, and proved that when the underlying Banach space X is a UMD space, $1 < p < \infty$ and $\{z^{1-\alpha}(z-1)^\alpha : |z| = 1, z \neq 1\} \subset \rho(T)$, then (1.1) has the ℓ^p -maximal regularity if and only if the set

$$\{z^{1-\alpha}(z-1)^\alpha[z^{1-\alpha}(z-1)^\alpha - T]^{-1} : |z| = 1, z \neq 1\}$$

is R-bounded [18]. In the case $1 < \alpha \leq 2$, $1 < p < \infty$ and X is a UMD space, Lizama and Arcila showed that (1.1) with the initial conditions $u(0) = u(1) = 0$, has the ℓ^p -maximal regularity if and only if $\{z^{2-\alpha}(z-1)^\alpha : |z| = 1, z \neq 1\} \subset \rho(T)$, and the set

$$\{z^{2-\alpha}(z-1)^\alpha[z^{2-\alpha}(z-1)^\alpha - T]^{-1} : |z| = 1, z \neq 1\}$$

is R-bounded [19]. See [6, 15], [17]-[21] for further results on the corresponding quasi-linear equations.

The fractional difference equation (1.1) when $0 < \alpha < 2$ was studied in [18, 19]. Meanwhile, when $2 < \alpha < 3$, the existence and uniqueness of solution of (1.1), as well as the ℓ^p -maximal regularity for (1.1), are open topics that deserve to be investigated, the objective of this paper is to solve this problem. When $2 < \alpha < 3$, we first introduce a sequence of bounded linear operators $P_\alpha(n)$ defined by the parameters T and α , that we call α -resolvent sequence, which will give an explicit representation of solution for (1.1). Precisely, the sequence $(P_\alpha(n))_{n \in \mathbb{N}_0}$ is defined by $P_\alpha(0) = P_\alpha(1) = P_\alpha(2) = I$, and

$$\begin{aligned} & P_\alpha(n+3) - 2P_\alpha(n+2) + P_\alpha(n+1) \\ &= T(k^{\alpha-2} * P_\alpha)(n) + k^{\alpha-2}(n+3)I + (1-\alpha)k^{\alpha-2}(n+2)I \\ &+ \frac{(\alpha-1)(\alpha-2)}{2}k^{\alpha-2}(n+1)I \end{aligned}$$

for $n \in \mathbb{N}_0$, where $k^{\alpha-2}$ is defined by (2.4). We show that when $f : \mathbb{N}_0 \rightarrow X$ is given, the fractional differential equation (1.1) has a unique solution $u : \mathbb{N}_0 \rightarrow X$ given by

$$u(n) = (h_\alpha * P_\alpha * f)(n-3)$$

for $n \geq 3$, where the function $h_\alpha : \mathbb{N}_0 \rightarrow \mathbb{R}$ is defined by

$$h_\alpha(n+3) = -(1-\alpha)h_\alpha(n+2) - (\alpha-1)(\alpha-2)h_\alpha(n+1)/2$$

for $n \in \mathbb{N}_0$, and $h_\alpha(0) = 1$, $h_\alpha(1) = \alpha - 1$, $h_\alpha(2) = \alpha(\alpha - 1)/2$. We notice that similar α -resolvent sequences have been used in the representation of solutions of (1.1) when $0 < \alpha \leq 2$ in [18, 19].

Concerning the ℓ^p -maximal regularity for (1.1), we show that when X is a UMD space, $1 < p < \infty$ and $\{z^{3-\alpha}(z-1)^\alpha : |z| = 1, z \neq \pm 1\} \subset \rho(T)$, then (1.1) has the ℓ^p -maximal regularity if and only if the set

$$\{z^{3-\alpha}(z-1)^\alpha[z^{3-\alpha}(z-1)^\alpha - T]^{-1} : |z| = 1, z \neq \pm 1\} \quad (1.2)$$

is R-bounded. Our main tool is the operator-valued Fourier multipliers theorems on $\ell^p(\mathbb{Z}; X)$ by Blunck [7], we will transform the ℓ^p -maximal regularity for (1.1) to an operator-valued Fourier multiplier problem on $\ell^p(\mathbb{Z}; X)$.

It is clear that the R-boundedness of the set (1.2) does not depend on the space parameter p . Thus when X is a UMD space and

$$\{z^{3-\alpha}(z-1)^\alpha : |z| = 1, z \neq \pm 1\} \subset \rho(T),$$

if (1.1) has the ℓ^p -maximal regularity for some $1 < p < \infty$, then (1.1) has the ℓ^p -maximal regularity for all $1 < p < \infty$. Since every norm bounded subset of $B(X)$ is actually R-bounded when X is a Hilbert space, we deduce that if X is a Hilbert space, $1 < p < \infty$, and $\{z^{3-\alpha}(z-1)^\alpha : |z|=1, z \neq \pm 1\} \subset \rho(T)$, then (1.1) has the ℓ^p -maximal regularity if and only if the set (1.2) is norm bounded.

Concerning applications of the fractional difference equation (1.1), it has been considered as a time-stepping scheme in many fields of sciences such as linear viscoelasticity theory for describing the behavior of polymeric materials, see [14] [22]. Our results also reveal a close relation between time-stepping scheme and linear viscoelasticity theory, which are supported and coincident with very recent research in mechanical engineering [9]. In terms of time-stepping scheme, the connection stated in our paper can be interpreted as a methodology to identify the desired (and probably best) time stepping scheme in terms not only of the mathematical model but also of the characteristics of the real material that it models.

This article is organized as follows: In section 2, we recall some basic concepts on UMD spaces, R-boundedness, fractional difference operators, the discrete time Fourier transform and Blunck’s Fourier multipliers theorems for operator-valued symbols on UMD spaces. Section 3 is devoted to the study of the α -resolvent sequence defined by T and α , which gives a representation of solution for (1.1). In the last section, we give a characterization of the ℓ^p -maximal regularity of (1.1) when $1 < p < \infty$ and X is a UMD space.

2. PRELIMINARIES

In this section, we briefly recall some basic notions about UMD spaces, R-boundedness, fractional difference operators, the discrete time Fourier transform and Blunck’s Fourier multipliers theorems, which will be fundamental in our investigation.

Let X be a Banach space. We denote by $S(\mathbb{N}_0; X)$ the set of all X -valued sequences $u : \mathbb{N}_0 \rightarrow X$. Similarly we denote by $S(\mathbb{Z}; X)$ the set consisting of all X -valued sequences $u : \mathbb{Z} \rightarrow X$. The forward Euler operator $\Delta : S(\mathbb{N}_0; X) \rightarrow S(\mathbb{N}_0; X)$ is defined as

$$\Delta u(n) := u(n+1) - u(n)$$

where $n \in \mathbb{N}_0$. For every $m \in \mathbb{N}$, we define recursively the m -th order forward difference operator $\Delta^m : S(\mathbb{N}_0; X) \rightarrow S(\mathbb{N}_0; X)$ by $\Delta^m = \Delta^{m-1}\Delta$.

Let $f \in S(\mathbb{N}_0; \mathbb{C})$ and $g \in S(\mathbb{N}_0; X)$. The finite convolution $f * g \in S(\mathbb{N}_0; X)$ is defined by

$$(f * g)(n) := \sum_{j=0}^n f(n-j)g(j) \tag{2.1}$$

for $n \in \mathbb{N}_0$. It is easy to verify that if $h \in S(\mathbb{N}_0; \mathbb{C})$, then

$$(f * h * g)(n) = ((f * h) * g)(n) = (f * (h * g))(n) = \sum_{i+j+k=n} f(i)h(j)g(k) \tag{2.2}$$

for $n \in \mathbb{N}_0$.

The following definition of fractional sum, used the previous works (see [1, 4, 5, 12]), was formally presented by Lizama in [17]. Let $0 < \beta \leq 1$ and $u \in S(\mathbb{N}_0; X)$

be given. The fractional sum of u of order β is defined by

$$\Delta^{-\beta}u(n) := (k^\beta * u)(n) = \sum_{j=0}^n k^\beta(n-j)u(j) \quad (2.3)$$

for $n \in \mathbb{N}_0$, where

$$k^\beta(j) := \frac{\Gamma(\beta+j)}{\Gamma(\beta)\Gamma(j+1)} \quad (2.4)$$

for $j \in \mathbb{N}_0$, where Γ is the Gamma function. It is clear that

$$k^\beta(0) = 1, \quad k^\beta(1) = \beta, \quad k^\beta(n) = \beta(\beta+1), \dots, (\beta+n-1)/n! \quad (2.5)$$

for $n \geq 2$.

Definition 2.1. Let $\alpha > 0$, $\alpha \notin \mathbb{N}$ and $u \in S(\mathbb{N}_0; X)$ be given. The fractional difference operator of order α of u is defined by

$$\Delta^\alpha u(n) := \Delta^m \Delta^{-(m-\alpha)} u(n) \quad (2.6)$$

for $n \in \mathbb{N}_0$, where $m \in \mathbb{N}$ is the unique integer m satisfying $m-1 < \alpha < m$. For more information about fractional difference operators, we refer the readers to [11]

Remark 2.2. We notice that when $0 < \beta \leq 1$ and $u \in S(\mathbb{N}_0; X)$, the value of $\Delta^{-\beta}u(n)$ defined by (2.3) depends on all $u(k)$ when $0 \leq k \leq n$. If $v \in S(\mathbb{N}_0; X)$ and $k \geq 1$ are given, we let $u \in S(\mathbb{N}_0; X)$ satisfy $u(n) = v(n-k)$ for $n \geq k$. Then

$$\Delta^{-\beta}u(n) = \sum_{j=0}^n k^\beta(j)u(n-j) = \Delta^{-\beta}v(n-k) + \sum_{j=n-k+1}^n k^\beta(j)u(n-j)$$

for $n \geq k$. Thus the equality

$$\Delta^{-\beta}u(n) = \Delta^{-\beta}v(n-k) \quad (2.7)$$

is not necessarily true when $n \geq k$. Meanwhile if $u(j) = 0$ where $0 \leq j \leq k-1$, then (2.7) is true. Similarly, let $\alpha > 0$, $\alpha \notin \mathbb{N}$, if $v \in S(\mathbb{N}_0; X)$ and $k \geq 1$ are given, we let $u \in S(\mathbb{N}_0; X)$ satisfy $u(n) = v(n-k)$ for $n \geq k$, then the equality

$$\Delta^\alpha u(n) = \Delta^\alpha v(n-k) \quad (2.8)$$

is not necessarily true when $n \geq k$. But if $u(j) = 0$ for $0 \leq j \leq k-1$, then (2.8) is true. This is the main reason that we only consider the equation (1.1) with the simpler initial conditions $u(0) = u(1) = u(2) = 0$, instead of the general initial conditions $u(0) = x_0, u(1) = x_1$ and $u(2) = x_2$.

Let $u \in S(\mathbb{Z}; X)$. The discrete time Fourier transform of u is

$$\hat{u}(z) := \sum_{j=-\infty}^{\infty} z^{-j}u(j)$$

for all $|z| = 1$, whenever it exists. We notice that the Fourier transform of \hat{u} is sometimes also denoted by $\mathcal{F}(u)$. It is clear that if $f \in S(\mathbb{N}_0; \mathbb{C})$ and $g \in S(\mathbb{N}_0; X)$, then

$$\widehat{(f * g)}(z) = \hat{f}(z)\hat{g}(z) \quad (2.9)$$

when both sides are well defined for all $|z| = 1$.

Let $0 < \alpha \leq 1$ and let k^α be defined by (2.4). It follows from (2.5) that the Fourier transform of k^α is given by

$$\hat{k}^\alpha(z) = \frac{z^\alpha}{(z-1)^\alpha} \tag{2.10}$$

for $|z| = 1$ and $z \neq 1$. This implies that if $0 < \alpha, \beta < 1$ and $\alpha + \beta = 1$, then

$$k^\alpha * k^\beta = k^1 \tag{2.11}$$

by (2.9).

We say that a Banach space X is a UMD space if for all $1 < p < \infty$, there exists a constant $C > 0$ (depending only on p and X) such that for any martingale $(g_n)_{n \geq 0} \subset L^p(\Omega, \Sigma, \mu; X)$ and all scalars $|\varepsilon_n| = 1, n = 1, 2, \dots, N$, the following inequality holds:

$$\left\| g_0 + \sum_{n=1}^N \varepsilon_n (g_n - g_{n-1}) \right\|_{L^p(\Omega, \Sigma, \mu; X)} \leq C \left\| g_N \right\|_{L^p(\Omega, \Sigma, \mu; X)}.$$

It is well known that L_p -spaces, Schatten class S_p , and Sobolev spaces $W^{m,p}$ are UMD spaces when $1 < p < \infty$. UMD spaces have played very important part in vector-valued harmonic analysis and probability theory, see [8].

Let X be a Banach space, we denote by $B(X)$ the space of all bounded linear operators on X . Let r_j be the j -th Rademacher function defined on $[0, 1]$ by $r_j(t) = \text{sgn}(\sin(2^j t))$ whenever $j \geq 1$.

Definition 2.3. Let X be a Banach spaces. A set $W \subset B(X)$ is said to be Rademacher bounded [10, 13], if there exists $C \geq 0$ such that

$$\left\| \sum_{j=1}^n r_j T_j x_j \right\|_{L^1([0,1]; X)} \leq C \left\| \sum_{j=1}^n r_j x_j \right\|_{L^1([0,1]; X)}$$

for all $T_1, T_2, \dots, T_n \in W, x_1, x_2, \dots, x_n \in X$ and $n \in \mathbb{N}$.

Remark 2.4. It is clear that if $W_1, W_2 \subset B(X)$ are R-bounded sets, then $W_1 W_2 := \{ST : S \in W_1, T \in W_2\}$ and $W_1 + W_2 := \{S + T : S \in W_1, T \in W_2\}$ are still R-bounded. It is easy to see that if W is a bounded subset of the complex plane, then the set $\{\mu I : \mu \in W\}$ is also R-bounded, where I stands for the identity operator. This follows easily from the Kahane’s contraction principle [16].

Let X be a Banach space and let $G : (-\pi, 0) \cup (0, \pi) \rightarrow B(X)$ be bounded and measurable. Let $f \in S(\mathbb{Z}; X)$ with finite support, i.e., the set $\{n \in \mathbb{Z} : f(n) \neq 0\}$ is finite. Then the function $t \rightarrow G(t)(\mathcal{F}f)(e^{it})$ defined on $(-\pi, 0) \cup (0, \pi)$ is bounded and measurable. Thus its inverse Fourier transform

$$\mathcal{F}^{-1}[G(\cdot)(\mathcal{F}f)(e^{i\cdot})](n) = \frac{1}{2\pi} \int_0^{2\pi} G(t)(\mathcal{F}f)(e^{it}) e^{int} dt$$

makes sense for all $n \in \mathbb{Z}$. Let $1 \leq p < \infty$ be given. We say that G is an ℓ^p -Fourier multiplier if there exists a constant $C > 0$ independent from f such that

$$\left(\sum_{n \in \mathbb{Z}} |\mathcal{F}^{-1}[G(\cdot)(\mathcal{F}f)(e^{i\cdot})](n)|^p \right)^{1/p} \leq C \left(\sum_{n \in \mathbb{Z}} |f(n)|^p \right)^{1/p}$$

for all $f \in S(\mathbb{Z}; X)$ with finite support. In this case, there exists a unique bounded linear operator $T_G \in B(\ell^p(\mathbb{Z}; X))$ such that

$$G(t)(\mathcal{F}f)(e^{it}) = \mathcal{F}(T_G f)(e^{it}) \tag{2.12}$$

for $t \in (-\pi, 0) \cup (0, \pi)$ and $f \in S(\mathbb{Z}; X)$ with finite support. Here we used the fact that the set of all elements $f \in S(\mathbb{Z}; X)$ with finite support is dense in $\ell^p(\mathbb{Z}; X)$.

We recall two results obtained by Blunck [7] concerning ℓ^p -Fourier multipliers which will be fundamental in our investigation.

Theorem 2.5. *Let X be a UMD space and let $1 < p < \infty$. Assume that $G : (-\pi, 0) \cup (0, \pi) \rightarrow B(X)$ is differentiable and the sets*

$$\{G(t) : t \in (-\pi, 0) \cup (0, \pi)\}, \quad \{(e^{it} - 1)(e^{it} + 1)G'(t) : t \in (-\pi, 0) \cup (0, \pi)\}$$

are R -bounded. Then G is an ℓ^p -Fourier multiplier.

Theorem 2.6. *Let X be a Banach space and $1 \leq p < \infty$. Let $G : (-\pi, 0) \cup (0, \pi) \rightarrow B(X)$ be continuous and bounded. Assume that G is an ℓ^p -Fourier multiplier. Then the set $\{G(t) : t \in (-\pi, 0) \cup (0, \pi)\}$ is R -bounded.*

3. REPRESENTATION OF SOLUTIONS

In this section, we introduce a special sequence of bounded operators, called α -resolvent sequence, which will give an explicit representation of the solution for the fractional difference equation (1.1).

3.1. α -Resolvent Sequence $(P_\alpha(n))_{n \in \mathbb{N}_0}$. We notice that the fractional difference equation (1.1) in the case $0 < \alpha \leq 1$ with the initial condition $u(0) = 0$ was previously studied by Lizama [18]. Later, Lizama and Murillo-Arcila [19] further studied the fractional difference equation (1.1) in the case $1 < \alpha < 2$ with initial conditions $u(0) = u(1) = 0$. We will introduce a similar α -resolvent sequence used in [18, 19].

Definition 3.1. Let X be a Banach space, $T \in B(X)$ and let $2 < \alpha < 3$. We let $(P_\alpha(n))_{n \in \mathbb{N}_0} \subset B(X)$ determined by:

- (i) $P_\alpha(0) = P_\alpha(1) = P_\alpha(2) = I$;
- (ii) $P_\alpha(n+3) - 2P_\alpha(n+2) + P_\alpha(n+1) = T(P_\alpha * k^{\alpha-2})(n) + k^{\alpha-2}(n+3)I + (1-\alpha)k^{\alpha-2}(n+2)I + \frac{(\alpha-1)(\alpha-2)}{2}k^{\alpha-2}(n+1)I$ for $n \in \mathbb{N}_0$.

$(P_\alpha(n))_{n \in \mathbb{N}_0}$ is called the α -resolvent sequence generated by T .

Remark 3.2. Let $2 < \alpha < 3$ and assume that $z^{3-\alpha}(z-1)^\alpha \in \rho(T)$ for all $|z| = 1$, $z \neq \pm 1$. Then the Fourier transform of P_α is

$$\hat{P}_\alpha(z) = [z^3 + (1-\alpha)z^2 + \frac{(\alpha-1)(\alpha-2)}{2}z] [z^{3-\alpha}(z-1)^\alpha - T]^{-1} \quad (3.1)$$

for $|z| = 1$, $z \neq \pm 1$. Indeed, taking the Fourier transform on both sides of (ii) in Definition 3.1 and using (2.5) and (2.9), we obtain

$$\begin{aligned} & z^3 \hat{P}_\alpha(z) - z^3 - z^2 - z - 2[z^2 \hat{P}_\alpha(z) - z^2 - z] + z \hat{P}_\alpha(z) - z \\ &= T \hat{P}_\alpha(z) \hat{k}^{\alpha-2}(z) + z^3 \hat{k}^{\alpha-2}(z) - z^3 - z^2(\alpha-2) - \frac{(\alpha-1)(\alpha-2)}{2}z \\ & \quad + (1-\alpha)[z^2 \hat{k}^{\alpha-2}(z) - z^2 - z(\alpha-2)] + \frac{(\alpha-1)(\alpha-2)}{2}[z \hat{k}^{\alpha-2}(z) - z] \end{aligned}$$

for $|z| = 1$, $z \neq \pm 1$. This implies that

$$[z^3 - 2z^2 + z - T \hat{k}^{\alpha-2}(z)] \hat{P}_\alpha(z) = [z^3 + (1-\alpha)z^2 + \frac{(\alpha-1)(\alpha-2)}{2}z] \hat{k}^{\alpha-2}(z)$$

for $|z| = 1, z \neq \pm 1$. It follows from (2.10) that

$$\hat{P}_\alpha(z) = [z^3 + (1 - \alpha)z^2 + \frac{(\alpha - 1)(\alpha - 2)}{2}z] [z^{3-\alpha}(z - 1)^\alpha - T]^{-1} \quad (3.2)$$

for $|z| = 1, z \neq \pm 1$.

For the proof of the main result of this section, we need the following results.

Lemma 3.3. *Let $2 < \alpha < 3, b : \mathbb{N}_0 \rightarrow \mathbb{C}$, and $P : \mathbb{N}_0 \rightarrow X$, where X is a Banach space. Then*

$$\begin{aligned} \Delta^\alpha(b * P)(n) &= (b * \Delta^\alpha P)(n) + b(n + 3)P(0) + b(n + 2)[P(1) - \alpha P(0)] \\ &\quad + b(n + 1)[P(2) - \alpha P(1) + \frac{\alpha(\alpha - 1)}{2}P(0)]. \end{aligned}$$

Proof. By (2.1)-(2.3) and (2.6), we have

$$\begin{aligned} \Delta^\alpha(b * P)(n) &= \Delta^3 \Delta^{-(3-\alpha)}(b * P)(n) \\ &= \Delta^{-(3-\alpha)}(b * P)(n + 3) - 3\Delta^{-(3-\alpha)}(b * P)(n + 2) \\ &\quad + 3\Delta^{-(3-\alpha)}(b * P)(n + 1) - \Delta^{-(3-\alpha)}(b * P)(n) \\ &= (k^{3-\alpha} * b * P)(n + 3) - 3(k^{3-\alpha} * b * P)(n + 2) \\ &\quad + 3(k^{3-\alpha} * b * P)(n + 1) - (k^{3-\alpha} * b * P)(n) \\ &= \sum_{j=0}^{n+3} b(j)(k^{3-\alpha} * P)(n + 3 - j) - 3 \sum_{j=0}^{n+2} b(j)(k^{3-\alpha} * P)(n + 2 - j) \\ &\quad + 3 \sum_{j=0}^{n+1} b(j)(k^{3-\alpha} * P)(n + 1 - j) - \sum_{j=0}^n b(j)(k^{3-\alpha} * P)(n - j) \end{aligned}$$

for $n \in \mathbb{N}_0$. It follows that

$$\begin{aligned} \Delta^\alpha(b * P)(n) &= \sum_{j=0}^n b(j)[(k^{3-\alpha} * P)(n + 3 - j) - 3(k^{3-\alpha} * P)(n + 2 - j) \\ &\quad + 3(k^{3-\alpha} * P)(n + 1 - j) - (k^{3-\alpha} * P)(n - j)] \\ &\quad + b(n + 1)(k^{3-\alpha} * P)(2) + b(n + 2)(k^{3-\alpha} * P)(1) \\ &\quad + b(n + 3)(k^{3-\alpha} * P)(0) - 3b(n + 1)(k^{3-\alpha} * P)(1) \\ &\quad - 3b(n + 2)(k^{3-\alpha} * P)(0) + 3b(n + 1)(k^{3-\alpha} * P)(0) \\ &= \sum_{j=0}^n b(j)\Delta^3(k^{3-\alpha} * P)(n - j) + b(n + 1)[k^{3-\alpha}(0)P(2) + k^{3-\alpha}(1)P(1) \\ &\quad + k^{3-\alpha}(2)P(0)] + b(n + 2)[k^{3-\alpha}(0)P(1) + k^{3-\alpha}(1)P(0)] \\ &\quad + b(n + 3)k^{3-\alpha}(0)P(0) - 3b(n + 1)[k^{3-\alpha}(0)P(1) + k^{3-\alpha}(1)P(0)] \\ &\quad - 3b(n + 2)k^{3-\alpha}(0)P(0) + 3b(n + 1)k^{3-\alpha}(0)P(0) \\ &= (\Delta^\alpha P * b)(n) + b(n + 3)P(0) + b(n + 2)[P(1) - \alpha P(0)] \end{aligned}$$

$$+ b(n+1) \left[P(2) - \alpha P(1) + \frac{\alpha(\alpha-1)}{2} P(0) \right]$$

where $n \in \mathbb{N}_0$. This completes the proof. \square

Lemma 3.4. *Let X be a Banach space, $T \in B(X)$, $2 < \alpha < 3$ and let $(P_\alpha(n))_{n \in \mathbb{N}_0}$ be the α -resolvent sequence given by Definition 3.1. Then*

$$\Delta^\alpha P_\alpha(n) = TP_\alpha(n) = T$$

where $0 \leq n \leq 2$.

Proof. By (2.1), (2.3) and (2.6), we have

$$\begin{aligned} \Delta^\alpha P_\alpha(0) &= \Delta^3 \Delta^{-(3-\alpha)} P_\alpha(0) \\ &= \Delta^{-(3-\alpha)} P_\alpha(3) - 3\Delta^{-(3-\alpha)} P_\alpha(2) + 3\Delta^{-(3-\alpha)} P_\alpha(1) - \Delta^{-(3-\alpha)} P_\alpha(0) \\ &= (k^{3-\alpha} * P_\alpha)(3) - 3(k^{3-\alpha} * P_\alpha)(2) + 3(k^{3-\alpha} * P_\alpha)(1) - (k^{3-\alpha} * P_\alpha)(0) \\ &= \sum_{j=0}^3 k^{3-\alpha} (3-j) P_\alpha(j) - 3 \sum_{j=0}^2 k^{3-\alpha} (2-j) P_\alpha(j) \\ &\quad + 3 \sum_{j=0}^1 k^{3-\alpha} (1-j) P_\alpha(j) - k^{3-\alpha} (0) P_\alpha(0) \\ &= P_\alpha(3) + k^{3-\alpha} (1) I - 2k^{3-\alpha} (2) I + k^{3-\alpha} (3) I - I. \end{aligned} \tag{3.3}$$

It follows from Definition 3.1 that

$$\begin{aligned} P_\alpha(3) &= I + T(k^{\alpha-2} * P_\alpha)(0) + k^{\alpha-2} (3) I + (1-\alpha) k^{\alpha-2} (2) I \\ &\quad + \frac{(\alpha-1)(\alpha-2)}{2} k^{\alpha-2} (1) I. \end{aligned} \tag{3.4}$$

Therefore, (3.3) and (3.4) imply that

$$\begin{aligned} \Delta^\alpha P_\alpha(0) &= I + T(k^{\alpha-2} * P_\alpha)(0) + k^{\alpha-2} (3) I \\ &\quad + (1-\alpha) k^{\alpha-2} (2) I + \frac{(\alpha-1)(\alpha-2)}{2} k^{\alpha-2} (1) I \\ &\quad + k^{3-\alpha} (1) I - 2k^{3-\alpha} (2) I + k^{3-\alpha} (3) I - I \\ &= T(k^{\alpha-2} * P_\alpha)(0) = TP_\alpha(0) = T. \end{aligned}$$

We have

$$\begin{aligned} P_\alpha(4) &= 2P_\alpha(3) - I + T(k^{\alpha-2} * P_\alpha)(1) + k^{\alpha-2} (4) I \\ &\quad + (1-\alpha) k^{\alpha-2} (3) I + \frac{(\alpha-1)(\alpha-2)}{2} k^{\alpha-2} (2) I. \end{aligned} \tag{3.5}$$

Applying (2.1), (2.3) and (2.6) again, we have

$$\begin{aligned}
 \Delta^\alpha P_\alpha(1) &= \Delta^3 \Delta^{-(3-\alpha)} P_\alpha(1) \\
 &= \Delta^{-(3-\alpha)} P_\alpha(4) - 3\Delta^{-(3-\alpha)} P_\alpha(3) + 3\Delta^{-(3-\alpha)} P_\alpha(2) - \Delta^{-(3-\alpha)} P_\alpha(1) \\
 &= (k^{3-\alpha} * P_\alpha)(4) - 3(k^{3-\alpha} * P_\alpha)(3) + 3(k^{3-\alpha} * P_\alpha)(2) - (k^{3-\alpha} * P_\alpha)(1) \\
 &= \sum_{j=0}^4 k^{3-\alpha}(4-j)P_\alpha(j) - 3 \sum_{j=0}^3 k^{3-\alpha}(3-j)P_\alpha(j) \\
 &\quad + 3 \sum_{j=0}^2 k^{3-\alpha}(2-j)P_\alpha(j) - \sum_{j=0}^1 k^{3-\alpha}(1-j)P_\alpha(j) \\
 &= P_\alpha(4) - \alpha P_\alpha(3) + k^{3-\alpha}(2)I - 2k^{3-\alpha}(3)I + k^{3-\alpha}(4)I + (\alpha - 1)I.
 \end{aligned} \tag{3.6}$$

By using (3.4), (3.5) and (3.6), we deduce that

$$\begin{aligned}
 \Delta^\alpha P_\alpha(1) &= T + k^{\alpha-2}(4)I + (3 - 2\alpha)k^{\alpha-2}(3)I + \frac{3(\alpha - 1)(\alpha - 2)}{2}k^{\alpha-2}(2)I \\
 &\quad - \frac{(\alpha - 1)(\alpha - 2)^2}{2}k^{\alpha-2}(1)I + k^{3-\alpha}(2)I - 2k^{3-\alpha}(3)I + k^{3-\alpha}(4)I = T.
 \end{aligned}$$

We have

$$\begin{aligned}
 P_\alpha(5) &= 2P_\alpha(4) - P_\alpha(3) + T(k^{\alpha-2} * P_\alpha)(2) + k^{\alpha-2}(5)I \\
 &\quad + (1 - \alpha)k^{\alpha-2}(4)I + \frac{(\alpha - 1)(\alpha - 2)}{2}k^{\alpha-2}(3)I.
 \end{aligned} \tag{3.7}$$

A similar argument used in (3.6) shows that

$$\begin{aligned}
 \Delta^\alpha P_\alpha(2) &= P_\alpha(5) - \alpha P_\alpha(4) + \frac{\alpha(\alpha - 1)}{2}P_\alpha(3) + k^{3-\alpha}(5)I - 2k^{3-\alpha}(4)I \\
 &\quad + k^{3-\alpha}(3)I - k^{3-\alpha}(2)I + 2k^{3-\alpha}(1)I - k^{3-\alpha}(0)I.
 \end{aligned} \tag{3.8}$$

By using (3.4), (3.5), (3.7) ,and (3.8), we deduce that

$$\begin{aligned}
 \Delta^\alpha P_\alpha(2) &= T + \frac{\alpha(\alpha - 3)}{2}I + k^{\alpha-2}(5)I + (3 - 2\alpha)k^{\alpha-2}(4)I + (\alpha - 2)(2\alpha - 3)k^{\alpha-2}(3)I \\
 &\quad + \frac{(2 - \alpha)(1 - \alpha)(5 - 2\alpha)}{2}k^{\alpha-2}(2)I - \frac{(\alpha - 1)(3 - \alpha)(\alpha - 2)^2}{4}k^{\alpha-2}(1)I \\
 &\quad + k^{3-\alpha}(5)I - 2k^{3-\alpha}(4)I + k^{3-\alpha}(3)I - k^{3-\alpha}(2)I + 2k^{3-\alpha}(1)I = T.
 \end{aligned}$$

This completes the proof. □

Lemma 3.5. *Let X be a Banach space, $T \in B(X)$, $2 < \alpha < 3$ and let $(P_\alpha(n))_{n \in \mathbb{N}_0}$ be the α -resolvent sequence given by Definition 3.1. Then*

$$\Delta^\alpha P_\alpha(n) = TP_\alpha(n)$$

for all $n \in \mathbb{N}_0$.

Proof. First note that

$$\Delta^\alpha k^{\alpha-2}(n) = \Delta^3 \Delta^{-(3-\alpha)} k^{\alpha-2}(n) = \Delta^3 (k^{3-\alpha} * k^{\alpha-2})(n) = \Delta^3 k^1(n) = 0 \tag{3.9}$$

where $n \in \mathbb{N}_0$. By (ii) of Definition 3.1, we obtain

$$\begin{aligned} & \sum_{j=0}^n k^{3-\alpha}(j)P_\alpha(n+3-j) - 2 \sum_{j=0}^n k^{3-\alpha}(j)P_\alpha(n+2-j) + \sum_{j=0}^n k^{3-\alpha}(j)P_\alpha(n+1-j) \\ &= T \sum_{j=0}^n k^{3-\alpha}(j)(k^{\alpha-2} * P_\alpha)(n-j) + \sum_{j=0}^n k^{3-\alpha}(j)k^{\alpha-2}(n+3-j)I \\ & \quad + (1-\alpha) \sum_{j=0}^n k^{\alpha-2}(n+2-j)k^{3-\alpha}(j)I \\ & \quad + \frac{(\alpha-1)(\alpha-2)}{2} \sum_{j=0}^n k^{\alpha-2}(n+1-j)k^{3-\alpha}(j)I \end{aligned}$$

where $n \in \mathbb{N}_0$. This implies that

$$\begin{aligned} & \Delta^{-(3-\alpha)}P_\alpha(n+3) - 2\Delta^{-(3-\alpha)}P_\alpha(n+2) + \Delta^{-(3-\alpha)}P_\alpha(n+1) \\ & \quad - P_\alpha(2)k^{3-\alpha}(n+1) - P_\alpha(1)k^{3-\alpha}(n+2) \\ & \quad - k^{3-\alpha}(n+3)P_\alpha(0) + 2k^{3-\alpha}(n+1)P_\alpha(1) \\ & \quad + 2k^{3-\alpha}(n+2)P_\alpha(0) - k^{3-\alpha}(n+1)P_\alpha(0) \\ &= T\Delta^{-(3-\alpha)}(k^{\alpha-2} * P_\alpha)(n) + \Delta^{-(3-\alpha)}k^{\alpha-2}(n+3)I \\ & \quad + (1-\alpha)\Delta^{-(3-\alpha)}k^{\alpha-2}(n+2)I \\ & \quad + \frac{(\alpha-1)(\alpha-2)}{2}\Delta^{-(3-\alpha)}k^{\alpha-2}(n+1)I - k^{\alpha-2}(0)k^{3-\alpha}(n+3)I \\ & \quad - k^{\alpha-2}(1)k^{3-\alpha}(n+2)I - k^{\alpha-2}(2)k^{3-\alpha}(n+1)I \\ & \quad - (1-\alpha)k^{\alpha-2}(1)k^{3-\alpha}(n+1)I - (1-\alpha)k^{\alpha-2}(0)k^{3-\alpha}(n+2)I \\ & \quad - \frac{(\alpha-1)(\alpha-2)}{2}k^{\alpha-2}(0)k^{3-\alpha}(n+1)I \end{aligned}$$

for $n \in \mathbb{N}_0$. Thus

$$\begin{aligned} & \Delta^{-(3-\alpha)}P_\alpha(n+3) - 2\Delta^{-(3-\alpha)}P_\alpha(n+2) + \Delta^{-(3-\alpha)}P_\alpha(n+1) \\ &= T\Delta^{-(3-\alpha)}(k^{\alpha-2} * P_\alpha)(n) + \Delta^{-(3-\alpha)}k^{\alpha-2}(n+3) \\ & \quad + (1-\alpha)\Delta^{-(3-\alpha)}k^{\alpha-2}(n+2) + \frac{(\alpha-1)(\alpha-2)}{2}\Delta^{-(3-\alpha)}k^{\alpha-2}(n+1) \end{aligned}$$

for $n \in \mathbb{N}_0$. Consequently

$$\Delta^\alpha P_\alpha(n+3) - 2\Delta^\alpha P_\alpha(n+2) + \Delta^\alpha P_\alpha(n+1) = T\Delta^\alpha(P_\alpha * k^{\alpha-2})(n)$$

by (3.9) for $n \in \mathbb{N}_0$. It follows from Lemma 3.3 that

$$\begin{aligned} \Delta^\alpha(k^{\alpha-2} * P_\alpha)(n) &= (\Delta^\alpha k^{\alpha-2} * P_\alpha)(n) + k^{\alpha-2}(0)P_\alpha(n+3) \\ & \quad + [k^{\alpha-2}(1) - \alpha k^{\alpha-2}(0)]P_\alpha(n+2) \\ & \quad + [k^{\alpha-2}(2) - \alpha k^{\alpha-2}(1) + \frac{\alpha(\alpha-1)}{2}k^{\alpha-2}(0)]P_\alpha(n+1) \\ &= P_\alpha(n+3) - 2P_\alpha(n+2) + P_\alpha(n+1) \end{aligned}$$

for $n \in \mathbb{N}_0$. We conclude that

$$\Delta^2 \Delta^\alpha P_\alpha(n+1) = \Delta^2 T P_\alpha(n+1)$$

for $n \in \mathbb{N}_0$. The conclusion follows easily from Lemma 3.4. The proof is complete. \square

3.2. Formula of Solution. With the help of the α -resolvent sequence $(P_\alpha(n))_{n \in \mathbb{N}_0}$, we are able to give the exact expression of the solution of (1.1).

Definition 3.6. For $2 < \alpha < 3$, we define the function $h_\alpha : \mathbb{N}_0 \rightarrow \mathbb{R}$ by $h_\alpha(0) = 1$, $h_\alpha(1) = \alpha - 1$, $h_\alpha(2) = \frac{\alpha(\alpha-1)}{2}$, and

$$h_\alpha(n+3) + (1-\alpha)h_\alpha(n+2) + \frac{(\alpha-1)(\alpha-2)}{2}h_\alpha(n+1) = 0 \tag{3.10}$$

for $n \geq 0$.

Remark 3.7. Let $(h_\alpha(n))_{n \in \mathbb{N}_0}$ be defined by (3.10). Then the Fourier transform of h_α is

$$\hat{h}_\alpha(z) = \frac{z^3}{z^3 + (1-\alpha)z^2 + \frac{(\alpha-1)(\alpha-2)}{2}z}. \tag{3.11}$$

Indeed, taking the Fourier transform on both sides of (3.10), we have

$$\begin{aligned} z^3 \hat{h}_\alpha(z) - z^3 - (\alpha-1)z^2 - \frac{\alpha(\alpha-1)}{2}z + (1-\alpha)[z^2 \hat{h}_\alpha(z) - z^2 - (\alpha-1)z] \\ + \frac{(\alpha-1)(\alpha-2)}{2}[z \hat{h}_\alpha(z) - z] = 0 \end{aligned}$$

when $|z| = 1$, which implies that

$$[z^3 + (1-\alpha)z^2 + \frac{(\alpha-1)(\alpha-2)}{2}z] \hat{h}_\alpha(z) = z^3 \tag{3.12}$$

for $|z| = 1$. Thus (3.11) holds for all $|z| = 1$.

Now we are ready to state the main result of this section.

Theorem 3.8. *Let X be a Banach space, $T \in B(X)$, $2 < \alpha < 3$ and let $f \in S(\mathbb{N}_0; X)$ be given. Then (1.1) has a unique solution $u \in S(\mathbb{N}_0; X)$ defined by $u(0) = u(1) = u(2) = 0$, and*

$$u(n) = (h_\alpha * P_\alpha * f)(n-3) \tag{3.13}$$

for $n \geq 3$.

Proof. Let $u \in S(\mathbb{N}_0; X)$. By Lemma 3.3, Lemma 3.5 and Definition 3.6, we have

$$\begin{aligned} &\Delta^\alpha (h_\alpha * P_\alpha)(n) \\ &= (h_\alpha * \Delta^\alpha P_\alpha)(n) + h_\alpha(n+3)P_\alpha(0) \\ &\quad + h_\alpha(n+2)[P_\alpha(1) - \alpha P_\alpha(0)] + h_\alpha(n+1)[P_\alpha(2) - \alpha P_\alpha(1) + \frac{\alpha(\alpha-1)}{2}P_\alpha(0)] \\ &= (Th_\alpha * P_\alpha)(n) + h_\alpha(n+3) + (1-\alpha)h_\alpha(n+2) + \frac{(\alpha-1)(\alpha-2)}{2}h_\alpha(n+1) \\ &= T(h_\alpha * P_\alpha)(n) \end{aligned} \tag{3.14}$$

for $n \in \mathbb{N}_0$. We deduce from Lemma 3.3, Lemma 3.5 and (3.14) that

$$\begin{aligned}
 & \Delta^\alpha(h_\alpha * P_\alpha * f)(n) \\
 &= (\Delta^\alpha(h_\alpha * P_\alpha) * f)(n) + (h_\alpha * P_\alpha)(0)f(n+3) \\
 & \quad + [(h_\alpha * P_\alpha)(1) - \alpha(h_\alpha * P_\alpha)(0)]f(n+2) \\
 & \quad + [(h_\alpha * P_\alpha)(2) - \alpha(h_\alpha * P_\alpha)(1) + \frac{\alpha(\alpha-1)}{2}(h_\alpha * P_\alpha)(0)]f(n+1) \\
 &= (\Delta^\alpha(h_\alpha * P_\alpha) * f)(n) + f(n+3) \\
 &= T(P_\alpha * h_\alpha * f)(n) + f(n+3)
 \end{aligned} \tag{3.15}$$

where $n \in \mathbb{N}_0$. We claim that

$$\Delta^\alpha u(n) = \Delta^\alpha(P_\alpha * h_\alpha * f)(n-3) \tag{3.16}$$

where $n \geq 3$. Indeed, since $u(0) = u(1) = u(2) = 0$, we have

$$\begin{aligned}
 \Delta^{-(3-\alpha)}u(n) &= (k^{3-\alpha} * u)(n) = \sum_{j=0}^n k^{3-\alpha}(j)u(n-j) \\
 &= \sum_{j=0}^{n-3} k^{3-\alpha}(j)(h_\alpha * P_\alpha * f)(n-3-j) \\
 &= \Delta^{-(3-\alpha)}(h_\alpha * P_\alpha * f)(n-3)
 \end{aligned}$$

for $n \geq 3$, which clearly implies that (3.16) is true as $\Delta^\alpha = \Delta^3 \Delta^{-(3-\alpha)}$ by (2.6). It follows from (3.15) and (3.16) that

$$\Delta^\alpha u(n) = Tu(n) + f(n) \tag{3.17}$$

for $n \geq 3$.

Now we are going to show that equality (3.17) remains valid when $0 \leq n \leq 2$. Indeed, it follows from (3.13) that

$$u(3) = (h_\alpha * P_\alpha * f)(0) = (h_\alpha * P_\alpha)(0)f(0) = h_\alpha(0)P_\alpha(0)f(0) = f(0); \tag{3.18}$$

$$\begin{aligned}
 u(4) &= (h_\alpha * P_\alpha * f)(1) = (h_\alpha * P_\alpha)(1)f(0) + (h_\alpha * P_\alpha)(0)f(1) \\
 &= [h_\alpha(1)P_\alpha(0) + h_\alpha(0)P_\alpha(1)]f(0) + h_\alpha(0)P_\alpha(0)f(1) \\
 &= \alpha f(0) + f(1);
 \end{aligned} \tag{3.19}$$

$$\begin{aligned}
 u(5) &= (h_\alpha * P_\alpha * f)(2) = (h_\alpha * P_\alpha)(2)f(0) + (h_\alpha * P_\alpha)(1)f(1) \\
 & \quad + (h_\alpha * P_\alpha)(0)f(2) \\
 &= [h_\alpha(2)P_\alpha(0) + h_\alpha(1)P_\alpha(1) + h_\alpha(0)P_\alpha(2)]f(0) \\
 & \quad + [h_\alpha(1)P_\alpha(0) + h_\alpha(0)P_\alpha(1)]f(1) + h_\alpha(0)P_\alpha(0)f(2) \\
 &= \frac{\alpha(\alpha+1)}{2}f(0) + \alpha f(1) + f(2).
 \end{aligned} \tag{3.20}$$

On the other hand, using the conditions $u(0) = u(1) = u(2) = 0$ and (3.18)-(3.20), we have

$$\Delta^\alpha u(0) = \Delta^3 \Delta^{-(3-\alpha)}u(0) = (k^{3-\alpha} * u)(3) = k^{3-\alpha}(0)u(3) = u(3) = f(0);$$

$$\begin{aligned} \Delta^\alpha u(1) &= \Delta^3 \Delta^{-(3-\alpha)} u(1) = (k^{3-\alpha} * u)(4) - 3(k^{3-\alpha} * u)(3) \\ &= k^{3-\alpha}(0)u(4) + k^{3-\alpha}(1)u(3) - 3k^{3-\alpha}(0)u(3) \\ &= u(4) - \alpha u(3) = f(1); \\ \Delta^\alpha u(2) &= \Delta^3 \Delta^{-(3-\alpha)} u(2) = (k^{3-\alpha} * u)(5) - 3(k^{3-\alpha} * u)(4) + 3(k^{3-\alpha} * u)(3) \\ &= k^{3-\alpha}(0)u(5) + k^{3-\alpha}(1)u(4) + k^{3-\alpha}(2)u(3) \\ &\quad - 3k^{3-\alpha}(0)u(4) - 3k^{3-\alpha}(1)u(3) + 3k^{3-\alpha}(0)u(3) \\ &= u(5) - \alpha u(4) + \frac{\alpha(\alpha - 1)}{2} u(3) = f(2). \end{aligned}$$

Thus $\Delta^\alpha u(n) = Tu(n) + f(n)$ where $0 \leq n \leq 2$ as $u(0) = u(1) = u(2) = 0$. We have shown that $u \in S(\mathbb{N}_0; X)$ given by (3.13) is a solution of (1.1).

It remains to show that the solution is unique. It is clear that we only need to show that $0 \in S(\mathbb{N}_0; X)$ is the unique solution of the equation

$$\begin{aligned} \Delta^\alpha u(n) &= Tu(n), \quad (n \in \mathbb{N}_0); \\ u(0) &= u(1) = u(2) = 0. \end{aligned} \tag{3.21}$$

Let $u \in S(\mathbb{N}_0; X)$ be a solution of (3.21). We first show that $u(3) = 0$. The identity $\Delta^\alpha u(0) = Tu(0)$ implies that $\Delta^\alpha u(0) = 0$, or, equivalently,

$$\begin{aligned} \Delta^3 \Delta^{-(3-\alpha)} u(0) &= \Delta^{-(3-\alpha)} u(3) - 3\Delta^{-(3-\alpha)} u(2) + 3\Delta^{-(3-\alpha)} u(1) + \Delta^{\alpha-3} u(0) \\ &= (k^{3-\alpha} * u)(3) - 3(k^{3-\alpha} * u)(2) + 3(k^{3-\alpha} * u)(1) + (k^{3-\alpha} * u)(0) \\ &= k^{3-\alpha}(0)u(3) = u(3) = 0 \end{aligned}$$

as $u(0) = u(1) = u(2) = 0$ by assumption.

Assume that $u(n) = 0$ for all $3 \leq n \leq k$ for some $k \geq 3$. We are going to show that $u(k + 1) = 0$. Since $k - 2 < k$, we have $u(k - 2) = 0$ by assumption. Thus $\Delta^\alpha u(k - 2) = Tu(k - 2) = 0$, or, equivalently,

$$\begin{aligned} \Delta^3 \Delta^{-(3-\alpha)} u(k - 2) &= \Delta^{-(3-\alpha)} u(k + 1) - 3\Delta^{-(3-\alpha)} u(k) + 3\Delta^{-(3-\alpha)} u(k - 1) + \Delta^{-(3-\alpha)} u(k - 2) \\ &= (k^{3-\alpha} * u)(k + 1) - 3(k^{3-\alpha} * u)(k) + 3(k^{3-\alpha} * u)(k - 1) + (k^{3-\alpha} * u)(k - 2) \\ &= k^{3-\alpha}(0)u(k + 1) = u(k + 1) = 0 \end{aligned}$$

as $u(k - 2) = u(k - 1) = u(k) = 0$ by assumption. Consequently $u(n) = 0$ for all $n \in \mathbb{N}_0$. This concludes the proof. \square

4. CHARACTERIZATION OF THE ℓ^p -MAXIMAL REGULARITY

Let $T \in B(X)$, where X is a Banach space. Let $f : \mathbb{N}_0 \rightarrow X$ be an X -valued sequence and $1 < p < \infty$. In this section, we will study the ℓ^p -maximal regularity for the discrete time evolution equation of fractional order

$$\begin{aligned} \Delta^\alpha u(n) &= Tu(n) + f(n), \quad (n \in \mathbb{N}_0); \\ u(0) &= u(1) = u(2) = 0, \end{aligned} \tag{4.1}$$

where $2 < \alpha < 3$ is given.

For all $f \in S(\mathbb{N}_0; X)$, the unique solution $u \in S(\mathbb{N}_0; X)$ of (4.1) is given by $u(0) = u(1) = u(2) = 0$, and

$$u(n) = (P_\alpha * h_\alpha * f)(n - 3), \tag{4.2}$$

for $n \geq 3$ by Theorem 3.8. This means that $\Delta^\alpha u(0) = \Delta^\alpha u(1) = \Delta^\alpha u(2) = 0$, and

$$\Delta^\alpha u(n) = T(P_\alpha * h_\alpha * f)(n - 3) + f(n) \tag{4.3}$$

for $n \geq 3$.

Now we introduce the following definition concerning maximal regularity, which is motivated by the case $\alpha = 1$ and $\alpha = 2$, see for instance [7].

Definition 4.1. Let $1 < p < \infty$. We say that (4.1) has the ℓ^p -maximal regularity if

$$(\mathcal{R}_\alpha f)(n) := T(P_\alpha * h_\alpha * f)(n) = T \sum_{j=0}^n (P_\alpha * h_\alpha)(n - j) f(j), \quad (n \in \mathbb{N}_0) \tag{4.4}$$

defines a bounded linear operator $\mathcal{R}_\alpha \in B(\ell^p(\mathbb{N}_0; X))$.

It is clear that (4.1) has the ℓ^p -maximal regularity if and only if for all $f \in \ell^p(\mathbb{N}_0; X)$, the unique solution u of (4.1) given by (4.2) satisfies $\Delta^\alpha u \in \ell^p(\mathbb{N}_0; X)$ by (4.3).

We say that $T \in B(X)$ satisfies the assumption (\mathcal{C}_α) if $z^{3-\alpha}(z - 1)^\alpha \in \rho(T)$ for all $|z| = 1, z \neq \pm 1$. Now we are ready to state the main result of this section.

Theorem 4.2. Let X be a UMD space, $2 < \alpha < 3$ and let $1 < p < \infty$. Assume that $T \in B(X)$ satisfies the assumption (\mathcal{C}_α) . Then the following statements are equivalent:

- (i) Equation(4.1) has the ℓ^p -maximal regularity;
- (ii) the set

$$\{e^{(3-\alpha)it}(e^{it} - 1)^\alpha [e^{(3-\alpha)it}(e^{it} - 1)^\alpha - T]^{-1} : t \in (-\pi, 0) \cup (0, \pi)\} \tag{4.5}$$

is R -bounded.

Proof. (ii) implies (i). Assume that the set (4.5) is R -bounded. We are going to show that (4.1) has the ℓ^p -maximal regularity. Let

$$g_\alpha(t) = e^{3it}(1 - e^{-it})^\alpha, \quad G(t) = g_\alpha(t)[g_\alpha(t) - T]^{-1}$$

for $t \in (-\pi, 0) \cup (0, \pi)$. Then

$$\begin{aligned} g'_\alpha(t) &= 3ig_\alpha(t) + \frac{\alpha ig_\alpha(t)}{e^{it} - 1} = (3i + \frac{\alpha i}{e^{it} - 1})g_\alpha(t) \\ G'(t) &= \frac{g'_\alpha(t)}{g_\alpha(t)}G(t) - \frac{g'_\alpha(t)}{g_\alpha(t)}G^2(t) = (3i + \frac{\alpha i}{e^{it} - 1})(G(t) - G^2(t)) \end{aligned}$$

for $t \in (-\pi, 0) \cup (0, \pi)$. Thus the sets

$$\{G(t) : t \in (-\pi, 0) \cup (0, \pi)\}, \quad \{(e^{it} - 1)(e^{it} + 1)G'(t) : t \in (-\pi, 0) \cup (0, \pi)\}$$

are R -bounded by assumption and Remark 2.4. Using Theorem 2.5, there exists an operator $T_\alpha \in B(\ell^p(\mathbb{Z}; X))$ such that

$$\widehat{T_\alpha f}(e^{it}) = G(t)\hat{f}(e^{it}) \tag{4.6}$$

when $t \in (-\pi, 0) \cup (0, \pi)$ for all $f \in \ell^p(\mathbb{Z}; X)$ with finite support. The trivial identity

$$T[g_\alpha(t) - T]^{-1} = g_\alpha(t)[g_\alpha(t) - T]^{-1} - I$$

for $t \in (-\pi, 0) \cup (0, \pi)$ together with (4.6) implies that

$$\begin{aligned} T[g_\alpha(t) - T]^{-1} \hat{f}(e^{it}) &= g_\alpha(t)[g_\alpha(t) - T]^{-1} \hat{f}(e^{it}) - \hat{f}(e^{it}) \\ &= G(t)\hat{f}(e^{it}) - \hat{f}(e^{it}) = \widehat{T_\alpha f}(e^{it}) - \hat{f}(e^{it}) \end{aligned} \tag{4.7}$$

when $t \in (-\pi, 0) \cup (0, \pi)$ for all $f \in \ell^p(\mathbb{Z}; X)$ with finite support. It follows from Remark 3.2 and Remark 3.7 that

$$\widehat{P_\alpha * h_\alpha}(e^{it}) = e^{3it}[g_\alpha(t) - T]^{-1} \tag{4.8}$$

for $t \in (-\pi, 0) \cup (0, \pi)$. This together with (4.8) implies that

$$\widehat{\mathcal{R}_\alpha f}(e^{it}) = e^{3it}T[g_\alpha(t) - T]^{-1} \hat{f}(e^{it})$$

when $t \in (-\pi, 0) \cup (0, \pi)$ for all $f \in \ell^p(\mathbb{N}_0; X)$ with finite support. Hence \mathcal{R}_α is a bounded and linear operator on $\ell^p(\mathbb{N}_0; X)$ by (4.7). Here we have used the fact that the set of all $f \in \ell^p(\mathbb{N}_0; X)$ with finite support is dense in $\ell^p(\mathbb{N}_0; X)$. We have shown that (4.1) has the ℓ^p -maximal regularity. Hence (ii) implies (i).

(i)implies (ii). Assume that (i) holds. Then

$$(\mathcal{R}_\alpha f)(n) = \begin{cases} T(P_\alpha * h_\alpha * f)(n), & n \geq 3, \\ 0, & \text{otherwise.} \end{cases} \tag{4.9}$$

defines a bounded linear operator $\mathcal{R}_\alpha \in B(\ell^p(\mathbb{N}_0; X))$. It follows from (4.8) that

$$\widehat{\mathcal{R}_\alpha f}(e^{it}) = e^{3it}T[g_\alpha(t) - T]^{-1} \hat{f}(e^{it}) \tag{4.10}$$

when $t \in (-\pi, 0) \cup (0, \pi)$ for all $f \in \ell^p(\mathbb{N}_0; X)$ with finite support. Since it is clear that \mathcal{R}_α is a convolution operator, \mathcal{R}_α is translation invariant on $\ell^p(\mathbb{Z}; X)$. Thus \mathcal{R}_α extends to a bounded linear operator on $\ell^p(\mathbb{Z}; X)$. In other words, the function $t \rightarrow e^{3it}T[g_\alpha(t) - T]^{-1}$ defined on $(-\pi, 0) \cup (0, \pi)$ is an ℓ^p -Fourier multiplier by (2.12). Hence the set

$$\{e^{3it}T[g_\alpha(t) - T]^{-1} : t \in (-\pi, 0) \cup (0, \pi)\}$$

is R-bounded by Theorem 2.6. The trivial equality

$$T[g_\alpha(t) - T]^{-1} + I = g_\alpha(t)[g_\alpha(t) - T]^{-1}$$

for $t \in (-\pi, 0) \cup (0, \pi)$ implies that the set

$$\{g_\alpha(t)[g_\alpha(t) - T]^{-1} : t \in (-\pi, 0) \cup (0, \pi)\}$$

is R-bounded. This completes the proof. □

Since the second condition in Theorem 4.2 does not depend on the parameter $1 < p < \infty$, we have the following immediate consequence.

Corollary 4.3. *Let X be a UMD space, $2 < \alpha < 3$ and let $T \in B(X)$ satisfy the assumption (\mathcal{C}_α) . If (4.1) has the ℓ^p -maximal regularity for some $1 < p < \infty$, then it has the ℓ^p -maximal regularity for all $1 < p < \infty$.*

Let H be a Hilbert space, then each bounded subset $W \subset B(H)$ is actually R-bounded [3]. This together with Theorem 4.2 gives the following result.

Corollary 4.4. *Let H be a Hilbert space, $2 < \alpha < 3$, $1 < p < \infty$ and let $T \in B(H)$ satisfy the assumption (C_α) . Then (4.1) has the ℓ^p -maximal regularity if and only if there exists $C > 0$ such that*

$$|e^{(3-\alpha)it}(e^{it} - 1)^\alpha [e^{(3-\alpha)it}(e^{it} - 1)^\alpha - T]^{-1}| \leq C \quad (4.11)$$

for all $t \in (-\pi, 0) \cup (0, \pi)$.

Now we give a concrete example that Corollary 4.4 is applicable. Let H be a Hilbert space, $1 < p < \infty$, $2 < \alpha < 3$ and let $T \in B(H)$ satisfy

$$\sigma(T) \subset \{z \in \mathbb{C} : |z| > 2^\alpha\}.$$

Then for every $f \in \ell^p(\mathbb{N}_0; H)$, there is a unique $u \in \ell^p(\mathbb{N}_0; H)$ such that (4.1) holds. Indeed, an easy calculation gives

$$\sup_{t \in [-\pi, \pi]} |(e^{it} - 1)^\alpha| \leq 2^\alpha.$$

This together with the assumption $\sigma(T) \subset \{z \in \mathbb{C} : |z| > 2^\alpha\}$ implies that $e^{(3-\alpha)it}(e^{it} - 1)^\alpha \in \rho(T)$ for all $t \in [-\pi, \pi]$, and since the function $t \rightarrow |(e^{(3-\alpha)it}(e^{it} - 1)^\alpha - T)^{-1}|$ is continuous on the closed interval $[-\pi, \pi]$, the condition (4.11) is valid.

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