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NORMALIZED GROUND STATE OF A MIXED DISPERSION NONLINEAR SCHRÖDINGER EQUATION WITH COMBINED POWER-TYPE NONLINEARITIES

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Abstract. We study the existence of normalized ground state solutions to a mixed dispersion fourth-order nonlinear Schrödinger equation with combined power-type nonlinearities. By analyzing the subadditivity of the ground state energy with respect to the prescribed mass, we employ a constrained minimization method to establish the existence of ground state that corresponds to a local minimum of the associated functional. Under certain conditions, by studying the monotonicity of ground state energy as the mass varies, we apply the constrained minimization arguments on the Nehari-Pohozaev manifold to prove the existence of normalized ground state solutions.

1. Introduction and main results

Consider the mixed dispersion nonlinear Schrödinger equation with combined power-type nonlinearities

$$i\partial_t\psi - \epsilon\Delta^2\psi + \gamma\Delta\psi + \mu|\psi|^{q-2}\psi + |\psi|^{p-2}\psi = 0, \tag{1.1}$$

where $N \geq 1, \mu \geq 0, \epsilon \geq 0, \gamma \in \mathbb{R}, \psi \in \mathbb{R}\times\mathbb{R}^N \to \mathbb{C}$ and $2 < q < p \leq 4^*$. Note that equation (1.1) becomes the well-known Schrödinger equation when $\epsilon = 0$ and $\gamma = 1$. This equation has been extensively studied as a partial differential equation, presenting various mathematical challenges from the perspective of mathematical physics [4, 6]. Over the past decades, a lot of attention has been paid to normalized solutions of the nonlinear Schrödinger equation with both pure and mixed nonlinearities [1, 7, 10, 11, 12, 13, 17, 18, 19, 22, 23, 26, 34, 35, 38] and the references therein. For the specific case $\mu = 0$, when $2 < p < 2 + \frac{4}{N}$, all solutions to (1.1) with $\epsilon = 0$ exist globally, and the associated standing waves are orbitally stable. However, for $p \geq 2 + \frac{4}{N}$, the solutions to equation (1.1) can exhibit singularity within a finite time. To address regularization and stabilization of these solutions, Karpman-Shagalov [21, 20] proposed the inclusion of a small fourth-order dispersion term $\epsilon\|\Delta u\|_2^2$ in the model. Through a combination of stability analysis and numerical simulations, they demonstrated the stable outcomes for $2 < p < 2 + \frac{8}{N}$, while noting the instability phenomena for $p \geq 2 + \frac{8}{N}$. Consequently, $\overline{p} = 2 + \frac{8}{N}$

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appears as a new mass critical exponent. Despite the significance of the mixed dispersion fourth-order nonlinear Schrödinger equation in physical contexts, it remains inadequately understood, as addressed in [4, 8, 15, 29, 30, 32].

In this article, we are concerned with equation (1.1) and its standing waves solutions of the form $\psi(t,x) = e^{i\omega t}u(x)$, where $\omega \in \mathbb{R}$ is a Lagrange multiplier and $u(x)$ satisfies

$$\epsilon\Delta^2 u - \gamma\Delta u + \omega u - \mu|u|^{q-2}u - |u|^{p-2}u = 0 \quad \text{in } \mathbb{R}^N. \tag{1.2}$$

When we consider solutions to (1.2), a possible choice is to consider a fixed value $\omega \in \mathbb{R}$ and search for solutions as the critical points of the action functional

$$A_{\omega,\mu}(u) = \frac{\epsilon}{2}\|\Delta u\|_2^2 + \frac{\gamma}{2}\|\nabla u\|_2^2 + \frac{\omega}{2}\|u\|_2^2 - \frac{\mu}{q}\|u\|_q^q - \frac{1}{p}\|u\|_p^p.$$

In this case, we focus on the existence of minimal action solutions, namely, solutions minimizing $A_{\omega,\mu}$ among all non-trivial solutions [6, 3].

Alternatively, we can search for solutions to (1.2) with a prescribed L^2-norm. Define the energy functional on $H^2 = H^2(\mathbb{R}^N, \mathbb{C})$ by

$$E_{p,q}(u) := \frac{\epsilon}{2}\|\Delta u\|_2^2 + \frac{\gamma}{2}\|\nabla u\|_2^2 - \frac{\mu}{q}\|u\|_q^q - \frac{1}{p}\|u\|_p^p.$$

It is standard to check that $E_{p,q}$ is of class C^1 and a critical point of $E_{p,q}$ restricted to the mass constraint

$$S(c) = \{u \in H^2 : \|u\|_2^2 = c\}$$

gives rise to a solution to (1.2) with $\|u\|_2^2 = c$.

If $\mu = 0$, the corresponding functional is denoted by E_p. When $\epsilon > 0$ and $\gamma > 0$, with a pure mass subcritical nonlinearity, i.e., $2 < p < \overline{p}$ as considered in [5], the functional E_p has been shown to be bounded from below on $S(c)$, and critical points of E can be sought as global minimizers for any $c > 0$. Bonheure et al [3] investigated the existence of normalized ground states of (1.2) by exploiting the constrained minimization method and explored the normalized solutions of equation (1.2) with pure mass-critical and mass-supcritical nonlinearity, i.e., $\overline{p} \le p < 4^*$.

When $\epsilon = 1$, $\gamma < 0$ and $\mu = 0$, Luo et al [24] used a profile decomposition technique to study the existence of ground states for (1.2) with $c = 1$ and $2 < p \le \overline{p}$. Boussaid et al [9] obtained the existence of normalized ground state solutions for all $c > 0$, $\gamma < 0$ and $2 < p \le \overline{p}$ without the restriction on c and γ imposed in [24]. For $\overline{p} < p < 4^*$, Luo-Yang [25] identified at least two radial normalized solutions: a ground state and an excited state, along with associated asymptotic properties. Recently, Fernández et al [14] utilized the Tomas-Stein inequality to develop a novel approach for establishing non-homogeneous Gagliardo-Nirenberg-type inequalities in $\mathbb{R}^N$. These inequalities play a crucial role in proving optimal results regarding the existence of global minimizers for $2 < q \le \overline{p}$. Additionally, for the case $2 < q \le \overline{p}$, they showed the existence of local minimizers in $H^2(\mathbb{R}^N)$ but not $H_r^2(\mathbb{R}^N)$.

When $\epsilon > 0$ and $\gamma = 0$, equation (1.1) becomes the biharmonic nonlinear Schrödinger equation, in which the stability of solitons in magnetic materials was investigated [16, 37]. Phan [33] presented the existence of normalized ground state solutions of (1.1) for $\epsilon > 0$ and $\gamma = 0$ with the pure mass-critical nonlinearity. The case involving mass supercritical nonlinearities was discussed in [27], where normalized ground states were shown to exist for $2 < q < \overline{p} < p = 4^*$. The existence of normalized ground state solutions for $\overline{p} \le q < p \le 4^*$ was shown in [28].

As for the case $\epsilon > 0, \gamma > 0$ and $\mu > 0$, however, as far as we know, very little has been known for the mixed dispersion fourth-order nonlinear Schrödinger equation with combined nonlinearities. This constitutes one of our primary motivations of study in the existence of normalized ground state solutions of (1.1) for $2 < q < 2 + \frac{4}{N} < \overline{p} < p \leq 4^*$ and $\overline{p} \leq q < p < 4^*$, respectively. For simplicity, we set $\epsilon = \gamma = 1$.

Definition 1.1. We say that a solution $u_c \in S(c)$ of equation (1.2) is a ground state solution to (1.2) if it possesses the minimal energy among all solutions in $S(c)$, i.e., if

$$E_{p,q}(u_c) = \inf\{E_{p,q}(u), u \in S(c), (E_{p,q}|_{S(c)})'(u) = 0\}.$$

We start with the case $2 < q < 2 + \frac{4}{N} < \overline{p} < p \leq 4^*$ by setting

$$V(c) := \{u \in S(c) : \|\Delta u\|_2^2 + |\nabla u\|_2^2 < \rho_0\},$$
$$\partial V(c) = \{u \in S(c) : \|\Delta u\|_2^2 + |\nabla u\|_2^2 = \rho_0\},$$

where ρ_0 is a suitable positive constant. For any given $\mu > 0$, we aim to determine a specific value $c_0 = c_0(\mu) > 0$ such that for any $c \in (0, c_0)$ it holds

$$m_{p,q}(c) := \inf_{u \in V(c)} E_{p,q}(u) < 0 < \inf_{u \in \partial V(c)} E_{p,q}(u).$$

Theorem 1.2. *Let $N \geq 5$, $\mu > 0$ and $2 < q < 2 + \frac{4}{N} < \overline{p} < p \leq 4^*$. For any $\mu > 0$, there exists $c_0 = c_0(\mu) > 0$ such that for any $c \in (0, c_0)$, the constraint functional $E_{p,q}|_{S(c)}$ admits a ground state, which corresponds to a local minimizer of $E_{p,q}$ in the set $V(c)$.*

As $p > \overline{p}$, it is evident that the constrained functional $E_{p,q}|_{S(c)}$ is unbounded from below. However, the presence of the lower order term $|u|^{q-2}u$ with $2 < q < 2 + \frac{4}{N}$ creates a geometry of local minima on $S(c)$ for sufficiently small $c > 0$. The challenge in establishing the existence of local minimizers arises from the lack of compactness of the bounded minimizing sequence $\{u_n\} \subset V(c)$ due to the noncompact embedding $H^2(\mathbb{R}^N) \hookrightarrow L^2(\mathbb{R}^N)$. By employing a minimization approach and incorporating the subadditivity of ground state energy, we overcome this obstacle and demonstrate the existence of local minima. Furthermore, we find that any ground state serves as a local minimum for the associated energy functional.

Theorem 1.3. *Let $N \geq 5$, $\mu > 0$ and $\overline{p} \leq q < p < 4^*$. If $q = \overline{p}$, we assume that $\mu c^{4/N} < \frac{N+4}{NC_{N,q}^q}$. Then there exists a sufficiently small $c^* > 0$ such that for any $c \in (0, c^*)$, the constrained functional $E_{p,q}|_{S(c)}$ possesses a critical point u at a positive level $E_{p,q}(u) > 0$ with the following properties: u satisfies (1.2) for some $\omega > 0$ and represents a normalized ground state of (1.2) on $S(c)$.*

We introduce the Nehari-Pohozaev set of $E_{p,q}|_{S(c)}$ as follows

$$\mathcal{Q}_{p,q}(c) = \{u \in S(c) : Q_{p,q}(u) = 0\},$$

where

$$Q_{p,q}(u) = \|\Delta u\|_2^2 + \frac{1}{2}\|\nabla u\|_2^2 - \mu\gamma_q\|u\|_q^q - \gamma_p\|u\|_p^p,$$
$$\gamma_r := \frac{N(r-2)}{4r} = \frac{N}{2}\big(\frac{1}{2} - \frac{1}{r}\big), \quad \forall r \in (2, 4^*].$$

It is easily seen that all critical points of $E_{p,q}|S(c)$ lie in $\mathcal{Q}_{p,q}(c)$.

To prove Theorem 1.3, we shall employ a direct minimization method for $E_{p,q}$ on $\mathcal{Q}_{p,q}(c)$. A crucial step is to show the convergence of a minimizing sequence $\{u_n\} \subset \mathcal{Q}_{p,q}(c)$ of $E_{p,q}$ at $m_{p,q}(c)$. The sign of the Lagrange multiplier $\omega \in \mathbb{R}$ plays a pivotal role in the analysis. However, tackling this issue is challenging because of the presence of the term $\|\nabla u\|_2$. As demonstrated in Lemma 4.5, we identify a sufficiently small $c^* > 0$ such that for any $c \in (0, c^*)$, the corresponding ω_c remains positive.

Another difficulty comes from weak limits of the minimizing sequence, which may violate the constraint due to the non-compactness of the embedding $H^2(\mathbb{R}^N) \hookrightarrow L^2(\mathbb{R}^N)$. Overcoming this obstacle, we need to show that the mapping $c \mapsto m_{p,q}(c)$ is strictly decreasing. This, together with the relationship between the energy functional $E_{p,q}$ and the Nehari-Pohozaev functional $Q_{p,q}$, leads to strong convergence of the minimizing sequence in $H^2(\mathbb{R}^N)$. Subsequently, by showing that $\mathcal{Q}_{p,q}(c)$ is a natural constraint, we observe that the minimizer of $E_{p,q}$ on $\mathcal{Q}_{p,q}(c)$ constitutes a normalized ground state solution of (1.2).

The paper is organized as follows. In Section 2, we provide some preliminary concepts and lemmas that will be utilized throughout the paper. We prove Theorem 1.2 in Section 3 and prove Theorem 1.3 in Section 4, respectively.

2. Preliminary results

Throughout this article, for $1 \le r < \infty$, $L^r(\mathbb{R}^N)$ denotes the standard Lebesgue space with norm $\|u\|_r^r := \int_{\mathbb{R}^N} |u|^r dx$. Additionally, the positive constants are denote by $C, C_1, C_2, \dots$, with values that may vary from line to line. The open ball in $\mathbb{R}^N$ is denoted as $B_R(x)$ with center at x and radius R.

In this section, we present some preliminary results which will be used in the next two sections. We start with recalling the well-known Gagliardo-Nirenberg inequality and Sobolev inequality.

Lemma 2.1 ([31])**.** *If $N \ge 5$ and $2 < r < 4^*$, then the Gagliardo-Nirenberg inequality*

$$\|u\|_r^r \le C_{N,r}^r \|\Delta u\|_2^{r\gamma_r} \|u\|_2^{r(1-\gamma_r)}$$

holds for $u \in H^2(\mathbb{R}^N)$, where $C_{N,r}$ denotes the sharp constant.

Lemma 2.2 ([36])**.** *When $N \ge 5$, we have*

$$\mathcal{S}\|u\|_{4^*}^2 \le \|\Delta u\|_2^2, \quad \forall u \in H^2(\mathbb{R}^N),$$

where $\mathcal{S} > 0$ depending only on N denotes an optimal constant.

Note that the following interpolation inequality holds:

$$\int_{\mathbb{R}^N} |\nabla u|^2 dx \le \Big(\int_{\mathbb{R}^N} |\Delta u|^2 dx \Big)^{1/2} \Big(\int_{\mathbb{R}^N} |u|^2 dx \Big)^{1/2}, \quad \forall u \in H^2(\mathbb{R}^N). \tag{2.1}$$

By similar arguments as those in [39], we can obtain the Lions' type lemma in $H^2(\mathbb{R}^N)$.

Lemma 2.3. *Assume that $\{u_n\}$ is bounded in $H^2(\mathbb{R}^N)$. For any $R > 0$, if*

$$\sup_{y \in \mathbb{R}^N} \int_{B_R(y)} |u_n|^2 dx \to 0 \quad \text{as } n \to \infty,$$

then $u_n \to 0$ in $L^r(\mathbb{R}^N)$ for $r \in (2, 4^*)$.

To understand the geometry of the constrained functional, we consider the function $f(c,\rho)$ defined on $\mathbb{R}^+\times\mathbb{R}^+$ by

$$f(c,\rho)=\frac{1}{2}-\frac{\mu}{q}C_{N,q}^{q}\rho^{\alpha_0}c^{\alpha_1}-\frac{C_{N,p}^{p}}{p}\rho^{\alpha_2}c^{\alpha_3},$$

and its restriction $g_c(\rho)$ is defined on $(0,\infty)$ by $\rho\mapsto g_c(\rho):=f(c,\rho)$ for each $c\in(0,\infty)$, where

$$\alpha_0=\frac{N(q-2)}{8}-1,\quad \alpha_1=\frac{2N-q(N-4)}{8},$$
$$\alpha_2=\frac{N(p-2)}{8}-1,\quad \alpha_3=\frac{2N-p(N-4)}{8}.$$

Note that for any $N\geq 5$ and $2<q<2+\frac{4}{N}<\overline{p}<p\leq 4^*$, we have $\alpha_0\in(-1,-\frac{1}{2}),\alpha_1\in(\frac{N+4}{2N},1)$, $\alpha_2\in(0,\frac{4}{N-4}]$, and $\alpha_3\in[0,\frac{4}{N})$.

Lemma 2.4. *For each $c>0$, the function $g_c(\rho)$ has a unique global maximum and the maximum value satisfies*

$$\max_{\rho>0}g_c(\rho)\begin{cases}>0 & \text{if } c<c_0,\\ =0 & \text{if } c=c_0,\\ \max_{\rho>0}g_c(\rho)<0 & \text{if } c>c_0,\end{cases}$$

where

$$c_0=(\frac{1}{2K})^{N/4}>0 \tag{2.2}$$

with

$$K=\frac{\mu}{q}C_{N,q}^{q}\Big[-\frac{\alpha_0}{\alpha_2}\frac{\mu p}{q}\frac{C_{N,q}^{q}}{C_{N,p}^{p}}\Big]^{\frac{\alpha_0}{\alpha_2-\alpha_0}}+\frac{C_{N,p}^{p}}{p}\Big[-\frac{\alpha_0}{\alpha_2}\frac{\mu p}{q}\frac{C_{N,q}^{q}}{C_{N,p}^{p}}\Big]^{\frac{\alpha_2}{\alpha_2-\alpha_0}}>0.$$

Proof. From the definition of $g_c(\rho)$ it follows that

$$g_c'(\rho)=-\alpha_0\frac{\mu}{q}C_{N,q}^{q}\rho^{\alpha_0-1}c^{\alpha_1}-\alpha_2\frac{1}{p}C_{N,p}^{p}\rho^{\alpha_2-1}c^{\alpha_3}.$$

Hence, the equation $g_c'(\rho)=0$ has a unique solution:

$$\rho_c=\Big[-\frac{\alpha_0}{\alpha_2}\frac{\mu p}{q}\frac{C_{N,q}^{q}}{C_{N,p}^{p}}\Big]^{\frac{1}{\alpha_2-\alpha_0}}c^{\frac{\alpha_1-\alpha_3}{\alpha_2-\alpha_0}}. \tag{2.3}$$

Taking into account that $g_c(\rho)\to-\infty$ as $\rho\to 0$ and $g_c(\rho)\to-\infty$ as $\rho\to\infty$, we obtain that ρ_c is the unique global maximum point of $g_c(\rho)$ and the maximum value is

$$\begin{aligned}\max_{\rho>0}g_c(\rho)&=\frac{1}{2}-\frac{\mu}{q}C_{N,q}^{q}\Big[-\frac{\alpha_0}{\alpha_2}\frac{\mu p}{q}\frac{C_{N,q}^{q}}{C_{N,p}^{p}}\Big]^{\frac{\alpha_0}{\alpha_2-\alpha_0}}c^{\frac{\alpha_0(\alpha_1-\alpha_3)}{\alpha_2-\alpha_0}}c^{\alpha_1}\\
&\quad-\frac{C_{N,p}^{p}}{p}\Big[-\frac{\alpha_0}{\alpha_2}\frac{\mu p}{q}\frac{C_{N,q}^{q}}{C_{N,p}^{p}}\Big]^{\frac{\alpha_2}{\alpha_2-\alpha_0}}c^{\frac{\alpha_2(\alpha_1-\alpha_3)}{\alpha_2-\alpha_0}}c^{\alpha_3}\\
&=\frac{1}{2}-\frac{\mu}{q}C_{N,q}^{q}\Big[-\frac{\alpha_0}{\alpha_2}\frac{\mu p}{q}\frac{C_{N,q}^{q}}{C_{N,p}^{p}}\Big]^{\frac{\alpha_0}{\alpha_2-\alpha_0}}c^{\frac{\alpha_1\alpha_2-\alpha_0\alpha_3}{\alpha_2-\alpha_0}}\\
&\quad-\frac{C_{N,p}^{p}}{p}\Big[-\frac{\alpha_0}{\alpha_2}\frac{\mu p}{q}\frac{C_{N,q}^{q}}{C_{N,p}^{p}}\Big]^{\frac{\alpha_2}{\alpha_2-\alpha_0}}c^{\frac{\alpha_1\alpha_2-\alpha_0\alpha_3}{\alpha_2-\alpha_0}}\end{aligned}$$

$$= \frac{1}{2} - Kc^{N/4}.$$

By the definition of c_0, we obtain $\max_{\rho>0} g_{c_0}(\rho) = 0$. □

Remark 2.5. When $p = 4^*$, we use $\mathcal{S}^{-4^*/2}$ instead of $C^p_{N,p}$, where $\mathcal{S}$ is the optimal constant given in Lemma 2.2.

Lemma 2.6. *Let $(c_1, \rho_1) \in \mathbb{R}^+ \times \mathbb{R}^+$ be such that $f(c_1, \rho_1) \geq 0$. Then for any $c_2 \in (0, c_1]$ we have*

$$f(c_2, \rho_2) \geq 0, \text{ if } \rho_2 \in [\frac{c_2}{c_1}\rho_1, \rho_1].$$

Proof. Since $c \to f(\cdot, \rho)$ is a non-increasing function, we have

$$f(c_2, \rho_1) \geq f(c_1, \rho_1) \geq 0.$$

Taking into account $\alpha_0 + \alpha_1 = \frac{q-2}{2}$ and $\alpha_2 + \alpha_3 = \frac{p-2}{2}$, we obtain

$$\begin{aligned}
&f(c_2, \frac{c_2}{c_1}\rho_1) - f(c_1, \rho_1)\\
&= \frac{\mu}{q} C^q_{N,q} \rho_1^{\alpha_1} c_1^{\alpha_1} (1 - (\frac{c_2}{c_1})^{\alpha_0+\alpha_1}) + \frac{1}{p} C^p_{N,p} \rho_1^{\alpha_1} c_1^{\alpha_3} (1 - (\frac{c_2}{c_1})^{\alpha_2+\alpha_3})\\
&= \frac{\mu}{q} C^q_{N,q} \rho_1^{\alpha_1} c_1^{\alpha_1} (1 - (\frac{c_2}{c_1})^{\frac{q-2}{2}}) + \frac{1}{p} C^p_{N,p} \rho_1^{\alpha_1} c_1^{\alpha_3} (1 - (\frac{c_2}{c_1})^{\frac{p-2}{2}}).
\end{aligned}$$

Since $c_2 < c_1$, $2 < q < 2 + \frac{4}{N}$ and $\overline{p} < p \leq 4^*$, we derive

$$f(c_2, \frac{c_2}{c_1}\rho_1) \geq f(c_1, \rho_1) \geq 0.$$

We claim that if $g_{c_2}(\frac{c_2}{c_1}\rho) \geq 0$ and $g_{c_2}(\rho_1) \geq 0$, then

$$f(c_2, \rho) = g_{c_2}(\rho) \geq 0, \quad \text{for } \rho \in [\frac{c_2}{c_1}\rho, \rho_1].$$

Indeed, if $g_{c_2}(\rho) < 0$ for some $\rho \in [\frac{c_2}{c_1}\rho, \rho_1]$, then there exists a local minimum point on $(\frac{c_2}{c_1}\rho, \rho_1)$. This contradicts the fact in Lemma 2.4 that the function $g_{c_2}(\rho)$ has a unique critical point which has to be its unique global maximum. □

Lemma 2.7. *For $\overline{p} < q < p < 4^*$, $a > 0, b \geq 0, c \geq 0$ and $d \geq 0$ with $c + d > 0$, which are independent of t, we denote*

$$H(a, b, c, d) = \max_{t>0} \left\{ a \cdot t^2 + b \cdot t - c \cdot t^{\frac{N(q-2)}{4}} - d \cdot t^{\frac{N(p-2)}{4}} \right\}.$$

Then the function $(a, b, c, d) \mapsto H(a, b, c, d)$ is continuous.

Proof. By making slight modifications to the proof of [2, Lemma 5.2], we can arrive at the desired result. So, we omit the details here. □

3. Case $2 < q < 2 + \frac{4}{N} < \overline{p} < p \leq 4^*$

In this section, we show that ground states of equation (1.2) exist which correspond to the local minima of the associated functional.

3.1. **Properties of mapping $c \mapsto m_{p,q}(c)$.** Let $c_0 > 0$ be determined by equation (2.2) and let $\rho_0 := \rho_{c_0} > 0$ be defined by equation (2.3). According to Lemmas 2.4 and 2.6, it follows that $f(c_0, \rho_0) = 0$, and $f(c, \rho_0) > 0$ for all $c \in (0, c_0)$. Set

$$B_{\rho_0} = \{u \in H^2(\mathbb{R}^N) : \|\Delta u\|_2^2 + \|\nabla u\|_2^2 < \rho_0\} \quad \text{and} \quad V(c) := S(c) \cap B_{\rho_0}.$$

For $c \in (0, c_0)$, we consider the local minimization problem:

$$m_{p,q}(c) = \inf_{u \in V(c)} E_{p,q}(u).$$

Lemma 3.1. *Let $c \in (0, c_0)$ and $2 < q < 2 + \frac{4}{N} < \overline{p} < p \le 4^*$. Then the following three assertions hold.*

1. $m_{p,q}(c) = \inf_{u \in V(c)} E_{p,q}(u) < 0 < \inf_{u \in \partial V(c)} E_{p,q}(u);$.
2. *The function $c \mapsto m_{p,q}(c)$ is a continuous mapping.*
3. *For all $\alpha \in (0, c)$, we have $m_{p,q}(c) \le m_{p,q}(\alpha) + m_{p,q}(c - \alpha)$. If $m_{p,q}(\alpha)$ or $m_{p,q}(c - \alpha)$ is attained, then the inequality is strict.*

Proof. (1) For any $u \in \partial V(c)$, we have $\|\Delta u\|_2^2 + \|\nabla u\|_2^2 = \rho_0$. Applying the Gagliardo-Nirenberg inequality leads to

$$\begin{aligned} E_{p,q}(u) &\ge \frac{1}{2}(\|\Delta u\|_2^2 + \|\nabla u\|_2^2) - \frac{\mu}{q} C_{N,q}^q (\|\Delta u\|_2^2 + \|\nabla u\|_2^2)^{\alpha_0 + 1} (\|u\|_2^2)^{\alpha_1} \\ &\quad - \frac{C_{N,p}^p}{p} (\|\Delta u\|_2^2 + \|\nabla u\|_2^2)^{\alpha_2 + 1} (\|u\|_2^2)^{\alpha_3} \\ &= (\|\Delta u\|_2^2 + \|\nabla u\|_2^2) f(\|u\|_2^2, \|\Delta u\|_2^2 + \|\nabla u\|_2^2) \\ &= \rho_0 f(c, \rho_0) > \rho_0 f(c_0, \rho_0) = 0. \end{aligned} \tag{3.1}$$

Let $u \in S(c)$ be arbitrary but fixed. For $s \in \mathbb{R}^+$, set $u_s(x) = s^{N/2} u(sx)$. Clearly, $u_s \in S(c)$ for any $s \in \mathbb{R}^+$. We define

$$\begin{aligned} \psi_u(s) &= E_{p,q}(u_s) \\ &= \frac{s^4}{2}\|\Delta u\|_2^2 + \frac{s^2}{2}\|\nabla u\|_2^2 - \frac{\mu}{q} s^{N(q-2)/2} \|u\|_q^q - \frac{1}{p} s^{N(p-2)/2} \|u\|_p^p, \end{aligned}$$

for all $s > 0$.

It is easily seen that $\psi_u(s) \to 0^-$ as $s \to 0$. Hence, there exists sufficiently small $s_0 > 0$ such that $\|\Delta u_{s_0}\|_2^2 + \|\nabla u_{s_0}\|_2^2 < \rho_0$ and $E_{p,q}(u_{s_0}) = \psi_u(s_0) < 0$. Consequently, we have $m_{p,q}(c) < 0$.

(2) Let $c \in (0, c_0)$ be arbitrary and $\{c_n\} \subset (0, c_0)$ be such that $c_n \to c$. By the definition of $m_{p,q}(c_n)$ with $m_{p,q}(c_n) < 0$, for any $\epsilon > 0$ small enough, there exists $u_n \in V(c)$ such that

$$E_{p,q}(u_n) \le m_{p,q}(c_n) + \epsilon \quad \text{and} \quad E_{p,q}(u_n) < 0. \tag{3.2}$$

Let $z_n = \sqrt{\frac{c}{c_n}} u_n$. Clearly, $z_n \in S(c)$. On the one hand, if $c_n \ge c$, then

$$\|\Delta z_n\|_2^2 + \|\nabla z_n\|_2^2 = \frac{c}{c_n}(\|\Delta u_n\|_2^2 + \|\nabla u_n\|_2^2) < \rho_0.$$

On the other hand, if $c_n < c$, by Lemma 2.6 and $f(c_n, \rho_0) \ge f(c_0, \rho_0) = 0$, we have $f(c_n, \rho) \ge 0$ for any $\rho \in [\frac{c_n}{c}\rho_0, \rho_0]$. However, from (3.1) and (3.2) it follows that $f(\|u_n\|_2^2, \|\Delta u_n\|_2^2 + \|\nabla u_n\|_2^2) < 0$. Hence, $\|\Delta u_n\|_2^2 + \|\nabla u_n\|_2^2 < \frac{c_n}{c}\rho_0$ and $\|\Delta z_n\|_2^2 + \|\nabla z_n\|_2^2 < \frac{c}{c_n} \cdot \frac{c_n}{c}\rho_0 = \rho_0$. Since $z_n \in V(c)$, we have

$$m_{p,q}(c) \le E_{p,q}(z_n)$$

$$
\begin{aligned}
&= E_{p,q}(u_n) + (E_{p,q}(z_n) - E_{p,q}(u_n)) \\
&= E_{p,q}(u_n) + \frac{1}{2}(\frac{c}{c_n} - 1)\|\Delta u_n\|_2^2 + \frac{1}{2}(\frac{c}{c_n} - 1)\|\nabla u_n\|_2^2 \\
&\quad - \frac{\mu}{q}[(\frac{c}{c_n})^{\frac{q}{2}} - 1]\|u_n\|_q^q - \frac{1}{p}[(\frac{c}{c_n})^{p/2} - 1]\|u_n\|_p^p.
\end{aligned}
$$

That is,

$$
m_{p,q}(c) \le E_{p,q}(z_n) = E_{p,q}(u_n) + o_n(1) \qquad \text{as } n \to \infty. \tag{3.3}
$$

Using (3.2) and (3.3) yields

$$
m_{p,q}(c) \le m_{p,q}(c_n) + \epsilon + o_n(1).
$$

Now, we let $u \in V(c)$ be such that

$$
E_{p,q}(u) \le m_{p,q}(c) + \epsilon \quad \text{and} \quad E_{p,q}(u) < 0.
$$

Set $u_n := \sqrt{\frac{c_n}{c}}u$. Then $u_n \in S(c_n)$, and $c_n \to c$ implies that $\|\Delta u_n\|_2^2 + \|\nabla u_n\|_2^2 < \rho_0$ for n large enough. So $u_n \in V(c_n)$. Note that $E_{p,q}(u_n) \to E_{p,q}(u)$. Thus, we obtain

$$
m_{p,q}(c_n) \le E_{p,q}(u) + (E_{p,q}(u_n) - E_{p,q}(u)) \le m_{p,q}(c) + \epsilon + o_n(1).
$$

Because of the arbitrariness of $\epsilon > 0$, we infer that $m_{p,q}(c_n) \to m_{p,q}(c)$.

(3) Given $\alpha \in (0, c)$, it suffices to prove that

$$
\forall \theta \in (1, \frac{c}{\alpha}] : m_{p,q}(\theta\alpha) \le \theta m_{p,q}(\alpha)
$$

and that, if $m_{p,q}(\alpha)$ is attained, the inequality is strict. Using (i), for any $\epsilon > 0$ small enough, there exists $u \in V(\alpha)$ such that

$$
E_{p,q}(u) \le m_{p,q}(\alpha) + \epsilon \quad \text{and} \quad E_{p,q}(u) < 0.
$$

From Lemma 2.6 and $f(\alpha, \rho_0) \ge f(c_0, \rho_0) = 0$, it follows that $f(\alpha, \rho) \ge 0$ for any $\rho \in [\frac{\alpha}{c}\rho_0, \rho_0]$. Hence, using (3.1) and (3.2) we obtain $f(\|u\|_2^2, \|\Delta u\|_2^2 + \|\nabla u\|_2^2) < 0$. That is,

$$
\|\Delta u\|_2^2 + \|\nabla u\|_2^2 < \frac{\alpha}{c}\rho_0.
$$

Set $v = \sqrt{\theta}u$. Then $\|v\|_2^2 = \theta\alpha$ and $\|\Delta v\|_2^2 + \|\nabla v\|_2^2 < \rho_0$. Thus $v \in V(\theta\alpha)$. A direct calculation yields

$$
\begin{aligned}
m_{p,q}(\theta\alpha) \le E_{p,q}(v) &< \frac{1}{2}\theta\|\Delta u\|_2^2 + \frac{1}{2}\theta\|\nabla u\|_2^2 - \frac{\mu}{q}\theta\|v\|_q^q - \frac{1}{p}\theta\|v\|_p^p \\
&= \theta E_{p,q}(u) \le \theta(m_{p,q}(\alpha) + \epsilon).
\end{aligned}
$$

Because of the arbitrariness of ϵ, we obtain $m_{p,q}(\theta\alpha) \le \theta m_{p,q}(\alpha)$. If $m_{p,q}(\alpha)$ is attained, we can choose $\epsilon = 0$. □

3.2. **Proof of Theorem 1.2.** We define

$$
\mathcal{M}_c = \{u \in V(c) : E_{p,q}(u) = m_{p,q}(c)\}.
$$

Lemma 3.2. *Let $2 < q < 2 + \frac{4}{N} < \overline{p} < p \le 4^*$. For any $c \in (0, c_0)$ and the sequence $\{u_n\} \subset B_{\rho_0}$ such that $\|u_n\|_2 \to c$ and $E_{p,q}(u_n) \to m_{p,q}(c)$, there exists a sequence $\{y_n\} \subset \mathbb{R}^N$ such that for some $R > 0$ it holds*

$$
\int_{B_R(y_n)} |u_n|^2 dx \ge \beta > 0. \tag{3.4}
$$

Proof. By way of contradiction, we assume that (3.4) does not hold. From $\{u_n\} \subset B_{\rho_0}$ and $\|u_n\|_2 \to c$ it follows that $\{u_n\}$ is bounded in $H^2(\mathbb{R}^N)$. For $2 < q < 2 + \frac{4}{N} < \overline{p} < p < 4^*$, by Lemma 2.3, we deduce that $\|u_n\|_q^q \to 0$ and $\|u_n\|_p^p \to 0$, as $n \to \infty$. At this point, it follows that $E_{p,q}(u_n) \geq o_n(1)$. If $p = 4^*$, in view of $f(c_0, \rho_0) = 0$, a straightforward computation yields

$$\begin{aligned} E_{p,q}(u_n) &= \frac{1}{2}\|\Delta u_n\|_2^2 + \frac{1}{2}\|\nabla u_n\|_2^2 - \frac{1}{4^*}\|u_n\|_{4^*}^{4^*} + o_n(1) \\ &\geq \frac{1}{2}\|\Delta u_n\|_2^2 + \frac{1}{2}\|\nabla u_n\|_2^2 - \frac{1}{4^*}\frac{1}{\mathcal{S}^{4^*/2}}(\|\Delta u_n\|_2^2 + \|\nabla u_n\|_2^2)^{\frac{4^*}{2}} + o_n(1) \\ &\geq (\|\Delta_n\|_2^2 + \|\nabla u_n\|_2^2)(\frac{1}{2} - \frac{1}{4^*}\frac{1}{\mathcal{S}^{4^*/2}}\rho_0^{\alpha_2}) + o_n(1) \\ &= (\|\Delta_n\|_2^2 + \|\nabla u_n\|_2^2)\frac{\mu}{q}C_{N,q}^q \rho_0^{\alpha_0} c_0^{\alpha_1} + o_n(1) > 0. \end{aligned}$$

Both cases contradict the fact $m_{p,q}(c) < 0$. Thus, we arrive at the desired result. $\square$

Proposition 3.3. *For any $c \in (0, c_0)$, if $\{u_n\} \subset B_{\rho_0}$ is such that $\|u_n\|_2^2 \to c$ and $E_{p,q}(u_n) \to m_{p,q}(c)$, then, up to translation, $u_n \longrightarrow u_c \in \mathcal{M}_c$ in $H^2(\mathbb{R}^N)$. In particular, the set $\mathcal{M}_c$ is compact in $H^2(\mathbb{R}^N)$, up to translation.*

The proof of the above proposition can be obtained by similar arguments as in [27] (see also [18]).

Proposition 3.4. *For any $c \in (0, c_0)$, if $m_{p,q}(c)$ is reached, then any ground state is contained in $V(c)$.*

Proof. For any $v \in S(c)$ and $s \in (0, \infty)$, we obtain

$$\psi_v'(s) = \frac{2}{s}Q(v_s),$$

which implies that if $w \in S(c)$ is a ground state solution, then there exist $v \in S(c)$ and $s_0 > 0$ such that $w = v_{s_0}, E_{p,q}(w) = \psi_v(s_0)$ and $\psi_v'(s_0) = 0$. To conclude the proof, it suffices to show that $\psi_v'(s)$ has at most two zeros. This is equivalent to showing that the function

$$s \mapsto \frac{\psi_v'(s)}{s}$$

has at most two zeros. Note that

$$\begin{aligned} \xi(s) = \frac{\psi_v'(s)}{s} &= 2s^2\|\Delta u\|_2^2 + \|\nabla u\|_2^2 - s^{\frac{N(q-2)}{2}-2}\frac{\mu N(q-2)}{2q}\|u\|_q^q \\ &\quad - s^{\frac{N(p-2)}{2}-2}\frac{N(p-2)}{2p}\|u\|_p^p \end{aligned}$$

and

$$\begin{aligned} \xi'(s) &= s[4\|\Delta u\|_2^2 - s^{\frac{N(q-2)}{2}-4} \cdot \frac{\mu N(q-2)}{2q}(\frac{N(q-2)}{2} - 2)\|u\|_q^q \\ &\quad - s^{\frac{N(p-2)}{2}-4} \cdot \frac{N(p-2)}{2p}(\frac{N(p-2)}{2} - 2)\|u\|_p^p] \\ &=: s[4\|\Delta u\|_2^2 - f(s)]. \end{aligned}$$

So we need to show that $\xi'(s)$ is the unique solution. Since $2 < q < 2 + \frac{4}{N} < \overline{p} < p \leq 4^*$, $N \geq 5$ and $s > 0$, it is easy to see that $s \to f(s)$ is a non-increasing function. Hence, $\xi'(s)$ has a unique solution and $\xi(s)$ has at most two zeros.

Now, since $\psi_v(s) \to 0^-, \|\Delta v_s\|_2^2 + \|\nabla v_s\|_2^2 \to 0$ as $s \to 0$ and $\psi_v(s) = E_{p,q}(v_s) > 0$, when $v_s \in \partial V(c)$, ψ_v' has a first zero $s_1 > 0$ corresponding to a local minima. Also, from $\psi_v(s_1) < 0, \psi_v(s) > 0$ when $v_s \in \partial V(c)$ and $\psi_v(s) \to -\infty$ as $s \to \infty$, ψ_v has a second zero $s_2 > s_1$ corresponding to a local maxima. In particular, $v_{s_1} \in V(c)$ and $E_{p,q}(v_{s_1}) = \psi_v(s_1) < 0$. Thus, if $m_{p,q}(c)$ is achieved, it is a ground state level. □

Proof of Theorem 1.2. The existence of a minimizer for $E_{p,q}$ on $V(c)$ follows from Proposition 3.3. By Proposition 3.4, this local minimizer is a ground state. □

4. Case $\overline{p} \le q < p < 4^*$

In this section, we present the proof of Theorem 1.3.

4.1. **Monotonicity of ground state energy $m_{p,q}(c)$.**

We start by showing some properties of $\mathcal{Q}_{p,q}(c)$ and the energy functional $E_{p,q}$ restricted on it. For any $u \in S(c)$ and $s \in (0, +\infty)$, we define

$$u_s(x) = s^{N/4} u(\sqrt{s}x), \quad \text{for a.e. } x \in \mathbb{R}^N.$$

Clearly, $u_s \in S(c)$ for any $s > 0$. It follows that

$$E_{p,q}(u_s) = \frac{s^2}{2}\|\Delta u\|_2^2 + \frac{s}{2}\|\nabla u\|_2^2 - \frac{\mu}{q} s^{\frac{N(q-2)}{4}}\|u\|_q^q - \frac{1}{p} s^{\frac{N(p-2)}{4}}\|u\|_p^p$$

and

$$Q_{p,q}(u_s) = s^2\|\Delta u\|_2^2 + \frac{s}{2}\|\nabla u\|_2^2 - \mu\gamma_q s^{\frac{N(q-2)}{4}}\|u\|_q^q - \gamma_p s^{\frac{N(p-2)}{4}}\|u\|_p^p.$$

Then, we have the following properties for $E_{p,q}(u_s)$ and $Q_{p,q}(u_s)$.

Lemma 4.1. . *Let $N \ge 5$, $c > 0$, $\mu > 0$ and $\overline{p} \le q < p < 4^*$. When $q = \overline{p}$, we assume that $\mu c^{4/N} < \frac{N+4}{NC_{N,q}^q}$. Then for any $u \in S(c)$, there exists a unique $s_u \in (0, +\infty)$ such that $u_{s_u} \in \mathcal{Q}_{p,q}(c)$ and s_u is the unique critical point of $E_{p,q}(u_s)$ such that $E_{p,q}(u_{s_u}) = \max_{s\in(0,+\infty)} E_{p,q}(u_s)$. The function $u \mapsto E_{p,q}(u_{s_u})$ is concave on $[s_u, +\infty)$. In particular, if $Q_{p,q}(u) \le 0$, then $s_u \in (0, 1]$. Moreover, the map $u \mapsto s_u$ is of class C^1.*

Since the proof is similar to the one of [28, Lemma 3.4], we omit it here. Under the same assumptions described in Lemma 4.1, we can obtain the following results concerning the Nehari-Pohozaev's type set $\mathcal{Q}_{p,q}(c)$ and the constrained functional $E_{p,q}$.

Lemma 4.2. *Let $N \ge 5$, $c > 0$, $\mu > 0$ and $\overline{p} \le q < p < 4^*$. When $q = \overline{p}$, we assume that $\mu c^{4/N} < \frac{N+4}{NC_{N,q}^q}$. Then we have*

1. $\mathcal{Q}_{p,q}(c) \neq \emptyset$;
2. $\inf_{u\in\mathcal{Q}_{p,q}(c)} \|\Delta u\|_2^2 + \frac{1}{2}\|\nabla u\|_2^2 > 0$ *and* $\inf_{u\in\mathcal{Q}_{p,q}(c)} \|\Delta u\|_2^2 > 0$;
3. $\inf_{u\in\mathcal{Q}_{p,q}(c)} E_{p,q}(u) > 0$;
4. $E_{p,q}$ *is coercive on* $\mathcal{Q}_{p,q}(c)$.

Proof. (1) By Lemma 4.1, for any $u \in S(c)$, there always exists $s_u > 0$ such that $u_{s_u} \in \mathcal{Q}_{p,q}(c)$, it follows that $\mathcal{Q}_{p,q}(c) \neq \emptyset$.

(2) For any $u \in \mathcal{Q}_{p,q}(c)$, using the Gagliardo-Nirenberg inequality yields

$$\|\Delta u\|_2^2 + \frac{1}{2}\|\nabla u\|_2^2 = \mu\gamma_q\|u\|_q^q + \gamma_p\|u\|_p^p$$

$$\begin{aligned}&\le \mu\gamma_q C_{N,q}^q(\sqrt{c})^{q(1-\gamma_q)}(\|\Delta u\|_2+\frac{1}{2}\|\nabla u\|_2^2)^{\frac{q\gamma_q}{2}}\\&\quad+\gamma_p C_{N,p}^p(\sqrt{c})^{p(1-\gamma_p)}(\|\Delta u\|_2+\frac{1}{2}\|\nabla u\|_2^2)^{\frac{p\gamma_p}{2}}.\end{aligned}$$

If $\overline{p}<q<p$, then $p\gamma_p>q\gamma_q>2$. If $\overline{p}=q<p$ and $\mu c^{4/N}<\frac{N+4}{NC_{N,q}^q}$, then $p\gamma_p>q\gamma_q=2$ and $\frac{\mu NC_{N,q}^q}{N+4}c^{4/N}<1$. In either case, there exists a constant $C>0$ such that $\|\Delta u\|_2^2+\frac{1}{2}\|\nabla u\|_2^2\ge C$, which implies $\inf_{u\in\mathcal{Q}_{p,q}(c)}\|\Delta u\|_2^2+\frac{1}{2}\|\nabla u\|_2^2>0$. By a similar argument, we can deduce that $\inf_{u\in\mathcal{Q}_{p,q}(c)}\|\Delta u\|_2^2>0$.

(3) For eachy $u\in\mathcal{Q}_{p,q}(c)$, we have

$$E_{p,q}(u)=\frac{q\gamma_q-2}{2q\gamma_q}\|\Delta u\|_2^2+\frac{q\gamma_q-1}{2q\gamma_q}\|\nabla u\|_2^2+\frac{p\gamma_p-q\gamma_q}{pq\gamma_q}\|u\|_p^p. \tag{4.1}$$

From (2) it follows that $\inf_{u\in\mathcal{Q}_{p,q}(c)}E_{p,q}(u)>0$.

(4) By (4.1), it is easily seen that (4) holds. □

For any fixed $c>0$, Lemma 4.2 indicates that

$$m_{p,q}(c)=\inf_{u\in\mathcal{Q}_{p,q}(c)}E_{p,q}(u)$$

is well-defined and strictly positive. We now analyze the behaviors of $m_{p,q}(c)$ when $c>0$ varies.

Lemma 4.3. *Let $\overline{p}\le p<q<4^*$. When $q=\overline{p}$, we assume that $\mu c^{4/N}<\frac{N+4}{NC_{N,q}^q}$. Then the function $c\mapsto m_{p,q}(c)$ is continuous for $c\in(0,+\infty)$.*

Proof. We define

$$\gamma(c)=\inf_{u\in S(c)}\max_{s>0}E_{p,q}(u_s). \tag{4.2}$$

To prove $\gamma(c)=m_{p,q}(c)$, for any $u\in\mathcal{Q}_{p,q}(c)$ we have $E_{p,q}(u)=\max_{s>0}E_{p,q}(u_s)$, which implies that $\gamma(c)\le m_{p,q}(c)$. On the other hand, for any $u\in S(c)$, by Lemma 4.1 there exists $s_u>0$ such that $u_{s_u}\in\mathcal{Q}_{p,q}(c)$ and $\max_{s>0}E_{p,q}(u_s)=E_{p,q}(u_{s_u})\ge m_{p,q}(c)$. Thus, we have $\gamma(c)=m_{p,q}(c)$.

For each fixed $c>0$, taking $\{c_n\}\subset\mathbb{R}^+$ such that $c_n\to c$, we shall prove $\lim_{n\to\infty}m_{p,q}(c_n)=m_{p,q}(c)$. For any $\epsilon>0$, by the definition of $m_{p,q}(c)$ there exists $v\in\mathcal{Q}_{p,q}(c)$ such that $E_{p,q}(v)\le m_{p,q}(c)+\frac{\epsilon}{2}$. Set $v_n:=\sqrt{\frac{c_n}{c}}v\in S(c_n)$. From the fact $\mu c^{4/N}<\frac{N+4}{NC_{N,\overline{p}}^{\overline{p}}}$, $c_n\to c$ and Lemma 2.7, it follows that

$$\begin{aligned}m_{p,q}(c_n)&\le\max_{s>0}E_{p,q}((v_n)_s)\\&=\max_{s>0}(\frac{s^2}{2}\|\Delta v_n\|_2^2+\frac{s}{2}\|\nabla v_n\|_2^2-\frac{\mu}{q}s^{\frac{N(q-2)}{4}}\|v_n\|_q^q-\frac{1}{p}s^{\frac{N(p-2)}{4}}\|v_n\|_p^p)\\&\le\max_{s>0}(\frac{s^2}{2}\|\Delta v\|_2^2+\frac{s}{2}\|\nabla v\|_2^2-\frac{\mu}{q}s^{\frac{N(q-2)}{4}}\|v\|_q^q-\frac{1}{p}s^{\frac{N(p-2)}{4}}\|v\|_p^p)+\frac{\epsilon}{2}\\&=\max_{s>0}E_{p,q}((v)_s)+\frac{\epsilon}{2}\\&=E_{p,q}(v)+\frac{\epsilon}{2}\le m_{p,q}(c)+\epsilon.\end{aligned}$$

That is,

$$\limsup_{n\to\infty}m_{p,q}(c_n)\le m_{p,q}(c). \tag{4.3}$$

Then we take $\{u_n\} \subset \mathcal{Q}_{p,q}(c_n)$ such that

$$E_{p,q}(u_n) \le m_{p,q}(c_n) + \frac{\epsilon}{2}. \tag{4.4}$$

In view of $Q_{p,q}(u_n) = 0$, for n large enough, from (4.3) and (4.4) it follows that

$$\begin{aligned}
&\big(\frac{1}{2} - \frac{1}{q\gamma_q}\big)\|\Delta u_n\|_2^2 + \frac{1}{2}\big(1 - \frac{1}{q\gamma_q}\big)\|\nabla u_n\|_2^2 + \big(\frac{\gamma_p}{q\gamma_q} - \frac{1}{p}\big)\|u_n\|_p^p \\
&\le E_{p,q}(u_n) \le m_{p,q}(c_n) + \frac{\epsilon}{2} \\
&\le m_{p,q}(c) + \frac{3\epsilon}{4}.
\end{aligned}$$

If $p\gamma_p > q\gamma_q > 2$, we can derive that $\{u_n\}$ is bounded in $H^2(\mathbb{R}^N)$. If $p\gamma_p > q\gamma_q = 2$, recalling Lemma 4.2 (2), we can see the same result.

Without loss of generality, as $n \to \infty$ we assume that

$$\|\Delta u_n\|_2^2 \to C_1, \quad \|\nabla u_n\|_2^2 \to C_2, \quad \|u_n\|_q^q \to C_3, \quad \|u_n\|_p^p \to C_4.$$

If follows from Lemma 4.2 (2) that $C_1 > 0$, $C_2 \ge 0$, and $C_3 \ge 0$, $C_4 \ge 0$ with $C_3 + C_4 > 0$.

Let $\tilde{u}_n := \sqrt{\frac{c}{c_n}}u_n$. Clearly, $\tilde{u}_n \in S(c)$. From Lemma 2.7 it follows that

$$\begin{aligned}
m_{p,q}(c) &\le \max_{s>0} E_{p,q}((\tilde{u}_n)_s) = \max_{s>0}\Big[\frac{s^2}{2}(\frac{c}{c_n})\|\Delta u_n\|_2^2 + \frac{s}{2}(\frac{c}{c_n})\|\nabla u_n\|_2^2 \\
&\quad - \frac{\mu}{q}s^{\frac{N(q-2)}{4}}(\frac{c}{c_n})^{\frac{q}{2}}\|u_n\|_q^q - \frac{1}{p}s^{\frac{N(p-2)}{4}}(\frac{c}{c_n})^{p/2}\|u_n\|_p^p\Big] \\
&\le \max_{s>0}\big(\frac{s^2}{2}\|\Delta u_n\|_2^2 + \frac{s}{2}\|\nabla u_n\|_2^2 \\
&\quad - \frac{\mu}{q}s^{\frac{N(q-2)}{4}}\|u_n\|_q^q - \frac{1}{p}s^{\frac{N(p-2)}{4}}\|u_n\|_p^p\big) + \frac{3\epsilon}{4} \\
&= \max_{s>0} E_{p,q}((u_n)_s) + \frac{3\epsilon}{4} \\
&= E_{p,q}(u_n) + \frac{3\epsilon}{4} \le m_{p,q}(c_n) + \epsilon.
\end{aligned}$$

That is,

$$m_{p,q}(c) \le \liminf_{n\to\infty} m_{p,q}(c_n).$$

Hence, we arrive at the desired result. □

Lemma 4.4. *Let $\overline{p} \le p < q < 4^*$. When $q = \overline{p}$, we assume that $\mu c^{4/N} < \frac{N+4}{NC_{N,q}^q}$. Then the function $c \mapsto m_{p,q}(c)$ is non-increasing for $c \in (0, +\infty)$.*

Proof. For $0 < c_1 < c_2 < +\infty$, we shall prove that $m_{p,q}(c_2) \le m_{p,q}(c_1)$. According to the definition of $\gamma(c)$ in 4.2, for any $\epsilon > 0$ there exists $u_1 \in \mathcal{Q}_{p,q}(c_1)$ such that

$$E_{p,q}(u_1) \le m_{p,q}(c_1) + \frac{\epsilon}{2} \quad \text{and} \quad \max_{\lambda>0} E_{p,q}((u_1)_\lambda) = E(u_1).$$

For $\kappa > 0$ and $\lambda \in (0,1)$, we define

$$w_\lambda^\kappa := u_1^\kappa + (v_0^\kappa)_\lambda.$$

We choose $u_1^\kappa \in H^2(\mathbb{R}^N)$ such that $\operatorname{supp} u_1^\kappa \subset B_{\frac{1}{\kappa}}(0)$ and $\|u_1^\kappa - u_1\| = o(\kappa)$, while $v_0^\delta := (c_2 - \|u_1^\kappa\|_2^2)^{1/2}\frac{v^\kappa}{\|v^\kappa\|_2}$, where $v^\kappa \in C_0^\infty(\mathbb{R}^N)$ such that $\operatorname{supp} v^\kappa \subset$

$B_{\frac{2}{\kappa}+1}(0)\backslash B_{\frac{2}{\kappa}}(0)$. It is obvious that $\mathrm{dist}(\mathrm{supp}(v_0^\kappa)_\lambda, \mathrm{supp}\, u_1^\kappa) \geq \frac{1}{\kappa}(\frac{2}{\sqrt{\lambda}}-1) > 0$. Hence, $\|w_\lambda^\kappa\|_2^2 = c_2$. By a standard argument, as $\lambda,\kappa \to 0$, we derive

$$\|\Delta w_\lambda^\kappa\|_2^2 \to \|\Delta u_1\|_2^2, \quad \|\nabla w_\lambda^\kappa\|_2^2 \to \|\nabla u_1\|_2^2, \quad \|w_\lambda^\kappa\|_q^q \to \|u_1\|_q^q, \quad \|w_\lambda^\kappa\|_p^p \to \|u_1\|_p^p.$$

Letting $(w_\lambda^\kappa)_t = t^{N/4} w_\lambda^\kappa(\sqrt{t}x)$, by Lemma 2.7 again, we can deduce that for $\lambda,\kappa > 0$ small enough, it holds

$$m_{p,q}(c_2) \leq \max_{t>0} E_{p,q}((w_\lambda^\kappa)_t) \leq \max_{t>0} E_{p,q}((u_1)_t) + \frac{\epsilon}{2} = E_{p,q}(u_1) + \frac{\epsilon}{2} \leq m_{p,q}(c_1) + \epsilon.$$

□

Lemma 4.5. *Let $\overline{p} \leq q < p < 4^*$. Assume that $u_c \in S(c)$ solves*

$$\Delta^2 u - \Delta u + \omega_c u = \mu |u|^{q-2}u + |u|^{p-2}u. \tag{4.5}$$

Then there exists $c^* > 0$ such that $\omega_c > 0$ for any $c \in (0, c^*)$.

Proof. By (4.5) we deduce $Q_{p,q}(u) = 0$ and

$$\|\Delta u_c\|_2^2 + \|\nabla u_c\|_2^2 + \omega_c \|u_c\|_2^2 - \mu \|u_c\|_q^q - \|u_c\|_p^p = 0.$$

Then

$$\omega_c \gamma_q c = (1-\gamma_q)\|\Delta u_c\|_2^2 + (\frac{1}{2} - \gamma_q)\|\nabla u_c\|_2^2 - (\gamma_p - \gamma_q)\|u_c\|_p^p. \tag{4.6}$$

For small $c > 0$, using the Gagliardo-Nirenberg inequality leads to

$$\begin{aligned} \|\Delta u_c\|_2^2 &= \gamma_p C_{N,q}^q \|\Delta u_c\|_2^{q\gamma_q} (\sqrt{c})^{q(1-\gamma_q)} + \gamma_p C_{N,p}^p \|\Delta u_c\|_2^{p\gamma_p} (\sqrt{c})^{p(1-\gamma_p)} \\ &\leq \gamma_p \max\{C_{N,q}^q, C_{N,p}^p\} (\sqrt{c})^{q(1-\gamma_q)} (\|\Delta u_c\|_2^{q\gamma_q} + \|\Delta u_c\|_2^{p\gamma_p}). \end{aligned}$$

Then, for $\overline{p} \leq q < p < 4^*$, as $c \to 0$ we obtain

$$\int_{\mathbb{R}^N} |\Delta u_c|^2 dx \to \infty. \tag{4.7}$$

On the other hand, we from (2.1) and (4.6) derive

$$\begin{aligned} \omega_c \gamma_q c &= (1-\gamma_q)\|\Delta u_c\|_2^2 + (\frac{1}{2} - \gamma_q)\|\nabla u_c\|_2^2 - (\gamma_p - \gamma_q)\|u_c\|_p^p \\ &> (1-\gamma_q)\|\Delta u_c\|_2^2 + (\frac{1}{2} - \gamma_q)\sqrt{c}\|\Delta u_c\|_2. \end{aligned}$$

From (4.7), it follows that $\omega_c > 0$ if $c > 0$ is small enough. □

Lemma 4.6. *Let $\overline{p} \leq q < p < 4^*$ and $c \in (0, c^*)$. When $q = \overline{p}$, we assume that $\mu c^{4/N} < \frac{N+4}{N C_{N,q}^q}$. Suppose that $u \in S(c)$ such that $E_{p,q}(u) = m_{p,q}(c)$ and*

$$\Delta^2 u - \Delta u + \omega u = \mu |u|^{q-2}u + |u|^{p-2}u.$$

Then the function $c \mapsto m_{p,q}(c)$ is strictly decreasing in a right neighborhood of c.

Proof. By Lemma 4.5, we know that $\omega > 0$. Set $u_{\lambda,t}(x) = t^{N/4}\sqrt{\lambda}u(\sqrt{t}x)$ for $\lambda, t > 0$. We define

$$\mathcal{K}(\lambda,t) = E_{p,q}(u_{\lambda,t}) = \frac{t^2}{2}\lambda\|\Delta u\|_2^2 + \frac{t}{2}\lambda\|\nabla u\|_2^2 - \frac{\mu \cdot t^{\frac{N(q-2)}{4}}}{q}\lambda^{\frac{q}{2}}\|u\|_q^q - \frac{t^{\frac{N(p-2)}{4}}}{p}\lambda^{p/2}\|u\|_p^p$$

and

$$\begin{aligned} \mathcal{M}(\lambda,t) &= Q_{p,q}(u_{\lambda,t}) \\ &= t^2\lambda\|\Delta u\|_2^2 + \frac{t}{2}\lambda\|\nabla u\|_2^2 - \mu\gamma_q t^{\frac{N(q-2)}{4}}\lambda^{\frac{q}{2}}\|u\|_q^q - \gamma_p t^{\frac{N(p-2)}{4}}\lambda^{p/2}\|u\|_p^p. \end{aligned}$$

By a direct calculation, we have

$$\begin{aligned}\frac{\partial \mathcal{K}}{\partial \lambda}(1,1) &= \frac{1}{2}\|\Delta u\|_2^2 + \frac{1}{2}\|\nabla u\|_2^2 - \frac{\mu}{2}\|u\|_q^q - \frac{1}{2}\|u\|_p^p = -\frac{1}{2}\omega c,\\ \frac{\partial \mathcal{K}}{\partial t}(1,1) &= \|\Delta u\|_2^2 + \frac{1}{2}\|\nabla u\|_2^2 - \mu\gamma_q\|u\|_q^q - \gamma_p\|u\|_p^p = 0,\\ \frac{\partial^2 \mathcal{K}}{\partial t^2}(1,1) &= \|\Delta u\|_2^2 - \mu\gamma_q(\frac{N(q-2)}{4} - 1)\|u\|_q^q - \gamma_p(\frac{N(p-2)}{4} - 1)\|u\|_p^p < 0,\end{aligned}$$

which yields for δ_t small enough and $\delta_\lambda > 0$,

$$\mathcal{K}(1+\delta_\lambda, 1+\delta_t) < \mathcal{K}(1,1) \quad \text{for } \omega > 0. \tag{4.8}$$

In addition, we observe that

$$\mathcal{M}(1,1) = Q_{p,q}(u) = \|\Delta u\|_2^2 + \frac{1}{2}\|\nabla u\|_2^2 - \mu\gamma_q\|u\|_q^q - \gamma_p\|u\|_p^p = 0.$$

We now claim that

$$\frac{\partial \mathcal{M}}{\partial t}(1,1) = 2\|\Delta u\|_2^2 + \frac{1}{2}\|\nabla u\|_2^2 - \mu\gamma_q\frac{N(q-2)}{4}\|u\|_q^q - \gamma_p\frac{N(p-2)}{4}\|u\|_p^p \neq 0.$$

Otherwise, we assume that

$$\frac{\partial \mathcal{M}}{\partial t}(1,1) = \|\Delta u\|_2^2 + \frac{1}{4}\|\nabla u\|_2^2 - \mu\gamma_q\frac{N(q-2)}{8}\|u\|_q^q - \gamma_p\frac{N(p-2)}{8}\|u\|_p^p = 0.$$

Then for any $\overline{p} \le q < p < 4^*$, we have that

$$\frac{1}{4}\|\nabla u\|_2^2 = \mu\gamma_q(1 - \frac{N(q-2)}{8})\|u\|_q^q + \gamma_p(1 - \frac{N(p-2)}{8})\|u\|_p^p,$$

which is impossible. According to the implicit function theorem, we deduce that there exists $\epsilon > 0$ and a continuous function $g : [1-\epsilon, 1+\epsilon] \mapsto \mathbb{R}$ satisfying $g(1) = 1$ such that $\mathcal{M}(\lambda, g(\lambda)) = 0$ for $\lambda \in [1-\epsilon, 1+\epsilon]$. This together with (4.8) gives

$$m_{p,q}((1+\epsilon)c) \le E_{p,q}(u_{1+\epsilon, g(1+\epsilon)}) < E_{p,q}(u) = m_{p,q}(c).$$

We have arrived at the desired result. □

4.2. **Ground states.** In this subsection, before presenting the proof of Theorem 1.3, we show the minimizer of $E_{p,q}(u)$ constrained on $\mathcal{Q}_{p,q}(c)$. For convenience, we set $f(s) = \mu|s|^{q-2}s + |s|^{p-2}s$, $F(s) = \frac{\mu}{q}|s|^q + \frac{1}{p}|s|^p$ and $H(s) = f(s)s - 2F(s)$.

Lemma 4.7. *Let $\overline{p} \le q < p < 4^*$ and $c \in (0, c^*)$. When $q = \overline{p}$, we assume that $\mu c^{4/N} < \frac{N+4}{NC_{N,q}^q}$. Then there exists $u_0 \in \mathcal{Q}_{p,q}(c)$ such that $E_{p,q}(u_0) = m_{p,q}(c)$.*

Proof. Using the Ekeland variational principle, there exists a minimizing sequence $\{u_n\} \subset \mathcal{Q}_{p,q}(c)$ such that

$$E_{p,q}(u_n) \to m_{p,q}(c) \quad \text{as } n \to +\infty. \tag{4.9}$$

By Lemma 4.2(4), it follows that $\{u_n\}$ is bounded in $H^2(\mathbb{R}^N)$. We claim that $\{u_n\}$ is non-vanishing. Indeed, if $\{u_n\}$ is vanishing, then it follows from Lemma 2.1 that

$$\int_{\mathbb{R}^N} |u_n|^r dx \to 0, \quad \text{for } r \in (2, 4^*).$$

Since $Q_{p,q}(u_n) = 0$ and $\overline{p} \le q < p < 4^*$, it follows that

$$|\Delta u_n|^2 + \frac{1}{2}|\nabla u_n|^2 = \mu\gamma_q\|u_n\|_q^q + \gamma_p\|u_n\|_p^p \to 0, \quad \text{as } n \to \infty,$$

which contradicts Lemma 4.2(2). Thus, up to a subsequence, we obtain that $u_n \rightharpoonup u_0 \neq 0$ in $H^2(\mathbb{R}^N)$. Denote $u_{n,0} = u_n - u_0$. It is easily seen that

$$\begin{gathered}
\|u_n\|_2^2 = \|u_0\|_2^2 + \|u_{n,0}\|_2^2 + o_n(1), \\
\|\nabla u_n\|_2^2 = \|\nabla u_0\|_2^2 + \|\nabla u_{n,0}\|_2^2 + o_n(1), \\
\|\Delta u_n\|_2^2 = \|\Delta u_0\|_2^2 + \|\Delta u_{n,0}\|_2^2 + o_n(1).
\end{gathered}$$

By the splitting properties of Brezis-Lieb we have

$$H(u_n) = H(u_0) + H(u_{n,0}) + o_n(1), \tag{4.10}$$

$$E_{p,q}(u_n) = E_{p,q}(u_0) + E_{p,q}(u_{n,0}) + o_n(1), \tag{4.11}$$

$$Q_{p,q}(u_n) = Q_{p,q}(u_0) + Q_{p,q}(u_{n,0}) + o_n(1). \tag{4.12}$$

We claim that $Q_{p,q}(u_0) \leq 0$. Up to a subsequence, we assume that $\delta_n := \int_{\mathbb{R}^N} |\Delta u_{n,0}|^2 dx + \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u_{n,0}|^2 dx \to \delta_0 \geq 0$. Now we need to consider two cases.

Case 1. $\delta_0 = 0$. By Lemma 2.3, for any $r \in (2, 4^*)$, we have $\int_{\mathbb{R}^N} |u_{n,0}|^r dx \to 0$. Then $Q_{p,q}(u_{n,0}) \to 0$ as $n \to +\infty$. Hence, from (4.12) we derive $Q_{p,q}(u_0) = 0$.

Case 2. $\delta_0 > 0$. By contradiction, we suppose that $Q_{p,q}(u_0) > 0$. From (4.12) it follows that $Q_{p,q}(u_{n,0}) \leq 0$. According to Lemma 4.1, there exists $s_{u_{n,0}} \in (0, 1]$ such that $Q_{p,q}((u_{n,0})_{s_{u_{n,0}}}) = 0$. In view of the fact that $\frac{H(s)}{|s|^{2+\frac{8}{N}}}$ is strictly increasing for $s \in (0, \infty)$, we deduce

$$\begin{aligned}
&E_{p,q}(u_{n,0}) - E_{p,q}((u_{n,0})_{s_{u_{n,0}}}) \\
&= \frac{1 - s_{u_{n,0}}^2}{2} \int_{\mathbb{R}^N} |\Delta u_{n,0}|^2 dx + \frac{1 - s_{u_{n,0}}}{2} \int_{\mathbb{R}^N} |\nabla u_{n,0}|^2 dx \\
&\quad - \int_{\mathbb{R}^N} F(u_{n,0}) dx + s_{u_{n,0}}^{-N/2} \int_{\mathbb{R}^N} F(s_{u_{n,0}}^{N/4} u_{n,0}) dx \\
&= \frac{1 - s_{u_{n,0}}^2}{2} Q_{p,q}(u_{n,0}) + (\frac{1 - s_{n,0}}{2} - \frac{1 - s_{n,0}^2}{4}) \int_{\mathbb{R}^N} |\nabla u_{n,0}|^2 dx \\
&\quad + \frac{1 - s_{n,0}^2}{2} \frac{N}{4} \int_{\mathbb{R}^N} (f(u_{n,0}) u_{n,0} - 2F(u_{n,0})) dx \\
&\quad - \int_{\mathbb{R}^N} F(u_{n,0}) dx + s_{u_{n,0}}^{-N/2} \int_{\mathbb{R}^N} F(s_{u_{n,0}}^{N/4} u_{n,0}) dx \\
&\geq \frac{1 - s_{n,0}^2}{2} \frac{N}{4} \int_{\mathbb{R}^N} (f(u_{n,0}) u_{n,0} - 2F(u_{n,0})) dx \\
&\quad - \int_{\mathbb{R}^N} F(u_{n,0}) dx + s_{u_{n,0}}^{-N/2} \int_{\mathbb{R}^N} F(s_{u_{n,0}}^{N/4} u_{n,0}) dx + \frac{1 - s_{u_{n,0}}^2}{2} Q_{p,q}(u_{n,0}) \\
&= \int_{\mathbb{R}^N} \int_{s_{n,0}}^1 \frac{N}{4} s |u_{n,0}|^{2+\frac{8}{N}} \Big(\frac{H(u_{n,0})}{|u_{n,0}|^{2+\frac{8}{N}}} - \frac{H(s^{N/4} u_{n,0})}{|s^{N/4} u_{n,0}|^{2+\frac{8}{N}}} \Big) ds dx \\
&\quad + \frac{1 - s_{u_{n,0}}^2}{2} Q_{p,q}(u_{n,0}) \\
&\geq \frac{1 - s_{u_{n,0}}}{2} Q_{p,q}(u_{n,0}).
\end{aligned}$$

We denote $c_{n,0} := \|u_{n,0}\|_2^2$. Clearly, $c_{n,0} \le c$. From Lemma 4.4 we derive

$$\begin{aligned}
m_{p,q}(c) &= \lim_{n\to+\infty} \Big(E_{p,q}(u_n) - \frac{1}{2}Q_{p,q}(u_n)\Big) \\
&= \lim_{n\to+\infty} \Big[\Big(\frac{N}{8}\int_{\mathbb{R}^N} H(u_n)dx - \int_{\mathbb{R}^N} F(u_n)dx\Big) + \frac{1}{4}\|\nabla u_n\|_2^2\Big] \\
&= \Big(\frac{N}{8}\int_{\mathbb{R}^N} H(u_0)dx - \int_{\mathbb{R}^N} F(u_0)dx + \frac{1}{4}\|\nabla u_0\|_2^2\Big) \\
&\quad + \lim_{n\to+\infty} \Big(\frac{N}{8}\int_{\mathbb{R}^N} H(u_{n,0})dx - \int_{\mathbb{R}^N} F(u_{n,0})dx + \frac{1}{4}\|\nabla u_{n,0}\|_2^2\Big) \\
&= \Big[\frac{N}{8}\int_{\mathbb{R}^N} \Big(f(u_0)u_0 - \big(2+\frac{8}{N}\big)F(u_0)\Big)dx + \frac{1}{4}\|\nabla u_0\|_2^2\Big] \\
&\quad + \lim_{n\to+\infty} \Big(E_{p,q}(u_{n,0}) - \frac{1}{2}Q_{p,q}(u_{n,0})\Big) \\
&\ge \lim_{n\to+\infty} \Big(E_{p,q}(u_{n,0}) - \frac{1}{2}Q_{p,q}(u_{n,0})\Big) \\
&\ge \lim_{n\to+\infty} \Big(E_{p,q}((u_{n,0})_{s_{u_{n,0}}}) - \frac{s_{u_{n,0}}^2}{2}Q_{p,q}(u_{n,0})\Big) \\
&\ge \lim_{n\to+\infty} E_{p,q}((u_{n,0})_{s_{u_{n,0}}}) \\
&\ge \lim_{n\to+\infty} m_{p,q}(c_{n,0}) \ge m_{p,q}(c).
\end{aligned}$$

This indicates that $\lim_{n\to+\infty} Q_{p,q}(u_{n,0}) = 0$ and

$$\lim_{n\to+\infty} E_{p,q}(u_{n,0}) = \lim_{n\to+\infty} m_{p,q}(c_{n,0}) = m_{p,q}(c). \tag{4.13}$$

On the other hand, combining (4.9) and (4.11) yields

$$m_{p,q}(c) = E_{p,q}(u_n) + o_n(1) = E_{p,q}(u_0) + E_{p,q}(u_{n,0}) + o_n(1).$$

In view of $E_{p,q}(u_0) > 0$, from (4.13) it follows that

$$m_{p,q}(c) > m_{p,q}(c) - E_{p,q}(u_0) = \lim_{n\to+\infty} E_{p,q}(u_{n,0}) = \lim_{n\to+\infty} m_{p,q}(c_{n,0}) = m_{p,q}(c).$$

This yields a contradiction.

Using $Q_{p,q}(u_0) \le 0$ and similar arguments as above, there exists $s_0 \in (0,1]$ such that $(u_0)_{s_0} \in \mathcal{Q}_{p,q}(c_0)$ and

$$E_{p,q}(u_0) - E_{p,q}((u_0)_{s_0}) \ge \frac{1-s_0^2}{2}Q_{p,q}(u_0). \tag{4.14}$$

We denote $c_0 = \|u_0\|_2^2$. Clearly, $c_0 \in (0,c]$. By (4.14) and Lemma 4.4 we have

$$\begin{aligned}
m_{p,q}(c) &= \lim_{n\to+\infty} \Big(E_{p,q}(u_n) - \frac{1}{2}Q_{p,q}(u_n)\Big) \\
&= \lim_{n\to+\infty} \Big[\Big(\frac{N}{8}\int_{\mathbb{R}^N} H(u_n)dx - \int_{\mathbb{R}^N} F(u_n)dx\Big) + \frac{1}{4}\|\nabla u_n\|_2^2\Big] \\
&= \lim_{n\to+\infty} \Big[\frac{N}{8}\int_{\mathbb{R}^N} \Big(f(u_{n,0})u_{n,0} - \big(2+\frac{8}{N}\big)F(u_{n,0})\Big)dx \\
&\quad + \frac{1}{4}\|\nabla u_{n,0}\|_2^2\Big] + \Big(E_{p,q}(u_0) - \frac{1}{2}Q_{p,q}(u_0)\Big) \\
&\ge E_{p,q}((u_0)_{s_0}) - \frac{s_0^2}{2}Q_{p,q}(u_0)
\end{aligned}$$

$$\geq m_{p,q}(c_0) \geq m_{p,q}(c),$$

which implies $m_{p,q}(c_0) = m_{p,q}(c)$ and $Q_{p,q}(u_0) = 0$, that is, $s_0 = 1$. Thus we have $u_0 \in \mathcal{Q}_{p,q}(c_0)$ and $E_{p,q}(u_0) = m_{p,q}(c_0)$. Using Lemma 4.6 at c_0 and $m_{p,q}(c_0) = m_{p,q}(c)$, we obtain $c_0 = c$ and thus $E_{p,q}(u_0) = m_{p,q}(c)$. □

Proof of Theorem 1.3. Consider the functional $\Psi(u) : S(c) \to \mathbb{R}$ defined by

$$\Psi(u) := E_{p,q}(u_{s_u}) = \frac{1}{2}s_u^2\|\Delta u\|_2^2 + \frac{s_u}{2}\|\nabla u\|_2^2 - \frac{\mu}{q}s_u^{\frac{N(q-2)}{4}}\|u\|_q^q - \frac{1}{p}s_u^{\frac{N(p-2)}{4}}\|u\|_p^p,$$

where s_u is given in Lemma 4.1 and $u_{s_u} \in \mathcal{Q}_{p,q}(c)$.

According to Lemma 4.7, we find $u_0 \in \mathcal{Q}_{p,q}(c)$ such that $E_{p,q}(u_0) = m_{p,q}(c)$. Then there exists $v_0 \in S(c)$ such that $(v_0)_{s_{v_0}} = u_0$ and $\Psi(v_0) = E_{p,q}((v_0)_{s_{v_0}}) = E_{p,q}(u_0) = m_{p,q}(c)$. This implies that v_0 is a minimizer of $E_{p,q}$ restricted on $S(c)$.

We claim that Ψ is of class C^1 and

$$d\Psi(u)[\varphi] = dE_{p,q}(u_{s_u})[\varphi_{s_u}] \tag{4.15}$$

for any $u \in S(c)$ and $\varphi \in T_uS(c)$. In fact, by the definition of Ψ we have

$$\Psi(u + t\varphi) - \Psi(u) = E_{p,q}((u + t\varphi)_{s_t}) - E_{p,q}(u_{s_0}),$$

where $|t|$ is small enough, $s_t = s_{u+t\varphi}$ and $s_0 = s_u$ is the unique maximum point of the functional $E_{p,q}(u_s)$. By the mean value theorem we obtain

$$\begin{aligned}
&E_{p,q}((u + t\varphi)_{s_t}) - E_{p,q}(u_{s_0})\\
&\leq E_{p,q}((u + t\varphi)_{s_t}) - E_{p,q}(u_{s_t})\\
&= \frac{s_t^2}{2}\Big(\int_{\mathbb{R}^N} 2t\Delta u \cdot \Delta\varphi + t^2|\Delta\varphi|^2 dx\Big) + \frac{s_t}{2}\Big(\int_{\mathbb{R}^N} 2t\nabla u \cdot \nabla\varphi + t^2|\nabla\varphi|^2 dx\Big)\\
&\quad - \mu s_t^{\frac{N(q-2)}{4}} \int_{\mathbb{R}^N} \Big(\int_0^1 |u + s\eta_t\varphi|^{q-2}(u + t\eta_t\varphi)t\varphi dt\Big)dx\\
&\quad - s_t^{\frac{N(p-2)}{4}} \int_{\mathbb{R}^N} \Big(\int_0^1 |u + t\eta_t\varphi|^{p-2}(u + t\eta_t\varphi)t\varphi dt\Big)dx,
\end{aligned} \tag{4.16}$$

where $\eta_t \in (0, 1)$. Similarly, we derive

$$\begin{aligned}
&E_{p,q}((u + t\varphi)_{s_t}) - E_{p,q}(u_{s_0})\\
&\geq E_{p,q}((u + t\varphi)_{s_0}) - E_{p,q}(u_{s_0})\\
&= \frac{s_0^2}{2}\Big(\int_{\mathbb{R}^N} 2t\Delta u \cdot \Delta\varphi + t^2|\Delta\varphi|^2 dx\Big) + \frac{s_0}{2}\Big(\int_{\mathbb{R}^N} 2t\nabla u \cdot \nabla\varphi + t^2|\nabla\varphi|^2 dx\Big)\\
&\quad - \mu s_0^{\frac{N(q-2)}{4}} \int_{\mathbb{R}^N} \Big(\int_0^1 |u + t\theta_t\varphi|^{q-2}(u + t\theta_t\varphi)t\varphi dt\Big)dx\\
&\quad - s_0^{\frac{N(p-2)}{4}} \int_{\mathbb{R}^N} \Big(\int_0^1 |u + t\theta_t\varphi|^{p-2}(u + t\theta_t\varphi)t\varphi dt\Big)dx,
\end{aligned} \tag{4.17}$$

where $\theta_t \in (0, 1)$. Since the map $u \mapsto s_u$ is of class C^1, from (4.16) and (4.17) it follows that

$$\begin{aligned}
&\lim_{t\to 0} \frac{\Psi(u + t\varphi) - \Psi(u)}{t}\\
&= s_u^2 \int_{\mathbb{R}^N} \Delta u \cdot \Delta\varphi dx + s_u \int_{\mathbb{R}^N} \nabla u \cdot \nabla\varphi dx
\end{aligned}$$

$$- s_u^{\frac{N(q-2)}{4}} \mu \int_{\mathbb{R}^N} |u|^{q-2} u \cdot \varphi dx - s_u^{\frac{N(p-2)}{4}} \mu \int_{\mathbb{R}^N} |u|^{p-2} u \cdot \varphi dx.$$

So the Gâteaux derivative of Ψ is bounded linear in φ and continuous in u. Therefore, Ψ is of class C^1. In particular, by changing variables in the integrals, we have

$$\begin{aligned} d\Psi(u)[\varphi] = s_u^2 &\int_{\mathbb{R}^N} \Delta u \cdot \Delta \varphi dx + s_u \int_{\mathbb{R}^N} \nabla u \cdot \nabla \varphi dx \\ &- s_u^{\frac{N(q-2)}{4}} \mu \int_{\mathbb{R}^N} |u|^{q-2} u \cdot \varphi dx - s_u^{\frac{N(p-2)}{4}} \int_{\mathbb{R}^N} |u|^{p-2} u \cdot \varphi dx \\ &= \int_{\mathbb{R}^N} \Delta u_{s_u} \cdot \Delta \varphi_{s_u} dx + \int_{\mathbb{R}^N} \nabla u_{s_u} \cdot \nabla \varphi_{s_u} dx \\ &- \mu \int_{\mathbb{R}^N} |u_{s_u}|^{q-2} u_{s_u} \cdot \varphi_{s_u} dx - \int_{\mathbb{R}^N} |u_{s_u}|^{p-2} u_{s_u} \cdot \varphi_{s_u} dx. \\ &= dE_{p,q}(u_{s_u})[\varphi_{s_u}]. \end{aligned}$$

So the claim (4.15) is true, from which we deduce

$$\begin{aligned} \|dE_{p,q}(u_0)\|_{(T_{u_0} S(c))^*} &= \sup_{\varphi \in T_{u_0} S(c), \|\varphi\| \le 1} |dE_{p,q}(u_0)[\varphi]| \\ &= \sup_{\varphi \in T_{u_0} S(c), \|\varphi\| \le 1} |dE_{p,q}((v_0)_{s_{v_0}})[(\varphi_{s_{v_0}^{-1}})_{s_{v_0}}]| \\ &= \sup_{\varphi \in T_{u_0} S(c), \|\varphi\| \le 1} |d\Psi(v_0)[\varphi_{s_{v_0}^{-1}}]| \\ &\le \|d\Psi(v_0)\|_{(T_{v_0} S(c))^*} \cdot \sup_{\varphi \in T_{u_0} S(c), \|\varphi\| \le 1} \|\varphi_{s_{v_0}^{-1}}\| \\ &\le \max\{s_{v_0}^{-1}, 1\} \|dE_{p,q}(v_0)\|_{(T_{v_0} S(c))^*} = 0. \end{aligned}$$

It follows that u_0 is a critical point of $E_{p,q}$ restricted on $S(c)$. By Lemma 4.5 for some $\omega > 0$, u_0 weakly solves (1.2). In view of $E_{p,q}(u_0) = m_{p,q}(c)$, we infer that u_0 is a normalized ground state solution of problem (1.2). □

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