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PROPERTIES OF THE SOLUTIONS TO PERIODIC CONFORMABLE NON-AUTONOMOUS NON-INSTANTANEOUS IMPULSIVE DIFFERENTIAL EQUATIONS

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Abstract. In this article, we study the properties of solutions to periodic non-autonomous conformable non-instantaneous impulsive differential equations. We use a conformable Cauchy matrix and obtain some basic properties of the periodic solution to the homogeneous and non-homogeneous problems. We consider the periodicity of solutions to nonlinear problem via a fixed theorem.

1. Introduction

Hernández and O'Regan [12] establish a non-instantaneous impulsive differential equation model depending on the current state and duration of action, that describes phenomena in engineering, physics, biology, and many other fields. With the development of research, there are many publications studying the existence, stability, controllability, and periodicity of solutions for non-instantaneous impulsive differential equations; see for example [1, 5, 7, 9, 11, 14, 15, 16, 18, 20, 22, 23, 24, 25]. Motivated by the results based on conformable derivatives, [2, 3, 4, 6, 8, 10, 13, 17, 19, 21], we consider the homogeneous linear conformable non-autonomous problem

$$
\begin{gathered}
\mathfrak{D}_{\tau}^{s_l}\beta(t) = \alpha(t)\beta(t), \quad t \in (s_l, t_{l+1}],\ l \in \mathbb{N}_0 := \{0,1,2,\dots\},\ 0<\tau<1, \\
\beta(t_l^+) = (\mathbb{E}+P_l)\beta(t_l^-), \quad l \in \mathbb{N} := \{1,2,\dots\}, \\
\beta(t) = \delta_l(t)\beta(t_l^+), \quad t \in (t_l, s_l],\ l \in \mathbb{N}, \\
\beta(s_l^+) = \beta(s_l^-), \quad l \in \mathbb{N}, \\
\beta(c) = \beta_c \in \mathbb{R}^n,
\end{gathered} \tag{1.1}
$$

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and the nonhomogeneous linear conformable non-autonomous problem

$$\begin{gathered}\mathfrak{D}_\tau^{s_l}\beta(t)=\alpha(t)\beta(t)+\zeta(t),\quad t\in(s_l,t_{l+1}],\ l\in\mathbb{N}_0:=\{0,1,2,\dots\},\ 0<\tau<1,\\ \beta(t_l^+)=(\mathbb{E}+P_l)\beta(t_l^-)+Q_l,\quad l\in\mathbb{N}:=\{1,2,\dots\},\\ \beta(t)=\delta_l(t)\beta(t_l^+),\quad t\in(t_l,s_l],\ l\in\mathbb{N},\\ \beta(s_l^+)=\beta(s_l^-),\quad l\in\mathbb{N},\\ \beta(c)=\beta_c\in\mathbb{R}^n,\end{gathered}\tag{1.2}$$

and the nonlinear conformable non-autonomous problem

$$\begin{gathered}\mathfrak{D}_\tau^{s_l}\beta(t)=\alpha(t)\beta(t)+\eta(t,\beta(t)),\quad t\in(s_l,t_{l+1}],\ l\in\mathbb{N}_0:=\{0,1,2,\dots\},\ 0<\tau<1,\\ \beta(t_l^+)=(\mathbb{E}+P_l)\beta(t_l^-)+Q_l,\quad l\in\mathbb{N}:=\{1,2,\dots\},\\ \beta(t)=\delta_l(t)\beta(t_l^+),\quad t\in(t_l,s_l],\ l\in\mathbb{N},\\ \beta(s_l^+)=\beta(s_l^-),\quad l\in\mathbb{N},\\ \beta(c)=\beta_c\in\mathbb{R}^n.\end{gathered}\tag{1.3}$$

The sequences $\{t_l\}_{l\in\mathbb{N}_0}$ and $\{s_l\}_{l\in\mathbb{N}_0}$ satisfy $c=s_0\leq\dots\leq t_l<s_l<t_{l+1}$ for $l\in\mathbb{N}$. Set $\mathbb{A}=\cup_{l=1}^{\infty}(s_l,t_{l+1}]$, $\mathbb{B}=\cup_{l=1}^{\infty}(t_l,s_l]$, $\alpha(\cdot):\mathbb{A}\to\mathbb{R}^{n\times n}$, $\delta_l(\cdot):\mathbb{B}\to\mathbb{R}^{n\times n}$, $\zeta(\cdot):\mathbb{A}\to\mathbb{R}^n$ and $\eta:\mathbb{A}\times\mathbb{R}^n\to\mathbb{R}^n$ are continuous functions. Also, $P_l\in\mathbb{R}^{n\times n}$, $Q_l\in\mathbb{R}^n$ with $P_{l+a}=P_l$ and $Q_{l+a}=Q_l$. What's more, $\mathbb{E}$ denotes the unit matrix. Note that for each $l\in\mathbb{N}$, the sequences t_l,s_l satisfy $t_{l+a}=t_l+\vartheta$ and $s_{l+a}=s_l+\vartheta$ where $a\in\mathbb{N}$ means the number of impulsive points of a periodic interval $(c,c+\vartheta)$ and ϑ is a fixed positive number.

This article consists of 6 sections. Section 2 presents basic theory and the conformable Cauchy matrix. Section 3 shows the properties of the conformable Cauchy matrix. Section 4 studies the stability and periodicity of the solution of (1.1). Section 5 proves the existence and boundedness of the periodic solution of (1.2). Section 6 proves the existence and uniqueness of periodic solutions of (1.3) using Brouwer's fixed point theorem.

2. Preliminaries

Set $\mathbb{C}=[c,+\infty)$ and $PC(\mathbb{C},\mathbb{R}^n):=\big\{\beta:\mathbb{C}\to\mathbb{R}^n:\beta\in C\big((t_l,t_{l+1}],\mathbb{R}^n\big)$, there exists $\beta(t_l^-)$ and $\beta(t_l^+)$, $l=1,2,\dots$ with $\beta(t_l^-)=\beta(t_l)\big\}$, where $C\big((t_l,t_{l+1}],\mathbb{R}^n\big)$ denotes the space of all continuous functions from $(t_l,t_{l+1}]$ into $\mathbb{R}^n$ endowed with the norm $\|\beta\|=\sup_{t\in\mathbb{C}}|\beta(t)|$. We introduce $PC_\vartheta(\mathbb{C},\mathbb{R}^n)=\{\beta\in PC(\mathbb{C},\mathbb{R}^n):\beta(t)=\beta(t+\vartheta),\ t\in\mathbb{C}\}$. We denote a vector $a=(a_1,\dots,a_n)^\top\in\mathbb{R}^n$ with its norm $\|a\|=\sum_{i=1}^n|a_i|$ and a matrix $b\in\mathbb{R}^{n\times n}$ with its norm $\|b\|=\max_{\|y\|=1}\|b\beta\|$.

Definition 2.1 ([3, Definition 2.1])**.** The conformable derivative with lower index c of a function $r:\mathbb{C}\to\mathbb{R}$ is defined as

$$\begin{gathered}\mathfrak{D}_\tau^c\beta(t)=\lim_{\varepsilon\to0}\frac{\beta(t+\varepsilon(t-c)^{1-\tau})-\beta(t)}{\varepsilon},\quad t>c,\ 0<\tau<1,\\ \mathfrak{D}_\tau^c\beta(c)=\lim_{t\to c^+}\mathfrak{D}_\tau^c\beta(t).\end{gathered}$$

Remark 2.2. For $t>c$ we note that the conformable derivative $\mathfrak{D}_\tau^c\beta(t)$ exists if and only if y is differentiable at t and $\mathfrak{D}_\tau^c\beta(t)=(t-c)^{1-\tau}y'(t)$.

Definition 2.3 ([3, Notation]). The conformable integral of a function $r : \mathbb{C} \to \mathbb{R}$ is written as

$$\mathfrak{I}_{\tau}^{c}\beta(t) = \int_{c}^{t} \beta(s) d_{\tau}(s,c) = \int_{c}^{t} (s-c)^{\tau-1}\beta(s)ds, \quad t \geq c, 0 < \tau < 1,$$

if $c = 0$, then we write $d_\tau(s,c)$ as $d_\tau(s)$.

Lemma 2.4. *Let $\beta : \mathbb{A} \to \mathbb{R}^n$ be a continuous function. A solution $\beta \in C(\mathbb{A}, \mathbb{R}^n)$ of the linear problem*

$$\mathfrak{D}_{\tau}^{s_l}\beta(t) = \alpha(t)\beta(t), \quad 0 < \tau < 1,$$
$$\beta(s) = \beta_s, \quad t > s \geq c.$$

has the form

$$\beta(t) = \Phi(t,s)\beta_s,$$

where $\Phi(\cdot,\cdot)$ is the Cauchy matrix of $\mathfrak{D}_{\tau}^{s_l}\beta(t) = \alpha(t)\beta(t)$.

We set $\|\Phi(t,s)\| \leq e^{\int_s^t \|\alpha(\theta)\|(\theta - s_l)^{\tau-1}d\theta}$ for $s_l \leq s \leq t \leq t_{l+1}$, and $\phi_l = \max_{t\in(s_l,t_{l+1}]} \|\alpha(t)\|$.

Lemma 2.5. *The solution $\beta(\cdot, s, \beta_s) \in PC(\mathbb{C}, \mathbb{R}^n)$ of (1.1) with $\beta(s) = \beta_s$ has the form*

$$\beta(t, s, \beta_s) = \Lambda(t,s)\beta_s, \ t \geq c,$$

in which

$$\Lambda(t,s) = \begin{cases} \Phi(t,s), \quad t,s \in (s_l, t_{l+1}], \ l = 0,1,2,\dots; \\ \delta_l(t)\delta_l^{-1}(s), \quad t,s \in (t_l, s_l], \ l = 1,2,\dots; \\ \delta_{\varsigma(c,t)}(t)\prod_{l=\varsigma(c,s)+2}^{\varsigma(c,t)}[(\mathbb{E}+P_l)\Phi(t_l,s_{l-1})\delta_{l-1}(s_{l-1}^-)](\mathbb{E}+P_{\varsigma(c,s)+1}) \\ \times\Phi(t_{\varsigma(c,s)+1},s), \quad s \in (s_{\varsigma(c,s)}, t_{\varsigma(c,s)+1}], \ t \in (t_{\varsigma(c,t)}, s_{\varsigma(c,t)}]; \\ \Phi(t, s_{\varsigma(c,t)})\delta_{\varsigma(c,t)}(s_{\varsigma(c,t)})\prod_{l=\varsigma(c,s)+1}^{\varsigma(c,t)}[(\mathbb{E}+P_l)\Phi(t_l,s_{l-1})\delta_{l-1}(s_{l-1})] \\ \times\delta_{\varsigma(c,s)}^{-1}(s), \quad s \in (t_{\varsigma(c,s)}, s_{\varsigma(c,s)}], \ t \in (s_{\varsigma(c,t)}, t_{\varsigma(c,t)+1}]; \\ \Phi(t, s_{\varsigma(c,t)})\delta_{\varsigma(c,t)}(s_{\varsigma(c,t)})\prod_{l=\varsigma(c,s)+2}^{\varsigma(c,t)}[(\mathbb{E}+P_l)\Phi(t_l,s_{l-1})\delta_{l-1}(s_{l-1})] \\ \times(\mathbb{E}+P_{\varsigma(c,s)+1})\Phi(t_{\varsigma(c,s)+1},s), \\ \quad s \in (s_{\varsigma(c,s)}, t_{\varsigma(c,s)+1}], \ t \in (s_{\varsigma(c,t)}, t_{\varsigma(c,t)+1}]; \\ \delta_{\varsigma(c,t)}(t)\prod_{l=\varsigma(c,s)+1}^{\varsigma(c,t)}[(\mathbb{E}+P_l)\Phi(t_l,s_{l-1})\delta_{l-1}(s_{l-1})]\delta_{\varsigma(c,s)}^{-1}(s), \\ \quad s \in (t_{\varsigma(c,s)}, s_{\varsigma(c,s)}], \ t \in (t_{\varsigma(c,t)}, s_{\varsigma(c,t)}], \end{cases} \tag{2.1}$$

when $\varsigma(c,t) = \varsigma(c,s)$ and $\varsigma(c,t)$ denotes the number of the impulsive points which belong to (c,t). $\prod_{l=\varsigma(c,s)}^{\varsigma(c,t)-1} = E$.

Proof. We consider 4 cases.

Case 1. No impulsive point between t and s.

(i) For $t,s \in (s_l, t_{l+1}]$, $l = 0,1,2,\dots,\varsigma(c,t)$, we have

$$\beta(t) = \Phi(t,s_l)\beta(s_l), \quad \text{and} \quad \beta(s) = \Phi(s,s_l)\beta(s_l),$$

so $\beta(t) = \Phi(t,s)\beta(s)$. We obtain

$$\Lambda(t,s) = \Phi(t,s).$$

(ii) For $t,s \in (t_l, s_l]$, $l = 1,2,\dots,\varsigma(c,t)$, we have

$$\beta(t) = \delta_l(t)\beta(t_l^+) \quad \text{and} \quad \beta(s) = \delta_l(s)\beta(t_l^+)$$

so $\Lambda(t,s)=\delta_l(t)\delta_l^{-1}(s),\ t,s\in(t_l,s_l]$.

(iii) For any $s\in(t_l,s_l]$ and any $t\in(s_l,t_{l+1}]$, we have

$$\beta(t)=\Phi(t,s_l)\beta(s_l^+)=\Phi(t,s_l)\beta(s_l^-)=\Phi(t,s_l)\delta_l(s_l)\delta_l^{-1}(s)\beta(s),$$

so $\Lambda(t,s)=\Phi(t,s_l)\delta_l(s_l)\delta_l^{-1}(s)$.

Case 2. One impulsive point between time t and s.

(i) For any $s\in(s_{l-1},t_l]$ and $t\in(t_l,s_l]$, we have

$$\beta(t)=\delta_l(t)\beta(t_l^+)=\delta_l(t)(\mathbb{E}+P_l)\beta(t_l^-)=\delta_l(t)(\mathbb{E}+P_l)\Phi(t_l,s)\beta(s),$$

so $\Lambda(t,s)=\delta_l(t)\Phi(t_l,s)$.

(ii) For any $s\in(s_{l-1},t_l]$ and any $t\in(s_l,t_{l+1}]$, we have

$$\begin{aligned}\beta(t)&=\Phi(t,s_l)\beta(s_l^+)\\&=\Phi(t,s_l)\beta(s_l^-)\\&=\Phi(t,s_l)\delta_l(s_l)\beta(t_l^+)\\&=\Phi(t,s_l)\delta_l(s_l)(\mathbb{E}+P_l)\beta(t_l^-)\\&=\Phi(t,s_l)\delta_l(s_l)(\mathbb{E}+P_l)\Phi(t_l,s)\beta(s),\end{aligned}$$

so

$$\Lambda(t,s)=\Phi(t,s_l)\delta_l(s_l)(\mathbb{E}+P_l)\beta(t_l^-)\Phi(t_l,s).$$

(iii) For any $s\in(t_{l-1},s_{l-1}]$ and any $t\in(t_l,s_l]$, we have

$$\begin{aligned}\beta(t)&=\delta_l(t)\beta(t_l^+)\\&=\delta_l(t)(\mathbb{E}+P_l)\beta(t_l^-)\\&=\delta_l(t)(\mathbb{E}+P_l)\Phi(t_l,s_{l-1})\beta(s_{l-1}^+)\\&=\delta_l(t)(\mathbb{E}+P_l)\Phi(t_l,s_{l-1})\beta(s_{l-1}^-)\\&=\delta_l(t)(\mathbb{E}+P_l)\Phi(t_l,s_{l-1})\delta_{l-1}(s_{l-1})\delta_{l-1}^{-1}(s)\beta(s),\end{aligned}$$

so

$$\Lambda(t,s)=\delta_l(t)(\mathbb{E}+P_l)\Phi(t_l,s_{l-1})\delta_{l-1}(s_{l-1})\delta_{l-1}^{-1}(s).$$

(iv) For any $s\in(t_{l-1},s_{l-1}]$ and any $t\in(s_l,t_{l+1}]$, we have

$$\begin{aligned}\beta(t)&=\Phi(t,s_l)\beta(s_l^+)\\&=\Phi(t,s_l)\beta(s_l^-)\\&=\Phi(t,s_l)\delta_l(s_l)\beta(t_l^+)\\&=\Phi(t,s_l)\delta_l(s_l)(\mathbb{E}+P_l)\beta(t_l^-)\\&=\Phi(t,s_l)\delta_l(s_l)(\mathbb{E}+P_l)\Phi(t_l,s_{l-1})\beta(s_l^+)\\&=\Phi(t,s_l)\delta_l(s_l)(\mathbb{E}+P_l)\Phi(t_l,s_{l-1})\beta(s_l^-)\\&=\Phi(t,s_l)\delta_l(s_l)(\mathbb{E}+P_l)\Phi(t_l,s_{l-1})\delta_{l-1}(s_{l-1})\delta_{l-1}^{-1}(s)\beta(s),\end{aligned}$$

so

$$\Lambda(t,s)=\Phi(t,s_l)\delta_l(s_l)(\mathbb{E}+P_l)\Phi(t_l,s_{l-1})\delta_{l-1}(s_{l-1})\delta_{l-1}^{-1}(s).$$

Case 3. Two impulsive points between t and s.

(i) For any $s\in(s_{l-2},t_{l-1}]$ and any $t\in(t_l,s_l]$, we have

$$\beta(t)=\delta_l(t)\beta(t_l^+)$$

$$
\begin{aligned}
&= \delta_l(t)(\mathbb{E}+P_l)\beta(t_l^-) \\
&= \delta_l(t)(\mathbb{E}+P_l)\Phi(t_l,s_{l-1})\beta(s_{l-1}^+) \\
&= \delta_l(t)(\mathbb{E}+P_l)\Phi(t_l,s_{l-1})\beta(s_{l-1}^-) \\
&= \delta_l(t)(\mathbb{E}+P_l)\Phi(t_l,s_{l-1})\delta_{l-1}(s_{l-1})\beta(t_{l-1}^+) \\
&= \delta_l(t)(\mathbb{E}+P_l)\Phi(t_l,s_{l-1})\delta_{l-1}(s_{l-1})(\mathbb{E}+P_{l-1})\beta(t_{l-1}^-) \\
&= \delta_l(t)(\mathbb{E}+P_l)\Phi(t_l,s_{l-1})\delta_{l-1}(s_{l-1})(\mathbb{E}+P_{l-1})\Phi(t_{l-1},s)\beta(s),
\end{aligned}
$$

so

$$
\Lambda(t,s) = \delta_l(t)(\mathbb{E}+P_l)\Phi(t_l,s_{l-1})\delta_{l-1}(s_{l-1})(\mathbb{E}+P_{l-1})\Phi(t_{l-1},s).
$$

(ii) For any $s \in (s_{l-2}, t_{l-1}]$ and any $t \in (s_l, t_{l+1}]$, we have

$$
\begin{aligned}
\beta(t) &= \Phi(t,s_l)\beta(s_l^+) \\
&= \Phi(t,s_l)\beta(s_l^-) \\
&= \Phi(t,s_l)\delta_l(s_l)\beta(t_l^+) \\
&= \Phi(t,s_l)\delta_l(s_l)(\mathbb{E}+P_l)\beta(t_l^-) \\
&= \Phi(t,s_l)\delta_l(s_l)(\mathbb{E}+P_l)\Phi(t_l,s_{l-1})\beta(s_{l-1}^+) \\
&= \Phi(t,s_l)\delta_l(s_l)(\mathbb{E}+P_l)\Phi(t_l,s_{l-1})\beta(s_{l-1}^-) \\
&= \Phi(t,s_l)\delta_l(s_l)(\mathbb{E}+P_l)\Phi(t_l,s_{l-1})\delta_{l-1}(s_{l-1})\beta(t_{l-1}^+) \\
&= \Phi(t,s_l)\delta_l(s_l)(\mathbb{E}+P_l)\Phi(t_l,s_{l-1})\delta_{l-1}(s_{l-1})(\mathbb{E}+P_{l-1})\beta(t_{l-1}^-) \\
&= \Phi(t,s_l)\delta_l(s_l)(\mathbb{E}+P_l)\Phi(t_l,s_{l-1})\delta_{l-1}(s_{l-1})(\mathbb{E}+P_{l-1})\Phi(t_{l-1},s)\beta(s),
\end{aligned}
$$

so

$$
\Lambda(t,s) = \Phi(t,s_l)\delta_l(s_l)(\mathbb{E}+P_l)\Phi(t_l,s_{l-1})\delta_{l-1}(s_{l-1})(\mathbb{E}+P_{l-1})\Phi(t_{l-1},s).
$$

(iii) For any $s \in (t_{l-2}, s_{l-2}]$ and any $t \in (t_l, s_l]$, we have

$$
\begin{aligned}
&\beta(t) \\
&= \delta_l(t)\beta(t_l^+) \\
&= \delta_l(t)(\mathbb{E}+P_l)\beta(t_l^-) \\
&= \delta_l(t)(\mathbb{E}+P_l)\Phi(t_l,s_{l-1})\beta(s_{l-1}^+) \\
&= \delta_l(t)(\mathbb{E}+P_l)\Phi(t_l,s_{l-1})\beta(s_{l-1}^-) \\
&= \delta_l(t)(\mathbb{E}+P_l)\Phi(t_l,s_{l-1})\delta_{l-1}(s_{l-1})\beta(t_{l-1}^+) \\
&= \delta_l(t)(\mathbb{E}+P_l)\Phi(t_l,s_{l-1})\delta_{l-1}(s_{l-1})(\mathbb{E}+P_{l-1})\beta(t_{l-1}^-) \\
&= \delta_l(t)(\mathbb{E}+P_l)\Phi(t_l,s_{l-1})\delta_{l-1}(s_{l-1})(\mathbb{E}+P_{l-1})\Phi(t_{l-1},s_{l-2})\beta(s_{l-2}^+) \\
&= \delta_l(t)(\mathbb{E}+P_l)\Phi(t_l,s_{l-1})\delta_{l-1}(s_{l-1})(\mathbb{E}+P_{l-1})\Phi(t_{l-1},s_{l-2})\beta(s_{l-2}^-) \\
&= \delta_l(t)(\mathbb{E}+P_l)\Phi(t_l,s_{l-1})\delta_{l-1}(s_{l-1})(\mathbb{E}+P_{l-1})\Phi(t_{l-1},s_{l-2})\delta_{l-2}(s_{l-2})\delta_{l-2}^{-1}(s)\beta(s),
\end{aligned}
$$

so

$$
\Lambda(t,s) = \delta_l(t)(\mathbb{E}+P_l)\Phi(t_l,s_{l-1})\delta_{l-1}(s_{l-1})(\mathbb{E}+P_{l-1})\Phi(t_{l-1},s_{l-2})\delta_{l-2}(s_{l-2})\delta_{l-2}^{-1}(s).
$$

(iv) For any $s \in (t_{l-2}, s_{l-2}]$ and any $t \in (s_l, t_{l+1}]$, we have

$$
\begin{aligned}
\beta(t) &= \Phi(t,s_l)\beta(s_l^+) \\
&= \Phi(t,s_l)\beta(s_l^-)
\end{aligned}
$$

$$
\begin{aligned}
&= \Phi(t,s_l)\delta_l(s_l)(\mathbb{E}+P_l)\beta(t_l^-)\\
&= \Phi(t,s_l)\delta_l(s_l)(\mathbb{E}+P_l)\Phi(t_l,s_{l-1})\beta(s_{l-1}^+)\\
&= \Phi(t,s_l)\delta_l(s_l)(\mathbb{E}+P_l)\Phi(t_l,s_{l-1})\beta(s_{l-1}^-)\\
&= \Phi(t,s_l)\delta_l(s_l)(\mathbb{E}+P_l)\Phi(t_l,s_{l-1})\delta_{l-1}(s_{l-1})\beta(t_{l-1}^+)\\
&= \Phi(t,s_l)\delta_l(s_l)(\mathbb{E}+P_l)\Phi(t_l,s_{l-1})\delta_{l-1}(s_{l-1})(\mathbb{E}+P_{l-1})\beta(t_{l-1}^-)\\
&= \Phi(t,s_l)\delta_l(s_l)(\mathbb{E}+P_l)\Phi(t_l,s_{l-1})\delta_{l-1}(s_{l-1})(\mathbb{E}+P_{l-1})\Phi(t_{l-1},s_{l-2})\beta(s_{l-2}^+)\\
&= \Phi(t,s_l)\delta_l(s_l)(\mathbb{E}+P_l)\Phi(t_l,s_{l-1})\delta_{l-1}(s_{l-1})(\mathbb{E}+P_{l-1})\Phi(t_{l-1},s_{l-2})\beta(s_{l-2}^-)\\
&= \Phi(t,s_l)\delta_l(s_l)(\mathbb{E}+P_l)\Phi(t_l,s_{l-1})\delta_{l-1}(s_{l-1})(\mathbb{E}+P_{l-1})\\
&\quad\times \Phi(t_{l-1},s_{l-2})\delta_{l-2}(s_{l-2})\delta_{l-2}^{-1}(s)\beta(s),
\end{aligned}
$$

so

$$
\begin{aligned}
\Lambda(t,s) &= \Phi(t,s_l)\delta_l(s_l)(\mathbb{E}+P_l)\Phi(t_l,s_{l-1})\delta_{l-1}(s_{l-1})(\mathbb{E}+P_{l-1})\Phi(t_{l-1},s_{l-2})\\
&\quad\times \delta_{l-2}(s_{l-2})\delta_{l-2}^{-1}(s).
\end{aligned}
$$

Case 4. Many impulsive points between t and s.

(i) For any $s\in(t_{\varsigma(c,s)},s_{\varsigma(c,s)}]$ and any $t\in(s_{\varsigma(c,t)},t_{\varsigma(c,t)+1}]$, we have

$$
\begin{aligned}
\Lambda(t,s) &= \Phi(t,s_{\varsigma(c,t)})\delta_{\varsigma(c,t)}(s_{\varsigma(c,t)})\\
&\quad\times \prod_{l=\varsigma(c,s)+1}^{\varsigma(c,t)}[(\mathbb{E}+P_l)\Phi(t_l,s_{l-1})\delta_{l-1}(s_{l-1})]\delta_{\varsigma(c,s)}^{-1}(s).
\end{aligned}
$$

(ii) For any $s\in(s_{\varsigma(c,s)},t_{\varsigma(c,s)+1}]$ and any $t\in(t_{\varsigma(c,t)},s_{\varsigma(c,t)}]$, we have

$$
\begin{aligned}
\Lambda(t,s) &= \delta_{\varsigma(c,t)}(t)\prod_{l=\varsigma(c,s)+2}^{\varsigma(c,t)}[(\mathbb{E}+P_l)\Phi(t_l,s_{l-1})\delta_{l-1}(s_{l-1}^-)]\\
&\quad\times(\mathbb{E}+P_{\varsigma(c,s)+1})\Phi(t_{\varsigma(c,s)+1},s).
\end{aligned}
$$

(iii) For any $s\in(s_{\varsigma(c,s)},t_{\varsigma(c,s)+1}]$ and any $t\in(s_{\varsigma(c,t)},t_{\varsigma(c,t)+1}]$, we have

$$
\begin{aligned}
\Lambda(t,s) &= \Phi(t,s_{\varsigma(c,t)})\delta_{\varsigma(c,t)}(s_{\varsigma(c,t)})\prod_{l=\varsigma(c,s)+2}^{\varsigma(c,t)}[(\mathbb{E}+P_l)\Phi(t_l,s_{l-1})\delta_{l-1}(s_{l-1})]\\
&\quad\times(\mathbb{E}+P_{\varsigma(c,s)+1})\Phi(t_{\varsigma(c,s)+1},s).
\end{aligned}
$$

(iv) For any $s\in(t_{\varsigma(c,s)},s_{\varsigma(c,s)}]$ and any $t\in(t_{\varsigma(c,t)},s_{\varsigma(c,t)}]$, we have

$$
\Lambda(t,s) = \delta_{\varsigma(c,t)}(t)\prod_{l=\varsigma(c,s)+1}^{\varsigma(c,t)}[(\mathbb{E}+P_l)\Phi(t_l,s_{l-1})\delta_{l-1}(s_{l-1})]\delta_{\varsigma(c,s)}^{-1}(s).
$$

Summarizing, we can write

$$\Lambda(t,s)=\begin{cases}\Phi(t,s), \quad t,s\in(s_l,t_{l+1}],\ l=0,1,2,\dots;\\ \delta_l(t)\delta_l^{-1}(s), \quad t,s\in(t_l,s_l],\ l=1,2,\dots;\\ \delta_{\varsigma(c,t)}(t)\prod_{l=\varsigma(c,s)+2}^{\varsigma(c,t)}[(\mathbb{E}+P_l)\Phi(t_l,s_{l-1})\delta_{l-1}(s_{l-1}^-)]\\ \times(\mathbb{E}+P_{\varsigma(c,s)+1})\Phi(t_{\varsigma(c,s)+1},s),\\ \quad s\in(s_{\varsigma(c,s)},t_{\varsigma(c,s)+1}],\ t\in(t_{\varsigma(c,t)},s_{\varsigma(c,t)}];\\ \Phi(t,s_{\varsigma(c,t)})\delta_{\varsigma(c,t)}(s_{\varsigma(c,t)})\prod_{l=\varsigma(c,s)+1}^{\varsigma(c,t)}[(\mathbb{E}+P_l)\Phi(t_l,s_{l-1})\delta_{l-1}(s_{l-1})]\\ \times\delta_{\varsigma(c,s)}^{-1}(s), \quad s\in(t_{\varsigma(c,s)},s_{\varsigma(c,s)}],\ t\in(s_{\varsigma(c,t)},t_{\varsigma(c,t)+1}];\\ \Phi(t,s_{\varsigma(c,t)})\delta_{\varsigma(c,t)}(s_{\varsigma(c,t)})\prod_{l=\varsigma(c,s)+2}^{\varsigma(c,t)}[(\mathbb{E}+P_l)\Phi(t_l,s_{l-1})\delta_{l-1}(s_{l-1})]\\ \times(\mathbb{E}+P_{\varsigma(c,s)+1})\Phi(t_{\varsigma(c,s)+1},s),\\ \quad s\in(s_{\varsigma(c,s)},\ t_{\varsigma(c,s)+1}],\ t\in(s_{\varsigma(c,t)},t_{\varsigma(c,t)+1}];\\ \delta_{\varsigma(c,t)}(t)\prod_{l=\varsigma(c,s)+1}^{\varsigma(c,t)}[(\mathbb{E}+P_l)\Phi(t_l,s_{l-1})\delta_{l-1}(s_{l-1})]\delta_{\varsigma(c,s)}^{-1}(s),\\ \quad s\in(t_{\varsigma(c,s)},s_{\varsigma(c,s)}],\ t\in(t_{\varsigma(c,t)},s_{\varsigma(c,t)}].\end{cases}$$

□

Definition 2.6. A function $\beta(\cdot,\beta_c)\in PC(\mathbb{C},\mathbb{R}^n)$ is ϑ-periodic if $\beta(t,\beta_c)=\beta(t+\vartheta,\beta_c)$ for all $t\geq c$.

Definition 2.7. System (1.1) is exponentially stable if there exist constants $\lambda_1\geq 1$ and $\lambda_2<0$ such that

$$\|\Lambda(t,s)\|\leq\lambda_1 e^{\lambda_2(t-s)}, \quad c\leq s\leq t.$$

3. Basic properties of $\Lambda(\cdot,\cdot)$

Set $\kappa=\sup_{l\geq1}\|\mathbb{E}+P_l\|$, $\mu=\sup_{l\geq0}\frac{(t_{l+1}-s_l)^\tau}{\tau}$, $\xi=\sup_{l\geq1}\max_{(t_l,s_l]}\|\delta_l(t)\|$, $\phi=\max_{l\geq0}\phi_l$. We use the following assumptions:

- (A1) $\alpha(t+\vartheta)=\alpha(t)$ for $t\in\mathbb{A}$;
- (A2) $\delta_{l+a}(t+\vartheta)=\delta_l(t)$ for $t\in\mathbb{B}$.

Theorem 3.1. *When $c\leq s\leq t$, we have*

$$\|\Lambda(t,c)\|\leq e^{\phi\mu+\varsigma(c,t)(\ln\kappa+\ln\xi+\phi\mu)},$$

or

$$\|\Lambda(t,c)\|\leq e^{\varsigma(c,t)(\ln\kappa+\ln\xi+\phi\mu)}.$$

Proof. For $t\in(s_{\omega(c,t)},t_{\omega(c,t)+1}]$, we have

$$\begin{aligned}\|\Lambda(t,c)\|&\leq\|\Phi(t,s_{\varsigma(c,t)})\delta_{\varsigma(c,t)}(s_{\varsigma(c,t)})\prod_{l=2}^{\varsigma(c,t)}[(\mathbb{E}+P_l)\Phi(t_l,s_{l-1})\delta_{l-1}(s_{l-1})]\\ &\quad\times(\mathbb{E}+P_1)\Phi(t_1,c)\|\\ &\leq e^{\frac{\phi}{\tau}(t-s_{\varsigma(c,t)})^\tau}\xi(\kappa e^{\phi\mu}\xi)^{\varsigma(c,t)-1}\kappa e^{\phi\mu}\\ &\leq e^{\phi\mu}(\kappa e^{\phi\mu}\xi)^{\varsigma(c,t)}\\ &=e^{\phi\mu+\varsigma(c,t)(\ln\kappa+\ln\xi+\phi\mu)}.\end{aligned}$$

For $t \in (t_{\omega(c,t)}, s_{\omega(c,t)}]$, we have

$$\begin{aligned}
\|\Lambda(t,c)\| &\leq \|\delta_{\varsigma(c,t)}(t) \prod_{l=2}^{\varsigma(c,t)} [(\mathbb{E}+P_l)\Phi(t_l, s_{l-1})\delta_{l-1}(s_{l-1})](\mathbb{E}+P_1)\Phi(t_1,c)\| \\
&\leq \xi(\kappa e^{\phi\mu}\xi)^{(\varsigma(c,t)-1)}\kappa e^{\phi\mu} \\
&\leq (\kappa e^{\phi\mu}\xi)^{\varsigma(c,t)} \\
&= e^{\varsigma(c,t)(\ln\kappa + \ln\xi + \phi\mu)}.
\end{aligned}$$

□

Theorem 3.2. *If $c \leq s < u < t$, then $\Lambda(t,s) = \Lambda(t,u)\Lambda(u,s)$.*

Proof. By the form of (2.1), when $s \in (s_{\varsigma(c,s)}, t_{\varsigma(c,s)+1}]$ and $t \in (s_{\varsigma(c,t)}, t_{\varsigma(c,t)+1}]$, we have

$$\begin{aligned}
&\Lambda(t,s) \\
&= \Phi(t, s_{\varsigma(c,t)})\delta_{\varsigma(c,t)}(s_{\varsigma(c,t)}) \prod_{l=\varsigma(c,u)+2}^{\varsigma(c,t)} [(\mathbb{E}+P_l)\Phi(t_l, s_{l-1})\delta_{l-1}(s_{l-1})](\mathbb{E}+P_{\varsigma(c,u)+1}) \\
&\quad \times \Phi(t_{\varsigma(c,u)+1}, u)\Phi(u, s_{\varsigma(c,u)})\delta_{\varsigma(c,u)}(s_{\varsigma(c,u)}) \prod_{l=\varsigma(c,s)+2}^{\varsigma(c,u)} [(\mathbb{E}+P_l)\Phi(t_l, s_{l-1})\delta_{l-1}(s_{l-1})] \\
&\quad \times (\mathbb{E}+P_{\varsigma(c,u)+1})\Phi(t_{\varsigma(c,u)+1}, u) \\
&= \Lambda(t,u)\Lambda(u,s), \quad u \in (s_{\varsigma(a,u)}, t_{\varsigma(a,u)+1}],
\end{aligned}$$

$$\begin{aligned}
&\Lambda(t,s) \\
&= \Phi(t, s_{\varsigma(c,t)})\delta_{\varsigma(c,t)}(s_{\varsigma(c,t)}) \prod_{l=\varsigma(c,u)+1}^{\varsigma(c,t)} [(\mathbb{E}+P_l)\Phi(t_l, s_{l-1})\delta_{l-1}(s_{l-1})]\delta^{-1}_{\varsigma(c,u)}(u) \\
&\quad \times \delta_{\varsigma(c,u)}(u) \prod_{l=\varsigma(c,s)+2}^{\varsigma(c,u)} [(\mathbb{E}+P_l)\Phi(u_l, s_{l-1})\delta_{l-1}(s_{l-1})](\mathbb{E}+P_{\varsigma(c,s)+1})\Phi(u_{\varsigma(c,s)+1}, s) \\
&= \Lambda(t,u)\Lambda(u,s), \quad u \in (t_{\varsigma(a,u)}, s_{\varsigma(a,u)}].
\end{aligned}$$

When $s \in (t_{\varsigma(c,s)}, s_{\varsigma(c,s)}]$ and $t \in (s_{\varsigma(c,t)}, t_{\varsigma(c,t)+1}]$, we have

$$\begin{aligned}
\Lambda(t,s) &= \Phi(t, s_{\varsigma(c,t)})\delta_{\varsigma(c,t)}(s_{\varsigma(c,t)}) \prod_{l=\varsigma(c,u)+2}^{\varsigma(c,t)} [(\mathbb{E}+P_l)\Phi(t_l, s_{l-1})\delta_{l-1}(s_{l-1})] \\
&\quad \times (\mathbb{E}+P_{\varsigma(c,u)+1})\Phi(t_{\varsigma(c,u)+1}, u)\Phi(u, s_{\varsigma(c,u)})\delta_{\varsigma(c,u)}(s_{\varsigma(c,u)}) \\
&\quad \times \prod_{l=\varsigma(c,s)+1}^{\varsigma(c,u)} [(\mathbb{E}+P_l)\Phi(t_l, s_{l-1})\delta_{l-1}(s_{l-1})]\delta^{-1}_{\varsigma(c,s)}(s) \\
&= \Lambda(t,u)\Lambda(u,s), \quad u \in (s_{\varsigma(c,u)}, t_{\varsigma(c,u)+1}],
\end{aligned}$$

$$\Lambda(t,s) = \Phi(t, s_{\varsigma(c,t)})\delta_{\varsigma(c,t)}(s_{\varsigma(c,t)}) \prod_{l=\varsigma(c,u)+1}^{\varsigma(c,t)} [(\mathbb{E}+P_l)\Phi(t_l, s_{l-1})\delta_{l-1}(s_{l-1})]\delta^{-1}_{\varsigma(c,u)}(u)$$

$$\times \delta_{\varsigma(c,u)}(u) \prod_{l=\varsigma(c,s)+1}^{\varsigma(c,u)} [(\mathbb{E}+P_l)\Phi(u_l,s_{l-1})\delta_{l-1}(s_{l-1})]\delta^{-1}_{\varsigma(c,s)}(s)$$
$$= \Lambda(t,u)\Lambda(u,s), \quad u \in (t_{\varsigma(c,u)}, s_{\varsigma(c,u)}].$$

When $s \in (s_{\varsigma(c,s)}, t_{\varsigma(c,s)+1}]$ and $t \in (t_{\varsigma(c,t)}, s_{\varsigma(c,t)}]$, we have

$$\begin{aligned}
&\Lambda(t,s)\\
&= \delta_{\varsigma(c,t)}(t) \prod_{l=\varsigma(c,u)+2}^{\varsigma(c,t)} [(\mathbb{E}+P_l)\Phi(t_l,s_{l-1})\delta_{l-1}(s_{l-1})](\mathbb{E}+P_{\varsigma(c,u)+1})\Phi(t_{\varsigma(c,u)+1},u)\\
&\quad \times \Phi(u,s_{\varsigma(c,u)})\delta_{\varsigma(c,u)}(s_{\varsigma(c,u)}) \prod_{l=\varsigma(c,s)+2}^{\varsigma(c,u)} [(\mathbb{E}+P_l)\Phi(t_l,s_{l-1})\delta_{l-1}(s_{l-1})]\\
&\quad \times (\mathbb{E}+P_{\varsigma(c,u)+1})\Phi(t_{\varsigma(c,u)+1},u)\\
&= \Lambda(t,u)\Lambda(u,s), \quad u \in (s_{\varsigma(c,u)}, t_{\varsigma(c,u)+1}],
\end{aligned}$$

$$\begin{aligned}
&\Lambda(t,s)\\
&= \delta_{\varsigma(c,t)}(t) \prod_{l=\varsigma(c,u)+1}^{\varsigma(c,t)} [(\mathbb{E}+P_l)\Phi(t_l,s_{l-1})\delta_{l-1}(s_{l-1})]\delta^{-1}_{\varsigma(c,u)}(u)\\
&\quad \times \ \delta_{\varsigma(c,u)}(u) \prod_{l=\varsigma(c,s)+2}^{\varsigma(c,u)} [(\mathbb{E}+P_l)\Phi(t_l,s_{l-1})\delta_{l-1}(s_{l-1})](\mathbb{E}+P_{\varsigma(c,s)+1})\Phi(t_{\varsigma(c,s)+1},s)\\
&= \Lambda(t,u)\Lambda(u,s), \quad u \in (t_{\varsigma(c,u)}, s_{\varsigma(c,u)}].
\end{aligned}$$

When $s \in (t_{\varsigma(c,s)}, s_{\varsigma(c,s)}]$ and $t \in (t_{\varsigma(c,t)}, s_{\varsigma(c,t)}]$, we have

$$\begin{aligned}
&\Lambda(t,s)\\
&= \delta_{\varsigma(c,t)}(t) \prod_{l=\varsigma(c,u)+2}^{\varsigma(c,t)} [(\mathbb{E}+P_l)\Phi(t_l,s_{l-1})\delta_{l-1}(s_{l-1})](\mathbb{E}+P_{\varsigma(c,u)+1})\Phi(t_{\varsigma(c,u)+1},u)\\
&\quad \times \Phi(u,s_{\varsigma(c,u)})\delta_{\varsigma(c,u)}(s_{\varsigma(c,u)}) \prod_{l=\varsigma(c,s)+1}^{\varsigma(c,u)} [(\mathbb{E}+P_l)\Phi(t_l,s_{l-1})\delta_{l-1}(s_{l-1})]\delta^{-1}_{\varsigma(c,s)}(s)\\
&= \Lambda(t,u)\Lambda(u,s), \quad u \in (s_{\varsigma(c,u)}, t_{\varsigma(c,u)+1}],
\end{aligned}$$

$$\begin{aligned}
\Lambda(t,s) &= \delta_{\varsigma(c,t)}(t) \prod_{l=\varsigma(c,u)+1}^{\varsigma(c,t)} [(\mathbb{E}+P_l)\Phi(t_l,s_{l-1})\delta_{l-1}(s_{l-1})]\delta^{-1}_{\varsigma(c,u)}(u)\\
&\quad \times \delta_{\varsigma(c,u)}(u) \prod_{l=\varsigma(c,s)+1}^{\varsigma(c,u)} [(\mathbb{E}+P_l)\Phi(u_l,s_{l-1})\delta_{l-1}(s_{l-1})]\delta^{-1}_{\varsigma(c,s)}(s)\\
&= \Lambda(t,u)\Lambda(u,s), \quad u \in (t_{\varsigma(c,u)}, s_{\varsigma(c,u)}].
\end{aligned}$$

□

Theorem 3.3. *If* (A1) *and* (A2) *hold, then*

$$\Lambda(\cdot+\vartheta,\cdot+\vartheta) = \Lambda(\cdot,\cdot), \quad N \in \mathbb{N}.$$

Proof. Equation (2.1) implies that for $s \in (s_{\varsigma(c,s)}, t_{\varsigma(c,s)+1}]$ and $t \in (s_{\varsigma(c,t)}, t_{\varsigma(c,t)+1}]$, we have

$$\begin{aligned}
&\Lambda(t+\vartheta, s+\vartheta)\\
&= \Phi(t+\vartheta, s_{\varsigma(c,t+\vartheta)})\delta_{\varsigma(c,t+\vartheta)}(s_{\varsigma(c,t+\vartheta)})\\
&\quad\times \prod_{l=\varsigma(c,s+\vartheta)+2}^{\varsigma(c,t+\vartheta)} [(\mathbb{E}+P_l)\Phi(t_l, s_{l-1})\delta_{l-1}(s_{l-1})](\mathbb{E}+P_{\varsigma(c,s+\vartheta)+1})\Phi(t_{\varsigma(c,s+\vartheta)+1}, s+\vartheta)\\
&= \Phi(t, s_{\varsigma(c,t)})\delta_{\varsigma(c,t)}(s_{\varsigma(c,t)})\\
&\quad\times \prod_{l=\varsigma(c,s)+2}^{\varsigma(c,t)} [(\mathbb{E}+P_l)\Phi(t_l, s_{l-1})\delta_{l-1}(s_{l-1})](\mathbb{E}+P_{\varsigma(c,s)+1})\Phi(t_{\varsigma(c,s)+1}, s)\\
&= \Lambda(t,s).
\end{aligned}$$

When $s \in (t_{\varsigma(c,s)}, s_{\varsigma(c,s)}]$ and $t \in (s_{\varsigma(c,t)}, t_{\varsigma(c,t)+1}]$, we have

$$\begin{aligned}
&\Lambda(t+\vartheta, s+\vartheta)\\
&= \Phi(t+\vartheta, s_{\varsigma(c,t+\vartheta)})\delta_{\varsigma(c,t+\vartheta)}(s_{\varsigma(c,t+\vartheta)})\\
&\quad\times \prod_{l=\varsigma(c,s+\vartheta)+1}^{\varsigma(c,t+\vartheta)} [(\mathbb{E}+P_l)\Phi(t_l, s_{l-1})\delta_{l-1}(s_{l-1})]\delta^{-1}_{\varsigma(c,s+\vartheta)}(s+\vartheta)\\
&= \Phi(t, s_{\varsigma(c,t)})\delta_{\varsigma(c,t)}(s_{\varsigma(c,t)}) \prod_{l=\varsigma(c,s)+1}^{\varsigma(c,t)} [(\mathbb{E}+P_l)\Phi(t_l, s_{l-1})\delta_{l-1}(s_{l-1})]\delta^{-1}_{\varsigma(c,s)}(s)\\
&= \Lambda(t,s).
\end{aligned}$$

When $s \in (s_{\varsigma(c,s)}, t_{\varsigma(c,s)+1}]$ and $t \in (t_{\varsigma(c,t)}, s_{\varsigma(c,t)}]$, we have

$$\begin{aligned}
&\Lambda(t+\vartheta, s+\vartheta)\\
&= \delta_{\varsigma(c,t+\vartheta)}(t+\vartheta) \prod_{l=\varsigma(c,s+\vartheta)+2}^{\varsigma(c,t+\vartheta)} [(\mathbb{E}+P_l)\Phi(t_l, s_{l-1})\delta_{l-1}(s_{l-1})]\\
&\quad\times (\mathbb{E}+P_{\varsigma(c,s+\vartheta)+1})\Phi(t_{\varsigma(c,s+\vartheta)+1}, s+\vartheta)\\
&= \delta_{\varsigma(c,t)}(t) \prod_{l=\varsigma(c,s)+2}^{\varsigma(c,t)} [(\mathbb{E}+P_l)\Phi(t_l, s_{l-1})\delta_{l-1}(s_{l-1})](\mathbb{E}+P_{\varsigma(c,s)+1})\Phi(t_{\varsigma(c,s)+1}, s)\\
&= \Lambda(t,s).
\end{aligned}$$

When $s \in (t_{\varsigma(c,s)}, s_{\varsigma(c,s)}]$ and $t \in (t_{\varsigma(c,t)}, s_{\varsigma(c,t)}]$, we have

$$\begin{aligned}
&\Lambda(t+\vartheta, s+\vartheta)\\
&= \delta_{\varsigma(c,t+\vartheta)}(t+\vartheta) \prod_{l=\varsigma(c,s+\vartheta)+1}^{\varsigma(c,t+\vartheta)} [(\mathbb{E}+P_l)\Phi(t_l, s_{l-1})\delta_{l-1}(s_{l-1})]\delta^{-1}_{\varsigma(c,s+\vartheta)}(s+\vartheta)\\
&= \delta_{\varsigma(c,t)}(t) \prod_{l=\varsigma(c,u)+1}^{\varsigma(c,t)} [(\mathbb{E}+P_l)\Phi(t_l, s_{l-1})\delta_{l-1}(s_{l-1})]\delta^{-1}_{\varsigma(c,s)}(s)\\
&= \Lambda(t,s).
\end{aligned}$$

□

Theorem 3.4. *Suppose that* (A1) *and* (A2) *hold. Then*

$$\Lambda(\cdot + N\vartheta, c) = \Lambda(\cdot, c)[\Lambda(c+\vartheta, c)]^N, \quad N \in \mathbb{N}.$$

Proof. From Theorems 3.2 and 3.3, we have

$$\begin{aligned}\Lambda(t+N\vartheta, c) &= \Lambda(t+N\vartheta, c+N\vartheta)\Lambda(c+N\vartheta, c)\\ &= \Lambda(t+N\vartheta, c+N\vartheta)\Lambda(c+N\vartheta, c+(N-1)\vartheta)\Lambda(c+(N-1)\vartheta, c)\\ &= \Lambda(t,c)\prod_{l=0}^{N-1}\Lambda(c+(N-l)\vartheta, c)\Lambda(c+(N-l-1)\vartheta, c)\\ &= \Lambda(t,c)[\Lambda(c+\vartheta,c)]^N,\ N\in\mathbb{N}.\end{aligned}$$

□

4. Homogeneous linear problem

Theorem 4.1. *If* (A1) *and* (A2) *hold, then one of the following 2 items is satisfied:*

(i) *Equation* (1.1) *has the trivial* ϑ*-periodic solution if and only if* $\operatorname{rank}(\mathbb{E} - \Lambda(c+\vartheta, c)) = n$.

(ii) Equation (1.1) *has at least one nontrivial* ϑ*-periodic solution if and only if* $\operatorname{rank}(\mathbb{E} - \Lambda(c+\vartheta, c)) < n$.

Proof. (i) If $\operatorname{rank}(\mathbb{E}-\Lambda(c+\vartheta, c)) = n$, then the only solution of $(\mathbb{E}-\Lambda(c+\vartheta, c))\beta = 0$ should be zero solution, which means the solution of (1.1) are trivial solution.

(ii) If $\operatorname{rank}(\mathbb{E} - \Lambda(c+\vartheta, c)) < n$, $(\mathbb{E} - \Lambda(c+\vartheta, c))\beta = 0$ has a nonzero solution which means (1.1) has ϑ-periodic nontrivial solution. □

Theorem 4.2. *If* (A1) *holds, then*

$$\lim_{t-s\to\infty}\frac{\varsigma(s,t)}{t-s} = \frac{a}{\vartheta}.$$

Proof. For $s \in [m\vartheta, (m+1)\vartheta]$ and $t \in [n\vartheta, (n+1)\vartheta]$ for $m \le n$, we have

$$(n-m-1)\vartheta \le t-s \le (n+1-m)\vartheta,$$

and

$$(n-m-1)a \le \varsigma(s,t) \le (n+1-m)a.$$

Hence,

$$\frac{(n-m-1)a}{(n+1-m)\vartheta} \le \frac{\varsigma(s,t)}{t-s} \le \frac{(n+1-m)a}{(n-m-1)\vartheta}.$$

It is obvious that $t-s\to\infty$ if and only if $n-m\to\infty$. So,

$$\frac{a}{\vartheta} \le \lim_{t-s\to\infty}\frac{\varsigma(s,t)}{t-s} \le \frac{a}{\vartheta},$$

and

$$\lim_{t-s\to\infty}\frac{\varsigma(s,t)}{t-s} = \frac{a}{\vartheta}.$$

□

Theorem 4.3. *If* $\ln\kappa + \ln\xi + \phi\mu < 0$*, then system* (1.1) *is exponentially stable.*

Proof. Combining Theorems 3.1 and 4.2, with $\varepsilon \in (c, \frac{a}{\vartheta})$ and $t \in (s_l, t_{l+1}]$, we obtain

$$\|\Lambda(t,c)\| \le e^{\phi\mu+\varsigma(c,t)(\ln\kappa+\ln\xi+\phi\mu)} \le e^{\phi\mu+(\frac{a}{\vartheta}-\varepsilon)(\ln\kappa+\ln\xi+\phi\mu)t},$$

in which $\lambda_1 = e^{\phi\mu} \ge 1$ and $\lambda_2 = (\frac{a}{\vartheta}-\varepsilon)(\ln\kappa+\ln\xi+\phi\mu) < 0$.

For $t \in (t_l, s_l]$, we obtaini

$$\|\Lambda(t,c)\| \le e^{\varsigma(c,t)(\ln\kappa+\ln\xi+\phi\mu)} \le e^{(\frac{a}{\vartheta}-\varepsilon)(\ln\kappa+\ln\xi+\phi\mu)t},$$

in which $\lambda_1 = 1$ and $\lambda_2 = (\frac{a}{\vartheta}-\varepsilon)(\ln\kappa+\ln\xi+\phi\mu) < 0$. From Definition 2.7, it follows that system (1.1) is exponentially stable. □

Theorem 4.4. *If a nontrivial solution $\beta(t,\beta_c)$ of* (1.1) *is ϑ-periodic, then $\ln\kappa + \ln\xi + \phi\mu \ge 0$.*

Proof. $\beta(c+\vartheta) = \Lambda(c+\vartheta,c)\beta_c$ implies $\beta(c) = \Lambda(c+\vartheta,c)\beta_c$. Hence, $\|\Lambda(c+\vartheta,c)\| \ge 1$ and

$$e^{\varsigma(c,t)(\ln\kappa+\ln\xi+\phi\mu)} \ge 1.$$

Then, there is $\ln\kappa+\ln\xi+\phi\mu \ge 0$. □

Corollary 4.5. *Suppose that* (A1) *and* (A2) *hold. If $\beta(t)$ is periodic and exponentially stable, then $\beta(t) = 0$.*

Example 4.6. Consider (1.1) and let $\tau = 1/2$, $s_0 = 0$, $s_l = l$, $t_l = l-\frac{1}{2}$, $l = 1,2,\dots$, $\vartheta = 1$, $a = 1$, $\beta_c = (1,0)^\top$. Set

$$\alpha(t) = \begin{pmatrix} \frac{3}{2}(t-s_{\varsigma(a,t)} & 0 \\ 0 & t-s_{\varsigma(a,t)} \end{pmatrix}, \quad P_l = \begin{pmatrix} -\frac{9}{10} & 0 \\ 0 & -\frac{9}{10} \end{pmatrix},$$

$$\delta_l(t) = \begin{pmatrix} \frac{1}{2} + \frac{t-t_l}{2(s_l-t_l)} & 0 \\ 0 & -1+\frac{t-t_l}{s_l-t_l} \end{pmatrix}.$$

So

$$\Phi(t,s) = \begin{pmatrix} e^{(t-s_l)^{3/2}-(s-s_l)^{3/2}} & 0 \\ 0 & e^{\frac{2}{3}(t-s_l)^{3/2}-\frac{2}{3}(s-s_l)^{3/2}} \end{pmatrix}, \quad t,s \in (s_l, t_{l+1}].$$

Next,

$$\begin{aligned}
\Lambda(t,0) &= \begin{pmatrix} e^{(t-s_{\varsigma(a,t)})^{3/2}} & 0 \\ 0 & e^{\frac{2}{3}(t-s_{\varsigma(a,t)})^{3/2}} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \\
&\quad \times \prod_{l=1}^{\varsigma(a,t)} \begin{pmatrix} \frac{1}{10} & 0 \\ 0 & \frac{1}{10} \end{pmatrix} \begin{pmatrix} e^{(t_l-s_{l-1})^{3/2}} & 0 \\ 0 & e^{\frac{2}{3}(t_l-s_{l-1})^{3/2}} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \\
&= \begin{pmatrix} e^{(t-s_l)^{3/2}}(\frac{1}{10}e^{\sqrt{2}/4})^{\varsigma(a,t)} & 0 \\ 0 & 0 \end{pmatrix}.
\end{aligned}$$

Then

$$\Lambda(t,0)\beta_c = \begin{pmatrix} e^{(t-s_{\varsigma(a,t)})^{3/2}}(\frac{1}{10}e^{\sqrt{2}/4})^{\varsigma(a,t)} \\ 0 \end{pmatrix}$$

and

$$\lim_{t\to\infty} \|\Lambda(t,0)\beta_c\| \le e^{(t-s_{\varsigma(a,t)})^{3/2}}(\frac{1}{10}e^{\sqrt{2}/4})^{\varsigma(a,t)} \le e^{\frac{\sqrt{2}}{4}+(\frac{\sqrt{2}}{4}-\ln 10)t} \to 0,$$

so β is exponentially stable.

Also, $\ln\kappa+\ln\xi+\phi\mu<0$ and Theorem 4.3 is verified. Furthermore, $\beta(t+1)\neq\beta(t)$ and

$$\mathbb{E}-\Lambda(1,0)=\begin{pmatrix}1-\frac{1}{10}e^{\sqrt{2}/4} & 0\\ 0 & 1\end{pmatrix},\quad \det(\mathbb{E}-\Lambda(1,0))\neq 0,$$

Equation (1.1) has only the trivial 1-periodic solution.

5. Nonhomogeneous linear problem

Theorem 5.1. *The solution of* (1.2) *has the form*

$$\begin{aligned}\beta(t)&=\Lambda(t,c)\beta_c+\sum_{l=0}^{\varsigma(c,t)-1}\int_{s_l}^{t_{l+1}}\Lambda(t,s)\zeta(s)(s-s_l)^{\tau-1}ds\\&\quad+\int_{s_{\varsigma(c,t)}}^{t}\Lambda(t,s)\zeta(s)(s-s_{\varsigma(c,t)})^{\tau-1}ds+\sum_{l=1}^{\varsigma(c,t)}\Lambda(t,s_l)\delta_l(s_l)Q_l.\end{aligned}$$

Proof. For $t\in[s_0,t_1]$, using the variation of constants method, one has

$$\beta(t)=\Lambda(t,c)\beta_c+\int_0^t\Lambda(t,s)\zeta(s)s^{\tau-1}ds.$$

If it holds for $t\in(s_{\varsigma(c,t)-1},t_{\varsigma(c,t)}]$, one has

$$\begin{aligned}\beta(t)&=\Lambda(t,c)\beta_c+\sum_{l=0}^{\varsigma(c,t)-2}\int_{s_l}^{t_{l+1}}\Lambda(t,s)\zeta(s)(s-s_l)^{\tau-1}ds\\&\quad+\int_{s_{\varsigma(c,t)-1}}^{t}\Lambda(t,s)\zeta(s)(s-s_{\varsigma(c,t)})^{\tau-1}ds+\sum_{l=1}^{\varsigma(c,t)-1}\Lambda(t,s_l)\delta_l(s_l)Q_l,\end{aligned}$$

and for $t\in(t_{\varsigma(c,t)},s_{\varsigma(c,t)}]$, we obtain

$$\begin{aligned}\beta(t)&=\delta_{\varsigma(c,t)}(t)(\mathbb{E}+P_{\varsigma(c,t)})\beta(t^-_{\varsigma(c,t)})+\delta_{\varsigma(c,t)}(t)Q_{\varsigma(c,t)}\\&=\delta_{\varsigma(c,t)}(t)(\mathbb{E}+P_{\varsigma(c,t)})[\Lambda(t^-_{\varsigma(c,t)},c)\beta_c\\&\quad+\sum_{l=0}^{\varsigma(c,t)-1}\int_{s_l}^{t_{l+1}}\Lambda(t^-_{\varsigma(c,t)},s)\zeta(s)(s-s_l)^{\tau-1}ds\\&\quad+\sum_{l=1}^{\varsigma(c,t)-1}\Lambda(t^-_{\varsigma(c,t)},s_l)\delta_l(s_l)Q_l]+\delta_{\varsigma(c,t)}(t)Q_{\varsigma(c,t)}.\end{aligned}$$

Next, for $t\in(s_{\varsigma(c,t)},t_{\varsigma(c,t)+1}]$, we have

$$\begin{aligned}&\beta(t)\\&=\Lambda(t,s_{\varsigma(c,t)})\beta(s_{\varsigma(c,t)})+\int_{s_{\varsigma(c,t)}}^{t}\Lambda(t,s)\zeta(s)(s-s_{\varsigma(c,t)})^{\tau-1}ds\\&=\Lambda(t,s_{\varsigma(c,t)})\delta_{\varsigma(c,t)}(s_{\varsigma(c,t)})(\mathbb{E}+P_{\varsigma(c,t)})\Lambda(t^-_{\varsigma(c,t)},c)\beta_c\\&\quad+\sum_{l=0}^{\varsigma(c,t)-1}\int_{s_l}^{t_{l+1}}\Lambda(t,s_{\varsigma(c,t)})\delta_{\varsigma(c,t)}(s_{\varsigma(c,t)})(\mathbb{E}+P_{\varsigma(c,t)})\Lambda(t^-_{\varsigma(c,t)},s)\zeta(s)(s-s_l)^{\tau-1}ds\end{aligned}$$

$$
\begin{aligned}
&+ \sum_{l=1}^{\varsigma(c,t)-1} \Lambda(t, s_{\varsigma(c,t)})\delta_{\varsigma(c,t)}(s_{\varsigma(c,t)})(\mathbb{E}+P_{\varsigma(c,t)})\Lambda(t^-_{\varsigma(c,t)}, s_l)\delta_l(s_l)Q_l \\
&+ \Lambda(t, s_{\varsigma(c,t)})\delta_{\varsigma(c,t)}(s_{\varsigma(c,t)})Q_{\varsigma(c,t)} \\
&+ \int_{s_{\varsigma(c,t)}}^{t} \Lambda(t,s)\zeta(s)(s-s_{\varsigma(c,t)})^{\tau-1}ds + \Lambda(t, s_{\varsigma(c,t)})\delta_{\varsigma(c,t)}(s_{\varsigma(c,t)})Q_{\varsigma(c,t)} \\
&= \Lambda(t,c)\beta_c + \sum_{l=0}^{\varsigma(c,t)-1} \int_{s_l}^{t_{l+1}} \Lambda(t,s)\zeta(s)(s-s_l)^{\tau-1}ds \\
&+ \int_{s_{\varsigma(c,t)}}^{t} \Lambda(t,s)\zeta(s)(s-s_{\varsigma(c,t)})^{\tau-1}ds + \sum_{l=1}^{\varsigma(c,t)} \Lambda(t,s_l)\delta_l(s_l)Q_l.
\end{aligned}
$$

By the mathematical induction method, we obtain

$$
\begin{aligned}
\beta(t) = \Lambda(t,c)\beta_c &+ \sum_{l=0}^{\varsigma(c,t)-1} \int_{s_l}^{t_{l+1}} \Lambda(t,s)\zeta(s)(s-s_l)^{\tau-1}ds \\
&+ \int_{s_{\varsigma(c,t)}}^{t} \Lambda(t,s)\zeta(s)(s-s_{\varsigma(c,t)})^{\tau-1}ds + \sum_{l=1}^{\varsigma(c,t)} \Lambda(t,s_l)\delta_l(s_l)Q_l. \qquad \square
\end{aligned}
$$

Now we introduce the following following assumption:

(A3) $\zeta(t+\vartheta) = \zeta(t)$, for $t \in \mathbb{A}$.

Theorem 5.2. *Suppose that* (A1)–(A3) *hold. If the solution of* (1.2) *is bounded, then it is a* ϑ*-solution.*

Proof. Let $\widetilde{\beta}(t)$ be a bounden solution of (1.2). Then $\widetilde{\beta}(c+n\vartheta)$ is bounded. Using Theorems 3.2 and 3.3, one obtains

$$
\begin{aligned}
&\widetilde{\beta}(c+(n+1)\vartheta) \\
&= \Lambda(c+(n+1)\vartheta, c)\beta_c + \sum_{l=0}^{(n+1)a-1} \int_{s_l}^{t_{l+1}} \Lambda(c+(n+1)\vartheta, s)\zeta(s)(s-s_l)^{\tau-1}ds \\
&+ \sum_{l=1}^{(n+1)a} \Lambda(c+(n+1)\vartheta, s_l)\delta_l(s_l)Q_l \\
&= \Lambda(c+(n+1)\vartheta, c+n\vartheta)\Big[\Lambda(c+n\vartheta, c)\beta_c + \sum_{l=0}^{na-1} \int_{s_l}^{t_{l+1}} \Lambda(c+n\vartheta, s)\zeta(s)(s-s_l)^{\tau-1}ds \\
&+ \sum_{l=1}^{na} \Lambda(c+n\vartheta, s_l)\delta_l(s_l)Q_l\Big] + \sum_{l=na}^{(n+1)a-1} \int_{s_l}^{t_{l+1}} \Lambda(c+(n+1)\vartheta, s)\zeta(s)(s-s_l)^{\tau-1}ds \\
&+ \sum_{l=na+1}^{(n+1)a} \Lambda(c+(n+1)\vartheta, s_l)\delta_l(s_l)Q_l \\
&= \Lambda(c+\vartheta, c)\widetilde{\beta}(c+n\vartheta) + \sum_{l=0}^{a-1} \int_{s_l}^{t_{l+1}} \Lambda(c+(n+1)\vartheta, s+n\vartheta)\zeta(s+n\vartheta)(s-s_l)^{\tau-1}ds
\end{aligned}
$$

$$
\begin{aligned}
&+\sum_{l=1}^{a}\Lambda(c+\vartheta,s_l)\delta_l(s_l)Q_l\\
&=\Lambda(c+\vartheta,c)\widetilde{\beta}(c+n\vartheta)+\sum_{l=0}^{a-1}\int_{s_l}^{t_{l+1}}\Lambda(c+\vartheta,s)\zeta(s)(s-s_l)^{\tau-1}ds\\
&+\sum_{l=1}^{a}\Lambda(c+\vartheta,s_l)\delta_l(s_l)Q_l\\
&=\Lambda(c+\vartheta,c)\widetilde{\beta}(c+n\vartheta)+\Lambda_c,
\end{aligned}
$$

where $\Lambda_c = \sum_{l=0}^{a-1}\int_{s_l}^{t_{l+1}}\Lambda(c+\vartheta,s)\zeta(s)(s-s_l)^{\tau-1}ds + \sum_{l=1}^{a}\Lambda(c+\vartheta,s_l)\delta_l(s_l)Q_l$. Hence,

$$
\widetilde{\beta}(c+n\vartheta)=\Lambda^a(c+\vartheta,c)\widetilde{\beta}(c)+\sum_{l=0}^{a-1}\Lambda^l(c+\vartheta,c)\Lambda_c.
$$

Then, $\widetilde{\beta}(t)$ is a ϑ-periodic solution that needs to be proven. If $\widetilde{\beta}(t)$ is not the ϑ-periodic of (1.2), then we can not find a $\beta_c\in\mathbb{R}^n$ such that

$$
(\mathbb{E}-\Lambda(c+\vartheta,c))\beta_c=\Lambda_c.
$$

By Fredholm alternative, we can find a $\mathcal{Z}\in\mathbb{R}^n$ such that

$$
(\mathbb{E}-\Lambda^\top(c+\vartheta,c))\mathcal{Z}=0,\quad \langle\Lambda_c,\mathcal{Z}\rangle\neq 0.
$$

Since $(\mathbb{E}-\Lambda^\top(c+\vartheta,c))\mathcal{Z}=0$, with each $n\in\mathbb{N}$, we have $[\Lambda^n(c+\vartheta,c))]^\top\mathcal{Z}=\mathcal{Z}$. Also,

$$
\begin{aligned}
\langle\widetilde{\beta}(c+n\vartheta),\mathcal{Z}\rangle&=\langle\Lambda^a(c+\vartheta,c)\widetilde{\beta}(c)+\sum_{l=0}^{a-1}\Lambda^l(c+\vartheta,c)\Lambda_c,\mathcal{Z}\rangle\\
&=\langle\widetilde{\beta}(c),[\Lambda^a(c+\vartheta,c)]^\top\mathcal{Z}\rangle+\sum_{l=0}^{a-1}\langle\Lambda_c,[\Lambda^l(c+\vartheta,c)]^\top\mathcal{Z}\rangle\\
&=\langle\widetilde{\beta}(c),\mathcal{Z}\rangle+a\langle\Lambda_c,\mathcal{Z}\rangle\to\infty,\quad\text{as } a\to\infty,
\end{aligned}
$$

which contradicts the boundness of $\widetilde{\beta}(t)$. So $\widetilde{\beta}(t)$ is a ϑ-periodic solution of (1.2). □

Now we introduce the following assumptions:

(A4) $\det(\mathbb{E}-\Lambda(c+\vartheta,c))\neq 0$;
(A5) $\det(\mathbb{E}-\Lambda(c+\vartheta,c))=0$.

Theorem 5.3. *If* (A1)–(A4) *hold, then* (1.2) *has a ϑ-periodic solution with*

$$
\beta_c=(\mathbb{E}-\Lambda(c+\vartheta,c))^{-1}\Lambda_c.
$$

Proof. $\det(\mathbb{E}-\Lambda(c+\vartheta,c))\neq 0$ implies

$$
\beta_c=(\mathbb{E}-\Lambda(c+\vartheta,c))^{-1}\Lambda_c
$$

to satisfy $\beta(c+\vartheta,\beta_c)=\beta_c$. With $\widetilde{\beta}(t)=\beta(t+\vartheta)$ and the properties of $\Lambda(t,s)$, we know $\widetilde{\beta}(t)=\beta(t+\vartheta)$ is the solution of (1.2) with $\widetilde{\beta}(c)=\beta(c+\vartheta)=\beta_c$. The uniqueness of the solution implies $\widetilde{\beta}(t)=\beta(t)$, i.e. $\beta(t+\vartheta,\beta_c)=\beta(t,\beta_c)$. □

We study the system

$$\begin{gathered} \mathfrak{D}_{\tau}^{s_l}\chi(t) = -\alpha^{\top}(t)\chi(t), \quad t \in (s_l, t_{l+1}],\ l \in \mathbb{N}_0 := \{0,1,2,\dots\},\ 0<\tau<1, \\ \chi(t_l^+) = -(\mathbb{E}+P_l^{\top})^{-1}P_l^{\top}\chi(t_l^-), \quad l \in \mathbb{N} := \{1,2,\dots\}, \\ \chi(t) = (\delta_l^{\top}(t))^{-1}\chi(t_l^+), \quad t \in (t_l, s_l],\ l \in \mathbb{N}, \\ \chi(s_l^+) = \chi(s_l^-), \quad l \in \mathbb{N}. \end{gathered} \tag{5.1}$$

Theorem 5.4. *If $\beta(t)$ is the solution of* (1.1)*, and $\chi(t)$ the solution of* (5.1)*, then $\langle \beta(t), \chi(t)\rangle$ is a constant.*

Proof. For $t \in (s_l, t_{l+1}]$, we have

$$\begin{aligned} \mathfrak{D}_{\tau}^{s_l}\langle \beta(t), \chi(t)\rangle &= \langle \mathfrak{D}_{\tau}^{s_l}\beta(t), \chi(t)\rangle + \langle \beta(t), \mathfrak{D}_{\tau}^{s_l}\chi(t)\rangle \\ &= \langle \alpha(t)\beta(t), \chi(t)\rangle + \langle \beta(t), -\alpha^{\top}\chi(t)\rangle \\ &= \langle \beta(t), \alpha(t)^{\top}\chi(t)\rangle + \langle \beta(t), -\alpha(t)^{\top}\chi(t)\rangle = 0. \end{aligned}$$

For $s \in (t_l, s_l]$, we have

$$\langle \beta(t), \chi(t)\rangle = \langle \delta_l(t)\beta(t_l^+), (\delta_l^{\top}(t))^{-1}\chi(t_l^+)\rangle = \langle \beta(t_l^+), \chi(t_l^+)\rangle.$$

For $t = t_l$, we have

$$\begin{aligned} \langle \beta(t_l^+), \chi(t_l^+)\rangle &= \langle (\mathbb{E}+P_l)\beta(t_l), [\mathbb{E} - (\mathbb{E}+P_l^{\top})^{-1}P_l^{\top}]\chi(t_l)\rangle \\ &= \langle (\mathbb{E}+P_l)\beta(t_l), (\mathbb{E}+P_l^{\top})^{-1}\chi(t_l)\rangle \\ &= \langle \beta(t_l), \chi(t_l)\rangle. \end{aligned}$$

Hence, $\langle \beta(t), s(t)\rangle$ is a constant. □

Theorem 5.5. *If* (A1)–(A3), (A5) *hold, then* (1.2) *has a ϑ-periodic solution if and only if $\langle \chi_c, \Lambda_c\rangle = 0$, where χ_c is the initial value of the ϑ-solution of* (5.1)*.*

Proof. Equation (1.2) has a ϑ-periodic solution if and only if there exists β_c such that

$$(\mathbb{E} - \Lambda(c+\vartheta, c))\beta_c = \Lambda_c.$$

Then

$$\begin{aligned} \langle \chi_c, \Lambda_c\rangle &= \langle \chi_c, (\mathbb{E} - \Lambda(c+\vartheta, c))\beta_c\rangle \\ &= \langle (\mathbb{E} - \Lambda(c+\vartheta, c))^{\top}\chi_c, \beta_c\rangle \\ &= \langle (\mathbb{E} - \Lambda^{\top}(c+\vartheta, c))\chi_c, \beta_c\rangle \\ &= \langle 0, \beta_c\rangle = 0. \end{aligned}$$

□

Example 5.6. let $\tau = 1/2$, $s_0 = 0$, $s_l = l$, $t_l = l - \frac{1}{2}$ for $l = 1, 2, \dots$, $\vartheta = 1$, and $a = 1$. Set $\alpha(t)$, P_l, $\delta_l(t)$ as in Example 4.6 and let

$$\zeta(t) = \begin{pmatrix} t - s_l \\ 0 \end{pmatrix}, \quad t \in (s_l, t_{l+1}],\ Q_l = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

So

$$\begin{aligned} \Lambda_c &= \sum_{l=0}^{a-1}\int_{s_l}^{t_{l+1}} \Lambda(c+\vartheta, s)\zeta(s)(s-s_l)^{\tau-1}ds + \sum_{l=1}^{a}\Lambda(c+\vartheta, s_l)\delta_l(s_l)Q_l \\ &= \begin{pmatrix} \frac{5}{3} - \frac{2}{3}e^{-\sqrt{2}/4} \\ 0 \end{pmatrix}. \end{aligned}$$

Then

$$\mathbb{E}-\Lambda(1,0)=\begin{pmatrix}1-\frac{1}{10}e^{\sqrt{2}/4} & 0\\ 0 & 1\end{pmatrix},$$

$$\beta_c=(\mathbb{E}-\Lambda(1,0))^{-1}\Lambda_c=\begin{pmatrix}\frac{\frac{5}{3}-\frac{2}{3}e^{-\sqrt{2}/4}}{1-\frac{1}{10}e^{\sqrt{2}/4}}\\ 0\end{pmatrix},$$

$$\begin{aligned}
&\Lambda(t,s)\\
&=\begin{pmatrix}e^{(t-s_{\varsigma(c,t)})^{3/2}} & 0\\ 0 & e^{\frac{2}{3}(t-s_{\varsigma(c,t)})^{3/2}}\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}\\
&\quad\times\prod_{l=\varsigma(c,s)+2}^{\varsigma(c,t)}\begin{pmatrix}\frac{1}{10} & 0\\ 0 & \frac{1}{10}\end{pmatrix}\begin{pmatrix}e^{(t_l-s_{l-1})^{3/2}} & 0\\ 0 & e^{\frac{2}{3}(t_l-s_{l-1})^{3/2}}\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}\begin{pmatrix}\frac{1}{10} & 0\\ 0 & \frac{1}{10}\end{pmatrix}\\
&\quad\times\begin{pmatrix}e^{(t_{\varsigma(c,s)+1}-s_{\varsigma(c,s)})^{3/2}-(s-s_{\varsigma(c,s)})^{3/2}} & 0\\ 0 & e^{\frac{2}{3}(t_{\varsigma(c,s)+1}-s_{\varsigma(c,s)})^{3/2}-\frac{2}{3}(s-s_{\varsigma(c,s)})^{3/2}}\end{pmatrix}\\
&=\begin{pmatrix}e^{(t-s_{\varsigma(c,t)})^{3/2}-(s-s_{\varsigma(c,s)})^{3/2}}(\frac{1}{10}e^{\sqrt{2}/4})^{\varsigma(c,t)-\varsigma(c,s)} & 0\\ 0 & 0\end{pmatrix}.
\end{aligned}$$

Then

$$\begin{aligned}
\beta(t)&=\Lambda(t,c)\beta_c+\sum_{l=0}^{\varsigma(c,t)-1}\int_{s_l}^{t_{l+1}}\Lambda(t,s)\zeta(s)(s-s_l)^{\tau-1}ds\\
&\quad+\int_{s_{\varsigma(c,t)}}^{t}\Lambda(t,s)\zeta(s)(s-s_{\varsigma(c,t)})^{\tau-1}ds+\sum_{l=1}^{\varsigma(c,t)}\Lambda(t,s_l)\delta_l(s_l)Q_l\\
&=\begin{pmatrix}e^{(t-s_{\varsigma(c,t)})^{3/2}}(\frac{1}{10}e^{\sqrt{2}/4})^{\varsigma(c,t)} & 0\\ 0 & 0\end{pmatrix}\begin{pmatrix}\frac{\frac{5}{3}-\frac{2}{3}e^{-\sqrt{2}/4}}{1-\frac{1}{10}e^{\sqrt{2}/4}}\\ 0\end{pmatrix}\\
&\quad+\sum_{l=0}^{\varsigma(c,t)-1}\int_{s_l}^{t_{l+1}}\begin{pmatrix}e^{(t-s_{\varsigma(c,t)})^{3/2}-(s-s_l)^{3/2}}(\frac{1}{10}e^{\sqrt{2}/4})^{\varsigma(c,t)-1-l}(s-s_l)^{1/2}\\ 0\end{pmatrix}ds\\
&\quad+\int_{s_{\varsigma(c,t)}}^{t}\begin{pmatrix}e^{(t-s_{\varsigma(c,t)})^{3/2}-(s-s_{\varsigma(c,t)})^{3/2}}(s-s_{\varsigma(c,t)})^{1/2}\\ 0\end{pmatrix}ds\\
&\quad+\sum_{l=1}^{\varsigma(c,t)}\begin{pmatrix}e^{(t-s_{\varsigma(c,t)})^{3/2}}(\frac{1}{10}e^{\sqrt{2}/4})^{\varsigma(c,t)-l}\\ 0\end{pmatrix}\\
&=\begin{pmatrix}\frac{\frac{5}{3}-\frac{2}{3}e^{-\sqrt{2}/4}}{1-\frac{1}{10}e^{\sqrt{2}/4}}e^{(t-s_{\varsigma(c,t)})^{3/2}}+\frac{2}{3}e^{(t-s_{\varsigma(c,t)})^{3/2}}-\frac{2}{3}\\ 0\end{pmatrix}.
\end{aligned}$$

So

$$\beta(t+1,0,\beta_0)=\beta(t,0,\beta_0),$$

$\beta(t,0,\beta_0)$ is a 1-periodic solution, and

$$\|\beta(t)\|\le\frac{\frac{5}{3}e^{\sqrt{2}/4}-\frac{2}{3}}{1-\frac{1}{10}e^{\sqrt{2}/4}}+\frac{2}{3}e^{\sqrt{2}/4}-\frac{2}{3}.$$

6. Nolinear problem

In this section, we study the ϑ-periodic solution of (1.3), using the following assumptions:

(A6) for $t \in \mathbb{A}$ and $\beta \in \mathbb{R}^n$, $\eta(t+\vartheta,\beta)=\eta(t,\beta)$;

(A7) for $t \in \mathbb{A}$ and $\beta \in \mathbb{R}^n$, there is a $\overline{\eta}>0$ such that $\|\eta(t,\beta)\| \le \overline{\eta}$.

We study the system

$$\mathfrak{D}_\tau^{s_l}\beta(t)=\alpha(t)\beta(t)+\eta(t,\beta(t)), \quad \beta(s_{l-1})=\beta_{l-1},\\ t\in(s_{l-1},t_l],\ 0<\tau<1,\ l\in\mathbb{N}_0,\ \beta_0=\beta(c),$$

whose solution is

$$\beta(t)=\Lambda(t,s_l)\beta_{l-1}+\int_{s_l}^t \Lambda(t,s)\eta(s,\beta(s))(s-s_l)^{\tau-1}ds. \tag{6.1}$$

We set the mapping

$$P_l(\beta_{l-1}):=\delta_l(s_l)\circ((\mathbb{E}+P_l)\circ\beta(t_l)+Q_l). \tag{6.2}$$

Equality (6.1) implies

$$\|\beta(t_l)\| \le e^{\frac{\phi}{\tau}(t_l-s_{l-1})^\tau}\|\beta_{l-1}\|+\frac{\overline{\eta}}{\phi}(e^{\frac{\phi}{\tau}(t_l-s_{l-1})^\tau}-1);$$

and (6.2) implies

$$\|P_l(\beta_{l-1})\| \le \kappa\xi e^{\frac{\phi}{\tau}(t_l-s_{l-1})^\tau}\|\beta_{l-1}\|+\kappa\xi\frac{\overline{\eta}}{\phi}(e^{\frac{\phi}{\tau}(t_l-s_{l-1})^\tau}-1)+\kappa\overline{Q},$$

where $\overline{Q}=\max_{l\in\mathbb{N}}\|Q_l\|$.

Then we construct the operator

$$P:=P_a\circ P_{a-1}\circ\dots\circ P_1,$$

and set $\overline{\phi}_l=e^{\frac{\phi}{\tau}(t_l-s_{l-1})^\tau}$, and $\varrho=\kappa\xi$.

Theorem 6.1. *If* (A7) *holds, then*

$$\begin{aligned}\|P(\beta_0)\| &\le \varrho^a\prod_{l=1}^a\overline{\phi}_l\|\beta_0\|+\frac{\overline{\eta}}{\phi}\sum_{l=1}^{a-1}\Big\{\varrho^{a-j+1}\prod_{j=l}^{a-1}\overline{\phi}_a\dots\overline{\phi}_{j+1}(\overline{\phi}_j-1)\Big\}\\ &\quad+\frac{\overline{\eta}\varrho}{\phi}(\overline{\phi}_a-1)+\Big[\sum_{l=2}^a\varrho^{a-j+1}\prod_{j=l}^a\overline{\phi}_a\dots\overline{\phi}_j+1\Big]\kappa\overline{Q}.\end{aligned} \tag{6.3}$$

Proof. For $l=1$, we have

$$\|\beta_1\|\le\varrho\overline{\phi}_1\|\beta_0\|+\frac{\varrho\overline{\eta}}{\phi}(\overline{\phi}_1-1)+\kappa\overline{Q}.$$

If (6.3) is satisfied with $l=a-1$, then for $l=a$, we have

$$\begin{aligned}&\|\beta_a\|\\ &\le\varrho\overline{\phi}_a\|\beta_{a-1}\|+\frac{\overline{\eta}\varrho}{\phi}(\overline{\phi}_a-1)+\kappa\overline{Q}\\ &\le\varrho\overline{\phi}_a\Big\{\varrho^{a-1}\prod_{l=1}^{a-1}\overline{\phi}_l\|\beta_0\|+\frac{\overline{\eta}}{\phi}\sum_{l=1}^{a-2}\Big\{\varrho^{a-j}\prod_{j=l}^{a-2}\overline{\phi}_{a-1}\dots\overline{\phi}_{j+1}(\overline{\phi}_j-1)\Big\}\end{aligned}$$

$$
\begin{aligned}
&+\frac{\overline{\eta}\varrho}{\phi}(\overline{\phi}_{a-1}-1)+\Big[\sum_{l=2}^{a-1}\varrho^{a-j}\prod_{j=l}^{a-1}\overline{\phi}_{a-1}\dots\overline{\phi}_{j}+1\Big]\kappa\overline{Q}\Big\}+\frac{\overline{\eta}\varrho}{\phi}(\overline{\phi}_{a}-1)+\kappa\overline{Q}\\
&=\varrho^{a}\prod_{l=1}^{a}\overline{\phi}_{l}\|\beta_0\|+\Big\{\frac{\overline{\eta}}{\phi}\sum_{l=1}^{a-2}\Big[\varrho^{a-j+1}\prod_{j=l}^{a-2}\overline{\phi}_{a}\overline{\phi}_{a-1}\dots\overline{\phi}_{j+1}(\overline{\phi}_{j}-1)\Big]+\frac{\overline{\eta}\varrho^{2}}{\phi}\overline{\phi}_{a}(\overline{\phi}_{a-1}-1)\Big\}\\
&+\Big[\sum_{l=2}^{a-1}\varrho^{a-j+1}\prod_{j=l}^{a-1}\overline{\phi}_{a}\overline{\phi}_{a-1}\dots\overline{\phi}_{j}\kappa\overline{Q}+\varrho\overline{\phi}_{a}\kappa\overline{Q}\Big]+\frac{\overline{\eta}\varrho}{\phi}(\overline{\phi}_{a}-1)+\kappa\overline{Q}\\
&=\varrho^{a}\prod_{l=1}^{a}\overline{\phi}_{l}\|\beta_0\|+\frac{\overline{\eta}}{\phi}\sum_{l=1}^{a-1}\Big\{\varrho^{a-j+1}\prod_{j=l}^{a-1}\overline{\phi}_{a}\dots\overline{\phi}_{j+1}(\overline{\phi}_{j}-1)\Big\}\\
&+\sum_{l=2}^{a}\varrho^{a-j+1}\prod_{j=l}^{a}\overline{\phi}_{a}\dots\overline{\phi}_{j}\kappa\overline{Q}+\frac{\overline{\eta}\varrho}{\phi}(\overline{\phi}_{a}-1)+\kappa\overline{Q}\\
&=\varrho^{a}\prod_{l=1}^{a}\overline{\phi}_{l}\|\beta_0\|+\frac{\overline{\eta}}{\phi}\sum_{l=1}^{a-1}\Big\{\varrho^{a-j+1}\prod_{j=l}^{a-1}\overline{\phi}_{a}\dots\overline{\phi}_{j+1}(\overline{\phi}_{j}-1)\Big\}\\
&+\Big[\sum_{l=2}^{a}\varrho^{a-j+1}\prod_{j=l}^{a}\overline{\phi}_{a}\dots\overline{\phi}_{j}+1\Big]\kappa\overline{Q}+\frac{\overline{\eta}\varrho}{\phi}(\overline{\phi}_{a}-1).
\end{aligned}
$$

□

Theorem 6.2. *If* (A1)–(A3), (A6), (A7) *hold, then* (1.3) *has a ϑ-periodic solution if and only if P has a fixed point.*

Proof. Sufficiency: If P has a fixed point β_0, there is

$$
\begin{aligned}
P(\beta_0) &:= P_a\circ P_{a-1}\circ\dots\circ P_1(\beta_0)\\
&=\Lambda(c+\vartheta,c)\beta_0+\sum_{l=0}^{a-1}\int_{s_l}^{t_{l+1}}\Lambda(c+\vartheta,s)\eta(s,\beta(s))(s-s_l)^{\tau-1}ds\\
&\quad+\sum_{l=1}^{a}\Lambda(c+\vartheta,s_l)\delta_l(s_l)Q_l.
\end{aligned}
$$

The above equality implies

$$\beta_c=\beta(c+\vartheta).$$

Next, we show that $\beta(\cdot+\vartheta)=\beta(\cdot)$. For $t=\widetilde{t}+N\vartheta$, Theorems 3.2, 3.3 and 3.4 imply that

$$\beta(t)=\beta(\widetilde{t}+N\vartheta)=\Lambda(\widetilde{t}+N\vartheta,c)\beta(c)=[\Lambda(\widetilde{t}+\vartheta,\widetilde{t})]^{N}\Lambda(\widetilde{t},c)\beta(c),$$

and

$$
\begin{aligned}
\beta(t+\vartheta)&=\beta(t+(N+1)\vartheta)\\
&=\Lambda(\widetilde{t}+(N+1)\vartheta,c)\beta_c\\
&=[\Lambda(\widetilde{t}+\vartheta,\widetilde{t})]^{N+1}\Lambda(\widetilde{t},c)\beta_c\\
&=[\Lambda(\widetilde{t}+\vartheta,\widetilde{t})]^{N}\Lambda(\widetilde{t}+\vartheta,c)\beta_c\\
&=[\Lambda(\widetilde{t}+\vartheta,\widetilde{t})]^{N}\beta(\widetilde{t}+\vartheta)\\
&=[\Lambda(\widetilde{t}+\vartheta,\widetilde{t})]^{N}\Lambda(\widetilde{t}+\vartheta,\vartheta)\beta(\vartheta)\\
&=[\Lambda(\widetilde{t}+\vartheta,\widetilde{t})]^{N}\Lambda(\widetilde{t},c)\beta_c,
\end{aligned}
$$

then $\beta(t+\vartheta)=\beta(t)$.

Necessity: if $\beta(t)$ is a ϑ-periodic solution of (1.3), then $P(\beta_0)=\beta_0$ and β_0 is a fixed point of P. □

Theorem 6.3. *Suppose that* (A1)–(A3), (A6), (A7) *hold. If*

$$\rho := \varrho^a \prod_{l=1}^{a} \overline{\phi}_l < 1,$$

then (1.3) *has at least one* ϑ*-periodic solution and* $\|\beta_0\| \le \omega := \frac{\widetilde{\rho}}{1-\rho}$, *where*

$$\begin{aligned}\widetilde{\rho} = \frac{\overline{\eta}}{\phi} \sum_{l=1}^{a-1} \Big\{ \varrho^{a-j+1} \prod_{j=l}^{a-1} \overline{\phi}_a \dots \overline{\phi}_{j+1}(\overline{\phi}_j - 1)\Big\} \\ + \frac{\overline{\eta}\varrho}{\phi}(\overline{\phi}_a - 1) + \Big[\sum_{l=2}^{a} \varrho^{a-j+1} \prod_{j=l}^{a} \overline{\phi}_a \dots \overline{\phi}_j + 1\Big]\kappa \overline{Q}.\end{aligned}$$

Proof. $\|\beta_0\| \le \frac{\widetilde{\rho}}{1-\rho}$ and (6.3) imply

$$\|P(\beta_0)\| \le \rho\|\beta_0\| + \widetilde{\rho} \le \frac{\widetilde{\rho}}{1-\rho}.$$

Then $P : \overline{B(0,\omega)} \to \overline{B(0,\omega)}$. Obviously, P is continuous. Next, Brouwer fixed point theorem implies there is a $\beta_0 \in \overline{B(0,\omega)}$ such that $P(\beta_0)=\beta_0$. □

Example 6.4. Consider (1.3) and let $\tau = \frac{1}{2}$, $s_0 = 0$, $s_l = l$, $t_l = l - \frac{1}{2}$, for $l = 1, 2, \dots$, $\vartheta = 1$, $a = 1$, $\beta_c = (0.1, 0)^\top$. Set

$$\begin{gathered}\alpha(t) = \begin{pmatrix} \frac{3}{2}(t - s_{\varsigma(a,t)} & 0 \\ 0 & t - s_{\varsigma(a,t)} \end{pmatrix}, \quad P_l = \begin{pmatrix} -\frac{9}{10} & 0 \\ 0 & -\frac{9}{10} \end{pmatrix}, \\ \delta_l(t) = \begin{pmatrix} \frac{1}{2} + \frac{t-t_l}{2(s_l - t_l)} & 0 \\ 0 & -1 + \frac{t-t_l}{s_l-t_l} \end{pmatrix}.\end{gathered}$$

Set $\alpha(t)$, P_l, $\delta_l(t)$ as in Example 4.6 and let

$$\eta(t,\beta) = \begin{pmatrix} (t-s_l)\cos\beta \\ 0 \end{pmatrix}, \quad t \in (s_l, t_{l+1}],\ Q_l = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Then, $\overline{\eta} = 1/2$, $\varrho = \kappa\xi = 1/10$, $\rho = \varrho\overline{\phi}_1 = \frac{e^{\frac{3\sqrt{2}}{8}}}{10} < 1$ and

$$\widetilde{\rho} = \frac{\overline{\eta}\varrho}{\phi}(\overline{\phi}_a - 1) + \kappa\overline{Q} = \frac{\sqrt{2}e^{\frac{3\sqrt{2}}{8}} - \sqrt{2} + 3}{30}.$$

Thus, (1.3) has at least one periodic solution and $0.1 = \|\beta_0\| < \frac{\widetilde{\rho}}{1-\rho} = 0.16$.

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