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A GLOBAL COMPACTNESS RESULT FOR QUASILINEAR ELLIPTIC PROBLEMS WITH CRITICAL SOBOLEV NONLINEARITIES AND HARDY POTENTIALS ON $\mathbb{R}^N$

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Abstract. In this article, we study the elliptic equation with critical Sobolev nonlinearity and Hardy potentials

$$(-\Delta)_p u + a(x)|u|^{p-1}u - \mu\frac{|u|^{p-1}u}{|x|^p} = |u|^{p^*-2}u + f(x,u), \quad u \in W^{1,p}(\mathbb{R}^N),$$

where $0 < \mu < \min\{\frac{(N-p)^p}{p^p}, \frac{N^{p-1}(N-p^2)}{p^p}\}$, $p^* = \frac{Np}{N-p}$ is the critical Sobolev exponent. Through a compactness analysis of the associated functional operator, we obtain the existence of positive solutions under certain assumptions on $a(x)$ and $f(x,u)$.

1. Introduction

For second-order semilinear elliptic differential equations on bounded domains, Brezis and Nirenberg [3] obtained an existence result of solutions for a class of elliptic equations with critical Sobolev nonlinearities. by verifying a sub-level which make the Palais-Smale conditions hold. A global compactness result for a semilinear elliptic problem with critical Sobolev nonlinearities on the bounded domains was obtained by Lions [19] and Struwe [27]. It was known that the sub-level which makes the Palais-Smale conditions hold is determined by a compact result (refer to [19, 27]). Alves [2] and Yan [29] generalized the result of Struwe [27] to the case of p-Laplacian with critical Sobolev terms. Alves [2] also obtained the global compactness result for the p-Laplace equation involving critical Sobolev terms on the whole space. As for the case, the global compactness results for the p-Laplacian with critical Sobolev terms were obtained by Saintier [22] on a smooth Riemannian manifold without boundary, and by Mercuri and Willem [20] on a smooth bounded domain respectively. For the semilinear elliptic equation with Hardy potentials and critical Sobolev terms, Cao and Peng [4] established global compactness results on bounded domains, also demonstrating some new blow-up phenomena. On the whole space, the global compactness result for the semilinear elliptic problem involving Hardy potentials, and critical Sobolev terms was discussed in [7, 14, 25]. It is worth noting that the equation discussed in [25] does not include sub-critical terms, whereas the

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equations discussed in [7, 14] include sub-critical terms caused new phenomena. As for the p-Laplace equation with Hardy potentials and critical Sobolev terms on bounded domains, the corresponding global compactness were proved in [13] and [17]. Over the past two decades, the loss of compactness has led to numerous interesting phenomena related to the existence and nonexistence of solutions for elliptic equations (see, for example, [1, 2, 3, 4, 5, 13, 6, 7, 10, 12, 22, 23, 24, 25, 26] and the references therein).

Motivated by [1, 7, 17, 20], we consider the nonlinear elliptic equation

$$\begin{gathered}(-\Delta)_p u + a(x)|u|^{p-1}u - \mu\frac{|u|^{p-1}u}{|x|^p} = |u|^{p^*-2}u + f(x,u),\\ u \in W^{1,p}(\mathbb{R}^N),\end{gathered} \tag{1.1}$$

where $0 < \mu < \min\{\frac{(N-p)^p}{p^p}, \frac{N^{p-1}(N-p^2)}{p^p}\}$, $p^* = \frac{Np}{N-p}$ is the critical Sobolev exponent.

The main feature for this type of problems is the presence of the singular potential $\frac{1}{|x|^p}$ related to the Sobolev-Hardy's inequality. We recall the Sobolev-Hardy's inequality,

$$\int_{\mathbb{R}^N} \frac{|u(x)|^p}{|x|^p}\,dx \le c\int_{\mathbb{R}^N} |\nabla u(x)|^p\,dx, \quad \forall u \in D^{1,p}(\mathbb{R}^N) \tag{1.2}$$

where c is a positive constant. The Sobolev embedding $D^{1,p}(\mathbb{R}^N) \hookrightarrow L^p(|x|^{-p}, \mathbb{R}^N)$ is not compact, even locally, in any neighborhood of zero. In addition to the inverse square potential, another motivation for our investigation of problem (1.1) is the presence of the critical Sobolev exponent and the unbounded domain, which result in the loss of compactness of embeddings $W^{1,p}(\mathbb{R}^N) \hookrightarrow L^p(\mathbb{R}^N)$ and $D^{1,p}(\mathbb{R}^N) \hookrightarrow L^{p^*}(\mathbb{R}^N)$. Therefore, considering the noncompactness of embedding, we encounter a triple loss of compactness, and their interaction introduces new challenges. To address the challenges arising from the lack of compactness, we conduct a noncompactness analysis, which allows us to distinctly identify and express all the elements responsible for non-compactness. To delve into more detail, in the context of Palais-Smale sequences associated with the variational functional corresponding to problem (1.1), we initially construct a comprehensive non-compact representation encompassing all instances of singular behavior resulting from the critical Sobolev-Hardy nonlinearity and the unbounded nature of the domain. Therefore, it can determine the energy level intervals corresponding to the Palais-Smale sequence. By leveraging the energy level intervals, we can more easily ascertain the existence of both minimal energy solutions and high-energy solutions. In this paper we only deduce the existence of minimal energy positive solutions for problem (1.1). Our methods are based on techniques from [7, 14, 18, 21, 25, 27, 29].

This article is structured as follows. In Section 2, we present the main results of the paper. In Section 3, we establish Theorem 2.1 through a meticulous analysis of the characteristics of a positive Palais-Smale sequence for I. Section 4 is dedicated to the proof of Theorem 2.3, achieved by employing both Theorem 2.1 and the Mountain Pass Theorem. Finally, in the last section, we provide some preliminary information as an appendix.

2. Main results

In this Section, we present the main results of this paper. For convenience, first we provide some important notation and assumptions.

Let $D^{1,p}(\mathbb{R}^N)$ be the homogeneous Sobolev space as the completion of $C_0^\infty(\mathbb{R}^N)$ under the norm

$$\|u\|_{D^{1,p}(\mathbb{R}^N)} = \|\nabla u\|_{L^p(\mathbb{R}^N)}, \tag{2.1}$$

and denote by $W^{1,p}(\mathbb{R}^N)$ the usual nonhomogeneous Sobolev space with the norm

$$\|u\|_{W^{1,p}(\mathbb{R}^N)} = \|\nabla u\|_{L^p(\mathbb{R}^N)} + \|u\|_{L^p(\mathbb{R}^N)}. \tag{2.2}$$

Let $u^+ = \max\{u, 0\}, u^- = u^+ - u$. Denote c and C as arbitrary constants which may change from line to line. Let $B(x, r)$ denote a ball centered at x with radius r and $B(x, r)^C = \mathbb{R}^N \setminus B(x, r)$.

A measurable function $u : \mathbb{R}^N \to \mathbb{R}$ belongs to the Morrey space with $p \in [1, \infty)$ and $\nu \in (0, N]$, if

$$\|u\|^p_{L^{p,\nu}(\mathbb{R}^N)} = \sup_{r>0, \bar{x}\in\mathbb{R}^N} r^{\nu-N} \int_{B(\bar{x},r)} |u(x)|^p \, dx < \infty.$$

By Hölder inequality, we can verify that

$$L^{r,r\frac{N-p}{p}}(\mathbb{R}^N) \hookrightarrow L^{p^*}(\mathbb{R}^N), \quad \text{for } 1 \le r < p^*, 1 < p < N. \tag{2.3}$$

Let X be a Banach space, $\Phi \in C^1(X, \mathbb{R})$, $c \in \mathbb{R}$, we call $\{u_n\} \subset X$ is a Palais-Smale sequence of Φ if

$$\Phi(u_n) \to c, \quad \Phi'(u_n) \to 0 \quad \text{as } n \to \infty. \tag{2.4}$$

Next we establish specific assumptions regarding the functions $a(x), f(x, u)$.

(A1) $a(x) \in C(\mathbb{R}^N)$, $\lim_{x\to\infty} a(x) = \bar{a} > 0$ and there exists a constant $\lambda_1 > 0$ such that

$$\int_{\mathbb{R}^N} \Big[\Big(1 - (\frac{p}{N-p})^p \mu\Big)|\nabla u|^p + a(x)|u|^p\Big] \, dx \geqslant \lambda_1 \int_{\mathbb{R}^N} \big(\bar{a} - a(x)\big)|u|^p \, dx, \tag{2.5}$$

for all $u \in W^{1,p}(\mathbb{R}^N)$. (Without loss of generality, we assume that $\bar{a} = 1$.)

(A2) $f(x, t)$ is differentiable with respect to $t \in [0, +\infty)$ for all $x \in \mathbb{R}^N$ and continuous with respect to $x \in \mathbb{R}^N$ for all $t \in [0, +\infty)$. Moreover, we extend $f(x, t) \equiv 0$ for all $t \in (-\infty, 0)$, $x \in \mathbb{R}^N$.

(A3) There exists a constant $q \in (p, \frac{Np}{N-p})$ such that $\lim_{t\to+\infty} \frac{f(x,t)}{t^{q-1}} = 0$ and $\lim_{t\to 0^+} \frac{f(x,t)}{t^{p-1}} = 0$ uniformly in $x \in \mathbb{R}^N$.

(A4) There exists a constant $\theta \in (0, p^* - p)$ such that $t\frac{\partial}{\partial t} f(x, t) \geqslant (p - 1 + \theta) f(x, t) > 0$, for all $x \in \mathbb{R}^N, t > 0$.

(A5) $\lim_{|x|\to+\infty} f(x, t) = \bar{f}(t)$ uniformly on any compact subset of $[0, \infty)$ and there exists a constant $\sigma > p(\frac{1}{p-1})^{\frac{1}{p}}$ such that for any $\varepsilon > 0$ we can find $C_\varepsilon > 0$ satisfying

$$f(x, t) - \bar{f}(t) \geqslant -e^{-\sigma|x|}(\varepsilon t^{p-1} + C_\varepsilon t^{q-1}) \quad \text{for all } x \in \mathbb{R}^N, \ t \geqslant 0,$$

where $q \in (p, \frac{Np}{N-p})$ is given by (A3).

As in [8], assumption (A1) implies that

$$\Big(\int_{\mathbb{R}^N}(|\nabla u|^p+a(x)u^p-\mu\frac{u^p}{|x|^p})\,dx\Big)^{1/p}$$

is an equivalent norm of $W^{1,p}(\mathbb{R}^N)$. Also in Lemma 5.8, we give the proof of (2.5) if $a(x)$ satisfies some specific conditions.

As an example of a function that satisfies (A2)–(A5), we have

$$f(x,t)=\begin{cases}(1-e^{-\sigma|x|})t^q,\ (p-1<q<p^*-1), & \text{for } t\geqslant 0, x\in\mathbb{R}^N,\\ 0, & \text{for } t<0, x\in\mathbb{R}^N,\end{cases}$$

In the following, we assume that $a(x), f(x,u)$ satisfy (A1)–(A5).

The energy functional associated with problem (1.1) is

$$\begin{aligned} I(u)&=\frac{1}{p}\int_{\mathbb{R}^N}\Big(|\nabla u|^p+a(x)|u|^p-\mu\frac{|u|^p}{|x|^p}\Big)\,dx\\ &\quad-\frac{1}{p^*}\int_{\mathbb{R}^N}\big(u^+\big)^{p^*}\,dx-\int_{\mathbb{R}^N}F(x,u)dx,\qquad \forall u\in W^{1,p}(\mathbb{R}^N), \end{aligned} \tag{2.6}$$

with

$$F(x,u)=\int_0^u f(x,t)\,dt.$$

Next, we present some problems associated with problem (1.1). The limit equation of (1.1) involving sub-critical terms is

$$(-\Delta)_p u+\bar{a}|u|^{p-1}u=\bar{f}(u)+|u|^{p^*-2}u,\quad u\in W^{1,p}(\mathbb{R}^N), \tag{2.7}$$

and its corresponding variational functional is

$$I^\infty(u)=\frac{1}{p}\int_{\mathbb{R}^N}\Big(|\nabla u|^p+\bar{a}|u|^p\Big)\,dx-\frac{1}{p^*}\int_{\mathbb{R}^N}\big(u^+\big)^{p^*}\,dx-\int_{\mathbb{R}^N}\bar{F}(u)\,dx,$$

for all $u\in W^{1,p}(\mathbb{R}^N)$, where $\bar{F}(u)=\int_0^u\bar{f}(t)\,dt$.

The limit equation of (1.1) involving the Sobolev critical nonlinear term is

$$(-\Delta)_p u=|u|^{p^*-2}u,\quad u\in D^{1,p}(\mathbb{R}^N), \tag{2.8}$$

and the corresponding variational functional is

$$I_0(u)=\frac{1}{p}\int_{\mathbb{R}^N}|\nabla u|^p\,dx-\frac{1}{p^*}\int_{\mathbb{R}^N}\big(u^+\big)^{p^*}\,dx,\quad \forall u\in D^{1,p}(\mathbb{R}^N).$$

The limit equation of (1.1) involving the Sobolev critical term and the Hardy term is

$$(-\Delta)_p u-\mu\frac{|u|^{p-1}u}{|x|^p}=|u|^{p^*-2}u,\quad u\in D^{1,p}(\mathbb{R}^N), \tag{2.9}$$

and its corresponding variational functional is

$$I_\mu(u)=\frac{1}{p}\int_{\mathbb{R}^N}\Big(|\nabla u|^p-\mu\frac{|u|^p}{|x|^p}\Big)\,dx-\frac{1}{p^*}\int_{\mathbb{R}^N}\big(u^+\big)^{p^*}\,dx,\quad \forall u\in D^{1,p}(\mathbb{R}^N).$$

Abdellaoui, Felli, and Peral [1] proved that all the positive solutions of problem (2.9) take the form

$$U_\mu^\varepsilon(x):=\varepsilon^{\frac{p-N}{p}}U_\mu(x/\varepsilon). \tag{2.10}$$

Additionally, $U_\mu(r)$ and $U'_\mu(r)$ have the following asymptotic properties

$$\begin{aligned}\lim_{r\to 0} r^{a(\mu)}U_\mu(r) &= c_1 > 0,\\ \lim_{r\to \infty} r^{b(\mu)}U_\mu(r) &= c_2 > 0,\\ \lim_{r\to 0} r^{a(\mu)+1}U'_\mu(r) &= c_1 a(\mu) > 0,\\ \lim_{r\to \infty} r^{b(\mu)+1}U'_\mu(r) &= c_2 b(\mu) > 0.\end{aligned} \tag{2.11}$$

Here, c_1 and c_2 are positive constants depending only on N and p, while $a(\mu)$ and $b(\mu)$ are the zeros of the function

$$g(t) = (p-1)t^p - (N-p)t^{p-1} + \mu, \quad t \geqslant 0 \quad (0 < \mu < \Lambda_{N,p} =: \big(\frac{N-p}{p}\big)^p),$$

and satisfy $0 < a(\mu) < b(\mu)$.

We need further information on $a(\mu), b(\mu)$, the two roots of $g(t) = 0$. After a direct calculation, we infer that $t_{\min} = \frac{N-p}{p}$ is the only minimal point of $g(t), t \geqslant 0$, and $g(\frac{N-p}{p}) = -\Lambda_{N,p} + \mu < 0$ for $0 < \mu < \Lambda_{N,p}$. Moreover, $g'(t) < 0$ for $0 < t < t_{\min}, g'(t) > 0$ for $t > t_{\min}$. That is, $g(t)$ is decreasing on the interval $(0, t_{\min})$ and increasing on the interval $(t_{\min}, \infty)$. Thus,

$$a(\mu) < \frac{N-p}{p} < b(\mu) \quad \text{for } 0 < \mu < \Lambda_{N,p}.$$

Furthermore, we obtain that

$$\begin{aligned}\frac{N}{p} < b(\mu) &\Longleftrightarrow -\frac{N^{p-1}(N-p^2)}{p^p} + \mu = g\big(\frac{N}{p}\big) < g\big(b(\mu)\big) = 0\\ &\Longleftrightarrow 0 < \mu < \frac{N^{p-1}(N-p^2)}{p^p} \quad (N > p^2).\end{aligned}$$

Moreover, $U_\mu^\varepsilon(x)$ are also minimizers for the quotient

$$S_\mu = \inf_{u\in D^{1,p}(\mathbb{R}^N)\setminus\{0\}} \frac{\int_{\mathbb{R}^N}\big(|\nabla u|^p - \mu\frac{|u|^p}{|x|^p}\big)\,dx}{\big(\int_{\mathbb{R}^N}|u|^{p^*}\,dx\big)^{p/p^*}}. \tag{2.12}$$

For the case that $\mu = 0$,

$$U_0 = \frac{1}{(1+|x|^{\frac{p}{p-1}})^{\frac{N-p}{p}}}. \tag{2.13}$$

We can define

$$J^\infty = \inf_{u\in\mathcal{N}} I^\infty(u), \tag{2.14}$$

with

$$\mathcal{N} = \big\{u \in W^{1,p}(\mathbb{R}^N)\setminus\{0\} : \int_{\mathbb{R}^N}\big(|\nabla u|^p + \bar{a}|u|^p - (u^+)^{p^*} - \bar{F}(u)\big)\,dx = 0\big\}. \tag{2.15}$$

It is well known that $\mathcal{N} \neq \emptyset$ since problem (2.7) has at least one positive solution if $N > p^2$ (see [15]). Moreover, the authors in [15] proved that J^∞ can be achieved by a function $w(x) \in \mathcal{N}$ satisfies following properties

$$c_1(1+|x|)^{-\frac{N-1}{p(p-1)}}e^{-(\frac{\bar{a}}{p-1})^{1/p}|x|} \le w(x) \le c_2(1+|x|)^{-\frac{N-1}{p(p-1)}}e^{-(\frac{\bar{a}}{p-1})^{1/p}|x|}. \tag{2.16}$$

For convenience, we define the quantities

$$D_0 = \int_{\mathbb{R}^N} \Big(\frac{1}{p}|\nabla U_0|^p - \frac{1}{p^*}|U_0|^{p^*}\Big)\,dx = \frac{1}{N}S_0^{N/p}, \tag{2.17}$$

$$D_\mu = \int_{\mathbb{R}^N} \Big[\frac{1}{p}\Big(|\nabla U_\mu|^p - \mu\frac{|U_\mu|^p}{|x|^p}\Big) - \frac{1}{p^*}|U_\mu|^{p^*}\Big]dx = \frac{1}{N}S_\mu^{N/p}. \tag{2.18}$$

The main result of our paper reads as follows.

Theorem 2.1. *Suppose $a(x)$, $f(x,u)$ satisfy* (A1)–(A5)*, $N > p^2$, and*

$$0 < \mu < \min\Big\{\frac{(N-p)^p}{p^p}, \frac{N^{p-1}(N-p^2)}{p^p}\Big\}.$$

Also assume that $\{u_n\}$ is a positive Palais-Smale sequence of I at level $d \ge 0$. Then there exist sequences $\{y_n^k\} \subset \mathbb{R}^N (1 \le k \le l_1)$, $\{\bar{R}_n^i\} \subset \mathbb{R}^+ (1 \le i \le l_2)$, $\{R_n^j\} \subset \mathbb{R}^+, \{x_n^j\} \subset \mathbb{R}^N$ $(1 \le j \le l_3)$ and $u_k \in W^{1,p}(\mathbb{R}^N)(1 \le k \le l_1), 0 \le u \in W^{1,p}(\mathbb{R}^N)$ $(l_1, l_2, l_3 \in \mathbb{N}^+)$ such that up to a subsequence:

$$d = I(u) + \sum_{k=1}^{l_1} I^\infty(u_k) + l_2 D_\mu + l_3 D_0 + o(1)$$

and

$$\Big\|u_n - u - \sum_{k=1}^{l_1} u_k(x - y_n^k) - \sum_{i=1}^{l_2} U^{\bar{R}_n^i} - \sum_{j=1}^{l_3} U_0^{R_n^j, x_n^j}\Big\|_{W^{1,p}(\mathbb{R}^N)} = o(1) \tag{2.19}$$

as $n \to \infty$, where u and $u_k(1 \le k \le l_1)$ satisfy

$$I'(u) = 0, \quad {I^\infty}'(u_k) = 0,$$
$$\bar{R}_n^i \to 0, \quad R_n^j \to 0, \quad \frac{|x_n^j|}{R_n^j} \to \infty.$$

In particular, if $u \not\equiv 0$, then u is a weakly solution of (1.1)*. Note that the corresponding sum in* (2.19) *will be treated as zero if $l_i = 0\,(i = 1, 2, 3)$.*

Remark 2.2. (1) Similar to [25, Corollary 3.3], one can demonstrate that any Palais-Smale sequence for I at a level that does not have the form $m_1 D_\mu + m_2 J^\infty + m_3 D_0$, $m_1, m_2, m_3 \in \mathbb{N}\bigcup\{0\}$, gives rise to a non-trivial weak solution of (1.1).

(2) To account for the lower-order terms in problem (1.1), it becomes necessary to impose the condition that $u \in W^{1,p}(\mathbb{R}^N)$ in order to ensure the well-defined nature of the functional $I(u)$. Specifically, when $u \in W^{1,p}(\mathbb{R}^N)$, the Sobolev inequality implies that $u \in L^q(\mathbb{R}^N)$ for $p \le q < p^*$. It is worth highlighting that the quantities $\|u\|_{L^p(\mathbb{R}^N)}$ and $\|u\|_{L^q(\mathbb{R}^N)}$ are influenced solely by translation invariance, while the integral $\int_{\mathbb{R}^N} \frac{|u|^p}{|x|^p}\,dx$ is affected by scaling invariance. Consequently, these considerations give rise to three limiting equations introducing intriguing new structures.

Using the compactness results and the Mountain Pass Theorem [3] we prove the following existence result.

Theorem 2.3. *Assume that $p < q < p^*$, $0 < \mu < \min\{\frac{(N-p)^p}{p^p}, \frac{N^{p-1}(N-p^2)}{p^p}\}$ and $N > p^2$. If $a(x), f(x,u)$ satisfy* (A1)–(A5)*, then problem* (1.1) *has a nontrivial* *solution $u \in W^{1,p}(\mathbb{R}^N)$ which satisfies*

$$I(u) < \min\Big\{\frac{1}{N} S_\mu^{N/p}, J^\infty\Big\}.$$

3. Non-compactness analysis

In this section, we prove Theorem 2.1 by using the Concentration-Compactness Principle and a delicate analysis of the Palais-Smale sequences of I. Firstly, we give the following Lemmas.

Lemma 3.1. *Let $\{u_n\} \subset D^{1,p}(\mathbb{R}^N)$ be a bounded sequence such that*

$$\inf_{n\in\mathbb{N}^+} \int_{\mathbb{R}^N} |u_n|^{p^*}\,dx \geq c > 0. \tag{3.1}$$

Then, up to subsequence, there exist two sequences $\{r_n\} \subset \mathbb{R}^+$ and $\{x_n\} \subset \mathbb{R}^N$ such that

$$\bar{u}_n \rightharpoonup \bar{u}_0 \not\equiv 0 \quad \text{in } D^{1,p}(\mathbb{R}^N), \tag{3.2}$$

where

$$\bar{u}_n = \begin{cases} r_n^{\frac{N-p}{p}} u_n(r_n x) & \text{if } \frac{|x_n|}{r_n} \text{ is bounded},\\ r_n^{\frac{N-p}{p}} u_n(r_n x + x_n) & \text{if } \frac{|x_n|}{r_n} \to \infty. \end{cases} \tag{3.3}$$

Proof. By [21, Theorem 2], we have

$$\|u_n\|_{L^{p^*}(\mathbb{R}^N)} \leq C\|u_n\|^\theta_{D^{1,p}(\mathbb{R}^N)} \|u_n\|^{1-\theta}_{L^{p,N-p}(\mathbb{R}^N)}, \tag{3.4}$$

where $\frac{p}{p^*} \leq \theta < 1$.

Then there exists a constant $c > 0$ such that

$$\|u_n\|^p_{L^{p,N-p}(\mathbb{R}^N)} = \sup_{\bar{x}\in\mathbb{R}^N,\, R\in\mathbb{R}^+} R^{-p} \int_{B(\bar{x},R)} |u_n|^p\,dx \geq c > 0. \tag{3.5}$$

From (3.5), we may find $r_n > 0$ and $x_n \in \mathbb{R}^N$ such that for n large enough,

$$r_n^{-p} \int_{B(x_n,r_n)} |u_n|^p\,dx \geq \|u_n\|^p_{L^{p,N-p}(\mathbb{R}^N)} - \frac{c}{2n} \geq \frac{c}{2} > 0. \tag{3.6}$$

We define

$$\bar{u}_n = \begin{cases} r_n^{\frac{N-p}{p}} u_n(r_n x) & \text{when } \frac{|x_n|}{r_n} \text{ is bounded},\\ r_n^{\frac{N-p}{p}} u_n(r_n x + x_n) & \text{when } \frac{|x_n|}{r_n} \to \infty. \end{cases} \tag{3.7}$$

Since $\{u_n\}$ is bounded in $D^{1,p}(\mathbb{R}^N)$, from the scaling and translation invariance of $D^{1,p}(\mathbb{R}^N)$, it follows that $\{\bar{u}_n\}$ is also bounded in $D^{1,p}(\mathbb{R}^N)$, therefore, up to a subsequence (still denoted by $\bar{u}_n$),

$$\bar{u}_n \rightharpoonup \bar{u}_0 \text{ in } D^{1,p}(\mathbb{R}^N) \quad \text{and} \quad \bar{u}_n \to \bar{u}_0 \text{ in } L^p_{\rm loc}(\mathbb{R}^N), \text{ as } n \to \infty.$$

If $|x_n|/r_n$ is bounded, there exists a constant $R > 1$ such that $B(\frac{x_n}{r_n}, 1) \subset B(0,R)$, then

$$\frac{c}{2} < \int_{B(\frac{x_n}{r_n},1)} |\bar{u}_n|^p\,dx \leq \int_{B(0,R)} |\bar{u}_n|^p\,dx \to \int_{B(0,R)} |\bar{u}_0(x)|^p\,dx. \tag{3.8}$$

If $|x_n|/r_n \to \infty$, then

$$\frac{c}{2} < \int_{B(0,1)} |\bar{u}_n|^p\,dx \leq \int_{B(0,R)} |\bar{u}_n|^p\,dx \to \int_{B(0,R)} |\bar{u}_0(x)|^p\,dx, \tag{3.9}$$

where $R > 1$. Obviously we have $\bar{u}_0 \not\equiv 0$. From (3.8) and (3.9), the proof is complete. $\square$

Lemma 3.2. *Let $\{v_n\} \subset W^{1,p}(\mathbb{R}^N)$ be a Palais-Smale sequence of I at level d and $v_n \rightharpoonup 0$ in $W^{1,p}(\mathbb{R}^N)$, $\|v_n\|_{L^q(\mathbb{R}^N)} \to 0$ for all $1 < q < p^*$, as $n \to \infty$. If there exist sequences $\{r_n\} \subset \mathbb{R}^+$, $\{x_n\} \subset \mathbb{R}^N$ with $r_n \to 0$, $|x_n|/r_n \to \infty$ as $n \to \infty$ such that $\bar{v}_n(x) := r_n^{\frac{N-p}{p}} v_n(r_n x + x_n)$ converges weakly in $D^{1,p}(\mathbb{R}^N)$ and almost everywhere to some $0 \neq v_0 \in D^{1,p}(\mathbb{R}^N)$ as $n \to \infty$, then v_0 solves problem (2.8) and the sequence $z_n := v_n - r_n^{\frac{p-N}{p}} v_0(\frac{x-x_n}{r_n})$ is a Palais-Smale sequence of I at level $d - I_0(v_0)$.*

Proof. First, we prove that v_0 solves problem (2.8). Fix a ball $B(0,r)$ and a test function $\phi \in C_0^\infty(B(0,r))$. Since

$$v_n \rightharpoonup 0,\ \bar{v}_n \rightharpoonup v_0 \text{ in } D^{1,p}(\mathbb{R}^N), \quad \|v_n\|_{L^q(\mathbb{R}^N)} \to 0, \text{ and } \frac{|x_n|}{r_n} \to \infty,$$

it follows that

$$\int_{\mathbb{R}^N} a(x)|v_n|^{p-2} v_n \phi_n \, dx = o(1), \quad \int_{\mathbb{R}^N} f(x, v_n) v_n \phi_n \, dx = o(1),$$
$$\mu \int_{\mathbb{R}^N} \frac{|\bar{v}_n|^{p-2} \bar{v}_n \phi}{|x + \frac{x_n}{r_n}|^p} \, dx = o(1),$$

where $\phi_n = r_n^{\frac{p-N}{p}} \phi(\frac{x-x_n}{r_n})$. It implies

$$\begin{aligned}
&\langle I_0'(v_0), \phi \rangle \\
&= \int_{\mathbb{R}^N} |\nabla v_0|^{p-2} \nabla v_0 \nabla \phi \, dx - \int_{\mathbb{R}^N} \big(v_0^+\big)^{p^*-1} \phi \, dx \\
&= \int_{\mathbb{R}^N} |\nabla \bar{v}_n|^{p-2} \nabla \bar{v}_n \nabla \phi \, dx - \mu \int_{\mathbb{R}^N} \frac{|\bar{v}_n|^{p-2} \bar{v}_n \phi}{|x + \frac{x_n}{r_n}|^p} \, dx - \int_{\mathbb{R}^N} \big(\bar{v}_n^+\big)^{p^*-1} \phi \, dx + o(1) \\
&= \int_{\mathbb{R}^N} |\nabla v_n|^{p-2} \nabla v_n \nabla \phi_n \, dx - \mu \int_{\mathbb{R}^N} \frac{|v_n|^{p-2} v_n \phi_n}{|x|^p} \, dx - \int_{\mathbb{R}^N} \big(v_n^+\big)^{p^*-1} \phi_n \, dx \\
&\quad + \int_{\mathbb{R}^N} a(x) \phi_n |v_n|^{p-2} v_n \, dx - \int_{\mathbb{R}^N} f(x, v_n) v_n \phi_n \, dx + o(1) = o(1)
\end{aligned} \tag{3.10}$$

as $n \to \infty$. The last equality in (3.10) holds since

$$\int_{\mathbb{R}^N} |\phi_n|^p \, dx = r_n^p \int_{\mathbb{R}^N} |\phi|^p \, dx = o(1),$$

and

$$\|\phi\|_{D^{1,p}(\mathbb{R}^N)} = \|\phi_n\|_{W^{1,p}(\mathbb{R}^N)} + o(1) \quad \text{as } n \to \infty.$$

Thus v_0 solves problem (2.8). From Lemma 5.6, (2.13) and $N > p^2$, it follows that

$$\int_{\mathbb{R}^N} |v_0(x)|^q \, dx \le c \int_{\mathbb{R}^N} \frac{1}{(1 + |x|^{\frac{p}{p-1}})^{\frac{q}{p}(N-p)}} \, dx \le c, \quad \forall q \ge p, \tag{3.11}$$

which implies that $v_0 \in L^p(\mathbb{R}^N)$.

Let

$$z_n(x) = v_n(x) - r_n^{\frac{p-N}{p}} v_0\Big(\frac{x - x_n}{r_n}\Big) \in W^{1,p}(\mathbb{R}^N).$$

Obviously $z_n \rightharpoonup 0$ in $W^{1,p}(\mathbb{R}^N)$ as $n \to \infty$. Now we prove that $\{z_n\}$ is a Palais-Smale sequence of I at level $d - I_0(v_0)$. From (3.11) it follows

$$\int_{\mathbb{R}^N} \big|r_n^{\frac{p-N}{p}} v_0\Big(\frac{x-x_n}{r_n}\Big)\big|^p \,dx = r_n^p \|v_0\|^p_{L^p(\mathbb{R}^N)} \to 0, \quad \text{as } n \to \infty, \tag{3.12}$$

by Brézis-Lieb Lemma and the weak convergence, similar to Lemma 5.7, we can prove that

$$I(z_n) = I(v_n) - I_0(v_0),$$

and $\langle I'(z_n), \phi\rangle = o(1)$ as $n \to \infty$. This completes the proof. □

Lemma 3.3. *Assume $0 < \mu < \min\{\frac{(N-p)^p}{p^p}, \frac{N^{p-1}(N-p^2)}{p^p}\}$. Let $\{v_n\} \subset W^{1,p}(\mathbb{R}^N)$ be a Palais-Smale sequence of I at level d and $v_n \rightharpoonup 0$ in $W^{1,p}(\mathbb{R}^N)$, $\|v_n\|_{L^q(\mathbb{R}^N)} \to 0$ for all $1 < q < p^*$, as $n \to \infty$. If there exists a sequence $\{r_n\} \subset \mathbb{R}^+$, with $r_n \to 0$ as $n \to \infty$ such that $\bar v_n(x) := r_n^{\frac{N-p}{p}} v_n(r_n x)$ converges weakly in $D^{1,p}(\mathbb{R}^N)$ and almost everywhere to some $0 \neq v_0 \in D^{1,p}(\mathbb{R}^N)$ as $n \to \infty$, then v_0 solves problem (2.9) and the sequence $z_n := v_n - r_n^{\frac{p-N}{p}} v_0(\frac{x}{r_n})$ is a Palais-Smale sequence of I at level $d - I_\mu(v_0)$.*

Proof. First, we prove that v_0 solves problem (2.9). Fix a ball $B(0,r)$ and a test function $\phi \in C_0^\infty(B(0,r))$. Since

$$v_n \rightharpoonup 0,\ \bar v_n \rightharpoonup v_0 \text{ in } D^{1,p}(\mathbb{R}^N), \quad \|v_n\|_{L^q(\mathbb{R}^N)} \to 0,$$

it follows that

$$\int_{\mathbb{R}^N} a(x)|v_n|^{p-2} v_n \phi_n \,dx = o(1), \quad \int_{\mathbb{R}^N} f(x,v_n) v_n \phi_n \,dx = o(1).$$

So, we obtain that

$$\begin{aligned}
&\langle I'_\mu(v_0), \phi\rangle \\
&= \int_{\mathbb{R}^N} |\nabla v_0|^{p-2}\nabla v_0 \nabla\phi \,dx - \mu\int_{\mathbb{R}^N} \frac{|v_0|^{p-2}v_0\phi}{|x|^p}\,dx - \int_{\mathbb{R}^N} \big(v_0^+\big)^{p^*-1}\phi\,dx \\
&= \int_{\mathbb{R}^N} |\nabla \bar v_n|^{p-2}\nabla \bar v_n \nabla\phi \,dx - \mu\int_{\mathbb{R}^N} \frac{|\bar v_n|^{p-2}\bar v_n\phi}{|x|^p}\,dx - \int_{\mathbb{R}^N} \big(\bar v_n^+\big)^{p^*-1}\phi\,dx + o(1) \\
&= \int_{\mathbb{R}^N} |\nabla v_n|^{p-2}\nabla v_n \nabla\phi_n \,dx - \mu\int_{\mathbb{R}^N} \frac{|v_n|^{p-2}v_n\phi_n}{|x|^p}\,dx - \int_{\mathbb{R}^N} \big(v_n^+\big)^{p^*-1}\phi_n\,dx \\
&\quad + \int_{\mathbb{R}^N} a(x)|v_n|^{p-2}v_n\phi_n\,dx - \int_{\mathbb{R}^N} f(x,v_n)v_n\phi_n\,dx + o(1) \\
&= o(1) \quad \text{as } n \to \infty,
\end{aligned} \tag{3.13}$$

where $\phi_n = r_n^{\frac{p-N}{p}}\phi(\frac{x}{r_n})$. The last equality in (3.13) holds since

$$\int_{\mathbb{R}^N} |\phi_n|^p \,dx = r_n^p \int_{\mathbb{R}^N} |\phi|^p\,dx = o(1),$$
$$\|\phi\|_{D^{1,p}(\mathbb{R}^N)} = \|\phi_n\|_{W^{1,p}(\mathbb{R}^N)} + o(1) \quad \text{as } n \to \infty.$$

Thus v_0 solves (2.9). From (2.11) and $\mu < \frac{N^{p-1}(N-p^2)}{p^p}$, it follows that

$$\int_{\mathbb{R}^N} |v_0(x)|^p\,dx \le c\int_{|x|\le 1} \frac{1}{|x|^{a(\mu)p}}\,dx + c\int_{|x|\ge 1}\frac{1}{|x|^{b(\mu)p}}\,dx \le c, \tag{3.14}$$

which implies that $v_0 \in L^p(\mathbb{R}^N)$.

Let

$$z_n(x) = v_n(x) - r_n^{\frac{p-N}{p}} v_0\Big(\frac{x}{r_n}\Big) \in W^{1,p}(\mathbb{R}^N).$$

Obviously $z_n \rightharpoonup 0$ in $W^{1,p}(\mathbb{R}^N)$ as $n \to \infty$. Now, we prove that $\{z_n\}$ is a Palais-Smale sequence of I at level $d - I_\mu(v_0)$. From (3.14) it follows that

$$\int_{\mathbb{R}^N} \Big|r_n^{\frac{p-N}{p}} v_0\Big(\frac{x}{r_n}\Big)\Big|^p \,dx = r_n^p \|v_0\|^p_{L^p(\mathbb{R}^N)} \to 0, \quad \text{as } n \to \infty. \tag{3.15}$$

By the Brézis-Lieb Lemma and the weak convergence, as in Lemma 5.7, we can prove that

$$I(z_n) = I(v_n) - I_\mu(v_0),$$
$$\langle I'(z_n), \phi\rangle = o(1)$$

as $n \to \infty$. This completes the proof. □

Lemma 3.4. *Let ν be a unit vector of $\mathbb{R}^N$ and w be that in* (2.16)*. There exist some constants $C_1 > 0$ and $C_2 > 0$ independent of $R \geqslant 1$ such that: (1)*

$$\int_{|x|\leqslant 1} (w(x - R\nu))^p \,dx \geqslant C_1 R^{-\frac{(N-1)}{p-1}} e^{-p(\frac{1}{p-1})^{\frac{1}{p}} R}, \quad \textit{for } R \geqslant 1,$$

and (2)

$$\int_{\mathbb{R}^N} e^{-\sigma|x|} (w(x - R\nu))^q \,dx \leqslant C_2 R^{-\frac{q(N-1)}{p(p-1)}} e^{-\min\{\sigma, q(\frac{1}{p-1})^{\frac{1}{p}} R\}}, \quad \textit{for } R \geqslant 1.$$

The above lemma can be proved by the similar arguments as that of [5, Lemma 3.6]. We omit its proof.

Proof of Theorem 2.1. By Lemma 5.4 in the appendix, we can assume that $\{u_n\}$ is bounded in $W^{1,p}(\mathbb{R}^N)$. Up to a subsequence, as $n \to \infty$, we assume that

$$u_n \rightharpoonup u \quad \text{in } W^{1,p}(\mathbb{R}^N),$$
$$u_n \to u \quad \text{in } L^q_{\rm loc}(\mathbb{R}^N) \quad \text{for } 1 < q < p^*,$$
$$u_n \to u \quad \text{a.e. in } \mathbb{R}^N.$$

We denote $v_n(x) = u_n(x) - u(x)$, then $\{v_n\}$ is a Palais-Smale sequence of I and

$$v_n \rightharpoonup 0 \quad \text{in } W^{1,p}(\mathbb{R}^N), \tag{3.16}$$

$$v_n \to 0 \quad \text{in } L^q_{\rm loc}(\mathbb{R}^N) \text{ for } 1 < q < p^*, \tag{3.17}$$

$$v_n \to 0 \quad \text{a.e. in } \mathbb{R}^N. \tag{3.18}$$

Then by Lemma 5.7, we know that

$$I(v_n) = I(u_n) - I(u) + o(1), \quad \text{as } n \to \infty, \tag{3.19}$$

$$I'(v_n) = o(1), \quad \text{as } n \to \infty, \tag{3.20}$$

$$\|v_n\|_{W^{1,p}(\mathbb{R}^N)} = \|u_n\|_{W^{1,p}(\mathbb{R}^N)} - \|u\|_{W^{1,p}(\mathbb{R}^N)} + o(1), \quad \text{as } n \to \infty. \tag{3.21}$$

Without loss of generality, we may assume that

$$\|v_n\|^p_{W^{1,p}(\mathbb{R}^N)} \to l > 0 \quad \text{as } n \to \infty.$$

In fact if $l = 0$, Theorem 2.1 is proved for $l_1 = 0, l_2 = 0, l_3 = 0$.

Step 1: Getting rid of the blowing up bubbles caused by unbounded domains. Suppose there exists a constant $0 < \delta < \infty$ such that

$$\|v_n\|_{L^p(\mathbb{R}^N)} \geq \delta > 0. \tag{3.22}$$

By Lemma 5.1, there exists a subsequence still denoted by $\{v_n\}$, such that one of the following two cases occurs.

(i) Vanishing occurs: for all $0 < R < \infty$,

$$\sup_{y\in\mathbb{R}^N} \int_{B(y,R)} \big(|\nabla v_n|^p + |v_n|^p\big)\, dx \to 0 \quad \text{as } n \to \infty.$$

By the Sobolev inequality, for $0 < R < \infty$, we have

$$\sup_{y\in\mathbb{R}^N} \int_{B(y,R)} |v_n|^r\, dx \leq \sup_{y\in\mathbb{R}^N} c \int_{B(y,R)} (|\nabla v_n|^p + |v_n|^p)\, dx \to 0 \quad \text{as } n \to \infty, \tag{3.23}$$

where $1 < r < p^*$. Since v_n is bounded in $W^{1,p}(\mathbb{R}^N)$, from (3.23) and Lemma 5.2 it follows that

$$\int_{\mathbb{R}^N} |v_n|^q\, dx \to 0 \text{ as } n \to \infty, \quad \forall 1 < q < p^*,$$

which contradicts (3.22).

(ii) Nonvanishing occurs: There exist $\beta > 0$, $0 < \bar{R} < \infty$, and $\{y_n\} \subset \mathbb{R}^N$ such that

$$\liminf_{n\to\infty} \int_{y_n + B_{\bar{R}}} \big(|\nabla v_n|^p + |v_n|^p\big) dx \geq \beta > 0. \tag{3.24}$$

We claim that there exists at least one $|y_n| \to \infty$ as $n \to \infty$. Otherwise, if any $\{y_n\}$ satisfying (3.24) is bounded, then there exists a $R > 0$ large enough such that

$$\|v_n\|_{W^{1,p}(B(0,R)^C)} \to 0 \quad \text{as } n \to \infty. \tag{3.25}$$

From the fact

$$v_n \to 0 \quad \text{in } L^q_{\rm loc}(\mathbb{R}^N) \text{ for } 1 < q < p^*,$$

(3.25), and the Sobolev inequality, it follows that $\|v_n\|_{L^p(\mathbb{R}^N)} \to 0$ as $n \to \infty$. This contradicts (3.22).

To proceed, we first construct the Palais-Smale sequences of I^∞. We denote $\bar{v}_n = v_n(x + y_n)$. Since $\|\bar{v}_n\|_{W^{1,p}(\mathbb{R}^N)} = \|v_n\|_{W^{1,p}(\mathbb{R}^N)} \leq C$, without loss of generality, we assume that as $n \to \infty$,

$$\begin{gathered} \bar{v}_n \rightharpoonup v_0 \quad \text{in } W^{1,p}(\mathbb{R}^N), \\ \bar{v}_n \to v_0 \quad \text{in } L^q_{\rm loc}(\mathbb{R}^N), \quad \forall 1 < q < p^*. \end{gathered}$$

Since for all $\phi \in C_0^\infty(\mathbb{R}^N)$, for n large enough,

$$\begin{aligned} \int_{\mathbb{R}^N} \frac{|\bar{v}_n|^{p-2}\bar{v}_n\phi}{|x+y_n|^p}\, dx &\leq \frac{2}{|y_n|^p} \int_{\mathbb{R}^N} |v_n|^{p-2} v_n \phi_n\, dx \\ &\leq \frac{2}{|y_n|^p} \Big(\int_{\mathbb{R}^N} |v_n|^p\, dx\Big)^{\frac{p-1}{p}} \Big(\int_{\mathbb{R}^N} |\phi_n|^p\, dx\Big)^{1/p} \end{aligned} \tag{3.26}$$

where $\phi_n = \phi(x - y_n)$. Obviously

$$\int_{\mathbb{R}^N} |\phi_n|^p\, dx = \int_{\mathbb{R}^N} |\phi|^p\, dx \leq c, \int_{\mathbb{R}^N} |v_n|^p\, dx \leq c. \tag{3.27}$$

Let $|y_n| \to \infty$, from (3.26) and (3.27), we have

$$\int_{\mathbb{R}^N} \frac{|\bar{v}_n|^{p-2}\bar{v}_n\phi}{|x+y_n|^p}\,dx = o(1) \quad \text{as } n \to \infty. \tag{3.28}$$

Since $v_n \rightharpoonup 0$ weakly in $W^{1,p}(\mathbb{R}^N)$ and $\lim_{n\to\infty} a(x+y_n) = \bar{a}$, by the Lebesgue convergence Theorem, as $n \to \infty$, we have

$$\begin{aligned}
&\int_{\mathbb{R}^N} a(x)|v_n|^{p-2}v_n\phi_n\,dx \\
&= \int_{\mathbb{R}^N} \bar{a}|v_n|^{p-2}v_n\phi\,dx + \int_{\mathbb{R}^N} \big[a(x+y_n)-\bar{a}\big]|v_n|^{p-2}v_n\phi\,dx \\
&= \int_{\mathbb{R}^N} \bar{a}|\bar{v}_n|^{p-2}\bar{v}_n\phi\,dx + o(1).
\end{aligned} \tag{3.29}$$

Similarly, we have

$$\begin{aligned}
\int_{\mathbb{R}^N} f(x,v_n)v_n\phi_n\,dx &= \int_{\mathbb{R}^N} \bar{f}(\bar{v}_n)\bar{v}_n\phi\,dx + \int_{\mathbb{R}^N} \big[f(x+y_n,\bar{v}_n)-\bar{f}(\bar{v}_n)\big]\bar{v}_n\phi\,dx \\
&= \int_{\mathbb{R}^N} \bar{f}(\bar{v}_n)\bar{v}_n\phi\,dx + o(1).
\end{aligned} \tag{3.30}$$

Recall that v_n is a Palais-Smale sequence of I, by (3.28)-(3.30) we have

$$\langle I'(v_n),\phi_n\rangle + o(1) = \langle {I^{\infty}}'(\bar{v}_n),\phi\rangle = o(1), \quad \text{as } n\to\infty. \tag{3.31}$$

This shows that $\bar{v}_n$ is a Palais-Smale sequence of $I^{\infty}(u)$, and v_0 is a weak solution of (2.7).

We claim that $v_0 \not\equiv 0$. From (3.22), we may assume there exists a sequence $\{y_n\}$ satisfying (3.24) and

$$\int_{B(y_n,R)} |v_n(x)|^p\,dx = b + o(1) > 0, \quad \text{as } n\to\infty, \tag{3.32}$$

where $b > 0$ is a constant. If $v_0 \equiv 0$, we have

$$\int_{B(R)} |\bar{v}_n|^p\,dx = \int_{B(y_n,R)} |v_n|^p\,dx = o(1) \text{ as } n\to\infty, \quad 0<R<\infty,$$

which contradicts (3.32).

We denote $z_n = v_n - v_0(x-y_n)$; therefore, as $n\to\infty$,

$$\|z_n\|_{W^{1,p}(\mathbb{R}^N)} = \|v_n\|_{W^{1,p}(\mathbb{R}^N)} - \|v_0\|_{W^{1,p}(\mathbb{R}^N)} + o(1), \tag{3.33}$$

$$I(z_n) = I(v_n) - I^{\infty}(v_0) + o(1). \tag{3.34}$$

Hence $z_n \rightharpoonup 0$ in $W^{1,p}(\mathbb{R}^N)$ as $n\to\infty$, and z_n is a Palais-Smale sequence of I. Then by Brézis-Lieb Lemma, we have

$$\begin{aligned}
\int_{\mathbb{R}^n} |z_n|^p\,dx &= \int_{\mathbb{R}^n} |v_n - v_0|^p\,dx + o(1) \\
&= \int_{\mathbb{R}^n} |v_n|^p\,dx - \int_{\mathbb{R}^n} |v_0|^p\,dx + o(1) \\
&\le \int_{\mathbb{R}^n} |v_n|^p\,dx - c,
\end{aligned} \tag{3.35}$$

where the last inequality follows from the fact $v_0 \not\equiv 0$. If $\|z_n\|_{L^p(\mathbb{R}^N)} \to \delta_2 > 0$ as $n\to\infty$, from (3.35) and the boundedness of $\|v_n\|_{L^p(\mathbb{R}^N)}$, then one can repeat Step

1 for finite times (l_1 times) since the amount of sequences satisfying (3.22) is finite.

Step 2: Getting rid of the blowing up bubbles caused by the Sobolev term. Suppose there exists $0 < \delta < \infty$ such that

$$\inf_{n\in\mathbb{N}^+} \int_{\mathbb{R}^N} \big(v_n^+\big)^{p^*}\, dx \geq \delta > 0. \tag{3.36}$$

It follows from Lemma 3.1 that there exist two sequences $\{r_n\} \subset \mathbb{R}^+$ and $\{x_n\} \subset \mathbb{R}^N$, such that

$$\bar{v}_n \rightharpoonup v_0 \neq 0 \quad \text{in } D^{1,p}(\mathbb{R}^N), \tag{3.37}$$

where

$$\bar{v}_n = \begin{cases} r_n^{\frac{N-p}{p}} v_n(r_n x) & \text{if } |x_n|/r_n \text{ is bounded},\\ r_n^{\frac{N-p}{p}} v_n(r_n x + x_n) & \text{if } |x_n|/r_n \to \infty. \end{cases} \tag{3.38}$$

Now we claim that $r_n \to 0$ as $n \to \infty$. In fact, there exists a $R_1 > 0$ such that

$$\int_{B(0,R_1)} |v_0|^p\, dx = \delta_1 > 0. \tag{3.39}$$

From the Sobolev compact embedding, (3.16)-(3.18), (3.37)-(3.39), for all $r > 0$ we have

$$\begin{aligned} &v_n \to 0 \text{ in } L^p(B(0,r)), \quad \bar{v}_n \to v_0 \text{ in } L^p(B(0,r)),\\ &0 \neq \|v_0\|^p_{L^p(B(0,R_1))} + o(1)\\ &= \int_{B(0,R_1)} |\bar{v}_n|^p\, dx\\ &= \begin{cases} r_n^{-p} \int_{B(0,r_nR_1)} |v_n|^p\, dx, & \text{if } |x_n|/r_n \text{ is bounded},\\ r_n^{-p} \int_{B(x_n,r_nR_1)} |v_n|^p\, dx, & \text{if } |x_n|/r_n \to \infty. \end{cases} \end{aligned} \tag{3.40}$$

From $\|v_n\|_{L^p(\mathbb{R}^N)} = o(1)$, (3.39) and (3.40), it follows that $r_n \to 0$.

For $|x_n|/r_n \to \infty$, we define $z_n = v_n - r_n^{\frac{p-N}{p}} v_0\big(\frac{x-x_n}{r_n}\big)$. Then $z_n \rightharpoonup 0$ in $W^{1,p}(\mathbb{R}^N)$. It follows from Lemma 3.3 that $\{z_n\}$ is a Palais-Smale sequence of I satisfying

$$I(z_n) = I(v_n) - I_0(v_0) + o(1), \text{ as } n \to \infty. \tag{3.41}$$

Since v_0 satisfies (2.8), from Lemma 3.1, (2.10) and (2.17) there exists $\varepsilon_1 > 0$ such that

$$v_0 = \varepsilon_1^{\frac{p-N}{p}} U_0\Big(\frac{x-\bar{x}_1}{\varepsilon_1}\Big), \quad I_0(v_0) = D_0. \tag{3.42}$$

Let $R_n^1 = r_n\varepsilon_1, x_n^1 = r_n\bar{x}_1 + x_n$, it follows that

$$r_n^{\frac{p-N}{p}} v_0\Big(\frac{x-x_n}{r_n}\Big) = (R_n^1)^{\frac{p-N}{p}} U_0\Big(\frac{x-x_n^1}{R_n^1}\Big) = U_0^{R_n^1,x_n^1}, \tag{3.43}$$

with $R_n^1 \to 0$, $|x_n^1|/R_n^1 \to \infty$. Then from (3.19) it follows that

$$\begin{gathered} z_n = v_n - U_0^{R_n^1,x_n^1} = u_n - u - U_0^{R_n^1,x_n^1},\\ I(z_n) = I(v_n) - D_0 + o(1) = I(u_n) - I(u) - D_0 + o(1) \end{gathered}$$

with $R_n^1 \to 0, |x_n^1|/R_n^1 \to \infty$. Obviously

$$\|z_n\|_{L^{p^*}(\mathbb{R}^N)} = \|v_n\|_{L^{p^*}(\mathbb{R}^N)} - \|U_0\|_{L^{p^*}(\mathbb{R}^N)} + o(1).$$

For $|x_n|/r_n$ bounded, we define $z_n = v_n - r_n^{\frac{p-N}{p}} v_0(\frac{x}{r_n})$. Then $z_n \rightharpoonup 0$ in $W^{1,p}(\mathbb{R}^N)$. It follows from Lemma 3.3 that $\{z_n\}$ is a Palais-Smale sequence of I satisfying

$$I(z_n) = I(v_n) - I_\mu(v_0) + o(1), \quad \text{as } n \to \infty. \tag{3.44}$$

Since v_0 satisfies (2.9), from (2.10) and (2.18) there exists $\varepsilon_1 > 0$ such that

$$v_0 = \varepsilon_1^{\frac{p-N}{p}} U_\mu\Big(\frac{x}{\varepsilon_1}\Big), \quad I_\mu(v_0) = D_\mu. \tag{3.45}$$

Let $\bar{R}_n^1 = r_n\varepsilon_1$, from (3.45), it follows that

$$r_n^{\frac{p-N}{p}} v_0\Big(\frac{x}{r_n}\Big) = (\bar{R}_n^1)^{\frac{p-N}{p}} U_\mu\Big(\frac{x}{\bar{R}_n^1}\Big) = U_\mu^{\bar{R}_n^1}, \tag{3.46}$$

with $\bar{R}_n^1 \to 0$. Then from (3.19) it follows that

$$\begin{gathered} z_n = v_n - U_\mu^{\bar{R}_n^1} = u_n - u - U_\mu^{\bar{R}_n^1}, \\ I(z_n) = I(v_n) - D_\mu + o(1) = I(u_n) - I(u) - D_\mu + o(1) \end{gathered} \tag{3.47}$$

with $\bar{R}_n^1 \to 0$. Obviously

$$\|z_n\|_{L^{p^*}(\mathbb{R}^N)} = \|v_n\|_{L^{p^*}(\mathbb{R}^N)} - \|U_\mu\|_{L^{p^*}(\mathbb{R}^N)} + o(1). \tag{3.48}$$

If still there exists a $\bar{\delta} > 0$ such that

$$\int_{\mathbb{R}^N} \big(z_n^+\big)^{p^*}\,dx \ge \bar{\delta} > 0,$$

then we repeat the previous argument. From (3.48) and that

$$\int_{\mathbb{R}^N} \big(z_n^+\big)^{p^*}\,dx \le \|z_n\|_{W^{1,p}(\mathbb{R}^N)}^{p^*} \le c,$$

we deduce that the iteration must stop after finite times. That is, from step 1 and step 2, there exist constants l_1, l_2, l_3 and a new Palais-Smale sequence of I, (without loss of generality) denoted by $\{v_n\}$, such that as $n \to \infty$,

$$d = I(v_n) + I(u) + \sum_{k=1}^{l_1} I^\infty(u_k) + l_2 D_\mu + l_3 D_0 + o(1), \tag{3.49}$$

$$\begin{gathered} v_n = u_n - u - \sum_{k=1}^{l_1} u_k(x - y_n^k) - \sum_{i=1}^{l_2} U^{\bar{R}_n^i} - \sum_{j=1}^{l_3} U_0^{R_n^j, x_n^j}, \\ \text{with } \bar{R}_n^i, \quad R_n^j \to 0, \quad \frac{|x_n^j|}{R_n^j} \to \infty, \end{gathered} \tag{3.50}$$

$$\|v_n\|_{L^q(\mathbb{R}^N)} \to 0, \quad \int_{\mathbb{R}^N} (v_n^+)^{p^*}\,dx \to 0 \tag{3.51}$$

as $n \to \infty$. Then from $\langle I'(v_n), v_n\rangle = o(1)$, it follows that

$$\begin{aligned} \|v_n\|_{W^{1,p}(\mathbb{R}^N)} &\le c \int_{\mathbb{R}^N} \Big(|\nabla v_n|^p + a(x)|v_n|^p - \mu\frac{|v_n|^p}{|x|^p}\Big)dx \\ &= c\Big(\int_{\mathbb{R}^N} f(x, v_n)v_n\,dx + \int_{\mathbb{R}^N} \big(v_n^+\big)^{p^*}\,dx\Big) \to 0 \end{aligned} \tag{3.52}$$

as $n \to \infty$. From (3.51) and (3.52), it gives that

$$I(v_n) = o(1). \tag{3.53}$$

From (3.49)-(3.53), the proof of Theorem 2.1 is complete. □

4. Proof of Theorem 2.3

For this proof we use Mountain Pass Theorem [3] and Theorem 2.1. From

$$I(tu) = \frac{t^p}{p}\Big[\int_{\mathbb{R}^N}\Big(|\nabla u|^p + a(x)|u|^p - \mu\frac{|u|^p}{|x|^p}\Big)\,dx\Big] - \frac{t^{p^*}}{p^*}\int_{\mathbb{R}^N}(u^+)^{p^*}\,dx - \int_{\mathbb{R}^N}F(x,tu)\,dx,$$

we deduce that for a fixed $u \not\equiv 0$ in $W^{1,p}(\mathbb{R}^N)$, $I(tu) \to -\infty$ if $t \to +\infty$. Since

$$\int_{\mathbb{R}^N} F(x,u)dx \le C\|u\|^q_{W^{1,p}(\mathbb{R}^N)} + \varepsilon\|u\|^p_{W^{1,p}(\mathbb{R}^N)}, \quad \int_{\mathbb{R}^N}|u|^{p^*}\,dx \le C\|u\|^{p^*}_{W^{1,p}(\mathbb{R}^N)},$$

we have

$$I(u) \ge c\|u\|^p_{W^{1,p}(\mathbb{R}^N)} - C\big(\|u\|^q_{W^{1,p}(\mathbb{R}^N)} + \|u\|^{p^*}_{W^{1,p}(\mathbb{R}^N)}\big).$$

Hence, there exists $r_0 > 0$ small such that $I(u)\big|_{\partial B(0,r_0)} \ge \rho > 0$ for $q, p^* > p$.

As a consequence, $I(u)$ satisfies the geometry structure of Mountain-Pass Theorem. Now define

$$c^* =: \inf_{\gamma\in\Gamma}\sup_{t\in[0,1]} I(\gamma(t)),$$

where $\Gamma = \{\gamma \in C([0,1], W^{1,p}(\mathbb{R}^N)) : \gamma(0) = 0, \gamma(1) = \psi_0 \in W^{1,p}(\mathbb{R}^N)\}$ with $I(t\psi_0) \le 0$ for all $t \ge 1$.

To complete the proof of Theorem 2.3, we need to verify that $I(u)$ satisfies the local Palais-Smale conditions. According to Remarks 2.2(1), we only need to verify that

$$c^* < \min\big\{\frac{1}{N}S_\mu^{N/p}, \frac{1}{N}S_0^{N/p}, J^\infty\big\} = \min\big\{\frac{1}{N}S_\mu^{N/p}, J^\infty\big\}. \tag{4.1}$$

Let $v_\varepsilon = \frac{U_\mu^\varepsilon}{(\int_{\mathbb{R}^N}|U_\mu^\varepsilon|^{p^*}\,d)^{1/p^*}}$, we claim that

$$\max_{t>0} I(tv_\varepsilon) < \frac{1}{N}S_\mu^{N/p}. \tag{4.2}$$

Since, U_μ^ε are the minimizers of S_μ, we have

$$\int_{\mathbb{R}^N}|\nabla v_\varepsilon|^p\,dx - \int_{\mathbb{R}^N}\mu\frac{|v_\varepsilon|^p}{|x|^p}\,dx = S_\mu. \tag{4.3}$$

From (2.11) (also refer to [6]), and $a(\mu) < \frac{N-p}{p}$, $\mu < \frac{N^{p-1}(N-p^2)}{p^p}$, it is easy to calculate the estimate

$$\begin{aligned}\int_{\mathbb{R}^N}|v_\varepsilon|^p\,dx &\le c\varepsilon^p\int_{\mathbb{R}^N}|U_\mu|^p\,dx\\ &\le c\varepsilon^p\int_{|x|\le 1}\frac{1}{|x|^{a(\mu)p}}\,dx + c\varepsilon^p\int_{|x|\ge 1}\frac{1}{|x|^{b(\mu)p}}\,dx = O(\varepsilon^p).\end{aligned} \tag{4.4}$$

Similarly,

$$\int_{\mathbb{R}^N}|v_\varepsilon|^q\,dx = O(\varepsilon^{\frac{(p-N)q}{p}+N}). \tag{4.5}$$

Since $p^* > q$, we have

$$O(\varepsilon^p) = o(\varepsilon^{\frac{(p-N)q}{p}+N}). \tag{4.6}$$

We denote by t_ε the attaining point of $\max_{t>0} I(tv_\varepsilon)$, similar to the proof of [6, Lemma 3.5] we can prove that t_ε is uniformly bounded. In fact, we consider the function

$$\begin{aligned}
h(t) &= I(tv_\varepsilon)\\
&= \frac{t^p}{p}\Big[\|\nabla v_\varepsilon\|^p_{L^p(\mathbb{R}^N)} + \int_{\mathbb{R}^N}\Big(a(x)|v_\varepsilon|^p - \mu\frac{|v_\varepsilon|^p}{|x|^p}\Big)dx\Big]\\
&\quad - \frac{t^{p^*}}{p^*}\int_{\mathbb{R}^N}|v_\varepsilon|^{p^*}\,dx - \int_{\mathbb{R}^N}F(x,tv_\varepsilon)dx\\
&\ge \frac{ct^p}{p}\|v_\varepsilon\|^p_{W^{1,p}(\mathbb{R}^N)} - \frac{ct^{p^*}}{p^*}\|v_\varepsilon\|^{p^*}_{W^{1,p}(\mathbb{R}^N)} - \delta t^p\|v_\varepsilon\|^p_{W^{1,p}(\mathbb{R}^N)}\\
&\quad - ct^q\|v_\varepsilon\|^q_{W^{1,p}(\mathbb{R}^N)}\\
&\ge \frac{(c-\delta p)t^p}{p}\|v_\varepsilon\|^p_{W^{1,p}(\mathbb{R}^N)} - \frac{ct^{p^*}}{p^*}\|v_\varepsilon\|^{p^*}_{W^{1,p}(\mathbb{R}^N)} - ct^q\|v_\varepsilon\|^q_{W^{1,p}(\mathbb{R}^N)},
\end{aligned} \tag{4.7}$$

where $\delta > 0$ small enough. Then $h(t) > 0$ when t is closed to 0, it follows that $\max_{t>0} h(t)$ is attained for $t_\varepsilon > 0$. From $\int_{\mathbb{R}^N}|v_\varepsilon|^{p^*}\,dx = 1$, it follows that

$$\begin{aligned}
0 &= h'(t_\varepsilon)\\
&= t_\varepsilon^{p-1}\Big[\|\nabla v_\varepsilon\|^p_{L^p(\mathbb{R}^N)} + \int_{\mathbb{R}^N}\Big(a(x)|v_\varepsilon|^p - \mu\frac{|v_\varepsilon|^p}{|x|^p}\Big)\,dx\Big]\\
&\quad - t_\varepsilon^{p^*-1} - \int_{\mathbb{R}^N} f(x,tv_\varepsilon)v_\varepsilon\,dx.
\end{aligned} \tag{4.8}$$

Since $f(x,v_\varepsilon) > 0$, from (4.3) and (4.4), for ε sufficiently small, we have

$$t_\varepsilon^{p^*-p} \le \|\nabla v_\varepsilon\|^p_{L^p(\mathbb{R}^N)} + \int_{\mathbb{R}^N}\Big(a(x)|v_\varepsilon|^p - \mu\frac{|v_\varepsilon|^p}{|x|^p}\Big)dx < 2S_\mu. \tag{4.9}$$

Then

$$\begin{aligned}
\frac{1}{2}S_\mu &< \|\nabla v_\varepsilon\|^p_{L^p(\mathbb{R}^N)} + \int_{\mathbb{R}^N}\Big(a(x)|v_\varepsilon|^p - \mu\frac{|v_\varepsilon|^p}{|x|^p}\Big)\,dx\\
&= t_\varepsilon^{p^*-p} + t_\varepsilon^{-p+1}\int_{\mathbb{R}^N} f(x,tv_\varepsilon)v_\varepsilon\,dx\\
&\le t_\varepsilon^{p^*-p} + O(\varepsilon^{N-q\frac{N-p}{p}}).
\end{aligned} \tag{4.10}$$

Choosing $\varepsilon > 0$ small enough, by (4.3)-(4.5), there exists a constant $\gamma > 0$ such that $t_\varepsilon > \gamma > 0$. Combining this with (4.9), it implies that t_ε is bounded for $\varepsilon > 0$ small enough. Hence, for $\varepsilon > 0$ small,

$$\begin{aligned}
\max_{t>0} I(tv_\varepsilon) &= I(t_\varepsilon v_\varepsilon)\\
&\le \max_{t>0}\Big\{\frac{t^p}{p}\int_{\mathbb{R}^N}\Big(|\nabla v_\varepsilon|^p - \mu\frac{|v_\varepsilon|^p}{|x|^p}\Big)\,dx - \frac{t^{p^*}}{p^*}\int_{\mathbb{R}^N}|v_\varepsilon|^{p^*}\,dx\Big\}\\
&\quad - O(\varepsilon^{\frac{(p-N)q}{p}+N}) + O(\varepsilon^p)\\
&< \frac{1}{N}S_\mu^{N/p} \quad \text{(by (4.6))}.
\end{aligned}$$

This completes the proof of (4.2). By the definition of c^*, we have $c^* < \frac{1}{N}S_\mu^{N/p}$.

Next we verify that

$$c^* < J^\infty. \tag{4.11}$$

We only need to verify that

$$\sup_{0\le t\le \bar t} I(tw_R) < J^\infty$$

for R large enough. Here $w_R = w(x - R\nu)$, ν is a unit vector of $\mathbb{R}^N$ and w be that in (2.16). Since $a(x) \in C(\mathbb{R}^N)$, we can choose a small $\tau \in (0, 1)$ such that

$$1 - a(x) + \frac{\mu}{|x|^p} \geqslant \frac{\mu}{2|x|^p}, \quad \forall |x| \leqslant \tau.$$

Then, we find that

$$\begin{aligned}
&\int_{\mathbb{R}^N} (1 - a(x) + \frac{\mu}{|x|^p}) w_R^p \, dx \\
&\geqslant \int_{|x|\leqslant\tau} \frac{\mu c_1^p}{2\tau^p} (|x - R\nu| + 1)^{-\frac{N-1}{p-1}} e^{-p(\frac{1}{p-1})^{\frac{1}{p}}|x-R\nu|} \, dx \\
&\geqslant c\frac{\mu}{\tau^p}(R + 1)^{-\frac{(N-1)}{p-1}} e^{-p(\frac{1}{p-1})^{\frac{1}{p}}(R+1)} \int_{|x|\leqslant\tau} dx \\
&\geqslant c\tau^{N-p} R^{-\frac{(N-1)}{p-1}} e^{-p(\frac{1}{p-1})^{\frac{1}{p}}R} = \bar c R^{-\frac{(N-1)}{p-1}} e^{-p(\frac{1}{p-1})^{\frac{1}{p}}R},
\end{aligned} \tag{4.12}$$

where $\bar c, c$ are positive constants. On the other hand, it follows from (A5) and Lemma 3.4 that

$$\begin{aligned}
&\int_{\mathbb{R}^N} \left[\bar F(tw_R) - F(x, tw_R)\right] \, dx \\
&= \int_{\mathbb{R}^N} \int_0^{tw_R} \left[\bar f(s) - f(x, s)\right] ds \, dx \\
&\leqslant \varepsilon \frac{t^p}{p} \int_{\mathbb{R}^N} e^{-\sigma|x|} w_R^p \, dx + C_\varepsilon \frac{t^q}{q} \int_{\mathbb{R}^N} e^{-\sigma|x|} w_R^q \, dx \\
&\leqslant \varepsilon c R^{-\frac{(N-1)}{p-1}} e^{-p(\frac{1}{p-1})^{\frac{1}{p}}R} + C_\varepsilon c R^{-\frac{(N-1)q}{p(p-1)}} e^{-\min\{\sigma, q(\frac{1}{p-1})^{\frac{1}{p}}R\}}
\end{aligned} \tag{4.13}$$

where c, C_ε are positive constants. Hence, noting $\sigma > p(\frac{1}{p-1})^{\frac{1}{p}}$, we see that for R large enough,

$$\begin{aligned}
&I(tw_R) \\
&\leqslant I^\infty(tw_R) - \frac{t^p}{p} \int_{\mathbb{R}^N} (1 - a(x) + \frac{\mu}{|x|^p}) w_R^p \, dx + \int_{\mathbb{R}^N} (\bar F(tw_R) - F(x, tw_R)) \, dx \\
&\leqslant J^\infty - cR^{-\frac{(N-1)}{p-1}} e^{-p(\frac{1}{p-1})^{R/p}} \\
&\quad + c\varepsilon R^{-\frac{(N-1)}{p-1}} e^{-p(\frac{1}{p-1})^{\frac{1}{p}}R} + C_\varepsilon c R^{-\frac{(N-1)q}{(p-1)p}} e^{-\min\{\sigma, q(\frac{1}{p-1})^{R/p}\}} < J^\infty.
\end{aligned}$$

5. Appendix

In this section, we give detailed proofs some lemmas used above.

Lemma 5.1 ([28, Lemma 2.1])**.** *Let $\{\rho_n\}_{n\ge1}$ be a sequence in $L^1(\mathbb{R}^N)$ satisfying*

$$\rho_n \ge 0 \quad \text{on } \mathbb{R}^N, \quad \lim_{n\to\infty} \int_{\mathbb{R}^N} \rho_n \, dx = \lambda > 0, \tag{5.1}$$

where $\lambda > 0$ is fixed. Then there exists a subsequence $\{\rho_{n_k}\}$ satisfying one of the following two possibilities: (i) Vanishing:

$$\lim_{k\to\infty} \sup_{y\in\mathbb{R}^N} \int_{B(y,R)} \rho_{n_k}\, dx = 0, \quad \text{for all } R < +\infty. \tag{5.2}$$

(ii) Nonvanishing: there exists $\alpha > 0$, $R < +\infty$ and $\{y_k\} \subset \mathbb{R}^N$ such that

$$\lim_{k\to+\infty} \int_{y_k+B_R} \rho_{n_k}\, dx \ge \alpha > 0.$$

Lemma 5.2 ([28, Lemma 2.3])**.** *Let $1 < p < \infty$, $1 \le q < \infty$, with $q \ne \frac{Np}{N-p}$ if $p < N$. Assume that $\{u_n\}$ is bounded in $L^q(\mathbb{R}^N)$, $\{|\nabla u_n|\}$ is bounded in $L^p(\mathbb{R}^N)$ and*

$$\sup_{y\in\mathbb{R}^N} \int_{y+B_R} |u_n|^q\, dx \to 0 \quad \text{for some } R > 0 \text{ as }, n \to \infty.$$

Then $u_n \to 0$ in $L^\alpha(\mathbb{R}^N)$, for α between q and $\frac{Np}{N-p}$.

Lemma 5.3. *Assume that $a(x)$ satisfies* (A1)*. It follows that*

$$C_1\|u\|^p_{W^{1,p}(\mathbb{R}^N)} \le \int_{\mathbb{R}^N} \Big(|\nabla u|^p + a(x)|u|^p - \mu\frac{|u|^p}{|x|^p}\Big)\, dx \le C_2\|u\|^p_{W^{1,p}(\mathbb{R}^N)}, \tag{5.3}$$

where C_1 and C_2 are positive constants.

The proof of the above Lemma is obtained based on condition (A1).

Lemma 5.4. *Let $\{u_n\}$ be a Palais-Smale sequence of I at level $d \in \mathbb{R}$. Then $d \ge 0$ and $\{u_n\} \subset W^{1,p}(\mathbb{R}^N)$ is bounded. Moreover, every Palais-Smale sequence for I at a level zero converges strongly to zero.*

Proof. From (A4), it follows that

$$\frac{1}{p+\theta} u_n f(u_n) \ge F(x, u_n), \quad \frac{1}{p+\theta} > \frac{1}{p^*}. \tag{5.4}$$

Thus from (5.3) and (5.4), we have

$$\begin{aligned} d + 1 + o(\|u_n\|) &\ge I(u_n) - \frac{1}{p+\theta}\langle I'(u_n), u_n\rangle \\ &= \Big(\frac{1}{p} - \frac{1}{p+\theta}\Big) \int_{\mathbb{R}^N} \Big(|\nabla u_n|^p - \mu\frac{|u_n|^p}{|x|^p} + a(x)|u_n|^p\Big)\, dx \\ &\quad + \frac{1}{p+\theta}\int_{\mathbb{R}^N} u_n f(x,u_n)\, dx - \int_{\mathbb{R}^N} F(x,u_n)\, dx \\ &\ge C\Big(\frac{1}{p} - \frac{1}{p+\theta}\Big) \int_{\mathbb{R}^N} \Big(|\nabla u_n|^p - \mu\frac{|u_n|^p}{|x|^p} + a(x)|u_n|^p\Big) dx \\ &\ge C\|u_n\|^p_{W^{1,p}(\mathbb{R}^N)}. \end{aligned} \tag{5.5}$$

It follows from (5.5) that $\{u_n\}$ is bounded in $W^{1,p}(\mathbb{R}^N)$. Since

$$d = \lim_{n\to\infty} I(u_n) - \frac{1}{p+\theta}\langle I'(u_n), u_n\rangle \ge C \limsup_{n\to\infty} \|u_n\|^p_{W^{1,p}(\mathbb{R}^N)},$$

then we have $d \ge 0$. Suppose now that $d = 0$, we obtain from the above inequality that

$$\lim_{n\to\infty} \|u_n\|_{W^{1,p}(\mathbb{R}^N)} = 0.$$

□

Lemma 5.5. *Let $\{u_n\}$ be a Palais-Smale sequence of I at level $d \in \mathbb{R}$. Then $\{u_n^+\}$ is also a Palais-Smale sequence of I at level d when $u_n^+ = \max\{u_n, 0\}$.*

Proof. By the definition of I, we have that as $n \to \infty$

$$\begin{aligned} I(u_n) &= \frac{1}{p}\int_{\mathbb{R}^N} \Big(|\nabla u_n|^p + a(x)|u_n|^p\Big)\,dx - \frac{\mu}{p}\int_{\mathbb{R}^N} \frac{|u_n|^p}{|x|^p}\,dx \\ &\quad - \frac{1}{p^*}\int_{\mathbb{R}^N} (u_n^+)^{p^*}\,dx - \int_{\mathbb{R}^N} F(x,u_n)u_n\,dx \to d, \end{aligned}$$

and

$$\begin{aligned} \langle I'(u_n),\phi\rangle &= \int_{\mathbb{R}^N} |\nabla u_n|^{p-2}\nabla u_n \nabla\phi\,dx + \int_{\mathbb{R}^N} a(x)|u_n|^{p-2}u_n\phi\,dx \\ &\quad - \mu\int_{\mathbb{R}^N} \frac{|u_n|^{p-2}u_n\phi}{|x|^p}\,dx - \int_{\mathbb{R}^N} f(x,u_n)\phi\,dx \\ &\quad - \int_{\mathbb{R}^N} \big(u_n^+\big)^{p^*-1}\phi\,dx \to 0, \quad \text{for all } \phi \in W^{1,p}(\mathbb{R}^N). \end{aligned}$$

Taking $\phi = -u_n^- = \min\{u_n, 0\}$, from

$$u_n = u_n^+ - u_n^-, \quad u_n^+u_n^- = 0, \tag{5.6}$$

we have

$$\begin{aligned} &\langle I'(u_n), -u_n^-\rangle \\ &= -\int_{\mathbb{R}^N} |\nabla u_n|^{p-2}\nabla u_n\nabla u_n^- dx - \int_{\mathbb{R}^N} a(x)|u_n|^{p-2}u_nu_n^-\,dx \\ &\quad + \mu\int_{\mathbb{R}^N} \frac{|u_n|^{p-2}u_nu_n^-}{|x|^p}\,dx - \int_{\mathbb{R}^N} f(x,u_n)u_n^- dx + \int_{\mathbb{R}^N} \big(u_n^+\big)^{p^*-1}u_n^-\,dx \\ &= \int_{\mathbb{R}^N} \Big(|\nabla u_n^-|^p + a(x)|u_n^-|^p\Big)dx - \mu\int_{\mathbb{R}^N} \frac{|u_n^-|^p}{|x|^p}\,dx \to 0. \end{aligned} \tag{5.7}$$

From (A1), (5.7), $u_n^+ \geq 0$, and $u_n^- \geq 0$, it follows that

$$\|u_n^-\|_{W^{1,p}(\mathbb{R}^N)} \to 0. \tag{5.8}$$

Thus

$$\begin{gathered} \lim_{n\to\infty} I(u_n^+) = \lim_{n\to\infty} I(u_n) = d, \\ I'(u_n^+,\phi) = I'(u_n,\phi) \to 0, \end{gathered}$$

as $n \to \infty$. This complete the proof. □

Lemma 5.6. *All nontrivial critical points of I_μ are the positive solutions.*

Proof. Let $u \not\equiv 0$ and $u \in W^{1,p}(\mathbb{R}^N)$ be a nontrivial critical point of I_μ. First, arguing as in the proof of Lemma 5.5 (similar to (5.7) and (5.8)), we can obtain that $\|u^-\|_{W^{1,p}(\mathbb{R}^N)} = 0$ which gives that $u \geq 0$ a.e. in $\mathbb{R}^N$. By the maximum principle we can obtain $u > 0$ in $\mathbb{R}^N$. □

Let $\{u_n\}$ be a Palais-Smale sequence. Up to a subsequence, we assume that

$$u_n \rightharpoonup u \quad \text{in } W^{1,p}(\mathbb{R}^N) \text{ as } n \to \infty.$$

Obviously, we have $I'(u) = 0$. Let $v_n = u_n - u$, then as $n \to \infty$,

$$v_n \rightharpoonup 0 \quad \text{in } W^{1,p}(\mathbb{R}^N), \tag{5.9}$$

$$v_n \to 0 \quad \text{in } L^q_{\rm loc}(\mathbb{R}^N) \text{ for all } 1 < q < p^*. \tag{5.10}$$

As a consequence, we have the following Lemma.

Lemma 5.7. *$\{v_n\}$ is a Palais-Smale sequence for I at level $d_0 = d - I(u)$.*

Proof. By the Brézis-Lieb Lemma in [3] and $v_n \rightharpoonup 0$ in $W^{1,p}(\mathbb{R}^N)$, as $n \to \infty$, we have

$$\int_{\mathbb{R}^N} F(x, v_n)dx = \int_{\mathbb{R}^N} F(x, u_n)dx - \int_{\mathbb{R}^N} F(x, u)dx + o(1), \tag{5.11}$$

$$\int_{\mathbb{R}^N} \frac{|v_n|^p}{|x|^p}\,dx = \int_{\mathbb{R}^N} \frac{|u_n|^p}{|x|^p}\,dx - \int_{\mathbb{R}^N} \frac{|u|^p}{|x|^p}\,dx + o(1), \tag{5.12}$$

$$\int_{\mathbb{R}^N} |\nabla v_n|^p\,dx = \int_{\mathbb{R}^N} |\nabla u_n|^p\,dx - \int_{\mathbb{R}^N} |\nabla u|^p\,dx + o(1). \tag{5.13}$$

Hence $I(v_n) = I(u_n) - I(u) + o(1) = d - I(u) + o(1)$.

For $\phi \in C_0^\infty(\mathbb{R}^N)$, there exists a $B(0, r)$ such that $\operatorname{supp}\phi \subset B(0, r)$. Then as $n \to \infty$,

$$\Big|\int_{\mathbb{R}^N} f(x, v_n)\phi\,dx\Big| \le c\Big|\int_{B(0,r)} (|v_n|^{q-1} + |v_n|^{p-1})\phi\,dx\Big| = o(1), \tag{5.14}$$

and from the Lebesgue convergence theorem

$$\Big|\int_{\mathbb{R}^N} \frac{|v_n|^{p^*-2}v_n\phi}{|x|^p}\,dx\Big| \le \Big|\int_{|x|\le r} \frac{|v_n|^{p^*-2}v_n\phi}{|x|^p}\,dx\Big| = o(1). \tag{5.15}$$

By (5.9), (5.14) and (5.15), we have $\langle \phi, I'(v_n)\rangle = o(1)$ as $n \to \infty$. □

Lemma 5.8. *The assumption* (A1) *holds naturally if $a(x) \in C(\mathbb{R}^N)$ satisfies*

1. *$a(x) \to \bar{a} > 0$ as $x \to \infty$;*
2. *$-m \le a(x)$ and the set $\{x \in \mathbb{R}^N : -m \le a(x) \le 0\}$ is nonempty and bounded, where $m \in (0, m^*)$ and m^* is a small positive constant.*

Proof. By assumptions (1) and (2), we can find $\rho > 0$ such that

$$\{x \in \mathbb{R}^N : a(x) \leqslant 0\} \subset B(0, \rho),$$

$$\inf_{\mathbb{R}^N \setminus B(0,\rho)} a(x) > \frac{\bar{a}}{2}.$$

We claim that

$$(|\mathbf{a}|^{\mathbf{2}} + |\mathbf{b}|^{\mathbf{2}} + \mathbf{2a} \cdot \mathbf{b})^{\mathbf{p/2}} - (|\mathbf{a}|^{\mathbf{2}})^{\mathbf{p/2}} \geqslant \mathbf{p}|\mathbf{a}|^{\mathbf{p-2}}\mathbf{a} \cdot \mathbf{b}.$$

From Cauchy's mean value theorem, we have

$$\big(|\mathbf{a}|^2 + |\mathbf{b}|^2 + 2\mathbf{a} \cdot \mathbf{b}\big)^{p/2} - \big(|\mathbf{a}|^2\big)^{p/2} = \frac{p}{2}\xi^{\frac{p-2}{2}}\big(|\mathbf{b}|^2 + 2\mathbf{a} \cdot \mathbf{b}\big).$$

If $|\mathbf{a} + \mathbf{b}|^2 \geqslant |\mathbf{a}|^2$, i.e., $|\mathbf{b}|^2 + 2\mathbf{a} \cdot \mathbf{b} \geqslant 0, |\mathbf{a}|^2 \leqslant \xi \leqslant |\mathbf{a} + \mathbf{b}|^2$, thus

$$\xi^{\frac{p-2}{2}}\Big(|\mathbf{b}|^2 + 2\mathbf{a} \cdot \mathbf{b}\Big) \geqslant |\mathbf{a}|^{p-2}(|\mathbf{b}|^2 + 2\mathbf{a} \cdot \mathbf{b}) \geqslant 2|\mathbf{a}|^{p-2}\mathbf{a} \cdot \mathbf{b}.$$

If $|\mathbf{a} + \mathbf{b}|^2 \leqslant |\mathbf{a}|^2$, i.e., $|\mathbf{b}|^2 + 2\mathbf{a} \cdot \mathbf{b} \leqslant 0$, $|\mathbf{a} + \mathbf{b}|^2 \leqslant \xi \leqslant |\mathbf{a}|^2$, thus

$$|\xi|^{\frac{p-2}{2}}(|\mathbf{b}|^2 + 2\mathbf{a} \cdot \mathbf{b}) \geqslant |\mathbf{a}|^{p-2}(|\mathbf{b}|^2 + 2\mathbf{a} \cdot \mathbf{b}) \geqslant 2|\mathbf{a}|^{p-2}\mathbf{a} \cdot \mathbf{b}.$$

For $R > \rho$, we can choose $\varphi(x) \in C_0^\infty(B(0,R))$ satisfying $0 \leqslant \varphi(x) \leqslant 1, \varphi(x) = 1$ in $B(0,\rho), \varphi(x) = 0$ in $\mathbb{R}^N \backslash B(0,R)$ and $|\nabla\varphi| \leqslant \frac{1}{2(R-\rho)p}$. For each $u \in W^{1,p}(\mathbb{R}^N)$, let

$$\mathbf{a} = \nabla(\varphi u), \quad \mathbf{b} = \nabla((1-\varphi)u).$$

Then we can derive that

$$\begin{aligned}
&\int_{\mathbb{R}^N} |\nabla u|^p \,dx \\
&\geqslant \int_{\mathbb{R}^N} |\nabla(\varphi u)|^p \,dx + p\int_{\mathbb{R}^N} |\nabla(\varphi u)|^{p-2}(\nabla u \cdot u \cdot \nabla\varphi(1-2\varphi) - |\nabla\varphi|^2 u^2)\,dx \\
&\geqslant \frac{1}{2}\int_{\mathbb{R}^N} |\nabla(\varphi u)|^p \,dx - \int_{\mathbb{R}^N} (|\nabla u||u||\nabla\varphi|)^{p/2} p^{p/2} 2^{p-2}\,dx - \int_{\mathbb{R}^N} |\nabla\varphi|^p |u|^p p^{p/2} 2^{p-2}\,dx \\
&\geqslant \frac{1}{2}\int_{B(0,R)} |\nabla(\varphi u)|^p \,dx - 2^{p-3}p^{p/2}\int_{\mathbb{R}^N} |\nabla u|^p \,dx - 3p^{p/2}2^{p-3}\int_{\mathbb{R}^N} |\nabla\varphi|^p |u|^p \,dx \\
&\geqslant \frac{\lambda_{1,p}(B(0,R))}{2}\int_{B(0,\rho)} |u|^p \,dx - \frac{3\cdot 2^{p-3}\cdot p^{p/2}}{(2(R-\rho)p)^p}\int_{\mathbb{R}^N} |u|^p \,dx - 2^{p-3}p^{p/2}\int_{\mathbb{R}^N} |\nabla u|^p \,dx \\
&\geqslant \frac{1}{2(p-1)R^p}\int_{B(0,\rho)} |u|^p \,dx - \frac{3}{8}\frac{1}{(R-\rho)^p}\cdot\frac{1}{p^{p/2}}\int_{\mathbb{R}^N} |u|^p \,dx - 2^{p-3}p^{p/2}\int_{\mathbb{R}^N} |\nabla u|^p \,dx,
\end{aligned}$$

where $\lambda_{1,p}(B(0,R)) \geq \frac{(2\pi)^p}{(p-1)(p\sin(\pi/p)2R)^p}$ is the first eigenvalue of the operator $(-\Delta)_p$ in $W_0^{1,p}(B(0,R))$ (refer to [16]).

We set $d = 1 - (\frac{p}{N-p})^p\mu$, choose R large enough such that

$$\frac{1}{2(p-1)R^p} - \frac{3}{8}\cdot\frac{1}{p^{p/2}}\frac{1}{(R-\rho)^p} \geqslant \frac{1}{8(p-1)R^p}, \quad \text{and} \quad \frac{3d}{8(R-\rho)^p p^{p/2}} < \frac{\bar{a}}{4}.$$

Then

$$\begin{aligned}
&\int_{\mathbb{R}^N} \big(d|\nabla u|^p + a(x)|u|^p\big)\,dx \\
&\geqslant \frac{d}{2(p-1)R^p}\int_{B(0,\rho)} |u|^p \,dx + \int_{\mathbb{R}^N} a(x)|u|^p \,dx \\
&\quad - \frac{3d}{8(R-\rho)^p p^{p/2}}\int_{\mathbb{R}^N} |u|^p \,dx - 2^{p-3}p^{p/2}d\int_{\mathbb{R}^N} |\nabla u|^p \,dx \\
&= \int_{B(0,\rho)} \big[\frac{d}{2(p-1)R^p} - \frac{3d}{8(R-\rho)^p p^{p/2}} + a(x)\big]|u|^p \,dx \\
&\quad + \int_{\mathbb{R}^N\backslash B(0,\rho)} \big[a(x) - \frac{3d}{8(R-\rho)^p p^{p/2}}\big]|u|^p \,dx - 2^{p-3}p^{p/2}d\int_{\mathbb{R}^N} |\nabla u|^p \,dx.
\end{aligned}$$

Therefore

$$\begin{aligned}
&(1 + 2^{p-3}p^{p/2})\int_{\mathbb{R}^N} (d|\nabla u|^p + a(x)|u|^p)\,dx \\
&\geqslant \int_{B_p(0)} \Big[\frac{d}{8(p-1)R^p} + (1 + 2^{p-3}p^{\frac{p}{2}})a(x)\Big]|u|^p \,dx + \frac{\bar{a}}{4}\int_{\mathbb{R}^N\backslash B(0,\rho)} |u|^p \,dx.
\end{aligned}$$

Let $m^* = \frac{1}{(1+2^{p-3}p^{\frac{p}{2}})}\cdot\frac{d}{16(p-1)R^p}$. For $0 \leqslant m \leqslant m^*$, it follows that

$$(1 + 2^{p-3}p^{p/2})\int_{\mathbb{R}^N} (d|\nabla u|^p + a(x)|u|^p)\,dx$$

$$\begin{aligned}&\geqslant \frac{d}{16(p-1)R^p}\int_{B(0,\rho)}|u|^p\,dx+\frac{\bar a}{4}\int_{\mathbb{R}^N\setminus B_\rho(0)}|u|^p\,dx\\&\geqslant \min\Big\{\frac{d}{16(p-1)R^p},\frac{\bar a}{4}\Big\}\int_{\mathbb{R}^N}|u|^p\,dx.\end{aligned}$$

If we set $\lambda^*=\frac{1}{1+2^{p-3}p^{\frac{p}{2}}}\min\Big\{\frac{d}{16(p-1)R^p},\frac{\bar a}{4}\Big\}$, then we have

$$\begin{aligned}\int_{\mathbb{R}^N}\big[(1-\big(\frac{p}{N-p}\big)^p\mu)|\nabla u|^p+a(x)|u|^p\big]\,dx&=\int_{\mathbb{R}^N}(d|\nabla u|^p+a(x)|u|^p)\,dx\\&\geqslant\lambda^*\int_{\mathbb{R}^N}|u|^p\,dx\\&=\frac{\lambda^*}{\bar a+m^*}\int_{\mathbb{R}^N}(\bar a+m^*)|u|^p\,dx\\&\geqslant\frac{\lambda^*}{\bar a+m^*}\int_{\mathbb{R}^N}(\bar a-a(x))|u|^p\,dx.\end{aligned}$$

We can complete the proof by taking $\lambda_1=\frac{\lambda^*}{\bar a+m^*}$. □

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