Electronic Journal of Differential Equations, Vol. 2025 (2025), No. 04, pp. 1–17.
ISSN: 1072-6691. URL: https://ejde.math.txstate.edu, https://ejde.math.unt.edu
DOI: 10.58997/ejde.2025.04

EXACT CONTROLLABILITY FOR DEGENERATE AND SINGULAR WAVE EQUATIONS

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Abstract. In this article, we study exact controllability for degenerate and singular wave equations with a general coefficient. We estimate the observability inequality by the multiplier method and determine the observability time. We also deduce the exact controllability of the corresponding degenerate and singular control problem at a sufficiently large time, employing the Hilbert uniqueness method.

1. Introduction

Control issues for non-degenerate parabolic and hyperbolic problems have been a mainstream topic over the past several years, and a lot of attention has led to numerous developments being pursued (see [10, 20, 22, 23, 25, 26, 28, 32]).

Let us recall that exact controllability for the nondegenerate wave equation, which is characterized by the system of equations

$$\begin{gathered} u_{tt} - u_{xx} = 0, \quad (t,x) \in (0,T) \times (0,1), \\ u(t,0) = 0, \; u(t,1) = f(t), \quad t \in (0,T), \\ u(0,x) = u_0(x), \quad x \in (0,1), \\ u_t(0,x) = u_1(x), \quad x \in (0,1), \end{gathered} \tag{1.1}$$

where u is the state, f acts as a boundary control and is used to drive the solution to zero at a given time T. To be more precise, for given the initial data (u_0, u_1) in a suitable space, we look for a control f such that

$$u(T,x) = u_t(T,x) = 0, \quad \forall x \in (0,1). \tag{1.2}$$

Because of the finite speed of propagation of solutions to the wave equation, exact controllability can only be achieved at a sufficiently large time T (in the parabolic case, we have null controllability at any final time T). As a general conclusion, we consider (1.1) to represent exact controllability if $T > 2$.

The degenerate wave equations began to receive some attention within the past decade, and had developed rapidly [4, 5, 7, 29]. Different from the case non-degenerate equations, the main difficulty with degenerate wave equations is introducing a suitable function space to deal with degenerate terms, requiring the

2020 *Mathematics Subject Classification.* 35L80, 35L81, 35L05, 93B05.
Key words and phrases. Degenerate wave equation; singular wave equation; controllability; boundary control.

Submitted September 18, 2024. Published January 9, 2025.

development of new rules for analyzing observability and controllability. Gueye [19] considered the boundary control about the degenerate wave equation

$$u_{tt} - (x^{\alpha} u_x)_x = 0, \quad (t,x) \in (0,T) \times (0,1). \tag{1.3}$$

We also refer to the work of Zhang and Gao [30, 31] for additional controllability results obtained through the use of a locally distributed control. Later, Alabau-Boussouira et al. [1] consider the equation

$$u_{tt} - (au_x)_x = 0, \quad (t,x) \in (0,T) \times (0,1), \tag{1.4}$$

where a is positive on $(0,1]$ and $a(0) = 0$. The degeneracy of (1.4) at $x = 0$ is measured by the parameter μ_a defined by

$$\mu_a := \sup_{0<x\leq 1} \frac{x|a'(X)|}{a(X)} < 2, \tag{1.5}$$

and say the function a is weakly degenerate (WD) if $\mu_a \in [0,1)$, strongly degenerate (SD) if $\mu_a \in [1,2)$. The authors establish observability inequalities for weakly as well as strongly degenerate equations, and prove a negative result when the diffusion coefficient degenerate too violently ($\mu_a \geq 2$). Moreover, the authors prove observability (or controllability) time blows up as μ_a approaches 2 from below. Finally, using the Hilbert Uniqueness Method (HUM), they deduce the exact controllability for corresponding control system when $\mu_a \in [0,2)$.

In recent years, great attention has been given to control issues for parabolic equations with both degenerate and singular terms. However, this paper will not delve into the details here; instead, we note that a common strategy to demonstrate controllability is through the proof of global Carleman estimates. For regular degenerate coefficients, the Carleman estimates and null controllability properties are discussed in [2, 6, 8, 9, 24], for non-smooth degenerate coefficients in [14, 15], and for equations with both degenerate and singular coefficients in [13, 16, 27].

In [3] the authors were the first to study the boundary controllability of the wave equation with degenerate and singular, where the singularity occurs at the same point as the degeneration of the leading coefficient. To be more precise, they consider the problem

$$\begin{gathered} y_{tt} - (x^{\alpha} y_x)_x - \frac{\mu}{x^{2-\alpha}} y = 0, \quad (t,x) \in Q := (0,T) \times (0,1), \\ y(t,1) = f, \quad t \in (0,T), \\ y(t,0) = 0, \quad \alpha \in [0,1),\ t \in (0,T), \\ x^{\alpha} y_x(t,0) = 0, \quad \alpha \in [1,2),\ t \in (0,T), \\ y(0,x) = y_0(x), \quad y_t(0,x) = y_1(x), \quad x \in (0,1), \end{gathered} \tag{1.6}$$

where α and μ are two parameters such that $\alpha \in [0,2)\backslash\{1\}$, $\mu \leq (1-\alpha)^2/4$. Furthermore, the problem is weakly degenerate if $\alpha \in [0,1)$, strongly degenerate if $\alpha \in [1,2)$. The authors prove the observability estimate for the corresponding adjoint system by means of the multiplier method and new Hardy-type inequalities. Moreover, the null controllability is proved for sufficiently large time by HUM. Note that the degeneracy and singularity coefficient are not general in (1.6). Therefore, the purpose of this paper is to study the controllability and boundary observability of a degenerate wave equation with a singular term(in fact, it is an open question in reference [3]). To be more precise, we consider the degenerate/singular wave equation

$$\begin{gathered} y_{tt}-(a(x)y_x)_x-\frac{\lambda}{b(x)}y=0, \quad (t,x)\in Q:=(0,T)\times(0,1),\\ y(t,1)=f, \quad t\in(0,T),\\ y(t,0)=0, \quad \text{if } K_a\in[0,1),\ t\in(0,T),\\ \lim_{x\to 0^+} ay_x(t,x)=0, \quad \text{if } K_a\in[1,2),\ t\in(0,T),\\ y(0,x)=y_0(x), \quad y_t(0,x)=y_1(x), \quad x\in(0,1), \end{gathered} \tag{1.7}$$

where a(degenerate coefficient) and b(singular coefficient) positive on $(0,1]$ and vanish at zero, $\lambda\in\mathbb{R}$ and u_0,u_1 are the initial values, the parameter K_g defined by (1.8). The control function f acts on non-degenerate and non-singular boundary which is used drive the solution to zero at a sufficiently large time T.

Definition 1.1. Let $g(x)\in\mathcal{C}^1((0,1])\cap\mathcal{C}([0,1])$ be a function satisfying $g(x)>0$ on $(0,1]$, $g(0)=0$ and

$$\sup_{0<x\le 1}\frac{x|g'(X)|}{g(X)}=K_g. \tag{1.8}$$

When the function g as a degenerate coefficient, we say that g is weakly degenerate at 0 if $K_g\in[0,1)$, g is considered strongly degenerate at 0 if $K_g\in[1,2)$.

Remark 1.2. Clearly, when $g(X)\sim x^K$, it is considered weakly degenerate if $K\in[0,1)$, strongly degenerate if $K\in[1,2)$. Furthermore, the case where $K_g\ge 2$ is not considered because it does not achieve controllability, as discussed in [1] when $\lambda=0$.

Finally, we point out that studies like ours are significant, particularly in the fields of medical research [18], materials science for invisible materials [12, 21], and climate science [17].

This article is organized as follows. Section 2 presents some function spaces and preliminary results. In Section 3, we employ the Lax-Milgram theorem and semigroup theory to address the dual problem and investigate the well-posedness of the associated problem under Dirichlet and Neumann boundary conditions. In Section 4, an energy estimate is established and the direct inequality is proven, which will eventually lead to the controllability result. Section 5 introduces the observable inequality and determines the observable time. In Section 6, the controllable result is obtained. We conclude the paper with a summary in the final section.

2. Preliminaries

Assumption 2.1. *The functions a,b belong to $\mathcal{C}^1((0,1])\cap\mathcal{C}([0,1])$ and satisfy*

$$\begin{gathered} a(x),b(x)>0\ \forall x\in(0,1],\ a(0)=b(0)=0,\\ K_a\in[0,2)\setminus\{1\},K_b\in[0,2] \text{ and } K_a+K_b\le 2. \end{gathered} \tag{2.1}$$

From Definition 1.1 and Assumption 2.1, it is easy to draw the following consequences.

Remark 2.2. By integrating the inequality

$$sg'(s)\le K_g g(s), \quad \forall s\in(0,1]$$

over $[x,1]$, we obtain

$$g(x) \geq g(1)x^{K_g}, \quad \forall x \in [0,1].$$

Hence, for all $x \in [0,1]$ we deduce that

$$a(x) \geq a(1)x^{K_a}, \quad b(x) \geq b(1)x^{K_b}. \tag{2.2}$$

Proposition 2.3 (Hardy-Poincare inequality)**.** *Under Assumption 2.1, there exists $C_{a,b} > 0$ such that*

$$\int_0^1 \frac{u^2}{b}\,dx \leq C_{a,b}\int_0^1 a(u')^2\,dx, \quad \forall u \in \mathcal{C}_c^\infty(0,1), \tag{2.3}$$

where

$$C_{a,b} = \frac{4}{a(1)b(1)(1-K_a)^2}.$$

Proof. By Remark 2.2 and generalized Hardy inequality [11, chap. 5.3],

$$\frac{(1-K_a)^2}{4}\int_0^1 \frac{u^2}{x^{2-K_a}}\,dx \leq \int_0^1 x^{K_a}u_x^2\,dx \quad \forall u \in \mathcal{C}_c^\infty(0,1),$$

we have

$$\begin{aligned} \int_0^1 \frac{u^2}{b}\,dx &\leq \frac{1}{b(1)}\int_0^1 \frac{u^2}{x^{K_b}}\,dx \leq \frac{1}{a(1)b(1)}\int_0^1 a\frac{u^2}{x^{K_a+K_b}}\,dx \\ &\leq \frac{1}{a(1)b(1)}\int_0^1 a\frac{u^2}{x^2}\,dx \leq \frac{4}{a(1)b(1)(1-K_a)^2}\int_0^1 a(u')^2\,dx. \end{aligned} \tag{2.4}$$

□

Assumption 2.4. The constant $\lambda \in \mathbb{R}$ satisfies

$$\lambda < \frac{1}{C_{a,b}}. \tag{2.5}$$

Under Assumptions 2.1 and 2.4, as in [1] and [3], we introduce the following spaces with related inner product

$$V_a^1(0,1) = \{u \in L^2(0,1) \cap H^1_{\rm loc}(0,1] : \sqrt{a}u' \in L^2(0,1)\},$$

$$\|u\|^2_{V_a^1(0,1)} = \int_0^1 u^2 + a(u')^2\,dx, \quad \forall u \in V_a^1(0,1),$$

$$\langle u,v\rangle_{V_a^1(0,1)} = \int_0^1 uv + au'v'\,dx, \quad \forall u,v \in V_a^1(0,1),$$

$$V_a^2(0,1) = \{u \in V_a^1(0,1) | au' \in H^1(0,1)\};$$

and

$$V_{a,b}^1(0,1) = \{u \in L^2(0,1) \cap H^1_{\rm loc}(0,1] | a(u')^2 - \frac{\lambda}{b}u^2 \in L^1(0,1)\},$$

$$\|u\|^2_{V_{a,b}^1(0,1)} = \int_0^1 u^2 + a(u')^2 - \frac{\lambda}{b}u^2\,dx, \quad \forall u \in V_{a,b}^1(0,1),$$

$$\langle u,v\rangle_{V_{a,b}^1(0,1)} = \int_0^1 uv + au'v' - \frac{\lambda}{b}uv\,dx, \quad \forall u,v \in V_{a,b}^1(0,1),$$

$$V_{a,b}^2(0,1) = \{u \in V_{a,b}^1(0,1) | (au')' - \frac{\lambda}{b}u \in L^2(0,1)\}.$$

Under the boundary conditions of (1.7), we introduce the space $H^1_{a,b}(0,1)$.

(i) If $K_a \in [0,1)$,

$$H^1_{a,b}(0,1) = \{u \in V^1_{a,b}(0,1) | u(0) = u(1) = 0\};$$

(ii) If $K_a \in (1,2)$,

$$H^1_{a,b}(0,1) = \{u \in V^1_{a,b}(0,1) | u(1) = 0\}.$$

Also, $H^{-1}_{a,b}(0,1)$ denotes the conjugate space of $H^1_{a,b}(0,1)$. We set

$$H^2_{a,b}(0,1) = V^2_{a,b}(0,1) \cap H^1_{a,b}(0,1).$$

From Assumption 2.4, when $\lambda > 0$ there exists $\theta \in (0,1)$ such that

$$\lambda = \frac{1}{C_{a,b}} - \frac{\theta}{C_{a,b}}. \tag{2.6}$$

Further, one can prove the next result.

Lemma 2.5. *Under Assumptions 2.1 and 2.4, we have*

$$\int_0^1 a(u')^2\,dx \le \frac{1}{C_\theta}\int_0^1 a(u')^2 - \frac{\lambda}{b}u^2\,dx, \tag{2.7}$$

where $C_\theta = \theta$ *if* $\lambda \in \Big(0, \frac{1}{C_{a,b}}\Big)$, $C_\theta = 1$ *if* $\lambda \le 0$.

Proof. (i) If $\lambda \in \Big(0, \frac{1}{C_{a,b}}\Big)$, then by (2.3) (2.6), we deduce that

$$\int_0^1 a(u')^2 - \frac{\lambda}{b}u^2\,dx \ge \int_0^1 a(u')^2\,dx - (1-\theta)\int_0^1 a(u')^2\,dx = \theta\int_0^1 a(u')^2\,dx.$$

(ii) If $\lambda \le 0$, obviously,

$$\int_0^1 a(u')^2 - \frac{\lambda}{b}u^2\,dx \ge \int_0^1 a(u')^2\,dx.$$

□

Assumption 2.6. Under Assumptions 2.1 and 2.4, the function

$$x \to \frac{x^{K_b}}{b(x)}$$

is nondecreasing in a right neighborhood of $x = 0$.

Remark 2.7. It is clear that, if Assumption 2.6 holds, then

$$\lim_{x\to 0^+} \frac{x^\gamma}{b(x)} = 0, \quad \gamma > K_b. \tag{2.8}$$

In particular,

$$\lim_{x\to 0^+} \frac{x^2}{b(x)} = 0. \tag{2.9}$$

Lemma 2.8. *Under Assumption 2.6, for all* $u \in H^1_{a,b}(0,1)$, *we have*

$$\lim_{x\to 0^+} \frac{x}{b(x)}u^2 = 0. \tag{2.10}$$

Proof. If $0 \le K_a < 1$, using that $u(0) = 0$, we have

$$|u(x)| \le \int_0^x |u(\xi)| d\xi \le \sqrt{x} \|u'\|_{L^2(0,1)}.$$

Then

$$\frac{x}{b(x)} u^2(x) \le \frac{x^2}{b(x)} \|u'\|_{L^2(0,1)}.$$

By equation (2.9), the lemma follows.

If $1 < K_a < 2$, then $0 \le K_b < 1$, the conclusion follows directly from Remark 2.7. □

3. Well-posedness

First, we consider the degenerate/singular wave problem with Dirichlet/Neumann boundary conditions:

$$\begin{gathered} u_{tt} - (a(x)u_x)_x - \frac{\lambda}{b(x)} u = 0, \quad (t,x) \in (0,T) \times (0,1), \\ u(t,1) = 0, \quad t \in (0,T), \\ u(t,0) = 0, \quad K_a \in [0,1),\ t \in (0,T), \\ \lim_{x \to 0^+} a u_x(t,x) = 0, \quad K_a \in (1,2),\ t \in (0,T), \\ u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x), \quad x \in (0,1). \end{gathered} \tag{3.1}$$

Let us recall the typical abstract setup of semigroup theory, which provides weak and classical solutions for the above system. Consider the Hilbert space $\mathcal{H} = H^1_{a,b}(0,1) \times L^2(0,1)$ endowed with the inner product

$$\langle (u,v), (\tilde{u}, \tilde{v}) \rangle_{\mathcal{H}} = \int_0^1 v\tilde{v} + a u' \tilde{u}' - \frac{\lambda}{b} u \tilde{u} \, dx, \quad \forall (u,v), (\tilde{u}, \tilde{v}) \in \mathcal{H}.$$

By Assumption 2.4 and the Hardy-Poincare inequality (2.3), we have

$$\langle (u,v), (u,v) \rangle_{\mathcal{H}} = \int_0^1 v^2 + a(u')^2 - \frac{\lambda}{b} u^2 \, dx \ge 0, \quad \forall (u,v) \in \mathcal{H}.$$

Thus, $\langle \cdot, \cdot \rangle_{\mathcal{H}}$ forms the scalar product.

Arguing as for the classical wave equation, the unbounded operator $\mathcal{A} : D(\mathcal{A}) \subset \mathcal{H} \to \mathcal{H}$ is defined by

$$\mathcal{A}(u,v) = \Big(v, a(u')' + \frac{\lambda}{b} u \Big), \quad \forall (u,v) \in D(\mathcal{A}) \tag{3.2}$$

with

$$D(\mathcal{A}) = H^2_{a,b}(0,1) \times H^1_{a,b}(0,1),$$

if $K_a \in [0,1)$, or

$$D(\mathcal{A}) = \{ (u,v) \in H^2_{a,b}(0,1) \times H^1_{a,b}(0,1) : a u_x(0) = 0 \},$$

provide $K_a \in (1,2)$.

Proposition 3.1. *Under Assumption 2.6, the operator $\mathcal{A}$ is maximally dissipative on $\mathcal{H}$.*

Proof. Let $(u,v)\in D(\mathcal{A})$. Then

$$\langle \mathcal{A}(u,v),(u,v)\rangle_{\mathcal{H}} = \int_0^1 \Big(a(u')'v + \frac{\lambda}{b}uv + au'v' - \frac{\lambda}{b}uv\Big)\,dx = 0.$$

Therefore, $\mathcal{A}$ is dissipative. It remains to be proved that the operator is maximally dissipative, which is equivalent to showing that $I-\mathcal{A}$ is surjective. Specifically, for any $(g_1,g_2)\in\mathcal{H}$, we need to find $(u,v)\in D(\mathcal{A})$ such that the problem

$$\begin{gathered} v = u - g_1,\\ u - a(u')' - \frac{\lambda}{b}u = g_1 + g_2. \end{gathered} \tag{3.3}$$

So far that, we consider the bilinear form $\beta: H^1_{a,b}(0,1)\times H^1_{a,b}(0,1)\to\mathbb{R}$ given by $\beta(u,\varphi)=\int_0^1\big(u\varphi + au'\varphi' - \frac{\lambda}{b}u\varphi\big)\,dx$, and the linear functional $L: H^1_{a,b}(0,1)\to\mathbb{R}$ given by $L\varphi=\int_0^1 (g_1+g_2)\varphi\,dx$. One can verify that β is a continuous and coercive bilinear functional on $\mathcal{H}$. Also, L is a continuous linear functional. Consequently, by the Lax-Milgram theorem, there exist a unique solution $u\in H^1_{a,b}(0,1)$ to the variational problem

$$\beta(u,\varphi) = L\varphi, \quad \forall\varphi\in H^1_{a,b}(0,1). \tag{3.4}$$

Since $\mathcal{C}_c^\infty(0,1)\subset H^1_{a,b}(0,1)$, we have

$$\int_0^1 \Big(u\varphi + au'\varphi' - \frac{\lambda}{b}u\varphi\Big)\,dx = \int_0^1 (g_1+g_2)\varphi\,dx, \quad \forall\varphi\in\mathcal{C}_c^\infty(0,1). \tag{3.5}$$

By duality, this implies that

$$u - a(u')' - \frac{\lambda}{b}u = g_1 + g_2$$

in the sense of distributions. Thus, $u\in H^2_{a,b}(0,1)$ and $u - a(u')' - \frac{\lambda}{b}u = g_1+g_2$ almost everywhere in $(0,1)$. Setting $v=u-g_1$, we conclude that $(u,v)\in D(\mathcal{A})$ and the problem (3.3) is solved. □

Therefore $\mathcal{A}$ is the generator of a contraction semigroup in $\mathcal{H}$, denoted by $e^{t\mathcal{A}}$. For any $U_0=(u_0,u_1)\in\mathcal{H}$, $U(t)=e^{t\mathcal{A}}U_0$ can be seen as the solution of the Cauchy problem

$$\begin{gathered} U'(t) = \mathcal{A}U(t),\\ U(0) = U_0. \end{gathered} \tag{3.6}$$

Hence, as in [1] or [3], we have the following conclusions.

Corollary 3.2. *Assume Assumption 2.6, for given $(u_0,u_1)\in H^1_{a,b}(0,1)\times L^2(0,1)$, there exist a unique mild solution u to the problem (3.1) satisfying*

$$u\in\mathcal{C}^1([0,T];L^2(0,1))\cap\mathcal{C}([0,T];H^1_{a,b}(0,1));$$

If $(u_0,u_1)\in D(\mathcal{A})$, then the solution u is classical, in the sense that

$$u\in\mathcal{C}^2([0,T];L^2(0,1))\cap\mathcal{C}^1([0,T];H^1_{a,b}(0,1)\cap\mathcal{C}([0,T];H^2_{a,b}(0,1)).$$

4. Energy estimate

In this section, we establish an estimate of the energy and a direct inequality associated to the solution of the initial value problem, which will be used to prove a controllability in Section 6.

Definition 4.1. Using Assumption 2.6, we define the generalized energy of a mild solution u of (3.1) as follows,

$$E_u(t) = \frac{1}{2}\int_0^1 \Big(u_t^2 + au_x^2 - \frac{\lambda}{b}u^2\Big)\,dx, \quad \forall t \geq 0. \tag{4.1}$$

Computations show that the conservation of the energy E_u remains valid in the degenerate and singular situation.

Proposition 4.2. *Under Assumption 2.6 and considering* $(u_0, u_1) \in H^1_{a,b}(0,1) \times L^2(0,1)$*, The energy* $E_u(t)$ *of the mild solution* u *of* (3.1) *is constant in time, that is,*

$$E_u(t) = E_u(0), \quad \forall t \geq 0. \tag{4.2}$$

Proof. First, suppose that u is a classical solution. Then, multiplying the equation by u_t and integrating over $(0,1)$, we obtain

$$\begin{aligned} 0 &= \int_0^1 u_t(t,x)\Big(u_{tt}(t,x) - (a(x)u_x(t,x))_x - \frac{\lambda}{b(x)}u(t,x)\Big)\,dx \\ &= \underbrace{\int_0^1 \Big(u_t(t,x)u_{tt}(t,x) - a(x)u_x(t,x)u_{tx}(t,x) - \frac{\lambda}{b(x)}u(t,x)u_t(t,x)\Big)\,dx}_{=\frac{d}{dt}E_u(t)} \\ &\quad - [a(x)u_x(t,x)u_t(t,x)]_0^1. \end{aligned} \tag{4.3}$$

Using the boundary conditions and Alabau-Boussouira et al. [1, Proposition 2.5], we have that the boundary $a(x)u_x(t,x)u_t(t,x)$ vanishes at $x = 1$ and $x = 0$. Now, let u be the mild solution associated with the initial data $(u_0, u_1) \in H^1_{a,b}(0,1) \times L^2(0,1)$. Consider a sequence $\{u_0^n, u_1^n\}_{n\in\mathbb{N}} \subset D(\mathcal{A}) = H^2_{a,b}(0,1) \times H^1_{a,b}(0,1)$ that approximates (u_0, u_1), and let u^n be the classical solution of (3.1) associated to (u_0^n, u_1^n). u^n satisfies (4.2) and u_x^n is a Cauchy sequence in $L^2(0,1)$. Therefore, we extend (4.2) to the mild solution. ☐

To facilitate the subsequent proof of controllability results, we prove the following direct inequality.

Proposition 4.3. *Under Assumption 2.6, if* u *is a classical solution of* (3.1)*, then*

$$\begin{aligned} a(1)\int_0^T u_x^2(t,1)dt &= \int_Q \Big(u_t^2 + (a - xa')u_x^2 + \lambda\frac{b - xb'}{b^2}\Big)\,dx\,dt \\ &\quad + 2\Big[\int_0^1 xu_xu_t dx\Big]_0^T. \end{aligned} \tag{4.4}$$

As a consequence, if u is a mild solution, then $u_x(\cdot,1) \in L^2(0,T)$ for every $T > 0$ and

$$
\begin{aligned}
&a(1)\int_0^T u_x^2(t,1)dt \\
&\le 4\max\{\frac{1}{a(1)C_\theta},1\}E_u(0) + \frac{2T}{C_\theta}\left(1+K_a+C_{a,b}|\lambda|(1+K_b)\right)E_u(0).
\end{aligned}
\tag{4.5}
$$

Proof. Suppose first that $(u_0,u_1) \in H^2_{a,b}(0,1)\times H^1_{a,b}(0,1)$, so that u is a classical solution of (3.1). Then, multiplying (3.1) by xu_x and integrating over Q, we obtain

$$
\begin{aligned}
0 &= \int_Q xu_x\Big(u_{tt} - (au_x)_x - \frac{\lambda}{b}u\Big)\,dx\,dt \\
&= \Big[\int_0^1 xu_xu_t dx\Big]_0^T - \int_Q xu_{xt}u_t\,dx\,dt \\
&\quad - \int_Q \Big(xa'u_x^2 + xau_xu_{xx} + \frac{\lambda}{b}xuu_x\Big)\,dx\,dt \\
&= \Big[\int_0^1 xu_xu_t dx\Big]_0^T - \int_Q xa'u_x^2\,dx\,dt \\
&\quad - \int_Q \Big(x\Big(\frac{u_t^2}{2}\Big)_x + xa\Big(\frac{u_x^2}{2}\Big)_x + \frac{\lambda}{2}\frac{x(u^2)_x}{b}\Big)\,dx\,dt.
\end{aligned}
\tag{4.6}
$$

Arguing as in the proof of Alabau-Boussouira et al. [1, Lemma 3.2],

$$
\Big[x\frac{u_t^2}{2}\Big]_0^1 = 0,
$$

$$
[xau_x^2]_0^1 = a(1)u_x^2(t,1).
$$

From the boundary conditions and Lemma 2.8, we have

$$
\Big[\frac{xu^2}{b}\Big]_0^1 = 0.
$$

Hence,

$$
\int_Q x\Big(\frac{u_t^2}{2}\Big)_x\,dx\,dt = -\frac{1}{2}\int_Q u_t^2\,dx\,dt, \tag{4.7}
$$

$$
\int_Q xa\Big(\frac{u_x^2}{2}\Big)_x\,dx\,dt = -\frac{1}{2}\int_Q (a+xa')u_x^2\,dx\,dt + \frac{1}{2}a(1)\int_0^T u_x^2(t,1)dt, \tag{4.8}
$$

$$
\frac{\lambda}{2}\int_Q \frac{x(u^2)_x}{b}\,dx\,dt = -\frac{\lambda}{2}\int_Q \frac{b-xb'}{b^2}u^2\,dx\,dt\,. \tag{4.9}
$$

Then (4.4) follows by inserting (4.7)–(4.9) into (4.6). Next, we estimate the term on the right side of equation (4.4) separately. According the Hölder inequality and Lemma 2.5, we have

$$
\begin{aligned}
2\int_0^1 xu_xu_t dx &\le \int_0^1 \Big(x^2u_x^2 + u_t^2\Big)dx \\
&\le \frac{1}{a(1)C_\theta}\int_0^1 \Big(au_x^2 - \frac{\lambda}{b}u^2\Big)\,dx + \int_0^1 u_t^2\,dx \\
&\le 2\max\Big\{\frac{1}{a(1)C_\theta},1\Big\}E_u(0).
\end{aligned}
\tag{4.10}
$$

Moreover, Using the definition of K_g and Hardy's inequality (2.3), we have

$$\int_0^1 (a+xa')u_x^2 dx \le (1+K_a)\int_0^1 au_x^2\,dx \le \frac{(1+K_a)}{C_\theta}E_u(0), \tag{4.11}$$

and

$$\begin{aligned}\lambda\int_0^1 \frac{b-xb'}{b^2}u^2\,dx &\le \int_0^1 \frac{\lambda}{b}\Big(1-\frac{xb'}{b}\Big)u^2dx\\ &\le \int_0^1 \frac{|\lambda|}{b}(1+K_b)u^2dx\\ &\le \frac{2C_{a,b}|\lambda|(1+K_b)}{C_\theta}E_u(0).\end{aligned} \tag{4.12}$$

Hence, by (4.4) and the inequalities (4.10)–(4.12), we obtain (4.5). As before, to extend (4.5) to the mild solution associated with the initial data $(u_0,u_1)\in H^1_{a,b}(0,1)\times L^2(0,1)$, it suffices to approximate such data by $(u_0^n,u_1^n)\in H^2_{a,b}(0,1)\times H^1_{a,b}(0,1)$, and thanks to (4.5), we can show that the normal derivatives of the corresponding classical solutions give a Cauchy sequence in $L^2(0,1)$. □

5. Boundary observability

Lemma 5.1. *Under Assumption 2.6, for any mild solution u of* (3.1) *and every $T\ge 0$, we have*

$$\int_Q \Big(a(x)u_x^2 - u_t^2 - \frac{\lambda}{b(x)}u^2\Big)\,dx\,dt + \Big[\int_0^1 uu_t\,dx\Big]_0^T = 0. \tag{5.1}$$

Proof. As before, suppose that u is the classical solution of (3.1). Multiplying (3.1) by u and integrating over the domain $Q=(0,T)\times(0,1)$, we obtain

$$\begin{aligned}0 &= \int_0^1 u\Big(u_{tt} - (a(x)u_x)_x - \frac{\lambda}{b(x)}u\Big)\,dx\\ &= \int_Q \Big(a(x)u_x^2 - u_t^2 - \frac{\lambda}{b(x)}u^2\Big)\,dx\,dt + \Big[\int_0^1 uu_t\,dx\Big]_0^T - \int_0^T [a(x)uu_x]_0^1\,dt.\end{aligned} \tag{5.2}$$

Thanks to the boundary conditions and Alabau-Boussouira et al. [12, Proposition 2.5], we have that auu_x also vanishes at $x=0$ and at $x=1$. The conclusion can be extended to mild solution by an approximation argument. □

Theorem 5.2. *Under Assumption 2.6, let u be a mild solution of* (3.1)*. Then, for every $T\ge 0$,*

$$\begin{aligned}a(1)\int_0^T u_x^2(t,1)dt \ge& -\Big(4\max\{\frac{1}{a(1)\theta},1\} + 2K_a\frac{1}{\sqrt{\theta a(1)}}\Big)E_u(0)\\ &+ T\{(2-K_a)+\lambda C_{a,b}\theta\,(2-K_a-K_b)\}E_u(0),\end{aligned} \tag{5.3}$$

for $\lambda\in(0,\frac{1}{C_{a,b}})$, and

$$\begin{aligned}a(1)\int_0^T u_x^2(t,1)dt \ge& -\Big(4\max\{\frac{1}{a(1)},1\} + 2K_a\frac{1}{\sqrt{a(1)}}\Big)E_u(0)\\ &+ T\{(2-K_a)-|\lambda|C_{a,b}\,(2-K_a-K_b)\}E_u(0),\end{aligned} \tag{5.4}$$

for $\lambda\in(-\infty,0]$, where the constants $C_{a,b}$ and θ are given in (2.3) *and* (2.6)*, respectively.*

Proof. As usual, let us suppose that u is a classical solution of (3.1). Multiplying both sides of equation (5.1) by $\frac{K_a}{2}$ and summing the corresponding ones to both side of equation (4.4), we obtain

$$\begin{aligned} &a(1)\int_0^T u_x^2(t,1)dt \\ &= 2\Big[\int_0^1 xu_xu_tdx\Big]_0^T + \frac{K_a}{2}\Big[\int_0^1 uu_t\,dx\Big]_0^T + \Big(1-\frac{K_a}{2}\Big)\int_Q u_t^2\,dx\,dt \\ &\quad + \int_Q \Big[\Big(1+\frac{K_a}{2}\Big)a - xa'\Big]u_x^2\,dx\,dt + \int_Q \frac{\lambda}{b}\Big(\frac{b-xb'}{b} - \frac{K_a}{2}\Big)u^2\,dx\,dt \\ &= \int_Q \Big(1-\frac{K_a}{2}\Big)u_t^2 + \Big[\Big(1+\frac{K_a}{2}\Big)a - xa'\Big]u_x^2 + \Big(1-\frac{K_a}{2}\Big)\frac{\lambda}{b}u^2\,dx\,dt \\ &\quad + 2\Big[\int_0^1 xu_xu_tdx\Big]_0^T + \frac{K_a}{2}\Big[\int_0^1 uu_t\,dx\Big]_0^T + \int_Q \frac{\lambda}{b}\Big(2-\frac{xb'}{b} - K_a\Big)u^2\,dx\,dt. \end{aligned} \tag{5.5}$$

Using Remark 2.2 and the inequality $xa' \le K_a a$, we have

$$\begin{aligned} &\int_Q \Big(1-\frac{K_a}{2}\Big)u_t^2 + \Big[\Big(1+\frac{K_a}{2}\Big)a - xa'\Big]u_x^2 + \Big(1-\frac{K_a}{2}\Big)\frac{\lambda}{b}u^2\,dx\,dt \\ &\ge \Big(1-\frac{K_a}{2}\Big)\int_Q \Big(u_t^2 + au_x^2 - \frac{\lambda}{b}u^2\Big)\,dx\,dt \\ &\ge (2-K_a)TE_u(0). \end{aligned} \tag{5.6}$$

By (4.10), we obtain

$$2\Big[\int_0^1 xu_xu_tdx\Big]_0^T \le 4\max\{\frac{1}{a(1)C_\theta}, 1\}E_u(0). \tag{5.7}$$

Furthermore, applying the Hardy inequality in its pure degenerate form (see [1]), we obtain

$$\int_0^1 u^2dx \le \frac{4}{a(1)}\int_0^1 au_x^2dx, \quad \forall u \in \mathcal{C}_c^\infty(0,1),$$

from which we can deduce that

$$\begin{aligned} \int_0^1 uu_t\,dx &\le \frac{1}{2}\int_0^1 \Big(\frac{\sqrt{C_\theta a(1)}}{2}u^2 + \frac{2}{\sqrt{C_\theta a(1)}}u_t^2\Big)\,dx \\ &\le \frac{1}{2}\int_0^1 \Big(2\sqrt{\frac{C_\theta}{a(1)}}au_x^2 + \frac{2}{\sqrt{C_\theta a(1)}}u_t^2\Big)\,dx \\ &\le \frac{1}{\sqrt{C_\theta a(1)}}\int_0^1 \Big(u_t^2 + au_x^2 - \frac{\lambda}{b}u^2\Big)dx \\ &= \frac{2}{\sqrt{C_\theta a(1)}}E_u(0). \end{aligned} \tag{5.8}$$

Finally, if $\lambda \in \Big(0, \frac{1}{C_{a,b}}\Big)$, we have

$$\begin{aligned} \int_Q \frac{\lambda}{b}\Big(2-\frac{xb'}{b} - K_a\Big)u^2\,dx\,dt &\ge \int_Q \frac{\lambda}{b}\left(2-K_b-K_a\right)u^2\,dx\,dt \\ &\ge \lambda C_{a,b}C_\theta\left(2-K_a-K_b\right)TE_u(0); \end{aligned} \tag{5.9}$$

If $\lambda \in (-\infty, 0]$, then

$$\begin{aligned} \int_Q \frac{\lambda}{b}\Big(2-\frac{xb'}{b}-K_a\Big)u^2\,dx\,dt &\le \int_Q \frac{|\lambda|}{b}\left(2-K_b-K_a\right)u^2\,dx\,dt \\ &\le |\lambda| C_{a,b}C_\theta\left(2-K_b-K_a\right)TE_u(0). \end{aligned} \tag{5.10}$$

Notice that $C_\theta = \theta$ if $\lambda \in (0, \frac{1}{C_{a,b}})$; $C_\theta = 1$ if $\lambda \in (-\infty, 0]$. Therefore, (5.3) and (5.4) by substituting (5.6)-(5.10) into (5.5). □

We recall that (3.1) is said to be observable in time $T \ge 0$ via the normal derivative at $x = 1$, if there exists a constant $C > 0$ such that for any $(u_0, u_1) \in H_a^1(0,1) \times L^2(0,1)$, the mild solution of (3.1) satisfies

$$\int_0^T u_x^2(t,1)\,dt \ge CE_u(0). \tag{5.11}$$

Moreover, any constant satisfying (5.11) is called an observability constant for (3.1) in time $T \ge 0$. The supremum of all observability constants for (3.1) is denoted by C_T, namely,

$$C_T := \sup\{C > 0, C \text{ satisfies } (5.11)\}.$$

We said that (3.1) is observable if

$$C_T = \inf_{(u_0,u_1)\neq(0,0)} \frac{\int_0^T u_x^2(t,1)dt}{E_u(0)} > 0. \tag{5.12}$$

From the definition of observability, and Theorem 5.2, we have following corollary.

Corollary 5.3. *Under Assumption 2.6 and $\lambda > 0$, (3.1) is observable in time T, provided that*

$$T > T_{a,b} := \frac{C_2}{C_1}.$$

In this case,

$$C_T \ge \frac{1}{a(1)}(C_1T - C_2),$$

where

$$\begin{gathered} C_1 = (2-K_a) + \lambda C_{a,b}\theta\left(2-K_a-K_b\right), \\ C_2 = \Big(4\max\{\frac{1}{a(1)\theta}, 1\} + 2K_a\frac{1}{\sqrt{\theta a(1)}}\Big). \end{gathered}$$

Corollary 5.4. *Under Assumption 2.6, $\frac{K_a-2}{C_{a,b}(2-K_b-K_a)} < \lambda \le 0$, and $\lambda \le 0$ if $K_a + K_b = 2$. We have that (3.1) is observable in time T, provided that*

$$T > T_{a,b} := \frac{C_4}{C_3}.$$

Moreover,

$$C_T \ge \frac{1}{a(1)}(C_3T - C_4),$$

where

$$\begin{gathered} C_3 = \{(2-K_a) - |\lambda| C_{a,b}\left(2-K_a-K_b\right)\}, \\ C_4 = \Big(4\max\{\frac{1}{a(1)}, 1\} + 2K_a\frac{1}{\sqrt{\theta a(1)}}\Big). \end{gathered}$$

6. Controllability

In this section, we study the problem of exact controllability for (1.7). By its linearity and reversibility, it is straightforward to verify that exact controllability will hold as long as it is valid for any initial data (y_0, y_1) and a zero final state. Equivalently, given $(y_0, y_1) \in H^1_{a,b}(0,1) \times L^2(0,1)$, we seek a control function $f \in L^2(0,T)$ such that the solution of (1.7) satisfies $(y, y')(T, \cdot) \equiv 0$.

Definition 6.1. Let $f \in L^2_{\rm loc}(0,T)$ and $(y_0, y_1) \in L^2(0,1) \times H^{-1}_{a,b}(0,1)$ be arbitrarily fixed. We say that y is a solution by transposition of (1.7) if
$y \in \mathcal{C}^1([0,T]; H^{-1}_{a,b}(0,1) \cap \mathcal{C}([0,T]; L^2(0,1))$ and for any $T > 0$,

$$\begin{aligned} &\langle y_t(T), w_T^0\rangle_{H_a^{-1}(0,1)\times H_a^1(0,1)} - \int_0^1 y(T) w_T^1\,dx \\ &= \langle y_1, w(0)\rangle_{H_{a,b}^{-1}(0,1)\times H_{a,b}^1(0,1)} - \int_0^1 y(0) w'(0)\,dx + a(1)\int_0^T f(t) w_x(t,1)\,dt \end{aligned} \tag{6.1}$$

for all $(w_T^0, w_T^1) \in H^1_{a,b}(0,1) \times L^2(0,1)$, where w is the solution of the backward equation

$$\begin{gathered} w_{tt} - (a(x) w_x)_x - \frac{\lambda}{b(x)} w = 0, \quad (t,x) \in (0,T) \times (0,1), \\ w(t,1) = 0, \quad t \in (0,T), \\ w(t,0) = 0, \quad K_a \in [0,1),\ t \in (0,T), \\ a(x) w_x(t,0) = 0, \quad K_a \in (1,2),]; t \in (0,T), \\ w(T,x) = w_T^0(x), \quad w_t(T,x) = w_T^1(x), \quad x \in (0,1). \end{gathered} \tag{6.2}$$

By setting $y(t,x) = w(T-t,x)$, we leverage the time reversibility of the wave equation to assert that the solution y maintains the same regularity as w for $t \le 0$. Consequently, the backward equation (6.2) admits a unique solution $w \in \mathcal{C}^1([0,T]; L^2(0,1) \cap \mathcal{C}([0,T]; H^1_{a,b}(0,1))$ which depends continuously on the initial data $W_T = (w_T^0, w_T^1) \in \mathcal{H}$. By Proposition 4.2, the energy $E_w(t)$ of w is conserved through time, which implies that the direct inequality (4.5) and observabilities inequality (5.3) and (5.4) remain valid for w. Therefore, there is a unique solution by transposition $w \in \mathcal{C}^1([0,T]; H^{-1}_{a,b}(0,1)) \cap \mathcal{C}([0,T]; L^2(0,1))$.

Now, consider the bilinear form $\Lambda : \mathcal{H} \times \mathcal{H} \to \mathbb{R}$ defined as

$$\Lambda(W_T, \tilde{W}_T) = a(1) \int_0^T w_x(t,1) \tilde{w}_x(t,1) dt, \tag{6.3}$$

where $w_x, \tilde{w}_x$ are the solution of (1.7) associated with the final data $W_T := (w_T^0, w_T^1)$, $\tilde{W}_T := (\tilde{w}_T^0, \tilde{w}_T^1)$, respectively. To prove that (1.7) is exactly controllable, the following Lemma is key.

Lemma 6.2. *Under Assumption 2.6, the bilinear form Λ is continuous and coercive.*

Proof. By the direct inequality and the result of energy conservation,

$$\begin{aligned}
|\Lambda(W_T, \tilde{W}_T)| &\le a(1)\int_0^T |w_x(t,1)\tilde{w}_x(t,1)|\,dt \\
&\le \Big(a(1)\int_0^T w_x^2(t,1)\,dt\Big)^{1/2}\Big(a(1)\int_0^T \tilde{w}_x^2(t,1)\,dt\Big)^{1/2} \\
&\le C E_{\mathrm{w}}^{1/2}(T) E_{\tilde{\mathrm{w}}}^{1/2}(T) \\
&\le C\|W_T\|_{\mathcal{H}}\|\tilde{W}_T\|_{\mathcal{H}}.
\end{aligned} \tag{6.4}$$

By the observability inequality, we have

$$\Lambda(W_T, W_T) = a(1)\int_0^T w_x^2(t,1)\,dt \ge C_T E_w(T) = C\|W_T\|^2_{\mathcal{H}}. \tag{6.5}$$

□

Theorem 6.3. *Under Assumption 2.6, for all $T > T_{a,b}$ and for all $(y_0, y_1) \in L^2(0,1) \times H^{-1}_{a,b}(0,1)$, there exists a control $f \in L^2(0,T)$ such that the solution (in the sense of transposition) satisfies*

$$y(T,x) = y_t(T,x) = 0, \quad \forall x \in (0,1).$$

Proof. We define the continuous linear map

$$\mathcal{L}(W_T) = \int_0^1 y_0 w_t(0)\,dx - \langle y_1, w(0)\rangle_{H^{-1}_{a,b}(0,1)\times H^1_{a,b}(0,1)}, \quad \forall W_T \in H^1_{a,b}(0,1)\times L^2(0,1).$$

Thanks to Lemma 6.2 and the Lax-Milgram theorem, there exists a unique $W_T \in H^1_{a,b}(0,1) \times L^2(0,1)$ such that

$$\Lambda(W_T, \tilde{W}_T) = L(\tilde{W}_T), \quad \forall \tilde{W}_T \in H^1_{a,b}(0,1) \times L^2(0,1). \tag{6.6}$$

We set $f = w_x(t,1)$ and denote by y the solution by transposition of (1.7). Then we have

$$\begin{aligned}
&a(1)\int_0^T f(t)\tilde{w}_x(t,1)\,dt \\
&= a(1)\int_0^T w_x(t,1)\tilde{w}_x(t,1)dt = \Lambda(W_T, \tilde{W}_T) = L(\tilde{W}_T) \\
&= \int_0^1 u_0\tilde{w}_t(0)\,dx - \langle u_1, \tilde{w}(0)\rangle_{H^{-1}_{a,b}(0,1)\times H^1_{a,b}(0,1)},
\end{aligned} \tag{6.7}$$

for all $(\tilde{w}_T^0, \tilde{w}_T^1) \in H^1_{a,b}(0,1) \times L^2(0,1)$. On the other hand, by the definition of the solution by transposition, for all $(\tilde{w}_T^0, \tilde{w}_T^1) \in H^1_{a,b}(0,1) \times L^2(0,1)$ we have

$$\begin{aligned}
a(1)\int_0^T f(t)\tilde{w}_x(t,1)\,dt &= \int_0^T y(T)\tilde{w}_T^1\,dt - \langle y_t(T), \tilde{w}_T^0\rangle_{H^{-1}_{a,b}(0,1)\times H^1_{a,b}(0,1)} \\
&\quad + \langle y_1, \tilde{w}(0)\rangle_{H_a^{-1}(0,1)\times H^1_{a,b}(0,1)} - \int_0^1 y(0)w_t(0)\,dx.
\end{aligned} \tag{6.8}$$

By equations (6.7) and (6.8), we deduce that

$$\langle y_t(T), \tilde{w}_T^0\rangle_{H^{-1}_{a,b}(0,1)\times H^1_{a,b}(0,1)} = \int_0^T y(T)\tilde{w}_T^1\,dt, \quad \forall(\tilde{w}_T^0, \tilde{w}_T^1) \in H^1_{a,b}(0,1) \times L^2(0,1).$$

Hence, we conclude that

$$y(T,x) = y_t(T,x) = 0, \quad \forall x \in (0,1). \qquad \square$$

7. Conclusion

In this article, we have considered the controllability of degenerate and singular wave equations, and obtained the exact controllability of the system under certain assumptions. We have adopted the coefficient settings from Reference [1], enhanced the equation system from Reference [3], and replaced the particular exponential form of the degenerate and singular coefficients presented in [3] with a more generalized form. Furthermore, we do not necessitate the relationship between the generalized coefficients to be as stringent as in [3]; it is only necessary that their sum falls within a certain range ($K_a + K_b \leq 2$). This allows for the application of diverse technical approaches when addressing the singular term, leading to various trade-offs in parameter λ selection. From this perspective, article [3] can be viewed as a special case of this paper, which is also one of the novel aspects of this work. To more intuitively illustrate this point, we have provided an example other than x^α as follows.

Example. Let $\theta \in (0,2)$ be given, we construct the functions $a(x), b(x)$ as follows

$$a(x) = \begin{cases} x^\theta \left(1 + \sin^2(\ln x^\alpha)\right) & \text{if } x \in (0,1] \\ 0 & \text{if } x = 0, \end{cases}$$

where $\alpha \in (0, 1 - \theta/2)$;

$$b(x) = \begin{cases} x^{2-\theta} \left(1 + \sin^2(\ln x^\beta)\right) & \text{if } x \in (0,1] \\ 0 & \text{if } x = 0, \end{cases}$$

where $\beta \in [\theta/2 - 1, -\alpha]$. Then the functions $a(x), b(x)$ satisfy Assumption 2.1. Indeed,

$$a'(x) = \theta x^{\theta-1} \left(1 + \sin^2(\ln x^\alpha)\right) + 2\alpha x^{\theta-1} \sin(\ln x^\alpha)\cos(\ln x^\alpha)$$

so that $K_a \leq \theta + 2\alpha < 2$. It is also easy to discover that K_a is not always equal to 1.

$$b'(x) = (2-\theta) x^{1-\theta} \left(1 + \sin^2(\ln x^\beta)\right) + 2\beta x^{1-\theta} \sin(\ln x^\beta)\cos(\ln x^\beta)$$

so that $K_b \leq 2 - \theta + 2\beta \leq 2$, and $K_a + K_b \leq 2 + 2\alpha + 2\beta \leq 2$.

We summarize the comparison between this paper and existing literature as follows.

	Degenerate term	Singular term	
In [3]	$x^\alpha,\ \alpha \in [0,2) \setminus \{1\}$	$x^{2-\alpha}$	$\mu \leq \frac{(1-\alpha)^2}{4}$
Here	$K_a := \sup_{0<x\leq 1} \frac{x\|a'(X)\|}{a(X)}$ $K_a \in [0,2) \setminus \{1\}$	$K_b := \sup_{0<x\leq 1} \frac{x\|b'(X)\|}{b(X)}$ $K_b \in [0,2]$ $K_a + K_b \leq 2$	$\frac{K_a - 2}{C_{a,b}(2 - K_b - K_a)} < \lambda < \frac{1}{C_{a,b}}$ or $\lambda < \frac{1}{C_{a,b}}$

Acknowledgments. This work was supported by the National Natural Science Foundation of China (Grant 12271316).

References

[1] F. Alabau-Boussouira, P. Cannarsa, G. Leugering; Control and stabilization of degenerate wave equations, *SIAM Journal on Control and Optimization*, 2017, **55**(3): 2052-2087.

[2] F. Alabau-Boussouira, P. Cannarsa, G. Fragnelli; Carleman estimates for degenerate parabolic operators with applications to null controllability, *Journal of Evolution Equations*, 2006, **6**: 161-204.

[3] B. Allal, M. Alhabib, S. Jawad; Boundary controllability for a degenerate and singular wave equation, *Mathematical Methods in the Applied Sciences*, 2022, **45**(17): 11526-11544.

[4] J. Bai, S. Chai; Exact controllability for a one-dimensional degenerate wave equation in domains with moving boundary, *Applied Mathematics Letters*, 2021, **119**: 107235.

[5] J. Bai, S. Chai; Exact controllability of Wave equations with interior degeneracy and one-sided boundary control, *Journal of Systems Science and Complexity*, 2023, **36**(2): 656-671.

[6] I. Boutaayamou, G. Fragnelli, L. Maniar; Carleman estimates for parabolic equations with interior degeneracy and Neumann boundary conditions. *Journal d'Analyse Mathématique*, 2018, **135**: 1-35.

[7] I. Boutaayamou, G. Fragnelli, D. Mugnai; Boundary controllability for a degenerate wave equation in nondivergence form with drift, *SIAM Journal on Control and Optimization*, 2023, **61**(4): 1934-1954.

[8] P. Cannarsa, P. Martinez, J. Vancostenoble; Null controllability of degenerate heat equations, *Adv. Differential Equations*, 2005, **10**: 153-190.

[9] P. Cannarsa, P. Martinez, J. Vancostenoble; Carleman estimates for a class of degenerate parabolic operators, *SIAM Journal on Control and Optimization*, 2008, **47**(1): 1-19.

[10] L. Cui, X. Liu, H. Gao; Exact controllability for a one-dimensional wave equation in non-cylindrical domains, *Journal of Mathematical Analysis and Applications*, 2013, **402**(2): 612-625.

[11] E. B. Davies; Spectral theory and differential operators, *Cambridge University Press*, 1995.

[12] S. N. Ethier; A class of degenerate diffusion processes occurring in population genetics, *Journal of Communications on Pure Applied Mathematics*, 2010, **29**(5): 483-493.

[13] G. Fragnelli; Interior degenerate/singular parabolic equations in nondivergence form: Well-posedness and Carleman estimates, *Journal of Differential Equations*, 2016, **260**: 1314-1371.

[14] G. Fragnelli, D. Mugnai; Carleman estimates for singular parabolic equations with interior degeneracy and non smooth coefficients. *Advances in Nonlinear Analysis*, 2017, **6**: 61-84.

[15] G. Fragnelli, D. Mugnai; Singular parabolic equations with interior degeneracy and non smooth coefficients: The Neumann case, *Discrete Continuous Dynamical Systems, Series S*, 2020, **13**(4): 1495-1511.

[16] M. Fotouhi, L. Salimi; Null controllability of degenerate/singular parabolic equations. *Journal of Dynamical and Control Systems*, 2012, **18**: 573-602.

[17] M. Ghil; Climate stability for a sellers type model. *Journal of the Atmospheric Science*, 2015, **16**(1): 3-20.

[18] A. Greenleaf, Y. Kurylev, M. Lassas, et al.; Cloaking devices, electromagnetic wormholes, and transformation optics, *SIAM Review*, 2009, **51**(1): 3-33.

[19] M. Gueye; Exact boundary controllability of 1-D parabolic and hyperbolic degenerate equations, *SIAM Journal on Control and Optimization*, 2014, **52**(4): 2037-2054.

[20] V. Komornik; Exact controllability and stabilization: the multiplier method. *Elsevier Masson*, 1994.

[21] K. H. Karlsen, N. H. Risebro; On the uniqueness and stability of entropy solutions of nonlinear degenerate parabolic equations with rough coefficients. *Discrete Continuous Dynamical Systems, Series S*, 2003, **9**(1): 1081-1104.

[22] J. L. Lions; Exact controllability, stabilization and perturbations for distributed systems. *SIAM Review*, 1988, **30**(1): 1-68.

[23] X. Liu, X. Zhang; Local controllability of multidimensional quasi-linear parabolic equations, *SIAM Journal on Control and Optimization*, 2012, **50**(4): 2046-2064.

[24] P. Martinez, J. Vancostenoble; Carleman estimates for one-dimensional degenerate heat equations, *Journal of Evolution Equations*, 2006, **6**: 325-362.

[25] L. Rosier; A survey of controllability and stabilization results for partial differential equations, *Journal Européen des Systèmes Automatisés*, 2007, **41**(3/4): 365.

[26] L. Russel; Controllability and stabilization theory for linear partial differential equations: recent progress and open questions, *SIAM Review*, 1978, **20**: 639-739.
[27] J. Vancostenoble; Improved Hardy-Poincare inequalities and sharp Carleman estimates for degenerate/singular parabolic problems, *Discrete Continuous Dynamical Systems, Series S*, 2011, **4**(1): 761-790.
[28] P. Yao; On the observability inequalities for exact controllability of wave equations with variable coefficients. *SIAM Journal on Control and Optimization*, 1999, **37**(5): 1568-1599.
[29] G. Zhang, S. Chai; Stabilization for some degenerate wave equations, *Evolution Equations and Control Theory*, 2024, **13**(2): 316-328.
[30] M. Zhang, H. Gao; Null controllability of some degenerate wave equations, *J. Syst. Sci. Complex.*, 2017, **29**: 1-15.
[31] M. Zhang, H. Gao; Persistent regional null controllability of some degenerate wave equations. *Math. Methods in the App. Sci.*, 2017, **40**: 5821-5830.
[32] E. Zuazua; Controllability of partial differential equations, *Optimization and Control*, 2006.

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