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EXACT BOUNDARY CONTROLLABILITY FOR WAVE EQUATIONS WITH FIXED AND MOVING BOUNDARIES IN TWO-DIMENSIONAL CONVEX-COMPLEMENTED DOMAINS

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ABSTRACT. The purpose of this article is to study exact boundary problems for the standard wave equation in domains that are the exterior of a convex compact set of $\mathbb{R}^2$, where both have a common boundary part. We consider two cases: first where the boundary domain is fixed, and where a part of the boundary is moving. In both cases we consider control problems with controls acting only one part of the boundary. For the fixed boundary case the control is of Neumann type, and for the moving boundary case the control is a conormal derivative type. The controllability method used here was developed by Russell [17].

1. Introduction

Let $A \subset \mathbb{R}^2$ be a bounded domain whose boundary ∂A is smooth by parts. Let $\gamma \subsetneq \partial A$ be a part of the boundary. The pair (A, γ) is called *convex-complemented* when there is a convex compact region $A^* \subseteq \mathbb{R}^2$ with

$$A \subseteq \mathbb{R}^2 - \overline{A^*} \quad \text{and} \quad \gamma \subseteq \partial A^*.$$

Let $\Omega \subset \mathbb{R}^2$ be a convex bounded domain whose boundary $\partial\Omega$ is smooth by parts with no cusps, with $\partial\Omega = \Gamma_0 \cup \Gamma_1$ where $\Gamma_0 \subsetneq \partial\Omega$, and $\overline{\Gamma}_0 \cap \Gamma_1 = \{P_0, P_1\}$ where $P_0, P_1 \in \mathbb{R}^2$. We also requires that Ω be in only one side of its boundary $\partial\Omega$ and the pair (Ω, Γ_0) be a convex-complemented with respect to the convex compact domain Ω^*. Here, we requires that the boundary $\partial\Omega^*$ be smooth, connected, and compact curve as illustrated in the Figure 1.

Now, let us consider the moving boundary domain $\Omega_t \subset \mathbb{R}^2$ whose boundary $\partial\Omega_t = \Gamma_0 \cup \Gamma_{1_t}$ for all $t \in \mathbb{R}$. We require that $\overline{\Gamma}_0 \cap \Gamma_{1_t} = \{P_0, P_1\}$, for all $t \in \mathbb{R}$. The moving part Γ_{1_t} of the boundary $\partial\Omega_t$ is such that $\Gamma_{1_0} = \Gamma_1$ and it may has two configurations.

Firstly, we can have

$$\Gamma_{1_t} = \{x \in \mathbb{R}^2 : x = \alpha(t)y,\, y \in \Gamma_1\}, \quad \text{for all } t \in \mathbb{R}.$$

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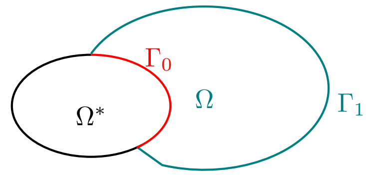

FIGURE 1. The pair (Ω, Γ_0) is convex-complemented. Ω^* is the convex complement.

In this case, Ω_t is illustrated in Figure 2.

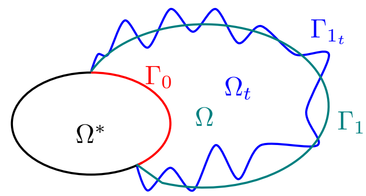

FIGURE 2. The pairs (Ω, Γ_0) and (Ω_t, Γ_0) are convex-complemented. In both cases, Ω^* is the convex complement.

Another configuration for Ω_t is

$$\Gamma_{1_t} = \{x \in \mathbb{R}^2 : x = \alpha(t)y,\ y \in \Gamma_1\} \cup \gamma_1 \cup \gamma_0\,, \quad \text{for all } t \in \mathbb{R},$$

where γ_0 and γ_1 are the straight segments joining the points P_0 to $\alpha(t)P_0$ and P_1 to $\alpha(t)P_1$ respectively.

Here $\alpha : \mathbb{R} \to \mathbb{R}^+ \cup \{0\}$ is a piecewise continuous bounded function, where α such that, for each $t \in \mathbb{R}$, the pair (Ω_t, Γ_0) is a convex-complemented with respect to Ω^*. An illustration for the domain Ω_t is shown in Figure 3.

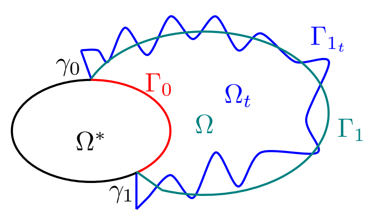

FIGURE 3. The pairs (Ω, Γ_0) and (Ω_t, Γ_0) are convex-complemented. In both cases, Ω^* is the convex complement.

For the well posedness of the initial boundary value problem we require

$$\Omega_t \times \mathbb{R} \subset \cup_{\overline{x} \in \Omega} \{(x,t) \in \mathbb{R}^2 \times \mathbb{R} : |x - \overline{x}|^2 \leq t^2\}. \tag{1.1}$$

The boundedness of movement function α implies in the existence of a bounded domain $\widetilde{\Omega} \subset \mathbb{R}^2$ such that $\Omega_t \subseteq \widetilde{\Omega}$, for all $t \in \mathbb{R}$. The boundary of $\widetilde{\Omega}$ is denoted by $\partial\widetilde{\Omega}$ and it is such that $\Gamma_0 \subset \partial\widetilde{\Omega}$. Furthermore, the pair $(\widetilde{\Omega}, \Gamma_0)$ has the convex-completed property with respect to the domain Ω^*.

Now, for $T > 0$, let us consider the non-cylindrical domain of $\mathbb{R}^{2+1}$,

$$Q_T = \cup_{0<t<T}\Omega_t \times \{t\}$$

whose the lateral boundary is $\Sigma_0 \cup \Sigma_T$, where $\Sigma_T = \cup_{0<t<T}\Gamma_{1_t} \times \{t\}$ and $\Sigma_0 = \Gamma_0 \times [0, T]$.

We denote by (ν_x, ν_t) the outward unit normal vector defined almost everywhere on $\Sigma_0 \cup \Sigma_T$. Particularly, when we consider the cylindrical boundary $\partial\Omega \times [0, T]$ its outward unit vector becomes $\nu = (\nu_x, 0)$.

Remark 1.1. Assumption (1.1) ensures that the surface $\Sigma_0 \cup \Sigma_t$ for $t \in \mathbb{R}$ is time-like which is a important property in order to guarantee that the initial boundary value problems is well posed on Q_T.

This studies two special exact boundary control problems for the standard wave equation. In the first problem we study a control problem for wave equation on the cylindrical domain $\Omega \times [0, T]$. In the second one we consider a control problem for wave equation on the non-cylindrical domain Q_T. In both problems we consider the null Neumann condition on Σ_0 and the controls acting on $\Sigma_1 = \Gamma_1 \times [0, T]$ and Σ_T respectively. Here we work the exact boundary controllability method established by Russell [16, 17]. One of the key elements of the Russell's method is the local energy decay of the system to be studied. Here, we use a local energy decay estimate for the standard wave equation in an exterior domain presented in [18].

From the point of view of applications there are many situations involving domains like Ω and Ω_t. For example, a flexible membrane which is attached to a rigid pillar by means of a part of its boundary. Without any variation in the temperature of the environment the membrane has no dilation and thus the complementary non fixed part of its boundary remains static, so such membrane represents the a domain like Ω. On the other hand, if there is a variation in the temperature, the membrane has a dilation or a contraction, causing the mobility of its non fixed boundary part, in this case the membrane represents a domain like Ω_t.

To state our results we need some essentials notation. Let $\mathcal{O} \subset \mathbb{R}^2$ be an arbitrary domain. We denote by $L^2(\mathcal{O})$ and $H^1(\mathcal{O})$ the Lebesgue and Sobolev spaces, with theirs usual norms $\|\cdot\|_{L^2(\mathcal{O})}$ and $\|\cdot\|_{H^1(\mathcal{O})}$ respectively (see [1]).

Now, let $(\mathcal{O}, \gamma)$ be a pair convex-complemented, we denote

$$\mathcal{H}^1(\mathcal{O}) = \big\{u(x) \in H^1(\mathcal{O}) : \partial_\nu u(x) = 0 \text{ if } x \in \gamma\big\}. \tag{1.2}$$

The topology of $\mathcal{H}^1(\mathcal{O})$ is induced from $H^1(\mathcal{O})$. Here, the space $\mathcal{H}^1_0(\mathcal{O})$ is the closure C_0^∞in $\mathcal{H}^1(\mathcal{O})$ provided with the norm of $H^1(\mathcal{O})$. Particularly, as the pair (Ω, Γ_0) is convex-complemented we have

$$\mathcal{H}^1(\Omega) = \big\{u(x) \in H^1(\Omega) : \partial_\nu u(x) = 0 \text{ if } x \in \Gamma_0\big\}, \quad \mathcal{H}^1_0(\Omega) = \overline{C_0^\infty(\Omega)}^{\|\cdot\|_{\mathcal{H}^1(\Omega)}}.$$

Theorem 1.2. *Let Ω be as defined above. Given $(f, g) \in \mathcal{H}^1(\Omega) \times L^2(\Omega)$, there exist $T > 0$ sufficiently large and a control function $h(\cdot, t) \in L^2(\partial\Omega \times [0, T])$ such that the solution $u \in \mathcal{H}^1_{\rm loc}(\Omega \times [0, T])$ of the problem*

$$\begin{gathered}
u_{tt} - \Delta u = 0 \quad \text{in } \Omega \times [0, T] \\
u(\cdot, 0) = f, \quad u_t(\cdot, 0) = g \quad \text{in } \Omega \\
\partial_\nu u(\cdot, t) = 0 \quad \text{on } \Sigma_0, \\
\partial_\nu u(\cdot, t) = h(\cdot, t) \quad \text{on } \Sigma_1,
\end{gathered} \tag{1.3}$$

satisfy the final condition

$$u(\cdot,T) = 0 = u_t(\cdot,T) \quad in\ \Omega. \tag{1.4}$$

Theorem 1.3. *Let Ω and Ω_t be as defined above. Given $(f,g) \in \mathcal{H}^1(\Omega) \times L^2(\Omega)$, there exist $T > 0$ sufficiently large and a control function $h(\cdot,t) \in L^2(\Gamma_{1_t} \times [0,T])$ such that the solution $u \in \mathcal{H}^1_{\rm loc}(Q_T)$ of the problem*

$$\begin{gathered} u_{tt} - \Delta u = 0 \quad in\ Q_T \\ u(\cdot,0) = f, \quad u_t(\cdot,0) = g \quad in\ \Omega \\ \partial_\nu u(\cdot,t) = 0 \quad on\ \Sigma_0, \\ \nu_t u_t - \nabla u \cdot \nu_x = h(\cdot,t) \quad on\ \Sigma_T, \end{gathered} \tag{1.5}$$

satisfy the final condition

$$u(\cdot,T) = 0 = u_t(\cdot,T) \quad in\ \Omega_T. \tag{1.6}$$

Note that in both problems (1.3) and (1.5) the controls acts only one part of the boundary. In the problem (1.3) the control is of Neumann type and acts on Σ_1 while in (1.5) the control acts on the moving boundary part Σ_T and the control is established via conormal derivative of the solution u.

Russell [17] developed a technique, based in [16], for studding an exact boundary control problem for wave equation with control acting only on a part of boundary of the domain. However, in [17] it is considered domains only fixed boundaries and with Dirichlet null condition on one part of the boundary. Here, it is used the Russell's technique but we go a step more by studying both exact boundary control problem with moving and fixed boundaries with Neumann null condition on the part of the boundary where the controls do not act.

In the literature there are also many works dealing with exact boundary control problems on non-cylindrical domains, using as HUM method (see [9]) as Russell's method, to cite a few see [2, 3, 4, 10, 11, 13, 14] and their references. When the boundary mobility is bounded, as considered here, we observe an advantage using the Russell's method instead of HUM because the original system do not need to be transformed into a system with variable coefficients as in [4, 11]. On the other hand, the applicability of Russell's method requires some properties of the system. Some of them are: linearity, time reversibility, local energy decay, and suitable trace theorems. With respect to local energy decay estimates for the problem considered here we use a decay estimate present in [18], which will presented in the next section. With respect to traces, the theorems we will use are presented in [19].

As seen above, the Russell's method requires many properties of the system but it has the advantage of requiring very little on the geometry of the domain. From this fact, we can consider the limiting function α, defined above, to be only piecewise smooth.

The rest of this artile is organized as follows. In Section 2 we presents a brief summary with respect to traces, extension and decay properties. Section 3 is dedicated to proof Theorem 1.2. In Section 4, we consider a special extension result. In Section 5, we prove Theorem 1.3.

2. Preliminaries results

In this section we state some preliminaries results which are essential for applying Russell's controllability method. We make a brief presentation about the trace, extension and local energy decay results which are fundamental ingredients in the Russell's method. We considering Ω as the domain as defined in the introductory function whose boundary is $\partial\Omega = \Gamma_0 \cup \Gamma_1$ where Γ_0 is a smooth curve. Also we consider the pair (Ω, Γ_0) being a convex-complemented. So, we let Ω^* be the a complementary convex domain associated with Ω.

Let us denote $\Omega^*_\infty = \mathbb{R}^2 - \overline{\Omega^*}$. It is clear that the pair $(\Omega^*_\infty, \Gamma_0)$ is convex complemented, because we have Ω^* a convex compact set in $\mathbb{R}^2$ such that $\Omega^*_\infty \subset \mathbb{R}^2 - \overline{\Omega^*}$ and $\Gamma_0 \subsetneq \partial\Omega^* = \partial\Omega^*_\infty$. So, from (1.2) it is possible to define the space $\mathcal{H}^1(\Omega^*_\infty)$.

For technical reasons to apply Russell's method under the following assumptions.
(A1) Let $B = B(0, R)$, with $R > 0$, be a disc where $\overline{\Omega \cup \Omega^*} \subset B$. Assume that there exist a bounded linear operator $P : \mathcal{H}^1(\Omega) \to \mathcal{H}^1(\Omega^*_\infty)$ such that for $u \in \mathcal{H}^1(\Omega)$ the extension $\widetilde{u} = Pu$ satisfy the following:

1. $\partial_\nu \widetilde{u} = 0$ in $\partial\Omega^*$;
2. $\widetilde{u} = u$ in Ω;
3. $\widetilde{u} = 0$ in $\mathbb{R}^2 - B$;
4. $\|Pu\|_{\mathcal{H}^1(\Omega^*_\infty)} \le C\|u\|_{\mathcal{H}^1(\Omega)}$, where C is a positive real constant independent on u.

Under the above assumptions, we proof the following extension result.

Lemma 2.1. *Let Ω, Ω^* and Ω^*_∞ be as above and consider the domain $\Omega_\delta = \{x \in \Omega^*_\infty : |x - y| \le \delta,\ \text{for } y \in \Omega\}$, where δ is a positive constant. Then there exists a bounded linear operator $E_1 : \mathcal{H}^1(\Omega) \to \mathcal{H}^1(\Omega^*_\infty)$, such that for each $f \in \mathcal{H}^1(\Omega)$ we have $E_1 f|_\Omega = f$ with $\operatorname{supp}(E_1 f) \subset \Omega_\delta$ and $\|E_1 f\|_{\mathcal{H}^1(\Omega^*_\infty)} \le C\|f\|_{\mathcal{H}^1(\Omega)}$ for some constant $C > 0$.*

Proof. Let $\varphi \in C_0^\infty(\Omega^*_\infty)$ be a function such that $\varphi = 1$ in Ω and φ in $\Omega^*_\infty - \Omega_{\frac{\delta}{2}}$. By considering the operator P of Assumption (A1), let us define $E_1 f = \varphi P f$ for $f \in \mathcal{H}^1(\Omega)$. Note that $E_1 : \mathcal{H}^1(\Omega) \to \mathcal{H}^1(\Omega^*_\infty)$ is linear, $E_1 f|_\Omega = f$ in Ω and $\operatorname{supp}(E_1 f) \subset \Omega_\delta$ for all $f \in \mathcal{H}^1(\Omega)$. Note now that $E_1 = M_\varphi \circ P$ where $M_\varphi : \mathcal{H}^1(\Omega) \to \mathcal{H}^1(\Omega^*_\infty)$ the multiplication operator defined by $M_\varphi(\phi) = \varphi\phi$ is a bounded linear operator. So, it follows that E_1 is a bounded linear operator. □

Another important ingredient in the application of Russell's controllability method is the trace regularity of the conormal derivative on time-like surfaces of the solutions of the initial boundary value problem to be studied. Next, we mention a result on the regularity of the traces of the solution of the wave equation which it is essential in the proof of Theorems 1.2 and 1.3.

Let us begin with some notation and definitions. Let $P(\xi, D)$ be a linear second order hyperbolic partial differential equation with C^∞ coefficients depending on ξ in some open bounded domain $\Xi \subset \mathbb{R}^N$. Being $\Sigma \subset \Xi$ an oriented smooth hypersurface which is time-like and non-characteristic with respect to $P(\xi, D)$. Let $\eta = (\eta_1, \cdots, \eta_N)$ be a normal unit to Σ. If $\sum a^{ij} \frac{\partial^2}{\partial\xi_i \partial\xi_j}$ is the principal part of $P(\xi, D)$, then the expression $\frac{\partial u}{\partial\eta} = \sum a^{ij} \frac{\partial u}{\partial\xi_i} \eta_j$ defines the conormal derivative of u relative to the $P(\xi, D)$ along Σ. An important fact is to know what the regularity of the traces of the conormal derivative on surfaces, for this purpose we turn to [19]. Considering $\Xi \subset \mathbb{R}^N$, with $N \ge 2$, [19, Theorem 2] shows that if $u \in H^1_{\text{loc}}(\Xi)$ is such that $P(\xi, D)u \in L^2_{\text{loc}}(\Xi)$ then $\frac{\partial u}{\partial\eta} \in L^2_{\text{loc}}(\Sigma)$.

Particularly, if we consider $P(\xi, D)$ as being the standard wave operator, its principal part will be $\frac{\partial^2}{\partial t^2} - \sum_{i=1}^{N} \frac{\partial^2}{\partial x_i^2}$. Now, if γ is a smooth hypersurface in $\mathbb{R}^N$ we consider the surface $\gamma \times \mathbb{R}$ whose unit normal vector is $\nu = (\nu_x, \nu_t)$, where $\nu_x = (\nu_1, \cdots, \nu_N)$. In this case the conormal derivative of u along $\gamma \times \mathbb{R}$ is $\frac{\partial u}{\partial \nu} = \nu_t u_t - \nabla u \cdot \nu_x$. Particularly, if we apply the trace result mentioned in the previous paragraph for the wave operator we obtain the following result.

Lemma 2.2. *Let $u \in \mathcal{H}^1_{\rm loc}(\Omega^*_\infty \times \mathbb{R})$ be the solution of the initial-boundary value problem*

$$\begin{gathered} u_{tt} - \Delta u = 0 \quad \text{in } \Omega^*_\infty \times \mathbb{R} \\ u(\cdot, 0) = u_0, \quad u_t(\cdot, 0) = u_1, \quad \text{in } \Omega^*_\infty \\ \partial_\nu u(\cdot, t) = 0, \quad \text{on } \partial\Omega^*_\infty \times \mathbb{R} \end{gathered} \tag{2.1}$$

with initial data $(u_0, u_1) \in \mathcal{H}^1(\Omega^*_\infty) \times L^2(\Omega^*_\infty)$, where $\operatorname{supp}(u_0), \operatorname{supp}(u_1) \subset \Omega^*_\infty$. Let γ be a smooth hypersurface in Ω^*_∞, with no self intersection and considers surface $\gamma \times \mathbb{R}$ whose the unit normal vector is $\nu = (\nu_x, \nu_t)$. Then the conormal derivative of u along $\gamma \times \mathbb{R}$ has trace $\nu_t u_t - \nabla u \cdot \nu_x \in L^2_{\rm loc}(\gamma \times \mathbb{R})$.

Remark 2.3. If the surface $\gamma \times \mathbb{R}$ in Lemma 2.2 is cylindrical then the component ν_t of the normal vector $\nu = (\nu_x, \nu_t)$ is null, that is $\nu_t = 0$. So, in this case, the conormal derivative coincides with normal derivative, that is, $\nu_t u_t - \nabla u \cdot \nu_x = -\nabla u \cdot \nu_x$.

2.1. **Local energy decay.** Local energy decay plays a fundamental role in the proof of the control problems proposed in this article. Particularly, we are interested in local energy decay estimates for the wave equation in exterior domains. With respect to this topic there are many paper available in the literature, to cite a few see [5, 6, 7, 12, 18, 20, 21] and references there in. In this paper, we are interested in a local decay estimate presented in [18]. To have such result let us consider the initial initial-boundary value problem

$$\begin{gathered} u_{tt} - \Delta u = 0 \quad \text{in } \Omega^*_\infty \times \mathbb{R} \\ u(\cdot, 0) = u_0, \quad u_t(\cdot, 0) = u_1, \quad \text{in } \Omega^*_\infty \\ \partial_\nu u(\cdot, t) = 0, \quad \text{on } \partial\Omega^*_\infty \times \mathbb{R} \end{gathered} \tag{2.2}$$

Let $\mathcal{O} \subset \Omega^*_\infty$ be a bounded domain. The energy of the solution u of the (2.2) confined in $\mathcal{O}$ is

$$E(t, \mathcal{O}, u) = \frac{1}{2} \int_{\mathcal{O}} [|\nabla u|^2 + |u_t|^2 + |u|^2](x, t) dx. \tag{2.3}$$

If there exist a positive constant C (independent on initial data) and a function $p(t)$ such that

$$E(t, \mathcal{O}, u) \le C p(t) E(0, \mathcal{O}, u), \tag{2.4}$$

with $p(t) \to 0$, as $t \to +\infty$, we say the energy of (2.2) decays locally. In [7, 12] the authors obtain local energy decay estimates for a exterior problem like (2.2) but with null Dirichlet condition on $\partial\Omega^*_\infty$. In this article we are interested in a local energy decay estimate but considering a null Neumann boundary condition $\partial\Omega^*_\infty$. So, we turn to a result presented in [18, Lemma 2.1] where with a little adaptation express the following local energy decay estimate.

Lemma 2.4. *Let* $(u_0, u_1) \in \mathcal{H}^1(\Omega^*_\infty) \times L^2(\Omega^*_\infty)$ *with* $\operatorname{supp}(u_0), \operatorname{supp}(u_1) \subset \mathcal{O} \subset \Omega^*_\infty$*, then there exist a positive real constant* K*, independent of* u_0 *and* u_1*, such that the solution* u *of* (2.2) *satisfies*

$$\begin{aligned} &\|u(\cdot,t)\|^2_{\mathcal{H}^1(\mathcal{O})} + \|u_t(\cdot,t)\|^2_{L^2(\mathcal{O})} \\ &\le\ K(1+t)^{-2}\big\{\|u(\cdot,0)\|^2_{\mathcal{H}^1(\mathcal{O})} + \|u_t(\cdot,0)\|^2_{L^2(\mathcal{O})}\big\}, \end{aligned} \tag{2.5}$$

for $t > 0$ *sufficiently large.*

The time reversibility of the wave operator has a central role to play in using Russell's controllability method. So, it is also necessary to know the local energy decay estimates for the system (2.2) in reverse time. The next result shows a estimate for the local energy in reverse time.

Lemma 2.5. *Let* T *a positive real number and* $(u_0, u_1) \in \mathcal{H}^1(\Omega^*_\infty) \times L^2(\Omega^*_\infty)$ *with* $\operatorname{supp}(u_0), \operatorname{supp}(u_1) \subset \mathcal{O} \subset \Omega^*_\infty$*. The solution* u *of the initial boundary value problem*

$$\begin{gathered} u_{tt} - \Delta u = 0 \quad \text{in } \Omega^*_\infty \times \mathbb{R} \\ u(\cdot,T) = u_0, \quad u_t(\cdot,T) = u_1, \quad \text{in } \Omega^*_\infty \\ \partial_\nu u(\cdot,t) = 0, \quad \text{on } \partial\Omega^*_\infty \times \mathbb{R} \end{gathered} \tag{2.6}$$

satisfies the estimate

$$\begin{aligned} &\|u(\cdot,0)\|^2_{\mathcal{H}^1(\mathcal{O})} + \|u_t(\cdot,0)\|^2_{L^2(\mathcal{O})} \\ &\le \overline{K}(1+T)^{-2}\big\{\|u(\cdot,T)\|^2_{\mathcal{H}^1(\mathcal{O})} + \|u_t(\cdot,T)\|^2_{L^2(\mathcal{O})}\big\}, \end{aligned} \tag{2.7}$$

where $\overline{K}$ *is a positive real constant independent on of the initial data* (u_0, u_1).

Proof. Let v be the solution of the problem

$$\begin{gathered} (v_{tt} - \Delta v)(\cdot,\tau) = 0 \quad \text{in } \Omega^*_\infty \times \mathbb{R} \\ v(\cdot,0) = u_0, \quad v_\tau(\cdot,0) = -u_1, \quad \text{in } \Omega^*_\infty \\ \partial_\nu v(\cdot,\tau) = 0, \quad \text{on } \partial\Omega^*_\infty \times \mathbb{R} \end{gathered} \tag{2.8}$$

by applying estimate (2.5) to v we obtain

$$\|v(\cdot,\tau)\|^2_{\mathcal{H}^1(\mathcal{O})} + \|v_\tau(\cdot,\tau)\|^2_{L^2(\mathcal{O})} \le \overline{K}(1+\tau)^{-2}\big\{\|v(\cdot,0)\|^2_{\mathcal{H}^1(\mathcal{O})} + \|v_\tau(\cdot,0)\|^2_{L^2(\mathcal{O})}\big\},$$

for $\tau > 0$ sufficiently great. Making $\tau = T - t$ in latest inequality and observing that $u(\cdot,t) = v(\cdot,T-t)$ is solution of (2.6) satisfying the estimate (2.7). □

3. Proof of Theorem 1.2

Let $\Omega, \Omega^*, \Omega^*_\infty$ and Ω_δ be domains as defined in the previous section. Given an arbitrary $(w_0, w_1) \in \mathcal{H}^1(\Omega) \times L^2(\Omega)$, according Lemma 2.1 and classical extension results in L^2 we can obtain bounded linear extension operator $E : \mathcal{H}^1(\Omega) \times L^2(\Omega) \to \mathcal{H}^1(\Omega^*_\infty) \times L^2(\Omega^*_\infty)$ such that the extension $(\widetilde{w}_0, \widetilde{w}_1)$ of (w_0, w_1), that is $(\widetilde{w}_0, \widetilde{w}_1) = E(w_0, w_1)$, satisfy $\operatorname{supp}(\widetilde{w}_0), \operatorname{supp}(\widetilde{w}_1) \subset \Omega_\delta$. Let w the solution of the initial boundary value problem

$$\begin{gathered} w_{tt} - \Delta w = 0 \quad \text{in } \Omega^*_\infty \times \mathbb{R} \\ w(\cdot,0) = \widetilde{w}_0, \quad w_t(\cdot,0) = \widetilde{w}_1, \quad \text{in } \Omega^*_\infty \\ \partial_\nu w(\cdot,t) = 0, \quad \text{in } \partial\Omega^*_\infty \times \mathbb{R}. \end{gathered} \tag{3.1}$$

Now, for $T > 0$ we define the bounded linear operator

$$S_T : \mathcal{H}^1_0(\Omega_\delta) \times L^2(\Omega_\delta) \to \mathcal{H}^1(\Omega^*_\infty) \times L^2(\Omega^*_\infty)$$

such that $S_T(w(\cdot,0),w_t(\cdot,0))=(w(\cdot,T),w_t(\cdot,T))$, where w is the solution of (3.1). From the decay estimate (2.5), with $\mathcal{O}=\Omega_\delta$, applied to w we obtain the estimate

$$\|(w(\cdot,T),w_t(\cdot,T))\|^2_{\mathcal{H}^1(\Omega^*_\infty)\times L^2(\Omega^*_\infty)} \le K(1+T)^{-2}\|(\widetilde{w}_0,\widetilde{w}_1)\|^2_{\mathcal{H}^1(\Omega_\delta)\times L^2(\Omega_\delta)}\,, \tag{3.2}$$

for $T>0$ sufficiently large and K is a constant independent of the data $(\widetilde{w}_0,\widetilde{w}_1)$.

In terms of the operator S_T, inequality (3.2) becomes

$$\begin{aligned}&\|S_T(\widetilde{w}_0,\widetilde{w}_1)\|^2_{\mathcal{H}^1(\Omega^*_\infty)\times L^2(\Omega^*_\infty)}\\ &\le K(1+T)^{-2}\|(\widetilde{w}_0,\widetilde{w}_1)\|^2_{\mathcal{H}^1(\Omega_\delta)\times L^2(\Omega_\delta)}\,,\end{aligned} \tag{3.3}$$

for $T>0$ sufficiently large and K is a constant independent on data $(\widetilde{w}_0,\widetilde{w}_1)$.

Now we consider the cut off function $\phi\in C_0^\infty(\Omega^*_\infty)$ such that $\phi\equiv 1$ in $\Omega_{\delta/2}$, and $\phi\equiv 0$ outside of Ω_δ. Then, for each $T>0$, we solve the backward initial boundary value problem

$$\begin{gathered} z_{tt}-\Delta z=0 \quad \text{in } \Omega^*_\infty\times\mathbb{R}\\ z(\cdot,T)=\phi w(\cdot,T),\quad z_t(\cdot,T)=\phi w_t(\cdot,T),\quad \text{in } \Omega^*_\infty\\ \partial_\nu z(\cdot,t)=0,\quad \text{in } \partial\Omega^*_\infty\times\mathbb{R},\end{gathered} \tag{3.4}$$

where the function w is the solution of problem (3.1). We define the linear operator $\overline{S}_T:\mathcal{H}^1_0(\Omega_\delta)\times L^2(\Omega_\delta)\to\mathcal{H}^1(\Omega^*_\infty)\times L^2(\Omega^*_\infty)$ by $\overline{S}_T(z(\cdot,T),z_t(\cdot,T))=(z(\cdot,0),z_t(\cdot,0))$. Applying again the decay estimate (2.7), with $\mathcal{O}=\Omega_\delta$, for function z, we obtain

$$\begin{aligned}&\|(z(\cdot,0),z_t(\cdot,0))\|^2_{\mathcal{H}^1(\Omega_\delta)\times L^2(\Omega_\delta)}\\ &\le K(1+T)^{-2}\|(z(\cdot,T),z_t(\cdot,T)))\|^2_{\mathcal{H}^1(\Omega_\delta)\times L^2(\Omega_\delta)},\end{aligned} \tag{3.5}$$

for $T>0$ sufficiently large and K is a constant independent on data (z_0,z_1).

In terms of the operator $\overline{S}_T$ the inequality (3.5) becomes

$$\begin{aligned}&\|\overline{S}_T(z(\cdot,T),z_t(\cdot,T))\|^2_{\mathcal{H}^1(\Omega_\delta)\times L^2(\Omega_\delta)}\\ &\le K(1+T)^{-2}\|(z(\cdot,T),z_t(\cdot,T)))\|^2_{\mathcal{H}^1(\Omega_\delta)\times L^2(\Omega_\delta)},\end{aligned} \tag{3.6}$$

for $T>0$ sufficiently large and K is a constant independent on data (z_0,z_1).

We define $\widetilde{v}(\cdot,t)=w(\cdot,t)-z(\cdot,t)$ and note that $\widetilde{v}$ satisfies

$$\begin{gathered} \widetilde{v}_{tt}-\Delta\widetilde{v}=0 \quad \text{in } \Omega^*_\infty\times\mathbb{R}\\ \widetilde{v}(\cdot,0)=w(\cdot,0)-z(\cdot,0),\quad \widetilde{v}_t(\cdot,0)=w_t(\cdot,0)-z_t(\cdot,0)\quad \text{in } \Omega^*_\infty\\ \partial_\nu\widetilde{v}(\cdot,t)=0,\quad \text{in } \partial\Omega^*_\infty\times\mathbb{R}\end{gathered} \tag{3.7}$$

and

$$\begin{gathered} \widetilde{v}(\cdot,T)=w(\cdot,T)-\phi w(\cdot,T)=(1-\phi)w(\cdot,T)=0 \quad \text{in } \Omega,\\ \widetilde{v}_t(\cdot,T)=w_t(\cdot,T)-\phi w_t(\cdot,T)=(1-\phi)w_t(\cdot,T)=0 \quad \text{in } \Omega,\end{gathered}$$

since $\phi=1$ in Ω.

Note that the function $\widetilde{v}$ solves the initial boundary value problem (3.7) and has the desirable final state $(\widetilde{v}(\cdot,T),\widetilde{v}_t(\cdot,T))=(0,0)$ in Ω. Now an important step it is to know if we can obtain $T>0$ such that $(\widetilde{v}(\cdot,0),\widetilde{v}_t(\cdot,0))$ extend the initial data (f,g) of the problem (1.3). That is, we wish establish solution for the equations

$$w(\cdot,0)-z(\cdot,0)=f,\quad w_t(\cdot,0)-z_t(\cdot,0)=g \quad \text{in } \Omega.$$

This latest two equations can be rewriten as

$$(w_0,w_1)-\mathcal{R}(z(\cdot,0),z_t(\cdot,0))=(f,g) \quad \text{in } \Omega, \tag{3.8}$$

where $\mathcal{R}$ denotes the restriction to Ω. So, we want to solve (3.8) for unknown $(w_0, w_1) \in \mathcal{H}^1(\Omega) \times L^2(\Omega)$. For this purpose we rewrite equation (3.8) in terms of the operators S_T and $\overline{S}_T$. Note that

$$\begin{aligned}(z(\cdot,0), z_t(\cdot,0)) &= \overline{S}_T(z(\cdot,T), \phi z_t(\cdot,T)) \\ &= \overline{S}_T(\phi w(\cdot,T), \phi w_t(\cdot,T)) \\ &= \overline{S}_T M_\phi (w(\cdot,T), w_t(\cdot,T)) \\ &= \overline{S}_T M_\phi S_T(w(\cdot,0), w_t(\cdot,0)) \\ &= [\overline{S}_T M_\phi S_T E](w_0, w_1),\end{aligned}$$

where M_ϕ is the operator multiplication by ϕ. Thus, (3.8) becomes

$$(w_0, w_1) - \mathcal{R}\overline{S}_T M_\phi S_T E(w_0, w_1) = (f, g) \quad \text{in } \Omega. \tag{3.9}$$

Denoting $\mathcal{R}\overline{S}_T M_\phi S_T E$ by K_T, equation (3.9) can be rewritten as

$$(I - K_T)(w_0, w_1) = (f, g) \quad \text{in } \Omega, \tag{3.10}$$

where I is the identity operator in $\mathcal{H}^1(\Omega) \times L^2(\Omega)$.

Now, for solving equation (3.10) it is sufficient to show that K_T is a contraction in $\mathcal{H}^1(\Omega) \times L^2(\Omega)$. It is in this point where the energy decay takes place, by considering inequalities (3.3) and (3.6) note that

$$\begin{aligned}\|K_T(w_0, w_1)\|^2_{\mathcal{H}^1(\Omega)\times L^2(\Omega)} &\le \|\overline{S}_T M_\phi S_T E(w_0, w_1))\|^2_{\mathcal{H}^1(\Omega_\delta)\times L^2(\Omega_\delta)} \\ &\le K(1+T)^{-2}\|M_\phi S_T E(w_0, w_1)\|^2_{\mathcal{H}^1(\Omega_\delta)\times L^2(\Omega_\delta)} \\ &\le C(1+T)^{-2}\|S_T E(w_0, w_1)\|^2_{\mathcal{H}^1(\Omega_\delta)\times L^2(\Omega_\delta)} \\ &\le CK(1+T)^{-4}\|E(w_0, w_1)\|^2_{\mathcal{H}^1(\Omega_\delta)\times L^2(\Omega_\delta)} \\ &\le C(1+T)^{-4}\|(w_0, w_1)\|^2_{\mathcal{H}^1(\Omega)\times L^2(\Omega)} \\ &\le \frac{C}{(1+T)^4}\|(w_0, w_1)\|^2_{\mathcal{H}^1(\Omega)\times L^2(\Omega)},\end{aligned}$$

where C in the above inequalities represents a positive real constant which may vary from line to line. So, from the above inequalities we obtain

$$\|K_T(w_0, w_1)\|_{\mathcal{H}^1(\Omega)\times L^2(\Omega)} \le \frac{\sqrt{C}}{(1+T)^2}\|(w_0, w_1)\|_{\mathcal{H}^1(\Omega)\times L^2(\Omega)}, \tag{3.11}$$

for $T > 0$ sufficiently large, being C a positive constant independent of the initial data. Now, choosing a $T > 0$ such that $\frac{\sqrt{C}}{(1+T)^2} \le c < 1$ and for such T, K_T is a contraction. After, for such T, we take the solution (w_0, w_1) for (3.10) and take it to the begin of the proof in order to obtain the function w, z and $\widetilde{v} = w - z$, where $\widetilde{v}$ solves (3.7) and has the desirable final condition $(\widetilde{v}(\cdot,T), \widetilde{v}(\cdot,T)) = (0,0)$. Besides, $\widetilde{v}(\cdot,0)$ and $\widetilde{v}_t(\cdot,0)$ extends f and g, respectively, from Ω to Ω^*_∞. To complete the proof note that $\widetilde{v}_{tt} - \Delta\widetilde{v} \in L^2_{\rm loc}(\Omega^*_\infty \times \mathbb{R})$, so, applying the trace regularity result of Lemma 2.2 we have that the trace of conormal derivative of $\widetilde{v}$ on surface $\partial\Omega \times [0,T]$ is well defined and it is locally square integrable. That is, $\widetilde{v}_t \nu_t - \nabla\widetilde{v} \cdot \nu_x \in L^2(\partial\Omega \times [0,T])$. As the surface $\partial\Omega \times [0,T]$ is cylindrical follows that the component ν_t of the normal vector (ν_x, ν_t) is null. So, the conormal derivative coincides to normal derivative $\partial_\nu \widetilde{v}$ on $\partial\Omega \times [0,T]$. To finish the proof, we defines $u := \widetilde{v}_{|\Omega\times[0,T]}$, the restriction of $\widetilde{v}$ to domain $\Omega \times [0,T]$ and $h := \partial_\nu \widetilde{v}$ on $\partial\Omega \times [0,T]$ and observing that the function u and h meet the conditions of Theorem 1.2.

4. A special extension theorem

Let us consider the domains Ω, Ω^*_∞, and $\widetilde{\Omega}$ as defined in the previous sections. In this section, we prove an important result which is of the fundamental importance in the proof of Theorem 1.3. Such result is stated in lemma below.

Lemma 4.1. *Let T be a positive real number. Each $(v_0, v_1) \in \mathcal{H}(\Omega) \times L^2(\Omega)$ can be extended to $(\widetilde{v}_0, \widetilde{v}_1) \in \mathcal{H}(\Omega^*_\infty) \times L^2(\Omega^*_\infty)$ such that the solution $v \in H^1_{\rm loc}(\Omega^*_\infty \times \mathbb{R})$ of the problem*

$$\begin{gathered} v_{tt} - \Delta v = 0 \quad \text{in } \Omega^*_\infty \times \mathbb{R} \\ v(\cdot, T) = \widetilde{v}_0, \quad v_t(\cdot, T) = \widetilde{v}_1 \quad \text{in } \Omega^*_\infty \\ \partial_\nu v = 0 \quad \text{on } \partial\Omega^*_\infty \times \mathbb{R}, \end{gathered} \tag{4.1}$$

satisfies the condition

$$v(\cdot, 0) = 0 = v_t(\cdot, 0) \quad \text{in } \widetilde{\Omega}. \tag{4.2}$$

Proof. Let δ be a positive real number, and $\widetilde{\Omega}_\delta = \{y \in \Omega^*_\infty : \exists x \in \widetilde{\Omega}; |x-y| < \delta\}$ be an open neighborhood of $\widetilde{\Omega}$. Given an arbitrary $(w_0, w_1) \in \mathcal{H}(\Omega) \times L^2(\Omega)$, according Lemma 2.1 and classical extension results in L^2 we can obtain bounded linear extension operator $E : \mathcal{H}^1(\Omega) \times L^2(\Omega) \to \mathcal{H}^1(\Omega^*_\infty) \times L^2(\Omega^*_\infty)$ such that the extension $(\widetilde{w}_0, \widetilde{w}_1)$ of (w_0, w_1), that is $(\widetilde{w}_0, \widetilde{w}_1) = E(w_0, w_1)$, satisfy $\operatorname{supp}(\widetilde{w}_0), \operatorname{supp}(\widetilde{w}_1) \subset \widetilde{\Omega}_\delta$. Let $w \in \mathcal{H}_{\rm loc}(\Omega^*_\infty \times \mathbb{R})$ the solution of the initial boundary value problem

$$\begin{gathered} w_{tt} - \Delta w = 0 \quad \text{in } \Omega^*_\infty \times \mathbb{R} \\ w(\cdot, T) = \widetilde{w}_0, \quad w_t(\cdot, T) = \widetilde{w}_1 \quad \text{in} \quad \Omega^*_\infty \\ \partial_\nu w = 0 \quad \text{on } \partial\Omega^*_\infty \times \mathbb{R}, \end{gathered} \tag{4.3}$$

Now, for $T > 0$ we define the bounded linear operator

$$\overline{S}_T : \mathcal{H}^1(\Omega^*_\infty) \times L^2(\Omega^*_\infty) \to \mathcal{H}^1(\Omega^*_\infty) \times L^2(\Omega^*_\infty)$$

such that $\overline{S}_T(w(\cdot, T), w_t(\cdot, T)) = (w(\cdot, 0), w_t(\cdot, 0))$, where w is the solution of (4.3). From the decay estimate (2.7), with $\mathcal{O} = \widetilde{\Omega}_\delta$, applied to w we obtain the estimate

$$\begin{aligned} &\|\overline{S}_T(w(\cdot, T), w_t(\cdot, T))\|^2_{\mathcal{H}^1(\Omega^*_\infty) \times L^2(\Omega^*_\infty)} \\ &\le K(1+T)^{-2} \|(w(\cdot, T), w_t(\cdot, T)\|^2_{\mathcal{H}^1(\Omega_\delta) \times L^2(\Omega_\delta)}, \end{aligned} \tag{4.4}$$

for $T > 0$ sufficiently large and K is a constant independent on data $(\widetilde{w}_0, \widetilde{w}_1)$.

Now we consider the cut off function $\phi \in C^\infty_0(\Omega^*_\infty)$ such that $\phi \equiv 1$ in $\widetilde{\Omega}_{\delta/2}$, and $\phi \equiv 0$ out side of $\widetilde{\Omega}_\delta$. Let $z \in \mathcal{H}_{\rm loc}(\Omega^*_\infty \times \mathbb{R})$ the solution of the initial boundary value problem

$$\begin{gathered} z_{tt} - \Delta z = 0 \quad \text{in } \Omega^*_\infty \times \mathbb{R} \\ z(\cdot, 0) = \phi w(\cdot, 0), \quad z_t(\cdot, 0) = \phi w_t(\cdot, 0) \quad \text{in } \Omega^*_\infty \\ \partial_\nu z = 0 \quad \text{on } \partial\Omega^*_\infty \times \mathbb{R}, \end{gathered} \tag{4.5}$$

We define the linear operator

$$S_T : \mathcal{H}^1(\Omega^*_\infty) \times L^2(\Omega^*_\infty) \to \mathcal{H}^1(\Omega^*_\infty) \times L^2(\Omega^*_\infty)$$

by $S_T(z(\cdot, 0), z_t(\cdot, 0)) = (z(\cdot, T), z_t(\cdot, T))$. Applying again the decay estimate (2.5), with $\mathcal{O} = \widetilde{\Omega}_\delta$, we obtain

$$\begin{aligned} &\|S_T(z(\cdot, 0), z_t(\cdot, 0))\|^2_{\mathcal{H}^1(\widetilde{\Omega}_\delta) \times L^2(\widetilde{\Omega}_\delta} \\ &\le K(1+T)^{-2} \|(z(\cdot, 0), z_t(\cdot, 0))\|^2_{\mathcal{H}^1(\widetilde{\Omega}_\delta) \times L^2(\widetilde{\Omega}_\delta)}, \end{aligned} \tag{4.6}$$

for $T > 0$ sufficiently large and K is a constant independent on data $(z(\cdot,0), z_t(\cdot,0))$.

We define $\widetilde{v}(\cdot,t) = w(\cdot,t) - z(\cdot,t)$ and see that $\widetilde{v} \in \mathcal{H}_{\rm loc}(\Omega^*_\infty \times \mathbb{R})$ satisfies

$$\begin{gathered}\widetilde{v}_{tt} - \Delta\widetilde{v} = 0 \quad \text{in } \Omega^*_\infty \times \mathbb{R}\\ \widetilde{v}(\cdot,T) = w(\cdot,T) - z(\cdot,T), \quad \text{in } \Omega^*_\infty\\ \widetilde{v}_t(\cdot,T) = w_t(\cdot,T) - z_t(\cdot,T) \quad \text{in } \Omega^*_\infty\\ \partial_\nu \widetilde{v} = 0 \quad \text{on } \partial\Omega^*_\infty \times \mathbb{R},\end{gathered} \tag{4.7}$$

and

$$\begin{gathered}\widetilde{v}(\cdot,0) = w(\cdot,0) - \phi w(\cdot,0) = (1-\phi)w(\cdot,0) = 0 \quad \text{in } \widetilde{\Omega},\\ \widetilde{v}_t(\cdot,0) = w_t(\cdot,0) - \phi w_t(\cdot,0) = (1-\phi)w_t(\cdot,0) = 0 \quad \text{in } \widetilde{\Omega},\end{gathered}$$

since $\phi = 1$ in $\widetilde{\Omega}$.

Note that the function $\widetilde{v}$ solves the homogeneous wave equation (4.7) and has the desirable final state $(\widetilde{v}(\cdot,0), \widetilde{v}_t(\cdot,0)) = (0,0)$ in $\widetilde{\Omega}$. Now, an important step it is to know if we may obtain $T > 0$ such that $(\widetilde{v}(\cdot,T), \widetilde{v}_t(\cdot,T))$ be a extension of the initial data (v_0, v_1) from $\mathcal{H}(\Omega) \times L^2(\Omega)$ to $\mathcal{H}(\Omega^*_\infty) \times L^2(\Omega^*_\infty)$. That is, we wish establish solution for the equations

$$w(\cdot,T) - z(\cdot,T) = v_0, \quad w_t(\cdot,T) - z_t(\cdot,T) = v_1 \quad \text{in } \Omega.$$

The two latest equations can be rewriting as

$$E(w_0,w_1) - (z(\cdot,T), z_t(\cdot,T)) = (v_0,v_1) \quad \text{in } \Omega. \tag{4.8}$$

We want to solve (4.8) for the unknown $(w_0,w_1) \in \mathcal{H}(\Omega) \times L^2(\Omega)$. For this purpose we rewrite equation (4.8) in terms of the operators S_T and $\overline{S}_T$. Note that

$$\begin{aligned}(z(\cdot,T), z_t(\cdot,T)) &= S_T(z(\cdot,0), z_t(\cdot,0))\\ &= S_T(\phi w(\cdot,0), \phi w_t(\cdot,0))\\ &= S_T M_\phi (w(\cdot,0), w_t(\cdot,0))\\ &= S_T M_\phi \overline{S}_T (w(\cdot,T), w_t(\cdot,T))\\ &= [S_T M_\phi \overline{S}_T E](w_0,w_1),\end{aligned}$$

where M_ϕ is the operator multiplication by ϕ. Thus, (4.8) becomes

$$(w_0,w_1) - \mathcal{R} S_T M_\phi \overline{S}_T E(w_0,w_1) = (v_0,v_1) \quad \text{in } \Omega, \tag{4.9}$$

where $\mathcal{R}$ denotes the restriction to Ω.

As in the latest section, by denoting $\mathcal{R} S_T M_\phi \overline{S}_T E$ by $\overline{K}_T$, equation (4.9) can be rewritten as

$$\left(I - \overline{K}_T\right)(w_0,w_1) = (v_0,v_1) \quad \text{in } \Omega, \tag{4.10}$$

where I is the identity operator in $\mathcal{H}(\Omega) \times L^2(\Omega)$. Equation (4.10) has (w_0,w_1) as its unknown. Note that $\overline{K}_T$ is a compact linear operator.

Proceeding analogously to the previous section and by considering inequalities (4.4) and (4.6) we obtain

$$\|\overline{K}_T(w_0,w_1)\|_{\mathcal{H}^1(\Omega)\times L^2(\Omega)} \le \frac{\sqrt{C}}{(1+T)^2} \|(w_0,w_1)\|_{\mathcal{H}^1(\Omega)\times L^2(\Omega)}, \tag{4.11}$$

for $T > 0$ sufficiently large, where C is a positive independent on initial data. So, choosing a $T > 0$ such that $\frac{\sqrt{C}}{(1+T)^2} \le c < 1$ and for such T, $\overline{K}_T$ is a contraction. Thus, we take the solution (w_0,w_1) for (4.10) and take it to the begin of the proof in order to obtain w, z and $\widetilde{v} = w - z$, where $\widetilde{v}$ solves (4.7) and has the desirable final condition $(\widetilde{v}(\cdot,0), \widetilde{v}(\cdot,0)) = (0,0)$. Besides, $(\widetilde{v}(\cdot,T), \widetilde{v}_t(\cdot,T)) \in \mathcal{H}^1(\Omega^*_\infty) \times L^2(\Omega^*_\infty)$ is extension of $(v_0, v_1) \in \mathcal{H}^1(\Omega) \times L^2(\Omega)$. By the end taken $\widetilde{v} := v$, we can see that v satisfy (4.1) and the condition (4.2) finalizing the proof of the Lemma 4.1. □

5. Proof of Theorem 1.3

Let Ω, Ω^*, Ω^*_∞ and $\widetilde{\Omega}$ as defined in the initial section. See that the $\widetilde{\Omega}$ is a domain with fixed boundary $\partial\widetilde{\Omega} = \widetilde{\Gamma} \cup \Gamma_0$ where the pair $(\widetilde{\Omega}, \Gamma_0)$ has the convex-complemented property with respect the convex domain Ω^*. Taking $(f,g) \in \mathcal{H}^1(\Omega) \times L^2(\Omega)$, let us consider the extensions $(\widetilde{f}, \widetilde{g}) \in \mathcal{H}^1(\Omega^*_\infty) \times L^2(\Omega^*_\infty)$ of (f,g), where $\operatorname{supp}(\widetilde{f}), \operatorname{supp}(\widetilde{g}) \subset \widetilde{\Omega}$. Let $\widetilde{u} \in \mathcal{H}^1_{\rm loc}(\Omega^*_\infty \times \mathbb{R})$ be the solution of the initial-boundary value problem

$$
\begin{gathered}
\widetilde{u}_{tt} - \Delta\widetilde{u} = 0 \quad \text{in } \Omega^*_\infty \times \mathbb{R} \\
\widetilde{u}(\cdot,0) = \widetilde{f}, \quad \widetilde{u}_t(\cdot,0) = \widetilde{g} \quad \text{in } \Omega^*_\infty \\
\partial_\nu \widetilde{u} = 0 \quad \text{on } \partial\Omega^*_\infty \times \mathbb{R},
\end{gathered}
\tag{5.1}
$$

Now, for a $T > 0$ sufficiently large, we take the state $(\widetilde{u}(\cdot,T), \widetilde{u}_t(\cdot,T)) \in \mathcal{H}^1(\widetilde{\Omega}) \times L^2(\widetilde{\Omega})$ and according to Lemma 4.1, changing Ω by $\widetilde{\Omega}$ the state $(\widetilde{u}(\cdot,T), \widetilde{u}_t(\cdot,T))$ can be extended to $\mathcal{H}^1(\Omega^*_\infty) \times L^2(\Omega^*_\infty)$ such that the solution $v \in \mathcal{H}^1_{\rm loc}(\Omega^*_\infty \times \mathbb{R})$ of the initial boundary value problem with extended data $(\widetilde{u}(\cdot,T), \widetilde{u}_t(\cdot,T))$,

$$
\begin{gathered}
v_{tt} - \Delta v = 0 \quad \text{in } \Omega^*_\infty \times \mathbb{R} \\
v(\cdot,T) = \widetilde{u}(\cdot,T), \quad v_t(\cdot,T) = \widetilde{u}_t(\cdot,T) \quad \text{in } \Omega^*_\infty \\
\partial_\nu v = 0 \quad \text{on } \partial\Omega^*_\infty \times \mathbb{R},
\end{gathered}
\tag{5.2}
$$

satisfies, at the instant $t = 0$, the condition

$$
v(\cdot,0) = 0 = v_t(\cdot,0) \quad \text{in } \widetilde{\Omega}. \tag{5.3}
$$

Now, by considering $\widetilde{u}$ and v the solutions of (5.1) and (5.2) respectively. Defining the function $\bar{u} = \widetilde{u} - v$ see that $\bar{u}(\cdot,0) = \widetilde{f}$, $\bar{u}_t(\cdot,0) = \widetilde{g}$ and

$$
\bar{u}(\cdot,T) = 0 = \bar{u}_t(\cdot,T) \quad \text{in } \Omega^*_\infty.
$$

Furthermore, the function u satisfy the initial-boundary value problem

$$
\begin{gathered}
\bar{u}_{tt} - \Delta\bar{u} = 0 \quad \text{in } \Omega^*_\infty \times \mathbb{R} \\
\bar{u}(\cdot,0) = \widetilde{f}, \quad \bar{u}_t(\cdot,0) = \widetilde{g} \quad \text{in } \Omega^*_\infty \\
\partial_\nu \bar{u} = 0 \quad \text{on } \partial\Omega^*_\infty \times \mathbb{R},
\end{gathered}
\tag{5.4}
$$

and the final condition

$$
\bar{u}(\cdot,T) = 0 = \bar{u}_t(\cdot,T) \quad \text{in } \Omega^*_\infty. \tag{5.5}
$$

Note that $\bar{u}_{tt} - \Delta\bar{u} \in L^2_{\rm loc}(\Omega^*_\infty \times \mathbb{R})$, so, applying the trace regularity result of Lemma 2.2 we have that the trace of conormal derivative of $\bar{u}$ on surface $\Sigma_T \times \mathbb{R}$ is well defined and it is locally square integrable. That is, $\bar{u}_t\nu_t - \nabla\bar{u} \cdot \nu_x \in L^2(\Sigma_T \times [0,T])$. So, we define $u := \bar{u}_{|Q_T}$, the restriction of $\bar{u}$ to domain $Q_T = \Omega_T \times [0,T]$

and $h := \bar{u}_t\nu_t - \nabla\bar{u}\cdot\nu_x$ on $\Sigma_T \times [0,T]$ and observing that the function u and satisfy the initial boundary value problem

$$\begin{gathered} u_{tt} - \Delta u = 0 \quad \text{in } Q_T \\ u(\cdot,0) = f,\ u_t(\cdot,0) = g \quad \text{in } \Omega \\ \partial_\nu u = 0 \quad \text{on } \Gamma_0 \times [0,T] \\ \nu_t u_t - \nabla u\cdot\nu_x = h(\cdot,t) \quad \text{on } \Sigma_T \end{gathered} \tag{5.6}$$

with the final condition

$$u(\cdot,T) = 0 = u_t(\cdot,T) \quad \text{in } \Omega_T, \tag{5.7}$$

finalizing the proof of the Theorem 1.3.

Remark 5.1. Theorems 1.2 and 1.3 establish only the existence of the control time T. But they do not provide a lower bound from which the control time can be taken. A manner for obtaining lower estimates for the control time, using the Russell's controllability method, is to follow the ideas of analytic extension given in [8, 14, 15]. In those papers it is shown the family of linear compact operators $\{K_T\}_{T\geq \mathrm{diam}(\Omega)}$ and $\{\bar{K}_T\}_{T\geq \mathrm{diam}(\Omega)}$ extent analytically, to a sector of complex plane Σ, to a family of compact linear operators $\{K_\zeta\}_{\zeta\in\Sigma}$ and $\{\bar{K}_\zeta\}_{\zeta\in\Sigma}$. In both papers the analyticity was obtained by handing the explicit formulas of the solution of the Cauchy problem in complete $\mathbb{R}^n\times\mathbb{R}$. But here we have a difficulty applying such technique because we do not have the explicit formulas for the solution to the initial-boundary value problem for the wave equation in an exterior domain. So, for this case, such analyticity must would be obtained by another manner.

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