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PERSISTENCE PROPERTIES OF SOLUTIONS FOR MULTI-COMPONENT NOVIKOV EQUATIONS

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Abstract. In this article, we investigate the asymptotic behavior of the solution for a multi-component Novikov equation in weighted Sobolev spaces. We introduce a set of weighted functions, and prove that the strong solution will retain the corresponding decay properties when the initial data $U_0(x)$ and its derivative $U_{0,x}(x)$ decay logarithmically, algebraically, and exponentially at infinity.

1. Introduction

In this article, we consider the initial value problem (IVP) for a multi-component Novikov equation

$$
\begin{aligned}
\partial_t m_k &= \sum_{i=1}^{N}(-2m_k v_i\partial_x u_i - m_k u_i\partial_x v_i - u_i v_i\partial_x m_k - m_i v_i\partial_x u_k + u_k m_i\partial_x v_i),\\
\partial_t n_k &= \sum_{i=1}^{N}(-2n_k u_i\partial_x v_i - n_k v_i\partial_x u_i - v_i u_i\partial_x n_k - n_i u_i\partial_x v_k + v_k n_i\partial_x u_i),\\
&(u_k(t,x), v_k(t,x))|_{t=0} = (u_{k,0}(x), v_{k,0}(x)),
\end{aligned}
\tag{1.1}
$$

where $m_k = u_k - \partial_x^2 u_k$, $n_k = v_k - \partial_x^2 v_k$, $t > 0$, $x \in \mathbb{R}$, $k = 1, 2, \dots, N$. Equation (1.1) was proposed by Li et al. [9] to derive its bi-Hamiltonian structure. Mi and Guo proved the local well-posedness of the system in a range of the Besov spaces using the Littlewood-Paley theory and transport equations [15]. Li and Wu et al. [11] deduced blow-up criteria for (1.1) and global existence of two-component case in $H^s(\mathbb{R})$, $s > 1/2$. Moreover, they verified that the system possesses peakons and periodic peakons.

For $N = 1$, equation (1.1) turns into the Geng-Xue (GX) equation

$$
\begin{aligned}
\partial_t m + 3vm\partial_x u + uv\partial_x m = 0, &\quad t > 0,\ x \in \mathbb{R},\\
\partial_t n + 3un\partial_x v + uv\partial_x n = 0, &\quad t > 0,\ x \in \mathbb{R},\\
m = u - \partial_x^2 u, n = v - \partial_x^2 v, &\quad t > 0,\ x \in \mathbb{R},\\
u(0,x) = u_0(x), v(0,x) = v_0(x), &\quad x \in \mathbb{R},
\end{aligned}
\tag{1.2}
$$

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which is a two-component CH-type equation constructed by Geng and Xue. They also showed that (1.2) admits multi-peakons and infinitely many conserved quantities [5]. In 2013, Li and Liu obtained a bi-Hamiltonian structure of this equation [10]. Luo and Yin [14] established local well-posedness of the system in Besov spaces $B^{s-1}_{l,r}\times B^{s}_{l,r}$ with $l,r\in[1,\infty]$, $s>\max\{1+\frac{1}{l},\frac{3}{2}\}$ by the Littlewood-Paley decomposition. Moreover, they introduced blow-up criteria for the system based on conservation laws. In 2018, Zhou and Li [25] investigated the persistence properties of strong solutions for two-component Novikov equation in weighted $L^p_\phi \doteq L^p(\mathbb{R},\phi^p(x)\,dx)$ spaces for a large class of moderate weights.

If we take $u=v$, then (1.2) becomes the Novikov equation

$$\begin{gathered}\partial_t m+u^2\partial_x m+3um\partial_x u=0,\quad t>0,\ x\in\mathbb{R},\\ m=(1-\partial_x^2)u,\quad t>0,\ x\in\mathbb{R},\\ u(0,x)=u_0(x),\quad x\in\mathbb{R}.\end{gathered}\tag{1.3}$$

This equation was discovered by Novikov [17]. Hone and Wang [7] showed that (1.3) has a bi-Hamiltonian structure and infinitely many conserved quantities. Also, it admits peakon solutions and conserves the H^1-norm as well as CH equation. The Cauchy problem of (1.3) has attracted a great deal of attention in [6, 16, 20, 21, 22]. Specifically, Ni and Zhou [16] showed (1.3) is locally well-posed in the Besov spaces $B^{3/2}_{2,1}(\mathbb{R})$. Furthermore, they verified two results on the persistence properties of the strong solution. Wu and Yin [21] proved (1.3) possesses a global strong solution on the initial value $u_0\in H^s(\mathbb{R})$ for $s>3/2$. They also proved the existence and uniqueness of global weak solutions to (1.3) with the initial data satisfying certain sign conditions [20]. Furthermore, they established the local well-posedness of (1.3) in Besov space $B^s_{p,r}(\mathbb{R})$, $p,r\in[1,\infty]$, $s>\max\{\frac{3}{2},1+\frac{1}{p}\}$ and proved the equation is ill-posed in $B^{3/2}_{2,\infty}(\mathbb{R})$ [22]. The global existence and blow-up for the weakly dissipative Novikov equation were considered in [24].

When $N=1, v=1$, equation (1.1) transforms into the Degasperis-Procesi (DP) equation

$$\begin{gathered}m_t+um_x+3u_xm=0,\quad t>0,\ x\in\mathbb{R},\\ m=u-u_{xx},\quad t>0,\ x\in\mathbb{R},\\ u(0,x)=u_0(x),\quad x\in\mathbb{R},\end{gathered}\tag{1.4}$$

where $m_t=\partial_t m(t,x)$, $m_x=\partial_x m(t,x)$, $u_x=\partial_x u(t,x)$. Equation (1.4) was presented by Degasperis et al. [2] through the construction of a Lax pair and considered as a model for nonlinear shallow water dynamics [3, 13]. Constantin and Ivanov [1] proved that (1.4) has a bi-Hamiltonian structure and an infinite many of conservation laws. Moreover, the authors presented the traveling wave solutions and classified all weak traveling wave solutions for the DP equation in [8, 18].

However, the persistence properties of solutions to (1.1) have not been studied yet. Inspired by the recent works [4, 19, 23], we study some new decay properties of solutions to (1.1) with a set of weighted functions, which comes form [23]. The focus of Theorems 3.3–3.10 is to estimate a class of norms like $\|IF(u_i,u_{ix})(s)\|_{L^p}$ and $\|IF_x(u_i,u_{ix})(s)\|_{L^p}$, essentially to investigate Lemma 3.1. The decay of the solution in this article covers and extents the results in [25].

The rest of this article is structured as follows. In Section 2, we recall the local well-posedness result and introduce several lemmas which are needed for later proofs. In Section 3, we establish persistence properties of strong solutions to (1.1) provided the initial data U_0 and $\partial_x U_0$ decay logarithmically, algebraically, and exponentially at infinity.

Notation. As all function spaces considered are over $\mathbb{R}$, for convenience, we simplify our notation by omitting $\mathbb{R}$ when there is no ambiguity. we denote by $*$ spatial convolution on $\mathbb{R}$ and use $A^\top$ stands for transpose of vector A. The notation $\doteq$ stands for the definition of functions. For $1 \le p \le \infty$, we denote the norm of the Banach space $L^p(\mathbb{R})$ by $\|\cdot\|_{L^p}$ and the norm in the classical Sobolev spaces $H^{s,p}(\mathbb{R})$ by $\|\cdot\|_{H^{s,p}}$, $s \in \mathbb{R}$. In addition, for constant $K \ge 0$, we denote

$$f(x) \sim \mathcal{O}(g(x)) \quad \text{as} |x| \to \infty, \quad \text{if } \lim_{x\to\infty} |\frac{f(x)}{g(x)}| \le K.$$

2. Preliminaries

In this section, we first recall the following local well-posedness result from [12], and present three key weighted functions, which will be used in Section 3.

Lemma 2.1 ([12]). *Let $p_1 \in (1,\infty)$ and $s > \max\{\frac{5}{2}, 2+\frac{1}{p_1}\}$. Assume that*

$$U_0 = (u_{1,0}, \dots, u_{N,0}, v_{1,0}, \dots, v_{N,0})^\top \in (H^{s,p_1})^{2N},$$

there exists a time $T > 0$ and a unique solution U of (1.1) such that

$$U = (u_1, \dots, u_N, v_1, \dots, v_N)^\top \in (L^\infty([0,T]; H^{s,p_1}) \cap Lip([0,T]; H^{s-1,p_1}))^{2N}.$$

Then the data-to-solution map of the initial value problem (1.1),

$$\Phi: \quad U_0 = (u_{1,0}, \dots, u_{N,0}, v_{1,0}, \dots, v_{N,0})^\top \mapsto U = (u_1, \dots, u_N, v_1, \dots, v_N)^\top,$$

is continuous from $(H^{s,p_1})^{2N}$ into $(L^\infty([0,T]; H^{s,p_1}) \cap Lip([0,T]; H^{s-1,p_1}))^{2N}$, where $c > 0$ is a constant depending on s, p_1.

Lemma 2.2 ([23]). *We define the weighted function*

$$I(x) = \begin{cases} (\ln(e^2 + |x|))^\alpha, & |x| \in [0, K], \\ (\ln(e^2 + K))^\alpha, & |x| \in (K, \infty), \end{cases}$$

where $\alpha \in [0,\infty)$ and $K \in \mathbb{R}^+$. Let $J(x) = (\ln(e^2 + |x|))^\alpha$, then we have $J(x+y) \le J(x)J(y)$. Furthermore, if there exists $C_0 \ge 0$ such that $I(x+y) \le C_0 J(x) I(y)$, then we say that the function $I(x)$ is $J(x)$-moderate.

Lemma 2.3 ([23]). *For $\theta \in [0,\infty)$ and $K \in \mathbb{R}^+$, we define the algebraic weighted function*

$$Q(x) = \begin{cases} (1 + |x|)^\theta, & |x| \in [0, K], \\ (1 + K)^\theta, & |x| \in (K, \infty). \end{cases}$$

Let $P(x) = (1+|x|)^\theta$, then $P(x+y) \le P(x)P(y)$. Additionally, if there exists $C_0 \ge 0$ such that $Q(x+y) \le C_0 P(x) Q(y)$, then the function $Q(x)$ is $P(x)$-moderate.

Lemma 2.4 ([23]). *Define the weighted function $\psi(x) = \min\{e^{|x|}, K\}$, where $x \in \mathbb{R}$ and $K \in \mathbb{R}^+$. Let $\phi(x) = e^{|x|}$. Then we have $\phi(x+y) \le \phi(x)\phi(y)$. If there exists $C_0 \ge 0$ such that $\psi(x+y) \le C_0 \phi(x) \psi(y)$, then the function $\psi(x)$ is $\phi(x)$-moderate. Moreover, $\psi^{1/2}(x)$ is $\phi^{1/2}(x)$-moderate.*

3. Logarithmic, algebraical, and exponential decay of solutions

In this section, using ideas from [23], we prove that the solution of (3.2) keep corresponding decay properties, provided the initial data decays logarithmically, algebraically, and exponentially at infinity.

For the sake of brevity, we let $m_i = u_i - \partial_x^2 u_i,\ n_i = v_i - \partial_x^2 v_i$. Then (1.1) can be rephrased as follows

$$\begin{gathered}
\partial_t u_i + \sum_{j=1}^{N} u_{ix}u_jv_j + F(u_i, u_{ix}) = 0, \quad t>0,\ x\in\mathbb{R},\\
\partial_t v_i + \sum_{j=1}^{N} v_{ix}u_jv_j + H(v_i, v_{ix}) = 0, \quad t>0,\ x\in\mathbb{R},\\
u_i(0,x) = u_{i,0}(x), v_i(0,x) = v_{i,0}(x), \quad x\in\mathbb{R},
\end{gathered} \tag{3.1}$$

where the nonlocal terms are

$$\begin{aligned}
F(u_i,u_{ix}) &= \Lambda^{-2}\partial_x \sum_{j=1}^{N} (u_iu_{jx}v_{jx} + u_{ix}u_jv_{jx})\\
&\quad + \Lambda^{-2}\sum_{j=1}^{N}(u_{ix}u_jv_j + 2u_iu_{jx}v_j - u_iu_{jx}v_{j,xx}),\\
H(v_i,v_{ix}) &= \Lambda^{-2}\partial_x \sum_{j=1}^{N} (v_iu_{jx}v_{jx} + v_{ix}u_{jx}v_j)\\
&\quad + \Lambda^{-2}\sum_{j=1}^{N}(v_{ix}u_jv_j + 2v_iu_jv_{jx} - v_iu_{j,xx}v_{jx}).
\end{aligned}$$

Note that $(1-\partial_x^2)^{-1}f = G*f$ for all $f\in L^p$, where $G(x) = \frac{1}{2}e^{-|x|}$, $x\in\mathbb{R}$. Then we can rewrite (3.1) as

$$\begin{gathered}
\partial_t u_i + \sum_{j=1}^{N} u_{ix}u_jv_j + G_x*(f_1+f_2) + G*(f_3+f_4+f_5) = 0,\\
\partial_t v_i + \sum_{j=1}^{N} v_{ix}u_jv_j + G_x*(h_1+h_2) + G*(h_3+h_4+h_5) = 0,\\
u_i(0,x) = u_{i,0}(x), v_i(0,x) = v_{i,0}(x),
\end{gathered} \tag{3.2}$$

where

$$\begin{gathered}
f_1 = \sum_{j=1}^{N} u_iu_{jx}v_{jx}, \quad f_2 = \sum_{j=1}^{N} u_{ix}u_jv_{jx}, \quad f_3 = \sum_{j=1}^{N} u_{ix}u_jv_j,\\
f_4 = 2\sum_{j=1}^{N} u_iu_{jx}v_j, \quad f_5 = -\sum_{j=1}^{N} u_iu_{jx}v_{j,xx},
\end{gathered} \tag{3.3}$$

$$\begin{gathered}
h_1=\sum_{j=1}^N v_i u_{jx} v_{jx}, \quad h_2=\sum_{j=1}^N v_{ix} u_{jx} v_j, \quad h_3=\sum_{j=1}^N v_{ix} u_j v_j, \\
h_4=2 \sum_{j=1}^N v_i u_j v_{jx}, \quad h_5=-\sum_{j=1}^N v_i u_{j,xx} v_{jx}.
\end{gathered} \tag{3.4}$$

First, we establish the logarithmic decay of the strong solution to (3.2), but before proving the result, we need to show a useful lemma.

Lemma 3.1. *Let $I(x)$ and $J(x)$ be the weighted functions defined in Lemma 2.2, and f_k, g_k be given by* (3.3) *and* (3.4)*. If $I(x)$ is $J(x)$-moderate, then for $p \geq 1$, $f, g \in L^p$, we have*

$$\|I(f * g)\|_{L^p} \leq \|J f\|_{L^1}\|I g\|_{L^p},$$

where $f=G$, G_x, $G(x)=\frac{1}{2} e^{-|x|}$, and $g=f_k, h_k$, $k=1,2, \ldots, 5$.

Proof. If $f=G$ and $g=f_1=\sum_{j=1}^N u_i u_{jx} v_{jx}$, by $I(x+y) \leq J(x) I(y)$ and the Young inequality to yield

$$\begin{aligned}
\|I(G * f_1)\|_{L^p} & =\|\int_{\mathbb{R}} I(x) G(x-y) f_1(y) d y\|_{L^p} \\
& \leq\|\int_{\mathbb{R}}| J(x-y) I(y) G(x-y) f_1(y)| d y \|_{L^p} \\
& =\|(J G) *(I f_1)\|_{L^p} \\
& \leq\|J G\|_{L^1}\|I f_1\|_{L^p},
\end{aligned}$$

which leads to $\|I(G * f_1)\|_{L^p} \leq\|J G\|_{L^1}\|I f_1\|_{L^p}$. Similarly, one can easily check the following inequalities

$$\begin{aligned}
\|I(G * f_k)\|_{L^p} & \leq\|J G\|_{L^1}\|I f_k\|_{L^p}, \quad k=1,2, \ldots, 4, \\
\|I(G_x * f_k)\|_{L^p} & \leq\|J G_x\|_{L^1}\|I f_k\|_{L^p}, \quad k=1,2, \ldots, 5, \\
\|I(G * h_k)\|_{L^p} & \leq\|J G\|_{L^1}\|I h_k\|_{L^p}, \quad k=1,2, \ldots, 5, \\
\|I(G_x * h_k)\|_{L^p} & \leq\|J G_x\|_{L^1}\|I h_k\|_{L^p}, \quad k=1,2, \ldots, 5.
\end{aligned}$$

This completes the proof. □

We shall estimate $\|I F(u_i, u_{ix})(s)\|_{L^p}$, $\|I F_x(u_i, u_{ix})(s)\|_{L^p}$, $\|I H(u_i, u_{ix})(s)\|_{L^p}$, and $\|I H_x(u_i, u_{ix})(s)\|_{L^p}$ in the following lemma, and will be referenced multiple times.

Lemma 3.2. *For $p>1$, the following estimates hold:*

$$\|I F(u_i, u_{ix})(s)\|_{L^p}+\|I F_x(u_i, u_{ix})(s)\|_{L^p} \leq c(\|u_i I\|_{L^p}+\|u_{ix} I\|_{L^p}), \tag{3.5}$$

$$\|I H(v_i, v_{ix})(s)\|_{L^p}+\|I H_x(v_i, v_{ix})(s)\|_{L^p} \leq c(\|v_i I\|_{L^p}+\|v_{ix} I\|_{L^p}), \tag{3.6}$$

where $c>0$ is a constant depending on α, N, M, for $i=1,2, \ldots, N$.

Proof. We will only give a proof of (3.5), the other can be proved similarly. For convenience, let

$$M=\sup_{t \in[0, T]}\{\|U(t, \cdot)\|_{H^{s, p_1}}\}.$$

The Sobolev embedding theorem leads to

$$\|u_i(t)\|_{L^{\infty}},\ \|u_{i,x}(t)\|_{L^{\infty}},\ \|u_{i,xx}(t)\|_{L^{\infty}},\ \|v_i(t)\|_{L^{\infty}},\ \|v_{i,x}(t)\|_{L^{\infty}},\ \|v_{i,xx}(t)\|_{L^{\infty}} \leq M.$$

Lemma 3.1 yields that

$$\begin{aligned}
&\|IF(u_i,u_{ix})(s)\|_{L^p}\\
&\le \|I\left(G_x*(f_1+f_2)\right)\|_{L^p}+\|I(G*(f_3+f_4+f_5))\|_{L^p}\\
&\le \|I(G_x*f_1)\|_{L^p}+\|I(G_x*f_2)\|_{L^p}+\|I(G*(f_3+f_4+f_5))\|_{L^p}\\
&\le \|JG_x\|_{L^1}(\|If_1\|_{L^p}+\|If_2\|_{L^p})+\|JG\|_{L^1}(\|If_3\|_{L^p}+\|If_4\|_{L^p}+\|If_5\|_{L^p}).
\end{aligned}\tag{3.7}$$

For the first term of (3.7), in view of the definition of f_1 in (3.3) and Hölder's inequality to arrive at

$$\begin{aligned}
\|JG_x\|_{L^1}\|If_1\|_{L^p}&=\sum_{j=1}^N\|JG_x\|_{L^1}\|u_iu_{jx}v_{jx}I\|_{L^p}\\
&\le\sum_{j=1}^N\|JG_x\|_{L^1}\|u_{jx}v_{jx}\|_{L^\infty}\|u_iI\|_{L^p}\\
&\le 2^{\alpha-1}(2+1/e^2)NM^2\|u_iI\|_{L^p}.
\end{aligned}$$

Using the same method, it follows that

$$\begin{aligned}
\|IF(u_i,u_{ix})(s)\|_{L^p}&\le 2^{\alpha-1}(2+1/e^2)NM^2(\|u_iI\|_{L^p}+\|u_{ix}I\|_{L^p})\\
&\le c(\|u_iI\|_{L^p}+\|u_{ix}I\|_{L^p}).
\end{aligned}$$

Since $G_{xx}*f=G*f-f$, $G(x)=\frac{1}{2}e^{-|x|}$, $x\in\mathbb{R}$, one obtains

$$\begin{aligned}
F_x(u_i,u_{ix})&=G_{xx}*(f_1+f_2)+G_x*(f_3+f_4+f_5)\\
&=G*(f_1+f_2)-(f_1+f_2)+G_x*(f_3+f_4+f_5).
\end{aligned}$$

Therefore, one can easily check that

$$\begin{aligned}
&\|IF_x(u_i,u_{ix})(s)\|_{L^p}\\
&\le\|I\left(G*(f_1+f_2)\right)\|_{L^p}+\|I(f_1+f_2)\|_{L^p}+\|I\left(G_x*(f_3+f_4+f_5)\right)\|_{L^p}\\
&\le c(\|u_iI\|_{L^p}+\|u_{ix}I\|_{L^p})+\sum_{j=1}^N\|I(u_iu_{jx}v_{jx}+u_{ix}u_jv_{jx})\|_{L^p}\\
&\le c(\|u_iI\|_{L^p}+\|u_{ix}I\|_{L^p}),
\end{aligned}$$

where the constant c depends on α, N, M, and the second inequality comes from

$$\|JG\|_{L^1},\|JG_x\|_{L^1}\le 2^{\alpha-1}(2+1/e^2).$$

The proof is complete. □

Theorem 3.3. *Assume the initial data $U_0=(u_0,v_0)^\top\in(H^{s,p_1})^{2N}$, $s>2+\frac{1}{p_1}$, $p_1\in(1,\infty)$, and $T>0$. Then there exists a unique solution $U(t,x)\in[C([0,T];H^{s,p_1}(\mathbb{R}))]^{2N}$ to (3.2) with the initial data U_0. For $p>1$, if the initial data satisfies for some $C>0$,*

$$\|U_0(x)I(x)\|_{L^p}+\|U_{0,x}(x)I(x)\|_{L^p}\le C,$$

then the solution satisfies

$$\|U(t,\cdot)I\|_{L^p}+\|U_x(t,\cdot)I\|_{L^p}\le C,$$

uniformly in the interval $[0,T]$*, where for* $\alpha \in [0,\infty)$ *and* $K \in \mathbb{R}^+$*, the weighted function* $I(x)$ *is given by*

$$I(x) = \begin{cases} (\ln(e^2 + |x|))^\alpha, & |x| \in [0,K], \\ (\ln(e^2 + K))^\alpha, & |x| \in (K,\infty). \end{cases}$$

In particular, if the initial data U_0 *and* $U_{0,x}$ *decay logarithmically as*

$$|U_0(x)| \sim \mathcal{O}\left((\ln(e^2 + |x|))^{-\alpha}\right), \quad \text{as } |x| \to \infty,$$
$$|U_{0,x}(x)| \sim \mathcal{O}\left((\ln(e^2 + |x|))^{-\alpha}\right), \quad \text{as } |x| \to \infty,$$

then the solution of (3.2) *decays logarithmically as*

$$|U(t,x)| \sim \mathcal{O}\left((\ln(e^2 + |x|))^{-\alpha}\right), \quad \text{as } |x| \to \infty,$$
$$|U_x(t,x)| \sim \mathcal{O}\left((\ln(e^2 + |x|))^{-\alpha}\right), \quad \text{as } |x| \to \infty,$$

uniformly in the interval $[0,T]$.

Proof. Multiplying the first equation in (3.1) by I, we obtain

$$(u_i I)_t + \sum_{j=1}^N u_{ix} u_j v_j I + F(u_i, u_{ix}) I = 0. \tag{3.8}$$

Then by this equality and $|u_i I|^{p-2}(u_i I)$ with $p > 1$, and integrating the result on $\mathbb{R}$ with respect to x-variable, it follows that

$$\begin{aligned} \frac{1}{p}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}} |u_i I|^p dx = &- \sum_{j=1}^N \int_{\mathbb{R}} u_{ix} u_j v_j I |u_i I|^{p-2}(u_i I) dx \\ &- \int_{\mathbb{R}} F(u_i, u_{ix}) I |u_i I|^{p-2}(u_i I) dx. \end{aligned} \tag{3.9}$$

Note that

$$\frac{1}{p}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}} |u_i I|^p dx = \frac{1}{p}\frac{\mathrm{d}}{\mathrm{d}t}\|u_i I\|_{L^p}^p = \|u_i I\|_{L^p}^{p-1}\frac{\mathrm{d}}{\mathrm{d}t}\|u_i I\|_{L^p}.$$

In view of $u_{ix} I = (u_i I)_x - u_i I_x$ and $0 \le I_x \le \beta I$, where $\beta = \frac{\alpha}{2e^2} > 0$, we have

$$\begin{aligned} &\Big|\int_{\mathbb{R}} u_{ix} u_j v_j I |u_i I|^{p-2}(u_i I) dx\Big| \\ &\le \int_{\mathbb{R}} u_j v_j [(u_i I)_x - u_i I_x] |u_i I|^{p-2}(u_i I) dx \\ &\le \int_{\mathbb{R}} u_j v_j (u_i I)_x |u_i I|^{p-2}(u_i I) dx + \int_{\mathbb{R}} u_j v_j u_i I_x |u_i I|^{p-2}(u_i I) dx \\ &\le \frac{1}{p}\int_{\mathbb{R}} u_j v_j (|u_i I|^p)_x dx + \beta \int_{\mathbb{R}} u_j v_j u_i I |u_i I|^{p-2}(u_i I) dx \\ &\le \frac{1}{p}\int_{\mathbb{R}} (u_j v_j)_x |u_i I|^p dx + \beta \|u_j v_j\|_{L^\infty} \|u_i I\|_{L^p}^p \\ &\le (2+\beta) M^2 \|u_i I\|_{L^p}^p, \end{aligned}$$

where M is a constant defined as Lemma 3.2. Using Holder's inequality, one obtains

$$\int_{\mathbb{R}} I F(u_i, u_{ix}) |u_i I|^{p-2}(u_i I) dx \le \|I F(u_i, u_{ix})\|_{L^p} \|u_i I\|_{L^p}^{p-1}.$$

Combining (3.8) with the above relations to yield

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u_iI\|_{L^p} \le (2+\beta)NM^2\|u_iI\|_{L^p} + \|IF(u_i,u_{ix})\|_{L^p}. \tag{3.10}$$

Differentiating the first equation in (3.1) with respect to x, and multiplying it by I, one has that

$$(u_{ix}I)_t + \sum_{j=1}^N (u_jv_ju_{i,xx} + u_{ix}u_{jx}v_j + u_{ix}u_jv_{jx})I + IF_x(u_i,u_{ix}) = 0. \tag{3.11}$$

Multiplying (3.11) by $|u_{ix}I|^{p-2}(u_{ix}I)$ and integrating the result on $\mathbb{R}$ with respect to x, one obtains

$$\begin{aligned}
&\|u_{ix}I\|_{L^p}^{p-1}\frac{\mathrm{d}}{\mathrm{d}t}\|u_{ix}I\|_{L^p}\\
&= -\sum_{j=1}^N\int_{\mathbb{R}}(u_{ix}u_jv_{jx} + u_{ix}u_{jx}v_j)I|u_{ix}I|^{p-2}(u_{ix}I)\,dx\\
&\quad -\sum_{j=1}^N\int_{\mathbb{R}}u_jv_ju_{i,xx}I|u_{ix}I|^{p-2}(u_{ix}I)\,dx - \int_{\mathbb{R}}IF_x(u_i,u_{ix})|u_{ix}I|^{p-2}(u_{ix}I)\,dx\\
&\doteq E_1+E_2+E_3.
\end{aligned} \tag{3.12}$$

Applying Hölder's inequality, it follows that

$$E_1 \le \sum_{j=1}^N(\|u_jv_{jx}\|_{L^\infty} + \|u_{jx}v_j\|_{L^\infty})\|u_{ix}I\|_{L^p}^p \le NM^2\|u_{ix}I\|_{L^p}^p.$$

From $u_{i,xx}I = (u_{ix}I)_x - u_{ix}I_x$ and $0 \le I_x \le \beta I$, where $\beta = \frac{\gamma}{2e^2} > 0$, we have

$$\begin{aligned}
|E_2| &= |-\sum_{j=1}^N\int_{\mathbb{R}}u_jv_j[(u_{ix}I)_x - u_{ix}I_x]|u_{ix}I|^{p-2}(u_{ix}I)\,dx|\\
&\le |\frac{1}{p}\sum_{j=1}^N\int_{\mathbb{R}}u_jv_j(|u_{ix}I|^p)_xdx + \sum_{j=1}^N\int_{\mathbb{R}}u_jv_ju_{ix}I_x|u_{ix}I|^{p-2}(u_{ix}I)\,dx|\\
&\le \frac{1}{p}\sum_{j=1}^N\int_{\mathbb{R}}(u_jv_j)_x|u_{ix}I|^pdx + \beta\sum_{j=1}^N\|u_jv_j\|_{L^\infty}\|u_{ix}I\|_{L^p}^p\\
&\le (2+\beta)NM^2\|u_{ix}I\|_{L^p}^p.
\end{aligned}$$

E_3 is treated similarly to obtain

$$\int_{\mathbb{R}}IF_x(u_i,u_{ix})|u_{ix}I|^{p-2}(u_{ix}I)\,dx \le \|IF_x(u_i,u_{ix})\|_{L^p}\|u_{ix}I\|_{L^p}^{p-1}.$$

Consequently, plugging the above inequalities into (3.12), we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u_{ix}I\|_{L^p} \le (3+\beta)NM^2\|u_{ix}I\|_{L^p} + \|IF_x(u_i,u_{ix})\|_{L^p}. \tag{3.13}$$

Taking into account (3.10) and (3.13) we have

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\left(\|u_iI\|_{L^p} + \|u_{ix}I\|_{L^p}\right) &\le (3+\beta)NM^2\left(\|u_iI\|_{L^p} + \|u_{ix}I\|_{L^p}\right)\\
&\quad + \|IF(u_i,u_{ix})\|_{L^p} + \|IF_x(u_i,u_{ix})\|_{L^p},
\end{aligned} \tag{3.14}$$

which along with Gronwall's inequality leads to

$$\begin{aligned} &\|u_i I\|_{L^p} + \|u_{ix} I\|_{L^p} \\ &\le e^{(3+\beta)NM^2 t} \left(\|u_{i0} I\|_{L^p} + \|u_{i0,x} I\|_{L^p}\right) \\ &\quad + e^{(3+\beta)NM^2 t} \int_0^t \Big(\|IF(u_i, u_{ix})(s)\|_{L^p} + \|IF_x(u_i, u_{ix})(s)\|_{L^p}\Big) ds. \end{aligned} \tag{3.15}$$

Analogously, one can easily deduce that

$$\begin{aligned} &\|v_i I\|_{L^p} + \|v_{ix} I\|_{L^p} \\ &\le e^{(3+\beta)NM^2 t} \left(\|v_{i0} I\|_{L^p} + \|v_{i0,x} I\|_{L^p}\right) \\ &\quad + e^{(3+\beta)NM^2 t} \int_0^t \Big(\|IH(v_i, v_{ix})(s)\|_{L^p} + \|IH_x(v_i, v_{ix})(s)\|_{L^p}\Big) ds. \end{aligned} \tag{3.16}$$

Let $U = (u, v)^\top$, where $u = (u_1, u_2, \dots, u_N)^\top, v = (v_1, v_2, \dots, v_N)^\top$ and we define

$$\|U(t)\|_{L^p} \doteq \|u(t)\|_{L^p} + \|v(t)\|_{L^p}.$$

Combining (3.15) with (3.16) and applying Lemma 3.2, one has

$$\begin{aligned} \|UI\|_{L^p} + \|U_x I\|_{L^p} &\le e^{(3+\beta)NM^2 t}(\|U_0 I\|_{L^p} + \|U_{0,x} I\|_{L^p}) \\ &\quad + e^{(3+\beta)NM^2 t} \int_0^t (\|UI\|_{L^p} + \|U_x I\|_{L^p}) ds. \end{aligned}$$

Assuming $Z(t) = \|U(t, \cdot)I\|_{L^p} + \|U_x(t, \cdot)I\|_{L^p}$, we deduce that

$$Z(t) \le e^{(3+\beta)NM^2 t}\Big(Z(0) + c\int_0^t Z(s) ds\Big). \tag{3.17}$$

Using Gronwall's inequality, one gets

$$Z(t) \le CZ(0) \le C\left(\|U_0(x)I(x)\|_{L^p} + \|U_{0,x}(x)I(x)\|_{L^p}\right), \tag{3.18}$$

where $C = C(c, \beta, M, N, T)$ is a positive constant.

If for some $C > 0$, the data U_0 and $U_{0,x}$ satisfy

$$\|U_0(x)I(x)\|_{L^p} + \|U_{0,x}(x)I(x)\|_{L^p} \le C,$$

then for all $t \in [0, T]$, we can show that the solution satisfies

$$\|U(t, \cdot)I\|_{L^p} + \|U_x(t, \cdot)I\|_{L^p} \le C.$$

Particularly, if the initial data U_0 and $U_{0,x}$ decay logarithmically as

$$\begin{aligned} |U_0(x)| &\sim \mathcal{O}\left((\ln(e^2 + |x|))^{-\alpha}\right), \quad \text{as } |x| \to \infty, \\ |U_{0,x}(x)| &\sim \mathcal{O}\left((\ln(e^2 + |x|))^{-\alpha}\right), \quad \text{as } |x| \to \infty. \end{aligned}$$

Taking the limit as $p \to \infty$ and $K \to \infty$ in the above inequality, we have

$$\begin{aligned} |U(t,x)| &\sim \mathcal{O}\left((\ln(e^2 + |x|))^{-\alpha}\right), \quad \text{as } |x| \to \infty, \\ |U_x(t,x)| &\sim \mathcal{O}\left((\ln(e^2 + |x|))^{-\alpha}\right), \quad \text{as } |x| \to \infty, \end{aligned}$$

uniformly in the interval $[0, T]$. This completes the proof. □

Corollary 3.4. *In fact, under the assumption of Theorem 3.3, if U_0, $U_{0,x}$ and $U_{0,xx}$ satisfy*

$$\|U_0(x)I(x)\|_{L^p} + \|U_{0,x}(x)I(x)\|_{L^p} + \|U_{0,xx}(x)I(x)\|_{L^p} \le C,$$

hat for some $C > 0$, then the solution satisfies

$$\|U(t,\cdot)I\|_{L^p} + \|U_x(t,\cdot)I\|_{L^p} + \|U_{xx}(t,\cdot)I\|_{L^p} \le C,$$

uniformly in the interval $[0,T]$. In particular, if the initial data U_0 satisfies

$$|U_0(x)|,\ |U_{0,x}(x)|,\ |U_{0,xx}(x)| \sim \mathcal{O}\left((\ln(e^2+|x|))^{-\alpha}\right), \quad \text{as } |x| \to \infty,$$

then the solution $U(t,x)$ decays logarithmically as

$$|U(t,x)|,\ |U_x(t,x)|,\ |U_{xx}(t,x)| \sim \mathcal{O}\left((\ln(e^2+|x|))^{-\alpha}\right), \quad \text{as } |x| \to \infty,$$

uniformly in the interval $[0,T]$.

Proof. Differentiating the first equation in (3.1) twice with respect to x, and multiplying it by I, applying the obtained result by $|u_{i,xx}I|^{p-2}(u_{i,xx}I)$, integration by parts, it yields that

$$\begin{aligned}
&\int_{\mathbb{R}} (u_{i,xx}I)_t |u_{i,xx}I|^{p-2}(u_{i,xx}I)\,dx \\
&= -\int_{\mathbb{R}} IF_{xx}(u_i,u_{ix},u_{i,xx})|u_{i,xx}I|^{p-2}(u_{i,xx}I)\,dx \\
&\quad - 2\sum_{j=1}^{N}\int_{\mathbb{R}} u_{i,xx}(u_{jx}v_j + u_j v_{jx})I|u_{i,xx}I|^{p-2}(u_{i,xx}I)\,dx \\
&\quad - 2\sum_{j=1}^{N}\int_{\mathbb{R}} u_{ix}u_{jx}v_{jx}I|u_{i,xx}I|^{p-2}(u_{i,xx}I)\,dx \\
&\quad - \sum_{j=1}^{N}\int_{\mathbb{R}} u_{ix}(u_{j,xx}v_j + u_j v_{j,xx})I|u_{i,xx}I|^{p-2}(u_{i,xx}I)\,dx \\
&\quad - \sum_{j=1}^{N}\int_{\mathbb{R}} u_{i,xxx}u_j v_j I|u_{i,xx}I|^{p-2}(u_{i,xx}I)\,dx.
\end{aligned} \tag{3.19}$$

By using Hölder's inequality, one can easily check that

$$\int_{\mathbb{R}} (u_{i,xx}I)_t |u_{i,xx}I|^{p-2}(u_{i,xx}I)\,dx = \|u_{i,xx}I\|_{L^p}^{p-1}\frac{\mathrm{d}}{\mathrm{d}t}\|u_{i,xx}I\|_{L^p},$$

$$\int_{\mathbb{R}} IF_{xx}(u_i,u_{ix},u_{i,xx})|u_{i,xx}I|^{p-2}(u_{i,xx}I)\,dx \le \|IF_{xx}(u_i,u_{ix},u_{i,xx})\|_{L^p}\|u_{i,xx}I\|_{L^p}^{p-1},$$

$$\begin{aligned}
&2\int_{\mathbb{R}} u_{i,xx}(u_{jx}v_j + u_j v_{jx})I|u_{i,xx}I|^{p-2}(u_{i,xx}I)\,dx \\
&\le 2(\|u_{jx}v_j\|_{L^\infty} + \|u_j v_{jx}\|_{L^\infty})\|u_{i,xx}I\|_{L^p}^p \\
&\le 4M^2\|u_{i,xx}I\|_{L^p}^p,
\end{aligned}$$

$$\begin{aligned}
2\int_{\mathbb{R}} u_{ix}u_{jx}v_{jx}I|u_{i,xx}I|^{p-2}(u_{i,xx}I)\,dx &\le 2\|u_{jx}v_{jx}\|_{L^\infty}\|u_{ix}I\|_{L^p}\|u_{i,xx}I\|_{L^p}^{p-1} \\
&\le 2M^2\|u_{ix}I\|_{L^p}\|u_{i,xx}I\|_{L^p}^{p-1},
\end{aligned}$$

$$\begin{aligned}
&\int_{\mathbb{R}} u_{ix}(u_{j,xx}v_j + u_j v_{j,xx})I|u_{i,xx}I|^{p-2}(u_{i,xx}I)\,dx \\
&\le (\|u_{j,xx}v_j\|_{L^\infty} + \|u_j v_{j,xx}\|_{L^\infty})\|u_{ix}I\|_{L^p}\|u_{i,xx}I\|_{L^p}^{p-1} \\
&\le 2M^2\|u_{ix}I\|_{L^p}\|u_{i,xx}I\|_{L^p}^{p-1}.
\end{aligned}$$

In view of $u_{i,xxx}I = (u_{i,xx}I)_x - u_{i,xx}I_x$ and $0 \le I_x \le \beta I$, where $\beta = \frac{\alpha}{2e^2} > 0$, we have

$$\begin{aligned}
&\int_{\mathbb{R}} u_{i,xxx} u_j v_j I |u_{i,xx}I|^{p-2}(u_{i,xx}I)\,dx \\
&= \int_{\mathbb{R}} u_j v_j [(u_{i,xx}I)_x - u_{i,xx}I_x] |u_{i,xx}I|^{p-2}(u_{i,xx}I)\,dx \\
&= \frac{1}{p}\int_{\mathbb{R}} u_j v_j (|u_{i,xx}I|^p)_x dx - \int_{\mathbb{R}} u_j v_j u_{i,xx} I_x |u_{i,xx}I|^{p-2}(u_{i,xx}I)\,dx \\
&\le \frac{1}{p}\int_{\mathbb{R}} (u_{jx}v_j + u_j v_{jx}) |u_{i,xx}I|^p dx + \beta \int_{\mathbb{R}} |u_j v_j| |u_{i,xx}I|^p dx \\
&\le \big(\frac{2}{p} + \beta\big) M^2 \|u_{i,xx}I\|_{L^p}^p \\
&\le (2+\beta)\, M^2 \|u_{i,xx}I\|_{L^p}^p.
\end{aligned}$$

Inserting the above relations into (3.19) yields

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\|u_{i,xx}I\|_{L^p} \le 4NM^2\|u_{ix}I\|_{L^p} + (6+\beta)\, NM^2 \|u_{i,xx}I\|_{L^p} \\
+ \|IF_{xx}(u_i, u_{ix}, u_{i,xx})\|_{L^p}.
\end{aligned} \tag{3.20}$$

Combining (3.14) with (3.20), it follows that

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left(\|u_iI\|_{L^p} + \|u_{ix}I\|_{L^p} + \|u_{i,xx}I\|_{L^p}\right) \\
&\le (6+\beta)\, NM^2 (\|u_iI\|_{L^p} + \|u_{ix}I\|_{L^p} + \|u_{i,xx}I\|_{L^p}) \\
&\quad + \|IF(u_i,u_{ix})\|_{L^p} + \|IF_x(u_i,u_{ix})\|_{L^p} + \|IF_{xx}(u_i,u_{ix},u_{i,xx})\|_{L^p}.
\end{aligned} \tag{3.21}$$

Using Gronwall's inequality, we obtain

$$\begin{aligned}
&\|u_iI\|_{L^p} + \|u_{ix}I\|_{L^p} + \|u_{i,xx}I\|_{L^p} \\
&\le e^{(6+\beta)NM^2 t}\left(\|u_{i0}I\|_{L^p} + \|u_{i0,x}I\|_{L^p} + \|u_{i0,xx}I\|_{L^p}\right) \\
&\quad + e^{(6+\beta)NM^2 t}\int_0^t \Big(\|IF(u_i,u_{ix})(s)\|_{L^p} + \|IF_x(u_i,u_{ix})(s)\|_{L^p} \\
&\quad + \|IF_{xx}(u_i,u_{ix},u_{i,xx})(s)\|_{L^p}\Big) ds.
\end{aligned}$$

Similarly, one can easily check that

$$\begin{aligned}
&\|v_iI\|_{L^p} + \|v_{ix}I\|_{L^p} + \|v_{i,xx}I\|_{L^p} \\
&\le e^{(6+\beta)NM^2 t}\left(\|v_{i0}I\|_{L^p} + \|v_{i0,x}I\|_{L^p} + \|v_{i0,xx}I\|_{L^p}\right) \\
&\quad + e^{(6+\beta)NM^2 t}\int_0^t \Big(\|IH(v_i,v_{ix})(s)\|_{L^p} \\
&\quad + \|IH_x(v_i,v_{ix})(s)\|_{L^p} + \|IH_{xx}(v_i,v_{ix},v_{i,xx})(s)\|_{L^p}\Big) ds.
\end{aligned}$$

From $G_{xx} * f = G * f - f$ and $G(x) = \frac{1}{2}e^{-|x|}$ for $x \in \mathbb{R}$, we have

$$\begin{aligned}
F_{xx}(u_i, u_{ix}, u_{i,xx}) &= G_x * (f_1 + f_2) - (f_{1x} + f_{2x}) + G_{xx} * (f_3 + f_4 + f_5) \\
&= G_x * (f_1 + f_2) + G * (f_3 + f_4 + f_5) \\
&\quad - (f_{1x} + f_{2x} + f_3 + f_4 + f_5).
\end{aligned}$$

Now, we just need to estimate $\|IF_{xx}(u_i,u_{ix},u_{i,xx})(s)\|_{L^p}$, by Lemma 3.1 to obtain

$$
\begin{aligned}
&\|IF_{xx}(u_i,u_{ix},u_{i,xx})(s)\|_{L^p}\\
&\le \|I\left(G_x*(f_1+f_2)\right)\|_{L^p}+\|I\left(G*(f_3+f_4+f_5)\right)\|_{L^p}\\
&\quad +\|I(f_{1x}+f_{2x}+f_3+f_4+f_5)\|_{L^p}\\
&\le \|JG_x\|_{L^1}(\|If_1\|_{L^p}+\|If_2\|_{L^p})+\|JG\|_{L^1}(\|If_3\|_{L^p}+\|If_4\|_{L^p}+\|If_5\|_{L^p})\\
&\quad +\|I(f_{1x}+f_{2x}+f_3+f_4+f_5)\|_{L^p}\\
&\le c(\|u_iI\|_{L^p}+\|u_{ix}I\|_{L^p}+\|u_{i,xx}I\|_{L^p}),
\end{aligned}
$$

where the constant c depends on α, N, M, the last inequality comes from

$$\|JG\|_{L^1},\|JG_x\|_{L^1}\le 2^{\alpha-1}(2+1/e^2).$$

Combining the above estimates, we obtain

$$
\begin{aligned}
&\|UI\|_{L^p}+\|U_xI\|_{L^p}+\|U_{xx}I\|_{L^p}\\
&\le e^{(6+\beta)NM^2t}(\|U_0I\|_{L^p}+\|U_{0,x}I\|_{L^p}+\|U_{0,xx}I\|_{L^p})\\
&\quad + e^{(6+\beta)NM^2t}\int_0^t(\|UI\|_{L^p}+\|U_xI\|_{L^p}+\|U_{xx}I\|_{L^p})ds.
\end{aligned}
$$

Setting $Y(t)=\|U(t,\cdot)I\|_{L^p}+\|U_x(t,\cdot)I\|_{L^p}+\|U_{xx}(t,\cdot)I\|_{L^p}$, we obtain

$$Y(t)\le e^{(6+\beta)NM^2t}\Big(Y(0)+c\int_0^tY(s)ds\Big). \tag{3.22}$$

Applying Gronwall's inequality, there exists a constant $C(c,\alpha,M,N,T)$ such that for all $t\in[0,T]$,

$$
\begin{aligned}
Y(t)&\le CY(0)\\
&\le C\left(\|U_0(x)I(x)\|_{L^p}+\|U_{0,x}(x)I(x)\|_{L^p}+\|U_{0,xx}I\|_{L^p}\right).
\end{aligned}\tag{3.23}
$$

If U_0, $U_{0,x}$ and $U_{0,xx}$ satisfy

$$\|U_0(x)I(x)\|_{L^p}+\|U_{0,x}(x)I(x)\|_{L^p}+\|U_{0,xx}(x)I(x)\|_{L^p}\le C$$

for some $C>0$, then for all $t\in[0,T]$, the solution satisfies

$$\|U(t,\cdot)I\|_{L^p}+\|U_x(t,\cdot)I\|_{L^p}+\|U_{xx}(t,\cdot)I\|_{L^p}\le C.$$

Particularly, if the initial data U_0 decays logarithmically as

$$|U_0(x)|,\ |U_{0,x}(x)|,\ |U_{0,xx}(x)|\sim\mathcal{O}\left((\ln(e^2+|x|))^{-\alpha}\right),\quad \text{as } |x|\to\infty.$$

Taking $K\to\infty$ and $p\to\infty$ in (3.23), the solution $U(t,x)$ decays logarithmically as

$$|U(t,x)|,\ |U_x(t,x)|,\ |U_{xx}(t,x)|\sim\mathcal{O}\left((\ln(e^2+|x|))^{-\alpha}\right),\quad \text{as } |x|\to\infty,$$

uniformly in the interval $[0,T]$. So, the proof is complete. $\square$

Secondly, we will show the algebraic decay of the solution to (3.2). As before, we first state the following lemma.

Lemma 3.5. *Let $Q(x)$ and $P(x)$ be the weighted functions defined in Lemma 2.3; and let f_k,g_k be given by* (3.3) *and* (3.4)*. If $Q(x)$ is $P(x)$-moderate, then for $p\ge 1$, $f,g\in L^p$, one has*

$$\|Q(f*g)\|_{L^p}\le\|Pf\|_{L^1}\|Qg\|_{L^p},$$

where $f=G,G_x$, $G(x)=\frac{1}{2}e^{-|x|}$, and $g=f_k,h_k$, $k=1,2,\dots,5$.

The proof of the above lemma is similar to the proof of Lemma 3.1 we omit its proof.

Theorem 3.6. *Suppose* $U_0 = (u_0, v_0)^\top \in (H^{s,p_1})^{2N}$, $s > 2 + \frac{1}{p_1}$, $p_1 \in (1,\infty)$. *Then there exist* $T > 0$ *and a unique solution* $U(t,x) \in [C([0,T]; H^{s,p_1}(\mathbb{R}))]^{2N}$ *to* (3.2). *Furthermore, if* U_0 *and* $U_{0,x}$ *satisfy*

$$\|U_0(x)Q(x)\|_{L^p} + \|U_{0,x}(x)Q(x)\|_{L^p} \le C,$$

for $p > 1$ *and some* $C > 0$, *then the solution satisfies*

$$\|U(t,\cdot)Q\|_{L^p} + \|U_x(t,\cdot)Q\|_{L^p} \le C,$$

uniformly in the interval $[0,T]$, *where for* $\theta \in [0,\infty)$ *and* $K \in \mathbb{R}^+$, *the weighted function* $Q(x)$ *is defined by*

$$Q(x) = \begin{cases} (1+|x|)^\theta, & |x| \in [0,K], \\ (1+K)^\theta, & |x| \in (K,\infty). \end{cases}$$

In particular, if the initial data U_0 *and* $U_{0,x}$ *decay algebraically as*

$$\begin{gathered} |U_0(x)| \sim \mathcal{O}\left((1+|x|)^{-\theta}\right), \quad \text{as } |x| \to \infty, \\ |U_{0,x}(x)| \sim \mathcal{O}\left((1+|x|)^{-\theta}\right), \quad \text{as } |x| \to \infty, \end{gathered}$$

then the solution $U(t,x)$ *decays algebraically as*

$$\begin{gathered} |U(t,x)| \sim \mathcal{O}\left((1+|x|)^{-\theta}\right), \quad \text{as } |x| \to \infty, \\ |U_x(t,x)| \sim \mathcal{O}\left((1+|x|)^{-\theta}\right), \quad \text{as } |x| \to \infty, \end{gathered}$$

uniformly in the interval $[0,T]$.

The proof of the above theorem is similar to that of Theorem 3.3. In view of Lemma 2.3 and Lemmas 3.2 and 3.5, we obtain the required statement.

As the proof of the following corollary is very similar to Corollary 3.4 just with a slight modification, thus we only show the result here.

Corollary 3.7. *Under the conditions of Theorem 3.6, if* U_0, $U_{0,x}$, *and* $U_{0,xx}$ *satisfy, for some* $C > 0$,

$$\|U_0(x)Q(x)\|_{L^p} + \|U_{0,x}(x)Q(x)\|_{L^p} + \|U_{0,xx}(x)Q(x)\|_{L^p} \le C,$$

then the solution satisfies

$$\|U(t,\cdot)Q\|_{L^p} + \|U_x(t,\cdot)Q\|_{L^p} + \|U_{xx}(t,\cdot)Q\|_{L^p} \le C,$$

uniformly in the interval $[0,T]$. *In particular, if the initial data* U_0 *satisfy*

$$|U_0(x)|,\ |U_{0,x}(x)|,\ |U_{0,xx}(x)| \sim \mathcal{O}\left((1+|x|)^{-\theta}\right), \quad \text{as } |x| \to \infty,$$

then the solution $U(t,x)$ *decays algebraically as*

$$|U(t,x)|,\ |U_x(t,x)|,\ |U_{xx}(t,x)| \sim \mathcal{O}\left((1+|x|)^{-\theta}\right), \quad \text{as } |x| \to \infty,$$

uniformly in the interval $[0,T]$.

To derive the exponential decay of the strong solution to (3.2), we first give the following lemma.

Lemma 3.8. *Let $\psi(x)$ and $\phi(x)$ be the weighted functions defined in Lemma 2.4. If the function $\psi^{1/2}(x)$ is $\phi^{1/2}(x)$-moderate, then for $p \geq 1$, $f, g \in L^p$, one has*

$$\|\psi^{1/2}(f * g)\|_{L^p} \leq C_0 \|\phi^{1/2} f\|_{L^1} \|\psi^{1/2} g\|_{L^p},$$

where $C_0 \geq 0$, $f = G, G_x$ and $g = f_k, h_k$, $k = 1, 2, \dots, 5$ are given by (3.3) *and* (3.4).

The proof of the above lemma is analogous to the proof of Lemma 3.1, we omit it. We now shall prove the exponential decay of the solution to (3.2).

Theorem 3.9. *Suppose that $U_0 = (u_0, v_0)^\top \in (H^{s,p_1})^{2N}$, $s > 2 + \frac{1}{p_1}$, $p_1 \in (1, \infty)$. Then there exist $T > 0$ and a unique solution $U(t, x) \in [C([0, T]; H^{s,p_1}(\mathbb{R}))]^{2N}$ to* (3.2)*. Additionally, if the initial data U_0 and $U_{0,x}$ admit $p > 1$ and some $C > 0$ such that*

$$\|U_0(x)\psi(x)\|_{L^p} + \|U_{0,x}(x)\psi(x)\|_{L^p} \leq C,$$

then the solution satisfies

$$\|U(t, \cdot)\psi\|_{L^p} + \|U_x(t, \cdot)\psi\|_{L^p} \leq C,$$

uniformly in the interval $[0, T]$, where for $K > 0$, the weighted function $\psi(x) = \min\{e^{|x|},\ K\}$.

The proof of the above theorem is similar to the proof of Theorem 3.3; in view of Lemmas 2.4, 3.2, and Lemma 3.8, the desired result follows.

Theorem 3.10. *Suppose $U_0 = (u_0, v_0)^\top \in (H^{s,p_1})^{2N}$, $s > 2 + \frac{1}{p_1}$, $p_1 \in (1, \infty)$. There exist $T > 0$ and a unique solution $U(t, x) \in [C([0, T]; H^{s,p_1}(\mathbb{R}))]^{2N}$ to* (3.2)*. Additionally, for $p > 1$ and some $C > 0$, if the initial data U_0 and $U_{0,x}$ satisfy*

$$\|U_0(x)\psi(x)\|_{L^\infty} + \|U_{0,x}(x)\psi(x)\|_{L^\infty} \leq C,$$

then the solution satisfies

$$\|U(t, \cdot)\psi\|_{L^\infty} + \|U_x(t, \cdot)\psi\|_{L^\infty} \leq C,$$

uniformly in the interval $[0, T]$. In particular, if the initial data U_0 and $U_{0,x}$ satisfy

$$\begin{aligned} |U_0(x)| &\sim \mathcal{O}\big(e^{-|x|}\big), \quad \text{as } |x| \to \infty, \\ |U_{0,x}(x)| &\sim \mathcal{O}\big(e^{-|x|}\big), \quad \text{as } |x| \to \infty, \end{aligned}$$

then we the solution $U(t, x)$ decays exponentially as

$$\begin{aligned} |U(t, x)| &\sim \mathcal{O}\big(e^{-|x|}\big), \quad \text{as } |x| \to \infty, \\ |U_x(t, x)| &\sim \mathcal{O}\big(e^{-|x|}\big), \quad \text{as } |x| \to \infty, \end{aligned}$$

uniformly in the interval $[0, T]$.

Proof. By Lemma 2.4, the function ψ is ϕ-moderate, and $\psi^{1/2}$ is $\phi^{1/2}$-moderate. Multiplying the first equation in (3.1) by $\psi^{1/2}$, by the method of estimate (3.14), one gets

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t} \Big(\|u_i \psi^{1/2}\|_{L^p} + \|u_{ix} \psi^{1/2}\|_{L^p} \Big) \\ &\leq (3 + \beta) N M^2 \Big(\|u_i \psi^{1/2}\|_{L^p} + \|u_{ix} \psi^{1/2}\|_{L^p} \Big) \\ &\quad + \|\psi^{1/2} F(u_i, u_{ix})\|_{L^p} + \|\psi^{1/2} F_x(u_i, u_{ix})\|_{L^p}. \end{aligned} \tag{3.24}$$

Next, we need to consider $\|\psi^{1/2}F(u_i,u_{ix})(\tau)\|_{L^p}$ and $\|\psi^{1/2}F_x(u_i,u_{ix})(\tau)\|_{L^p}$. Note that

$$F(u_i,u_{ix}) = G_x*(f_1+f_2)+G*(f_3+f_4+f_5),$$
$$F_x(u_i,u_{ix}) = G*(f_1+f_2)-(f_1+f_2)+G_x*(f_3+f_4+f_5).$$

In view of $|G|,|G_x|,|G_{xx}|\le \frac{1}{2}e^{-|x|}$, and $\psi^{1/2}$ is $\phi^{1/2}$-moderate, one can easily check that

$$\begin{aligned}
&\|\psi^{1/2}F(u_i,u_{ix})(s)\|_{L^p}\\
&\le \Big(\|\psi^{1/2}\left(G_x*(f_1+f_2)\right)\|_{L^p}+\|\psi^{1/2}(G*(f_3+f_4+f_5))\|_{L^p}\Big)\\
&\le \|\psi^{1/2}(G_x*f_1)\|_{L^p}+\|\psi^{1/2}(G_x*f_2)\|_{L^p}+\|\psi^{1/2}(G*(f_3+f_4+f_5))\|_{L^p}\\
&\le \|\phi^{1/2}G_x\|_{L^1}\Big(\|\psi^{1/2}f_1\|_{L^p}+\|\psi^{1/2}f_2\|_{L^p}\Big)\\
&\quad+\|\phi^{1/2}G\|_{L^1}\Big(\|\psi^{1/2}f_3\|_{L^p}+\|\psi^{1/2}f_4\|_{L^p}+\|\psi^{1/2}f_5\|_{L^p}\Big)\\
&\le c\Big(\|u_i\psi^{1/2}\|_{L^p}+\|u_{ix}\psi^{1/2}\|_{L^p}\Big),
\end{aligned}$$

$$\begin{aligned}
&\|\psi^{1/2}F_x(u_i,u_{ix})(s)\|_{L^p}\\
&\le \|\psi^{1/2}\left(G*(f_1+f_2)\right)\|_{L^p}+\|\psi^{1/2}(f_1+f_2)\|_{L^p}+\|\psi^{1/2}\left(G_x*(f_3+f_4+f_5)\right)\|_{L^p}\\
&\le \|\phi^{1/2}G\|_{L^1}\Big(\|u_i\psi^{1/2}\|_{L^p}+\|u_{ix}\psi^{1/2}\|_{L^p}\Big)+\sum_{j=1}^N\|\psi^{1/2}(u_iu_{jx}v_{jx}+u_{ix}u_jv_{jx})\|_{L^p}\\
&\quad+\|\phi^{1/2}G_x\|_{L^1}\Big(\|u_i\psi^{1/2}\|_{L^p}+\|u_{ix}\psi^{1/2}\|_{L^p}\Big)\\
&\le c\Big(\|u_i\psi^{1/2}\|_{L^p}+\|u_{ix}\psi^{1/2}\|_{L^p}\Big),
\end{aligned}$$

where in the last inequality we have used the estimates $\|\phi^{1/2}G\|_{L^1},\|\phi^{1/2}G_x\|_{L^1}\le 2$, and the constant c depends on M,N.

Inserting the above estimates into (3.24), it follows that

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big(\|u_i\psi^{1/2}\|_{L^p}+\|u_{ix}\psi^{1/2}\|_{L^p}\Big)\le c\Big(\|u_i\psi^{1/2}\|_{L^p}+\|u_{ix}\psi^{1/2}\|_{L^p}\Big). \tag{3.25}$$

By Gronwall's inequality, we derive that

$$\|u_i\psi^{1/2}\|_{L^p}+\|u_{ix}\psi^{1/2}\|_{L^p}\le ce^{ct}\Big(\|u_{i0}\psi^{1/2}\|_{L^p}+\|u_{i0x}\psi^{1/2}\|_{L^p}\Big). \tag{3.26}$$

Note that ψ is ϕ-moderate, it yields that

$$\begin{aligned}
&\|\psi F(u_i,u_{ix})(s)\|_{L^\infty}\\
&\le \|\psi\left(G_x*(f_1+f_2)\right)\|_{L^\infty}+\|\psi(G*(f_3+f_4+f_5))\|_{L^\infty}\\
&\le \|\psi(G_x*f_1)\|_{L^\infty}+\|\psi(G_x*f_2)\|_{L^\infty}+\|\psi(G*(f_3+f_4+f_5))\|_{L^\infty}\\
&\le \|\phi G_x\|_{L^\infty}(\|\psi f_1\|_{L^1}+\|\psi f_2\|_{L^1})+\|\phi G\|_{L^\infty}(\|\psi f_3\|_{L^1}+\|\psi f_4\|_{L^1}+\|\psi f_5\|_{L^1})\\
&\le c\sum_{j=1}^N\Big(\|u_i\psi^{1/2}\|_{L^2}\|u_jv_j\psi^{1/2}\|_{L^2}+\|u_{ix}\psi^{1/2}\|_{L^2}\|u_jv_{jx}\psi^{1/2}\|_{L^2}\\
&\quad+\|u_{ix}\psi^{1/2}\|_{L^2}\|u_jv_{jx}\psi^{1/2}\|_{L^2}\Big)
\end{aligned}$$

$$
\begin{aligned}
&+ c\sum_{j=1}^{N}\Big(\|u_i\psi^{1/2}\|_{L^2}\|u_{jx}v_j\psi^{1/2}\|_{L^2}+\|u_i\psi^{1/2}\|_{L^2}\|u_{jx}v_{j,xx}\psi^{1/2}\|_{L^2}\Big)\\
&\le ce^{ct},
\end{aligned}\tag{3.27}
$$

$$
\begin{aligned}
&\|\psi F_x(u_i,u_{ix})(s)\|_{L^\infty}\\
&\le \|\psi\left(G*(f_1+f_2)\right)\|_{L^\infty}+\|\psi(f_1+f_2)\|_{L^\infty}\\
&\quad+\|\psi\left(G_x*(f_3+f_4+f_5)\right)\|_{L^\infty}\\
&\le \|\psi(G*f_1)\|_{L^\infty}+\|\psi(G*f_2)\|_{L^\infty}+\|\psi(f_1+f_2)\|_{L^p}\\
&\quad+\|\psi(G_x*(f_3+f_4+f_5))\|_{L^\infty}\\
&\le \|\phi G\|_{L^\infty}(\|\psi f_1\|_{L^1}+\|\psi f_2\|_{L^1})+\|\phi G_x\|_{L^\infty}(\|\psi f_3\|_{L^1}\\
&\quad+\|\psi f_4\|_{L^1}+\|\psi f_5\|_{L^1})+c(\|u_i\psi\|_{L^p}+\|u_{ix}\psi\|_{L^p})\\
&\le c(\|u_i\psi\|_{L^p}+\|u_{ix}\psi\|_{L^p})+ce^{ct},
\end{aligned}\tag{3.28}
$$

where we have applied (3.26) with $p=2$ in the fourth inequality. By replacing $\psi^{1/2}$ with ψ in (3.24), it follows that

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\left(\|u_i\psi\|_{L^p}+\|u_{ix}\psi\|_{L^p}\right)&\le (3+\beta)NM^2\left(\|u_i\psi\|_{L^p}+\|u_{ix}\psi\|_{L^p}\right)\\
&\quad+\|\psi F(u_i,u_{ix})\|_{L^p}+\|\psi F_x(u_i,u_{ix})\|_{L^p}.
\end{aligned}\tag{3.29}
$$

Taking (3.27), (3.28) into (3.29), and letting $p\to\infty$, we obtain

$$
\frac{\mathrm{d}}{\mathrm{d}t}\left(\|u_i\psi\|_{L^\infty}+\|u_{ix}\psi\|_{L^\infty}\right)\le c\left(\|u_i\psi\|_{L^\infty}+\|u_{ix}\psi\|_{L^\infty}\right)+ce^{ct}.\tag{3.30}
$$

Similarly, it implies that

$$
\frac{\mathrm{d}}{\mathrm{d}t}\left(\|v_i\psi\|_{L^\infty}+\|v_{ix}\psi\|_{L^\infty}\right)\le c\left(\|v_i\psi\|_{L^\infty}+\|v_{ix}\psi\|_{L^\infty}\right)+ce^{ct}.\tag{3.31}
$$

By combining (3.30) with (3.31), it follows that

$$
\frac{\mathrm{d}}{\mathrm{d}t}\left(\|U\psi\|_{L^\infty}+\|U_x\psi\|_{L^\infty}\right)\le c(\|U\psi\|_{L^\infty}+\|U_x\psi\|_{L^\infty})+ce^{ct}.
$$

By Gronwall's inequality, one obtains

$$
\|U(x)\psi\|_{L^\infty}+\|U_x(x)\psi\|_{L^\infty}\le C.\tag{3.32}
$$

Particularly, if the initial data U_0 and $U_{0,x}$ decay exponentially as

$$
\begin{gathered}
|U_0(x)|\sim\mathcal{O}\big(e^{-|x|}\big),\quad \text{as } |x|\to\infty,\\
|U_{0,x}(x)|\sim\mathcal{O}\big(e^{-|x|}\big),\quad \text{as } |x|\to\infty.
\end{gathered}
$$

Taking the limit as $K\to\infty$ in inequality (3.32) to derive the result of theorem. □

Corollary 3.11. *Suppose $U_0=(u_0,v_0)^\top\in(H^{s,p_1})^{2N}$, $s>2+\frac{1}{p_1}$, $p_1\in(1,\infty)$. Then there exist $T>0$ and a unique solution $U(t,x)\in[C([0,T];H^{s,p_1}(\mathbb{R}))]^{2N}$ to (3.2). For $p>1$, if U_0, $U_{0,x}$ and $U_{0,xx}$ satisfying*

$$
\|U_0(x)\psi(x)\|_{L^p}+\|U_{0,x}(x)\psi(x)\|_{L^p}+\|U_{0,xx}(x)\psi(x)\|_{L^p}\le C,
$$

for some $C>0$, then the solution satisfies

$$
\|U(t,\cdot)\psi\|_{L^p}+\|U_x(t,\cdot)\psi\|_{L^p}+\|U_{xx}(t,\cdot)\psi\|_{L^p}\le C,
$$

uniformly in the interval $[0,T]$, where the weighted function $\psi(x)=\min\big\{e^{|x|},\ K\big\}$.

In addition, for $p \in (1, \infty)$, if the initial data U_0 satisfy

$$|U_0(x)|,\ |U_{0,x}(x)|,\ |U_{0,xx}(x)| \sim \mathcal{O}\big(e^{-|x|}\big), \quad \text{as } |x| \to \infty,$$

then the solution $U(t,x)$ decays exponentially as

$$|U(t,x)|,\ |U_x(t,x)|,\ |U_{xx}(t,x)| \sim \mathcal{O}\big(e^{-|x|}\big), \quad \text{as } |x| \to \infty,$$

uniformly in the interval $[0,T]$.

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