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LONG-TIME BEHAVIOR OF SOLUTIONS TO THE 2D MAGNETIC BÉNARD PROBLEM IN POROUS MEDIA ON UNBOUNDED DOMAINS

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Abstract. In this article, we study the long time behavior of solutions to the 2D magnetic Bénard problem in porous media, considering on an arbitrary (bounded or unbounded) domain satisfying Poincaré inequality. We first prove the existence of a weak solution and a global attractor for the problem. For $r = 1, 2, 3$, we derive estimates for Hausdorff as well as fractal dimensions of the global attractors. We then show an upper semicontinuity of global attractors and final study the exponential stability of a stationary solution to the problem.

1. Introduction

The study of fluid flow through porous media frequently employs Darcy's Law to model the momentum balance, which establishes a linear relationship between the flow rate and the pressure drop within the medium. This relationship is described as $-\frac{k}{\mu}\nabla p = u$, where k is the permeability, μ is the dynamic viscosity, and p is the pressure. However, in scenarios involving high velocities or non-Newtonian fluids, deviations from this linearity occur, necessitating more comprehensive models such as the Darcy-Forchheimer law. This law introduces a nonlinear correction to account for increased pressure drops at higher velocities and leads to the so-called Brinkman-Forchheimer-extended-Darcy equations (a generalisation actually) read

$$\begin{gathered}\partial_t u - \nu\Delta u + (u\cdot\nabla)u + \alpha|u|^{2q}u + \beta|u|^{2r}u + \nabla p = f,\\ \nabla\cdot u = 0.\end{gathered}$$

This system has been extensively studied, with investigations on its long-time behavior, the existence of global attractors, and various qualitative properties under different boundary conditions and parameter settings (see, e.g., [5, 27, 33, 34]).

In the context of magnetohydrodynamic (MHD) flows with thermal effects, we extend this framework to study the following problem on an arbitrary (bounded or

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unbounded) domain $\Omega \subset \mathbb{R}^2$ with boundary $\partial\Omega$:

$$
\begin{gathered}
\partial_t u - R_e^{-1}\Delta u + (u\cdot\nabla)u - S(B\cdot\nabla)B + \alpha u + \beta|u|^{r-1}u \\
+ \nabla\Big(p + \frac{S}{2}|B|^2\Big) = \theta e_2 + f, \quad \text{in } \Omega, t>0, \\
\partial_t B + R_m^{-1}\nabla^\perp(\operatorname{curl} B) + (u\cdot\nabla)B - (B\cdot\nabla)u = \Psi, \quad \text{in } \Omega, t>0, \\
\partial_t\theta - \kappa\Delta\theta + (u\cdot\nabla)\theta = u_2 + h, \quad \text{in } \Omega, t>0, \\
\nabla\cdot u = \nabla\cdot B = 0, \quad \text{in } \Omega, t>0, \\
u = \mathbf{0}, \quad B\cdot n = 0, \quad \operatorname{curl} B = 0, \quad \theta = 0 \quad \text{on } \partial\Omega, t>0, \\
u(x,0) = u_0(x), \quad B(x,0) = B_0(x), \quad \theta(x,0) = \theta_0(x) \quad \text{in } \Omega.
\end{gathered} \tag{1.1}
$$

In this system, the unknowns are the fluid particle velocity $u = (u_1, u_2)$, the magnetic field $B = (B_1, B_2)$, the temperature θ (or the density in the modeling of geophysical fluids), and the fluid pressure $p = p(x,t)$. The pressure is the standard pressure and can be obtained by applying the divergence-free condition for velocity, taking the divergence, and then inverting the Laplacian operator. The term $|B|^2/2$ represents the magnetic pressure, $e_2 = (0,1)$ denoting the unit vector in the direction of gravity. The unit outward normal on $\partial\Omega$ is represented by n, and the parameters R_e and R_m denote the Reynolds and magnetic Reynolds numbers, respectively. The parameter $S = M^2/(R_eR_m)$ incorporates the Hartman number M, while $\kappa > 0$ signifies the heat conductivity coefficient. Additionally, f, Ψ and h represent external time-dependent forces, respectively, while the forcing term θe_2 describes the buoyancy force's action on fluid motion. And

$$
\begin{gathered}
\operatorname{curl} u = \frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2}, \quad \text{for every vector function } u, \\
\nabla^\perp\phi = \Big(\frac{\partial\phi}{\partial x_2}, -\frac{\partial\phi}{\partial x_1}\Big), \quad \text{for every scalar function } \phi, \\
\nabla^\perp(\operatorname{curl} u) = \nabla(\nabla\cdot u) - \Delta u.
\end{gathered}
$$

In mathematics, the system (1.1) can be considered a modification (by an absorption term $\alpha u + \beta|u|^{r-1}u$) of the classical Bénard problem combine with Maxwell's equations of electromagnetism. Herein, the positive constants α and β hold significance as the Darcy and Forchheimer coefficients, respectively, representing the permeability of porous medium and its porosity-related proportionality. The parameter r, constituting the absorption exponent, is in $[1,\infty)$. It models the convection of an incompressible flow, which occurs in a horizontal layer of conductive fluid heated from below, with the presence of a magnetic field. Moreover, the model (1.1) is recognized to be more accurate when the flow velocity is too large for the Darcy's law to be valid alone, and in addition, the porosity is not too small. The nonlinearity of the form $|u|^{r-1}u$ can be found in tidal dynamics as well as non-Newtonian fluid flows (see [4] and the references therein). In instances where thermal effects on the fluid can be negligible (i.e. $\theta \equiv 0$) and $\alpha = \beta = 0$, then (1.1) reduces to the Magnetohydrodynamics (MHD) equations, which govern the dynamics of the velocity and the magnetic field in electrically conducting fluids such as plasmas (as expounded in [17]) and reflect the basic physics conservation laws. In the past years, the existence and long-time behavior of solutions to the MHD have attracted the attention of many mathematicians. There are many results on the existence of solutions and existence of attractors for MHD, see e.g. [3, 13, 18, 37, 42] and references therein. Furthermore, the regularity of solutions has been studied extensively in recent years (see e.g. [12, 21, 25, 26, 28]). We also refer the interested reader to [39] for recent results on time optimal control problem associated with the two-dimensional MHD equations with memory. In case $\theta \equiv 0$, $\alpha = 0$ and $4 \leq r < 5$, $\beta > 0$, the authors in [31] have recently proved the existence of global attractor for 3D MHD with damping.

On the other hand, in case where the fluid remains unaffected by the magnetic field (i.e. $B \equiv 0$) and $\alpha = \beta = 0$, system (1.1) transforms into the Bénard problem. Numerous studies have been dedicated to exploring the global well-posedness and the existence of attractors for the Bénard problem, as evidenced by works such as [2, 11, 20, 24, 32, 41] and related references.

Turning attention to the magnetic Bénard problem without velocity damping (i.e. $\alpha = \beta = 0$), significant attention has been focused on the 2D case. Bian *et al.* established the global well-posedness of weak or strong solutions for the initial boundary value problems under various boundary conditions and providing a comprehensive analysis of stability and instability within a fully nonlinear, dynamical setting from a mathematical point of view as stated in the references [7, 8, 9, 43]. Furthermore, a comprehensive analysis of the long-time behavior of solutions to an optimal control problem for the magnetic Bénard problem in a two-dimensional bounded domain, with a focus on distributed control adjustments, is presented in [38].

The study of long-time behavior of nonlinear dynamical system is an interesting branch of applied mathematics and it is essential in understanding many natural phenomena. For an extensive study on infinite dimensional dynamical systems in mathematical physics, we refer to [14, 40]. And as is well known, a useful way for studying the long-time behavior of solutions is to use the theory of attractors. The classical global attractor for autonomous dynamical systems is an invariant compact set which attracts all bounded sets and contains some important information about the long-time behavior of the solutions.

The aim of this article is to continue the study of the long-time behavior of weak solution to problem (1.1) in some domains not necessarily bounded. More precisely, the domain Ω can be an arbitrary bounded or unbounded open set in $\mathbb{R}^2$ without any regularity assumption on its boundary $\partial\Omega$ and with the assumption that the Poincaré inequality holds on it, i.e., there exists $\lambda_1 > 0$ such that

$$\int_\Omega |\phi(x)|^2 dx \leq \frac{1}{\lambda_1} \int_\Omega |\nabla \phi(x)|^2 dx \quad \text{for all } \phi \in H_0^1(\Omega). \tag{1.2}$$

We also require that the domain Ω satisfy the cone condition so that Lemma 2.2 in Section 2 is valid on Ω (see [1, Chapter 5] for details). We will discuss the existence and long-time behavior of solutions in terms of the existence of a global attractor and the stability of stationary solutions. Here, the existence and uniqueness of solutions are studied by Galerkin approximation method. To prove the existence of a global attractor, the usual approach is to obtain a bounded absorbing set in a more regular space and then use the compactness of the Sobolev embeddings. However, because the domain considered may be unbounded, the Sobolev embedding is no longer compact, and therefore the usual method in bounded domains no longer works. To overcome this difficulty, we exploit the energy equation method introduced by Ball in [6]. Next, following the general lines of the approach in [36], we show that the solution map $\mathbb{S}(t) : H \to H$ is Fréchet differentiable with respect to the initial data for the absorption exponent $r = 1, 2, 3$ and hence we obtain estimates for the Hausdorff and Fractal dimensions of such attractors under an additional condition on the domain. Then, we establish an upper semicontinuity of global attractors for the problem (1.1). We take an expanding sequence of simply connected, bounded and smooth subdomains $\{\Omega_m\}_m \subset \Omega$. If $\mathcal{A}_m$ and $\mathcal{A}$ are the global attractors of (1.1) corresponding to Ω_m and Ω, respectively, then we show that for large enough m, the global attractor $\mathcal{A}_m$ enters into any neighborhood $\mathcal{U}(\mathcal{A})$ of $\mathcal{A}$. Finally, the existence of a stationary solution is established by a corollary of the Brouwer fixed point theorem.

This article is organized as follows. In Section 2, for the convenience of the readers, we recall and prove some auxiliary results on the 2D magnetic Bénard problem in porous media. In the next Section, we discuss the existence and uniqueness of a weak solution. In Section 4, for $(f, \Psi, h) \in V'$, we show that problem (1.1) possesses a global attractor by using the energy equation method. For the absorption exponent $r = 1, 2, 3$, the estimates for Hausdorff as well as Fractal dimensions of the global attractor for the problem (1.1) is obtained in Section 5. In Section 6, we prove an upper semicontinuity of global attractors for the problem (1.1). The existence and exponential stability of a stationary solution is shown in the last Section.

2. Preliminaries

We first recall several function spaces necessary to write system (1.1). We denote

$$\mathbb{L}^p(\Omega) = L^p(\Omega)^2, \quad \mathbb{H}^1(\Omega) = H^1(\Omega)^2, \quad \mathbb{H}^1_0(\Omega) = H^1_0(\Omega)^2.$$

The spaces used in the theory of the magnetic Bénard problem are a combination of spaces used for the Bénard problem and spaces used in the theory of Maxwell equations. They are

$$\begin{aligned}
&\mathcal{V}_1 = \{u \in \mathcal{C}_0^\infty(\Omega)^2 : \nabla \cdot u = 0\},\\
&V_1 = \text{ closure of } \mathcal{V}_1 \text{ in the } \mathbb{H}^1_0(\Omega) \text{ norm} = \{u \in \mathbb{H}^1_0(\Omega), \nabla \cdot u = 0\},\\
&H_1 = \text{ closure of } \mathcal{V}_1 \text{ in the } \mathbb{L}^2(\Omega) \text{ norm} = \{u \in \mathbb{L}^2(\Omega), \nabla \cdot u = 0 \text{ and } u \cdot n|_{\partial\Omega} = 0\},\\
&\mathcal{V}_2 = \{B \in \mathcal{C}^\infty(\bar{\Omega})^2 : \nabla \cdot B = 0 \text{ and } B \cdot n|_{\partial\Omega} = 0\},\\
&V_2 = \text{ closure of } \mathcal{V}_2 \text{ in the } \mathbb{H}^1(\Omega) \text{ norm} = \{B \in \mathbb{H}^1(\Omega), \nabla \cdot B = 0 \text{ and } B \cdot n|_{\partial\Omega} = 0\},\\
&H_2 = \text{ closure of } \mathcal{V}_2 \text{ in the } \mathbb{L}^2(\Omega) \text{ norm} = H_1,\\
&V_3 = H^1_0(\Omega), \quad H_3 = L^2(\Omega),\\
&V = V_1 \times V_2 \times V_3, \quad H = H_1 \times H_2 \times H_3,\\
&\tilde{L}^p = \text{ closure of } \mathcal{V}_1 \text{ in the } \mathbb{L}^p(\Omega) \text{ norm} = \{u \in \mathbb{L}^p(\Omega), \nabla \cdot u = 0 \text{ and } u \cdot n|_{\partial\Omega} = 0\}.
\end{aligned}$$

For $i = 1, 2$, we define

$$(u, v)_i := \int_\Omega \sum_{j=1}^2 u_j v_j dx, \quad \forall u, v \in H_i,$$
$$(\theta, \eta)_3 := \int_\Omega \theta\eta dx, \quad \forall \theta, \eta \in H_3,$$

and the associated norms $|\cdot|_j^2 = (\cdot,\cdot)_j$, $j = \overline{1, 3}$.

The inner product and norm in V_1 are

$$((u,\tilde{u}))_1 = \sum_{i=1}^{2} \int_\Omega \nabla u_i \cdot \nabla \tilde{u}_i . dx, \quad \forall u, \tilde{u} \in V_1,$$
$$\|u\|_1 = ((u,u))_1^{1/2}, \quad \forall u \in V_1,$$

which is a norm in $\mathbb{H}_0^1(\Omega)$, thanks to Poincaré inequality. The inner product and norm in V_2 are

$$((B,\tilde{B}))_2 = \int_\Omega \operatorname{curl} B \cdot \operatorname{curl} \tilde{B} dx, \quad \forall B, \tilde{B} \in V_2,$$
$$\|B\|_2 = ((B,B))_2^{1/2}, \quad \forall B \in V_2.$$

Since the domain Ω is simply-connected, the above bilinear form is actually a scalar product on V_2, it defines a norm which is equivalent to that induced by $\mathbb{H}^1(\Omega)$ on V_2 (see [18]). The inner product and norm in V_3 are

$$((\theta,\tilde{\theta}))_3 = \int_\Omega \nabla \theta \cdot \nabla \tilde{\theta} . dx, \quad \forall \theta, \tilde{\theta} \in V_3,$$
$$\|\theta\|_3 = ((\theta,\theta))_3^{1/2}, \quad \forall \theta \in V_3.$$

Using the same notation again for simplicity, we define the inner product and norm in V by

$$((z,\tilde{z})) = ((u,\tilde{u}))_1 + S.((B,\tilde{B}))_2 + \gamma((\theta,\tilde{\theta}))_3, \quad \forall z = (u,B,\theta), \tilde{z} = (\tilde{u},\tilde{B},\tilde{\theta}) \in V,$$
$$\|z\| = ((z,z))^{1/2}, \quad \forall z \in V.$$

The inner products and norms in H_1, H_2 and H_3 are the usual ones inherited from $\mathbb{L}^2(\Omega)$ and $L^2(\Omega)$, respectively. We define the inner product and norm in H by

$$(z,\tilde{z}) = (u,\tilde{u}) + S.(B,\tilde{B}) + \gamma.(\theta,\tilde{\theta}), \quad \forall z = (u,B,\theta), \tilde{z} = (\tilde{u},\tilde{B},\tilde{\theta}) \in H,$$
$$|z| = (z,z)^{1/2}, \quad \forall z \in H.$$

We define $\gamma > 0$ so that

$$\gamma \geq \frac{4R_e}{\lambda_1^2 \kappa}. \tag{2.1}$$

This constant γ is chosen so that an operator to be defined later (related to the linear part of the system of equations) is coercive under the norm defined. Moreover, since S and γ are positive, the inner products and norms defined above for H and V are equivalent to the usual ones defined on these product spaces.

It follows from (1.2) and the equivalence of norms in $\mathbb{H}^1$ and V_2 that there exists $c_0 > 0$ such that

$$\lambda_1 |u|^2 \leq \|u\|_1^2, \quad c_0 |B|^2 \leq \|B\|_2^2, \quad \lambda_1 |\theta|^2 \leq \|\theta\|_3^2 \tag{2.2}$$

for all $u \in V_1$, $B \in V_2$ and $\theta \in V_3$. Furthermore, by applying the Riesz representation theorem, we can identify the dual space H' with H and obtain the following relation: $V \subset H \equiv H' \subset V'$, where the injections are continuous and each space is dense in the following ones. We also use $\langle \cdot, \cdot \rangle$ to denote the induced duality between the spaces V and its dual $V' = V_1' \times V_2' \times V_3'$ as well as $\tilde{L}^p$ and its dual $\tilde{L}^{p'}$, where $\frac{1}{p} + \frac{1}{p'} = 1$.

Let us define the bilinear forms $a_i : V_i \times V_i \to \mathbb{R}$, for $i = 1, 2, 3$ and $a : V \times V \to \mathbb{R}$ by

$$
\begin{gathered}
a_1(u, \tilde{u}) = ((u, \tilde{u}))_1 = \int_\Omega \sum_{i=1}^{2} \nabla u_i \cdot \nabla \tilde{u}_i \, dx, \\
a_2(B, \tilde{B}) = ((B, \tilde{B}))_2 = \int_\Omega \operatorname{curl} B \cdot \operatorname{curl} \tilde{B} \, dx, \\
a_3(\theta, \tilde{\theta}) = ((\theta, \tilde{\theta}))_3 = \int_\Omega \nabla \theta \cdot \nabla \tilde{\theta} \, dx, \\
a(z, \tilde{z}) = R_e^{-1} a_1(u, \tilde{u}) + R_m^{-1} S a_2(B, \tilde{B}) + \kappa\gamma a_3(\theta, \tilde{\theta}).
\end{gathered}
$$

The bilinear form a is coercive since

$$
\min\big(R_e^{-1}, R_m^{-1}, \kappa\big)\|z\|^2 \le a(z, z) \le \max\big(R_e^{-1}, R_m^{-1}, \kappa\big)\|z\|^2. \tag{2.3}
$$

We define $\sigma : V \times V \to \mathbb{R}$ by

$$
\sigma(z, \tilde{z}) = -\int_\Omega (\theta e_2 \cdot \tilde{u} + \gamma u_2 \tilde{\theta}) dx.
$$

By using the Hölder and Poincaré inequalities, we deduce that

$$
|\sigma(z, \tilde{z})| \le c_\sigma \|z\| \|\tilde{z}\|. \tag{2.4}
$$

To obtain the coercivity of bilinear form σ, we assume that for all $u \in V_1$ and $\theta \in V_3$,

$$
|(u_2, \theta)| \le \epsilon \|u\|_1 \|\theta\|_3, \tag{2.5}
$$

where ϵ is a positive constant such that

$$
\epsilon \le \Big(\frac{R_e^{-1}\kappa}{4\gamma}\Big)^{1/2} \le \frac{\lambda_1 R_e^{-1}\kappa}{4}. \quad \text{(from (2.1))} \tag{2.6}
$$

Lemma 2.1. *For all $z \in V$, we have the estimate*

$$
a(z, z) + \sigma(z, z) \ge \frac{\delta}{2}\|z\|^2
$$

where $\delta := \min(R_e^{-1}, R_m^{-1}, \kappa)$.

Proof. Using Young's inequality in (2.5), we have

$$
\gamma|(u_2, \theta)| \le \gamma\epsilon \|u\|_1 \|\theta\|_3 \le \frac{\gamma\epsilon^2}{\kappa}\|u\|_1^2 + \frac{\gamma\kappa}{4}\|\theta\|_3^2.
$$

From (2.6), we obtain

$$
\gamma|(u_2, \theta)| \le \frac{R_e^{-1}\|u\|_1^2 + \gamma\kappa\|\theta\|_3^2}{4}. \tag{2.7}
$$

From the definition of γ in (2.1), by using (2.2) and the Hölder inequality, we obtain

$$
\begin{aligned}
|(\theta e_2, u)| &\le |\theta||u| \le \frac{1}{\lambda_1}\|u\|_1\|\theta\|_3 \\
&\le 2^{-1}(R_e)^{-1/2}\|u\|_1(\gamma\kappa)^{1/2}\|\theta\|_3 \\
&\le \frac{R_e^{-1}\|u\|_1^2 + \gamma\kappa\|\theta\|_3^2}{4}.
\end{aligned}
$$

From (2.7) and the inequality above, we deduce that

$$|\sigma(z,z)| = |(\theta e_2, u)| + \gamma|(u_2,\theta)| \le \frac{R_e^{-1}\|u\|_1^2 + \gamma\kappa\|\theta\|_3^2}{2}.$$

Using the definition of a and the inequality above, we obtain

$$\begin{aligned} a(z,z) + \sigma(z,z) \ge a(z,z) - |\sigma(z,z)| &\ge \frac{R_e^{-1}}{2}\|u\|_1^2 + R_m^{-1}S\|B\|^2 + \frac{\kappa}{2}\gamma\|\theta\|_3^2 \\ &\ge \frac{1}{2}\min(R_e^{-1}, R_m^{-1}, \kappa)\|z\|^2. \end{aligned}$$

The proof is complete. □

We next define the trilinear forms

$$b(u,v,w) = \sum_{i,j=1}^{2}\int_\Omega u_i \frac{\partial v_j}{\partial x_i} w_j\,dx, \quad \bar{b}(u,\theta,\eta) = \sum_{i=1}^{2}\int_\Omega u_i \frac{\partial \theta}{\partial x_i}\eta\,dx,$$

whenever the integrals make sense, and $\mathbb{B} : V \times V \times V \to \mathbb{R}$ by

$$\begin{aligned} \mathbb{B}(z_1,z_2,z_3) &= b(u_1,u_2,u_3) + \gamma\bar{b}(u_1,\theta_2,\theta_3) - Sb(B_1,B_2,u_3) \\ &\quad + Sb(u_1,B_2,B_3) - Sb(B_1,u_2,B_3), \quad \forall z_i = (u_i,B_i,\theta_i) \in V. \end{aligned}$$

It is easy to check that if $u,v,w \in V_i, i = 1,2$ and $\theta,\eta \in V_3$, then

$$b(u,v,w) = -b(u,w,v) \quad \text{and} \quad \bar{b}(u,\theta,\eta) = -\bar{b}(u,\eta,\theta). \tag{2.8}$$

Hence

$$b(u,v,v) = 0 \quad \text{and} \quad \bar{b}(u,\theta,\theta) = 0. \tag{2.9}$$

From the relation of $\mathbb{B}(z_1,z_2,z_2)$ and from (2.8), (2.9), we obtain

$$\begin{gathered} \mathbb{B}(z_1,z_2,z_2) = 0, \quad \forall z_1,z_2 \in V, \\ \mathbb{B}(z_1,z_2,z_3) = -\mathbb{B}(z_1,z_3,z_2), \quad \forall z_1,z_2,z_3 \in V. \end{gathered} \tag{2.10}$$

The following results is well-known.

Lemma 2.2 ([1, Theorem 5.8])**.** *Let $\Omega \subset \mathbb{R}^2$ satisfying the cone condition. Then there exists the constant K depending on the dimensions of the cone C providing the cone condition for Ω such that for all $\phi \in H^1(\Omega)$,*

$$\|\phi\|_{L^4(\Omega)} \le K\|\phi\|_{L^2(\Omega)}^{1/2}\|\phi\|_{H^1(\Omega)}^{1/2}.$$

By using similar arguments as in [3, Lemma 2.3] for MHD equations, we have the following result.

Lemma 2.3. *For any open set $\Omega \subset \mathbb{R}^2$ satisfying the cone condition and $z,\tilde{z} \in V$, we have*

$$|\mathbb{B}(z,z,\tilde{z})| \le c_b|z|\|z\|\|\tilde{z}\|.$$

Next, for $r > 0$ and for all $u,v \in \tilde{L}^{r+1}$, we define $\mathcal{C}_r(u,v) = |u|^{r-1}v$ and $\mathcal{C}_r(u) = \mathcal{C}_r(u,u)$. We have the following crucial properties of the nonlinearity $\mathcal{C}_r$.

Lemma 2.4. *For every $r \ge 1$ and for all functions $u,v \in \tilde{L}^{r+1}$*

$$\langle \mathcal{C}_r(u) - \mathcal{C}_r(v), u - v\rangle \ge 0. \tag{2.11}$$

Proof. We have

$$
\begin{aligned}
&\langle \mathcal{C}_r(u)-\mathcal{C}_r(v), u-v\rangle\\
&=\langle |u|^{r-1}u-|v|^{r-1}v, u-v\rangle\\
&=\langle |u|^{r-1}, |u-v|^2\rangle+\langle |v|^{r-1}, |u-v|^2\rangle+\langle v|u|^{r-1}-u|v|^{r-1}, u-v\rangle\\
&=\||u|^{\frac{r-1}{2}}(u-v)\|_{\tilde{L}^2}^2+\||v|^{\frac{r-1}{2}}(u-v)\|_{\tilde{L}^2}^2\\
&\quad+\langle u\cdot v, |u|^{r-1}+|v|^{r-1}\rangle-\langle |u|^2, |v|^{r-1}\rangle-\langle |v|^2, |u|^{r-1}\rangle.
\end{aligned}
\tag{2.12}
$$

Moreover, we also have

$$
\begin{aligned}
&\langle u\cdot v, |u|^{r-1}+|v|^{r-1}\rangle-\langle |u|^2, |v|^{r-1}\rangle-\langle |v|^2, |u|^{r-1}\rangle\\
&=\frac{1}{2}\langle |u|^{r-1}-|v|^{r-1}, |u|^2-|v|^2\rangle-\frac{1}{2}\||u|^{\frac{r-1}{2}}(u-v)\|_{\tilde{L}^2}^2-\frac{1}{2}\||v|^{\frac{r-1}{2}}(u-v)\|_{\tilde{L}^2}^2\\
&\ge-\frac{1}{2}\||u|^{\frac{r-1}{2}}(u-v)\|_{\tilde{L}^2}^2-\frac{1}{2}\||v|^{\frac{r-1}{2}}(u-v)\|_{\tilde{L}^2}^2,
\end{aligned}
$$

since $\langle |u|^{r-1}-|v|^{r-1}, |u|^2-|v|^2\rangle\ge 0$ for all $u,v\in\tilde{L}^{r-1}$. Applying the inequality above to (2.12) we deduce

$$
\langle \mathcal{C}_r(u)-\mathcal{C}_r(v), u-v\rangle\ge\frac{1}{2}\||u|^{\frac{r-1}{2}}(u-v)\|_{\tilde{L}^2}^2+\frac{1}{2}\||v|^{\frac{r-1}{2}}(u-v)\|_{\tilde{L}^2}^2.
$$

By using Hölder's inequality, from $u,v\in\tilde{L}^{r+1}$ we obtain

$$
|u|^{\frac{r-1}{2}}(u-v) \text{ and } |v|^{\frac{r-1}{2}}(u-v) \text{ are bounded in } \tilde{L}^2
$$

and it yields (2.11). □

Furthermore, for all $u\in\tilde{L}^{r+1}$, $\mathcal{C}_r(u)$ is Gateaux differentiable with Gateaux derivative

$$
\mathcal{C}_r'(u)v=\begin{cases}
v, & \text{for } r=1,\\
|u|v+\frac{u}{|u|}(u\cdot v), & \text{if } u\neq\mathbf{0},\ \text{for } r=2,\\
0, & \text{if } u=\mathbf{0},\ \text{for } r=2,\\
|u|^{r-1}v+(r-1)\left(u|u|^{r-3}(u\cdot v)\right), & \text{for } r\ge 3,
\end{cases}
\tag{2.13}
$$

for $v\in\tilde{L}^{r+1}$. For $u,v\in\tilde{L}^{r+1}$, it can be easily seen that

$$
\langle \mathcal{C}_r'(u)v, v\rangle=\int_\Omega |u(x)|^{r-1}|v(x)|^2dx+(r-1)\int_\Omega |u(x)|^{r-3}|u(x)\cdot v(x)|^2dx\ge 0 \tag{2.14}
$$

for $r\ge 1$. Note that for $r=2$ also the same result holds, since in that case, the second integral

$$
\int_{x\in\Omega:u(x)\neq 0}\frac{1}{|u(x)|}|u(x)\cdot v(x)|^2dx\ge 0.
$$

Moreover, for $r \geq 3$, $\mathcal{C}_r(u)$ is twice Gateaux differentiable with second order Gateaux derivative

$$
\begin{aligned}
&\mathcal{C}''(u)(v\otimes w)\\
&= \begin{cases}
(r-1)|u|^{r-3}[(u\cdot w)v+(u\cdot v)w+(w\cdot v)u] \\
+(r-1)(r-3)\frac{u}{|u|^{5-r}}(u\cdot v)(u\cdot w), & \text{for } u\neq \mathbf{0},\ 3\leq r<5 \\
(r-1)(r-3)\frac{u}{|u|^{5-r}}(u\cdot v)(u\cdot w), & \text{for } u\neq \mathbf{0},\ 3\leq r<5, \\
(r-1)|u|^{r-3}[(u\cdot w)v+(u\cdot v)w+(w\cdot v)u] \\
+(r-1)(r-3)|u|^{r-5}(u\cdot v)(u\cdot w)u, & \text{for } r\geq 5,
\end{cases}
\end{aligned}
\tag{2.15}
$$

for all $u, v, w \in \tilde{L}^{r+1}$.

We now define $g : V \to \mathbb{R}$ by

$$
g(z) = \langle f, u\rangle_{V_1',V_1} + S\langle \Psi, B\rangle_{V_2',V_2} + \gamma\langle h,\theta\rangle_{V_3',V_3} = \langle \Phi, z\rangle_{V',V},
$$

where $\Phi = (f, \Psi, h)$. By using Schwarz's inequality, we have

$$
|g(z)| \leq \|\Phi\|_{V'}\|z\|
$$

with $\|\Phi\|_{V'}^2 = \|f\|_{V_1'}^2 + S\|\Psi\|_{V_2'}^2 + \gamma\|h\|_{V_3'}^2$. Finally, to prove the existence of a stationary solution, we need the following lemma.

Lemma 2.5 ([10]). *Let X be a finite dimensional Hilbert space with scalar product $[\cdot,\cdot]$ and norm $[\cdot]$ and let P be a continuous mapping from X into itself such that*

$$
[P(\xi), \xi] > 0 \quad \textit{for } [\xi] = k > 0.
$$

Then there exists $\xi \in X$, $[\xi] < k$, such that $P(\xi) = 0$.

3. Existence and uniqueness of a weak solution

Taking the inner product of the first equation of (1.1) with $\tilde{u} \in V_1$, we obtain

$$
\begin{aligned}
&(\partial_t u, \tilde{u}) + R_e^{-1}((u,\tilde{u}))_1 + b(u,u,\tilde{u}) - Sb(B,B,\tilde{u}) + \alpha(u,\tilde{u}) + \beta\langle \mathcal{C}_r(u),\tilde{u}\rangle - (\theta e_2,\tilde{u})\\
&= \langle f, \tilde{u}\rangle_{V_1',V_1}.
\end{aligned}
$$

We take the inner product of the second equation of (1.1) with $S\tilde{B}$ ($\tilde{B} \in V_2$) to obtain

$$
S(\partial_t B, \tilde{B}) + SR_m^{-1}((B,\tilde{B}))_2 + Sb(u,B,\tilde{B}) - Sb(B,u,\tilde{B}) = S\langle \Psi, \tilde{B}\rangle_{V_2',V_2}.
$$

And taking the inner product of the third equation of (1.1) with $\gamma\tilde{\theta} \in V_3$, we obtain

$$
\gamma(\partial_t\theta,\tilde{\theta}) + \gamma\kappa((\theta,\tilde{\theta}))_3 + \gamma\bar{b}(u,\theta,\tilde{\theta}) - \gamma(u_2,\tilde{\theta}) = \gamma\langle h,\tilde{\theta}\rangle_{V_3',V_3}.
$$

This suggests the following weak formulation of problem (1.1).

Problem For $z_0 = (u_0, B_0, \theta_0) \in H$ given, find weak solution $z = (u, B, \theta)$ such that

$$
\begin{gathered}
z \in L^2(0,T;V)\cap L^\infty(0,T;H), \quad u \in L^{r+1}(0,T;\tilde{L}^{r+1}), \quad \partial_t z \in L^{\frac{r+1}{r}}(0,T;V'),\\
\forall T>0,\\
\begin{aligned}
&(\partial_t z(t),\tilde{z}) + a(z(t),\tilde{z}) + \sigma(z(t),\tilde{z}) + \mathbb{B}(z(t),z(t),\tilde{z})\\
&+\alpha(u,\tilde{u}) + \beta\langle \mathcal{C}_r(u),\tilde{u}\rangle = \langle \Phi,\tilde{z}\rangle, \quad \forall \tilde{z} = (\tilde{u},\tilde{B},\tilde{\theta}) \in V,\ \text{a.e. } t\in(0,T),
\end{aligned}\\
z(0) = z_0 \quad \text{in } H.
\end{gathered}
\tag{3.1}
$$

We have the result of the existence of a weak solution to Problem (3.1).

Theorem 3.1. *For $z_0 = (u_0, B_0, \theta_0) \in H$, $T > 0$ and $(f, \Psi, h) \in L^2(0,T;V')$, there exists a unique weak solution $z = (u, B, \theta)$ to the system* (1.1) *in the sense of* (3.1).

Proof. The existence of a weak solution to Problem (3.1) in $(0,T)$ is based on Galerkin approximations, a priori estimates, and the compactness method. As it is standard, we only provide some basic a priori estimates that we shall frequently use later.

From the first equation in (3.1), by using Lemma 2.1 and (2.10) to obtain

$$\frac{1}{2}\frac{d}{dt}|z(t)|^2 + \frac{\delta}{2}\|z(t)\|^2 + \alpha|u(t)|^2 + \beta\|u(t)\|^{r+1}_{\tilde{L}^r+1} \le \frac{1}{\delta}\|\Phi\|^2_{V'} + \frac{\delta}{4}\|z(t)\|^2. \tag{3.2}$$

Integrating over $[0,t] \subset [0,T]$ the inequality above, we obtain

$$\begin{aligned} &|z(t)|^2 + \frac{\delta}{2}\int_0^t \|z(s)\|^2 ds + 2\alpha\int_0^t |u(s)|^2 ds + 2\beta\int_0^t \|u(s)\|^{r+1}_{\tilde{L}^r+1} ds \\ &\le |z(0)|^2 + \frac{2}{\delta}\int_0^t \|\Phi\|^2_{V'} ds, \quad \forall t \in [0,T]. \end{aligned} \tag{3.3}$$

This implies the estimates of z in the space $L^\infty(0,T;H) \cap L^2(0,T;V)$ and $u \in L^{r+1}(0,T;\tilde{L}^{r+1})$.

Next we estimate the boundedness of $\partial_t z$. From (2.3), (2.4) and Lemma 2.3, it can be easily seen that $\partial_t z$ is bounded in $L^2(0,T;V')$. Moreover,

$$\||u|^{r-1}u\|^{\frac{r+1}{r}}_{\tilde{L}^{\frac{r+1}{r}}} = \|u\|^{r+1}_{\tilde{L}^{r+1}}$$

yields that

$$|u|^{r-1}u \quad \text{remains bounded in} \quad L^{\frac{r+1}{r}}(0,T;\tilde{L}^{\frac{r+1}{r}}).$$

Note that

- $V_1 \hookrightarrow \tilde{L}^{r+1}$ for every $r \ge 1$, then $\tilde{L}^{\frac{r+1}{r}} \hookrightarrow V_1'$,
- $L^2(0,T) \hookrightarrow L^{\frac{r+1}{r}}(0,T)$ for every $r > 0$,

thus $\partial_t u$ is bounded in $L^{\frac{r+1}{r}}(0,T;V_1')$. Therefore $\partial_t z$ is bounded in $L^{\frac{r+1}{r}}(0,T;V')$.

To prove the uniqueness, assume that $z_1 = (u_1, B_1, \theta_1)$ and $z_2 = (u_2, B_2, \theta_2)$ are two solutions of our problem, and set $y = (w, \phi, \eta) = (u_1 - u_2, B_1 - B_2, \theta_1 - \theta_2)$. Then $y = (w, \phi, \eta)$ satisfies

$$\begin{gathered} \partial_t w - R_e^{-1}\Delta w + (u_1\cdot\nabla)u_1 - (u_2\cdot\nabla)u_2 + S\big[(B_2\cdot\nabla)B_2 - (B_1\cdot\nabla)B_1\big] \\ + \alpha w + \beta\left(\mathcal{C}_r(u_1) - \mathcal{C}_r(u_2)\right) = \eta e_2, \\ \partial_t\phi + R_m^{-1}\nabla^\perp(\operatorname{curl}\phi) + (u_1\cdot\nabla)B_1 - (u_2\cdot\nabla)B_2 + (B_2\cdot\nabla)u_2 - (B_1\cdot\nabla)u_1 = 0, \\ \partial_t\eta - \kappa\Delta\eta + (u_1\cdot\nabla)\theta_1 - (u_2\cdot\nabla)\theta_2 = w_2, \\ \nabla\cdot w = \nabla\cdot\phi = 0, \\ w(x,0) = u_{10}(x) - u_{20}(x), \quad \phi(x,0) = B_{10}(x) - B_{20}(x), \\ \eta(x,0) = \theta_{10}(x) - \theta_{20}(x). \end{gathered} \tag{3.4}$$

We multiply the first, second and third equations of (3.4) by w, $S\phi$ and $\gamma\eta$, respectively, then integrate over Ω and add the resulting equations, using the definition of $\mathbb{B}$ and Lemma 2.1 to obtain

$$\frac{1}{2}\frac{d}{dt}|y(t)|^2+\frac{\delta}{2}\|y(t)\|^2+\alpha|w(t)|^2+\beta\langle\mathcal{C}_r(u_1)-\mathcal{C}_r(u_2),w\rangle\le\mathbb{B}(z_2,z_2,y)-\mathbb{B}(z_1,z_1,y).$$

By Lemma 2.3, we obtain

$$|\mathbb{B}(z_2,z_2,y)-\mathbb{B}(z_1,z_1,y)|=|-\mathbb{B}(y,z_1,y)|\le c_b|y|\|z_1\|\|y\|\le\frac{\delta}{2}\|y\|^2+\frac{c_b^2}{2\delta}|y|^2\|z_1\|^2.$$

From this with (2.11) we deduce that

$$\frac{d}{dt}|y(t)|^2\le\frac{c_b^2}{2\delta}|y|^2\|z_1\|^2\quad\text{or}\quad|y(t)|^2\le|y(0)|^2\exp\left(\frac{c_b^2}{2\delta}\int_0^t\|z_1(s)\|^2ds\right).$$

The last estimate implies the uniqueness (if $z_{10}=z_{20}$) and the continuous dependence of solutions on the initial data. □

Remark 3.2. (i) It can be easily seen that $z\in L^\infty(0,T;H)$ implies $z\in L^\infty(0,T;V')$. Thus, z and $\partial_t z\in L^{\frac{r+1}{r}}(0,T;V')$ and then using [19, Theorem 2, Section 5.9.2], we have $u\in C([0,T];V')$. The reflexivity of the space H and [15, Proposition 1.7.1] gives $u\in C_w([0,T];H)$.

(ii) Note that $\partial_t z\in L^2(0,T;V')$, for $r=2$ and $\partial_t z\in L^{4/3}(0,T;V')$, for $r=3$. Thus applying [19, Theorem 3, Section 5.9.2], we have $z\in C([0,T];H)$, for $r=2$. For $r=3$, one can show that $z\in C([0,T];H)$ satisfying the energy equality (3.3) (see [22, Theorem 4.1]).

4. Existence of a global attractor

In this section, we discuss the existence of a global attractor for Problem (3.1) in two dimensional bounded domains. We assume that $\Phi\in V'$ is independent of t in (3.1). Thanks to Theorem 3.1 we can define a continuous semigroup $\{\mathbb{S}(t)=(S_1(t),S_2(t),S_3(t))\}_{t\ge0}$ in H by

$$\mathbb{S}(t)(u_0,B_0,\theta_0)=\big(S_1(t)u_0,S_2(t)B_0,S_3(t)\theta_0\big)=(u(t),B(t),\theta(t)),\quad t\ge0$$

where $z(0)=(u(0),B(0),\theta(0))=(u_0,B_0,\theta_0)=z_0\in H$. It is easy to see that the map $\mathbb{S}(t):H\to H$, for $t\ge0$, is Lipschitz continuous on bounded subsets of H.

Moreover, by setting $z_n(t)=\mathbb{S}(t)(u_{0_n},B_{0_n},\theta_{0_n})$ and $z(t)=\mathbb{S}(t)(u_0,B_0,\theta_0)$, the following lemma shows weak continuity of the semigroup $\{\mathbb{S}(t)\}_{t\ge0}$, which is needed to prove the asymptotic compactness of the semigroup by using the energy equation method introduced by Ball [6].

Lemma 4.1. *Let $z_{0_n}=(u_{0_n},B_{0_n},\theta_{0_n})$ be a sequence in H converging weakly to an element $z_0\in H$. Then*

$$z_n(t)\to z(t)\quad\text{weakly in } H,\quad\forall t\ge0,\tag{4.1}$$

$$z_n(t)\to z(t)\quad\text{weakly in } L^2(0,T;V),\quad\forall T>0,\tag{4.2}$$

$$u_n(t)\to u(t)\quad\text{weakly in } L^{r+1}(0,T;\tilde{L}^{r+1}),\quad\forall T>0.\tag{4.3}$$

The proof of the above lemma closely follows the argument presented in [36, Lemma 2.1], with only slight adjustments. Therefore, we omit the proof here for conciseness.

We now prove that the semigroup $\mathbb{S}(t)$ has a compact global attractor $\mathcal{A}$. For the general theory of global attractors, we refer the reader [29, 35, 40] for details. The main result in this section is as follows.

Theorem 4.2. *Under the conditions of Theorem 3.1, there exists a global attractor $\mathcal{A}$ in H for the semigroup $\{\mathbb{S}(t)\}_{t\geq 0}$ associated with Problem* (3.1).

To prove this theorem, we need to show that the semigroup $\{\mathbb{S}(t)\}_{t\geq 0}$ possesses an absorbing set $\mathcal{B}$ bounded in H and is asymptotically compact in H, that is,

(1) given a bounded set $B \subset H$, there exists an entering time $t_B > 0$ such that $\mathbb{S}(t)B \subset \mathcal{B}$ for all $t \geq t_B$,
(2) for $\{z_n\}_n$ is bounded and $t_n \to \infty$, then $\{\mathbb{S}(t_n)z_n\}_n$ is precompact in H.

We first prove the existence of a bounded absorbing set for semigroup $\{\mathbb{S}(t)\}_{t\geq 0}$ generated by Problem (3.1) in H. For the sake of brevity, in the following lemma we only give some formal calculations, the rigorous proof is done by using Galerkin approximations.

Lemma 4.3. *The semigroup $\{\mathbb{S}(t)\}_{t\geq 0}$ generated by Problem* (3.1) *has a bounded absorbing set in H, that is, there exists a positive constant ρ and a time $t_0(|z_0|, \lambda)$ such that for the solution $z(t) = \mathbb{S}(t)z_0$,*

$$|z(t)| \leq \rho \quad \text{for all } t \geq t_0(|z_0|, \delta, \lambda),$$

where δ is in Lemma 2.1 and $\lambda = \min(\lambda_1, c_0)$.

Proof. From (3.2), by using (2.2), we have

$$\frac{d}{dt}|z(t)|^2 + \frac{\delta\lambda}{2}|z(t)|^2 \leq \frac{2}{\delta}\|\Phi\|_{V'}^2,$$

and an application of Gronwall's inequality yields

$$|z(t)|^2 \leq |z_0|^2 e^{-\frac{\delta\lambda}{2}t} + \frac{2}{\delta^2\lambda}\|\Phi\|_{V'}^2.$$

Therefore, if we choose $\rho^2 = \frac{4}{\delta^2\lambda}\|\Phi\|_{V'}^2$, then

$$|z(t)| \leq \frac{2}{\delta\sqrt{\lambda}}\|\Phi\|_{V'}$$

for all $t \geq t_0(|z_0|, \delta, \lambda)$, and so the proof is complete. □

We next show the asymptotic compactness of the semigroup $\{\mathbb{S}(t)\}_{t\geq 0}$ generated by Problem (3.1). Let us define a symmetric bilinear form $[\cdot,\cdot] : V \times V \to \mathbb{R}$ by

$$[z, \tilde{z}] = a(z, \tilde{z}) + \sigma(z, \tilde{z}) - \frac{\delta\lambda}{4}(z, \tilde{z}), \quad \forall z, \tilde{z} \in V. \tag{4.4}$$

From (2.3) and (2.4), we have

$$[z, z] + \frac{\delta\lambda}{4}|z|^2 = a(z, \tilde{z}) + \sigma(z, \tilde{z}) \leq \left(\max\left(R_e^{-1}, R_m^{-1}, \kappa\right) + c_\sigma\right)\|z\|^2.$$

Thus,

$$[z]^2 \equiv [z, z] \leq \left(\max\left(R_e^{-1}, R_m^{-1}, \kappa\right) + c_\sigma\right)\|z\|^2. \tag{4.5}$$

By using the definition of $|z|$ and (2.2) to obtain

$$\begin{aligned}
\frac{\delta\lambda}{4}|z|^2 &= \frac{\delta\lambda}{4}(|u|^2 + S|B|^2 + \gamma|\theta|^2) \\
&\leq \frac{\delta}{4}(\lambda_1|u|^2 + Sc_0|B|^2 + \gamma\lambda_1|\theta|^2) \\
&\leq \frac{\delta}{4}(\|u\|_1^2 + S\|B\|_2^2 + \gamma\|\theta\|_3^2) = \frac{\delta}{4}\|z\|^2.
\end{aligned}$$

Using this and Lemma 2.1 we obtain

$$[z]^2 \geq \frac{\delta}{2}\|z\|^2 - \frac{\delta\lambda}{4}|z|^2 \geq \frac{\delta}{4}\|z\|^2. \tag{4.6}$$

Putting together (4.5) and (4.6) we obtain

$$\frac{\delta}{4}\|z\|^2 \leq [z]^2 \leq \left(\max\left(R_e^{-1}, R_m^{-1}, \kappa\right) + c_\sigma\right)\|z\|^2, \quad \forall z \in V. \tag{4.7}$$

Thus, $[\cdot,\cdot]$ defines an inner product in V with norm $[\cdot] = [\cdot,\cdot]^{1/2}$ equivalent to $\|\cdot\|$.

Lemma 4.4. *The semigroup $\{\mathbb{S}(t)\}_{t\geq 0}$ generated by Problem* (3.1) *is asymptotically compact in H.*

Proof. Let B be a bounded subset of H, and consider $\{z_n\}_n \subset B$ and $\{t_n\}_n$, $t_n \geq 0$, $t_n \to \infty$. Set

$$\mathcal{B} = \{z \in H : |z| \leq \rho\}, \tag{4.8}$$

where ρ is the positive constant in Lemma 4.3, and then we find that there exists a time $t_B > 0$ such that

$$\mathbb{S}(t)B \subset \mathcal{B}, \quad \text{for all } t \geq t_B.$$

Hence for t_n large enough ($t_n \geq t_B$), we have

$$\mathbb{S}(t_n)z_n \in \mathcal{B}. \tag{4.9}$$

Thus, the sequence $\{\mathbb{S}(t_{n_k})z_{n_k}\}_{n_k}$ is weakly precompact in H, and since $\mathcal{B}$ is closed and convex, we have

$$\{\mathbb{S}(t_{n_k})z_{n_k}\}_{n_k} \to w \quad \text{weakly in } H, \tag{4.10}$$

for some subsequence $\{\mathbb{S}(t_{n_k})z_{n_k}\}_{n_k}$ of $\{\mathbb{S}(t_n)z_n\}_n$ and $w \in \mathcal{B}$. Similarly, for each $T > 0$, one can show that $\mathbb{S}(t_n - T)z_n \in \mathcal{B}$, for all $t_n \geq T + t_B$. Terefore, we obtain that $\mathbb{S}(t_n - T)z_n$ is precompact in H, and by using a diagonal argument and passing to a further subsequence (if necessary), we can assume that

$$\{\mathbb{S}(t_{n_k} - T)z_{n_k}\}_{n_k} \to w_T \quad \text{weakly in } H,$$

for all $T \in \mathbb{N}$ with $w_T = (u_T, B_T, \theta_T) \in \mathcal{B}$. Using the weak continuity of $\mathbb{S}(t)$ established in Lemma 4.1 and (4.10), for all $\xi \in H$, we have

$$\begin{aligned}
(w,\xi) &= \lim_{k\to\infty}\Big(\mathbb{S}(t_{n_k})z_{n_k}, \xi\Big) \\
&= \lim_{k\to\infty}\Big(\mathbb{S}(T)\mathbb{S}(t_{n_k} - T)z_{n_k}, \xi\Big) \\
&= \Big(\mathbb{S}(T)\lim_{k\to\infty}\mathbb{S}(t_{n_k} - T)z_{n_k}, \xi\Big) = \Big(\mathbb{S}(T)w_T, \xi\Big).
\end{aligned}$$

Therefore $w = \mathbb{S}(T)w_T$ for all $T \in \mathbb{N}$. Moreover, by using the weakly lower-semicontinuity property, from (4.10) we obtain

$$|w| \leq \liminf_{k\to\infty} |\mathbb{S}(t_{n_k})z_{n_k}|.$$

Our next aim is to show that

$$\limsup_{k\to\infty} |\mathbb{S}(t_{n_k})z_{n_k}| \leq |w|. \tag{4.11}$$

Then we have

$$\lim_{k\to\infty} |\mathbb{S}(t_{n_k})z_{n_k}| = |w|,$$

and this, together with the weak convergence, will imply the strong convergence in H of $\mathbb{S}(t_{n_k})z_{n_k}$ to w.

Now, setting $\tilde{z} = z$ in (3.1), using (2.10) and (4.4) to obtain

$$\frac{1}{2}\frac{d}{dt}|z(t)|^2 + \frac{\delta\lambda}{4}|z(t)|^2 = \langle \Phi, z\rangle - \Big(\alpha|u(t)|^2 + \beta\|u(t)\|^{r+1}_{\tilde{L}^{r+1}} + [z]^2\Big),$$

then, using the variation of constant formula, we obtain

$$\begin{aligned}|z(T)|^2 = e^{-\frac{\delta\lambda}{2}T}|z_0|^2 + 2\int_0^T e^{-\frac{\delta\lambda}{2}(T-s)}\Big[\langle \Phi, z(s)\rangle \\ - \Big(\alpha|u(s)|^2 + \beta\|u(s)\|^{r+1}_{\tilde{L}^{r+1}} + [z(s)]^2\Big)\Big]ds\end{aligned} \tag{4.12}$$

which can be written as

$$\begin{aligned}|\mathbb{S}(T)z_0|^2 = e^{-\frac{\delta\lambda}{2}T}|z_0|^2 + 2\int_0^T e^{-\frac{\delta\lambda}{2}(T-s)}\Big[\langle \Phi, \mathbb{S}(s)z_0\rangle \\ - \Big(\alpha|S_1(s)u_0|^2 + \beta\|S_1(s)u_0\|^{r+1}_{\tilde{L}^{r+1}} + [\mathbb{S}(s)z_0]^2\Big)\Big]ds,\end{aligned} \tag{4.13}$$

for all $z_0 \in H$ and $t \geq 0$. Thus, for $T \in \mathbb{N}$ and $t_n > T$, we have by letting $z_0 = \mathbb{S}(t_n - T)z_n$ in (4.13), we obtain

$$\begin{aligned}&|\mathbb{S}(t_n)z_n|^2 \\ &= |\mathbb{S}(T)\mathbb{S}(t_n - T)z_n|^2 \\ &= e^{-\frac{\delta\lambda}{2}T}|\mathbb{S}(t_n - T)z_n|^2 + 2\int_0^T e^{-\frac{\delta\lambda}{2}(T-s)}\big\langle \Phi, \mathbb{S}(s)\mathbb{S}(t_n - T)z_n\big\rangle ds \\ &\quad - 2\int_0^T e^{-\frac{\delta\lambda}{2}(T-s)}\Big[\alpha|S_1(s)S_1(t_n - T)u_n|^2 + \beta\|S_1(s)S_1(t_n - T)u_n\|^{r+1}_{\tilde{L}^{r+1}}\Big]ds \\ &\quad - 2\int_0^T e^{-\frac{\delta\lambda}{2}(T-s)}[\mathbb{S}(s)\mathbb{S}(t_n - T)z_n]^2 ds.\end{aligned} \tag{4.14}$$

We now estimate the terms in (4.14). In the same way that we have obtained in (4.9), for each $T > 0$,

$$\mathbb{S}(t_n - T)z_n \in \mathcal{B}, \quad \forall t_n \geq T + t_B.$$

From (4.8), we find that

$$\limsup_{k\to\infty} e^{-\frac{\delta\lambda}{2}T}|\mathbb{S}(t_n - T)z_n|^2 \leq e^{-\frac{\delta\lambda}{2}T}\rho^2. \tag{4.15}$$

Next, by using the weak continuity result in (4.2) and the convergence $\{\mathbb{S}(t_{n_k} - T)z_{n_k}\}_k \to w_T$ weakly in H, we have

$$\mathbb{S}(\cdot)\mathbb{S}(t_{n_k} - T)z_{n_k} \to \mathbb{S}(\cdot)w_T \quad \text{weakly in } L^2(0, T; V). \tag{4.16}$$

Now, we consider

$$\int_0^T \|e^{-\frac{\delta\lambda}{2}(T-s)}\Phi\|^2_{V'}ds = \int_0^T e^{-\delta\lambda(T-s)}\|\Phi\|^2_{V'}ds = \|\Phi\|^2_{V'}\Big(\frac{1-e^{-\delta\lambda T}}{\delta\lambda}\Big) < \infty$$

and hence the mapping $s \mapsto e^{-\frac{\delta\lambda}{2}(T-s)}\Phi \in L^2(0, T; V')$. Thus, we find that

$$\lim_{k\to\infty}\int_0^T e^{-\frac{\delta\lambda}{2}(T-s)}\big\langle \Phi, \mathbb{S}(s)\mathbb{S}(t_{n_k} - T)z_{n_k}\big\rangle ds = \int_0^T e^{-\frac{\delta\lambda}{2}(T-s)}\big\langle \Phi, \mathbb{S}(s)w_T\big\rangle ds. \tag{4.17}$$

From (4.7), the norm $[\cdot]$ is equivalent to $\|\cdot\|$. Also

$$0 < e^{-\frac{\delta\lambda}{2}T} \le e^{-\frac{\delta\lambda}{2}(T-s)} \le 1, \quad \forall s \in [0,T]$$

and therefore $\big(\int_0^T e^{-\frac{\delta\lambda}{2}(T-s)}[\cdot]^2 ds\big)^{1/2}$ defines a norm on $L^2(0,T;V)$ equivalent to the norm $\big(\int_0^T \|\cdot\|^2 ds\big)^{1/2}$. Using (4.16) and the weakly lower-semicontinuity property of the norm, we obtain

$$\int_0^T e^{-\frac{\delta\lambda}{2}(T-s)}[\mathbb{S}(s)w_T]^2 ds \le \liminf_{k\to\infty} \int_0^T e^{-\frac{\delta\lambda}{2}(T-s)}[\mathbb{S}(s)\mathbb{S}(t_{n_k}-T)z_{n_k}]^2 ds.$$

Therefore,

$$\begin{aligned}
&\limsup_{k\to\infty} \Big[-2\int_0^T e^{-\frac{\delta\lambda}{2}(T-s)}[\mathbb{S}(s)\mathbb{S}(t_{n_k}-T)z_{n_k}]^2 ds\Big] \\
&= -2\liminf_{k\to\infty} \int_0^T e^{-\frac{\delta\lambda}{2}(T-s)}[\mathbb{S}(s)\mathbb{S}(t_{n_k}-T)z_{n_k}]^2 ds \\
&\le -2\int_0^T e^{-\frac{\delta\lambda}{2}(T-s)}[\mathbb{S}(s)w_T]^2 ds.
\end{aligned} \tag{4.18}$$

We also have that

$$\Big(\int_0^T e^{-\frac{\delta\lambda}{2}(T-s)}|\cdot|^2 ds\Big)^{1/2} \quad \text{and} \quad \Big(\int_0^T e^{-\frac{\delta\lambda}{2}(T-s)}\|\cdot\|_{\tilde{L}^{r+1}}^{r+1} ds\Big)^{\frac{1}{r+1}}$$

define norms on $L^2(0,T;H)$ and $L^{r+1}(0,T;\tilde{L}^{r+1})$, which are equivalent to the norms $\big(\int_0^T |\cdot|^2 ds\big)^{1/2}$ and $\big(\int_0^T \|\cdot\|_{\tilde{L}^{r+1}}^{r+1} ds\big)^{\frac{1}{r+1}}$, respectively. Using (4.3) and the weakly lower-semicontinuity property of the norm, we obtain

$$\begin{aligned}
&\limsup_{k\to\infty} \Big[-2\int_0^T e^{-\frac{\delta\lambda}{2}(T-s)}\big[\alpha|S_1(s)S_1(t_{n_k}-T)u_{n_k}|^2 \\
&+ \beta\|S_1(s)S_1(t_{n_k}-T)u_{n_k}\|_{\tilde{L}^{r+1}}^{r+1}\big] ds\Big] \\
&= -2\liminf_{k\to\infty} \int_0^T e^{-\frac{\delta\lambda}{2}(T-s)}\big[\alpha|S_1(s)S_1(t_{n_k}-T)u_{n_k}|^2 \\
&\quad + \beta\|S_1(s)S_1(t_{n_k}-T)u_{n_k}\|_{\tilde{L}^{r+1}}^{r+1}\big] ds \\
&\le -2\int_0^T e^{-\frac{\delta\lambda}{2}(T-s)}\big[\alpha|S_1(s)u_T|^2 + \beta\|S_1(s)u_T\|_{\tilde{L}^{r+1}}^{r+1}\big] ds.
\end{aligned} \tag{4.19}$$

Collecting (4.15) and (4.17)-(4.19) in (4.14) and then taking lim sup in (4.14), we obtain

$$\begin{aligned}
&\limsup_{k\to\infty} |\mathbb{S}(t_{n_k})z_{n_k}|^2 \\
&\le e^{-\frac{\delta\lambda}{2}T}\rho^2 + 2\int_0^T e^{-\frac{\delta\lambda}{2}(T-s)}\langle \Phi, \mathbb{S}(s)w_T\rangle ds \\
&\quad - 2\int_0^T e^{-\frac{\delta\lambda}{2}(T-s)}\Big(\alpha|S_1(s)u_T|^2 + \beta\|S_1(s)u_T\|_{\tilde{L}^{r+1}}^{r+1} + [\mathbb{S}(s)w_T]^2\Big) ds.
\end{aligned} \tag{4.20}$$

On the other hand, we obtain in (4.13) applied to $w = S(T)w_T$ that

$$\begin{aligned} |w|^2 &= |\mathbb{S}(T)w_T|^2 \\ &= e^{-\frac{\delta\lambda}{2}T}|w_T|^2 + 2\int_0^T e^{-\frac{\delta\lambda}{2}(T-s)}\langle\Phi, \mathbb{S}(s)w_T\rangle ds \\ &\quad - 2\int_0^T e^{-\frac{\delta\lambda}{2}(T-s)}\Big(\alpha|S_1(s)u_T|^2 + \beta\|S_1(s)u_T\|^{r+1}_{\tilde{L}^{r+1}} + [\mathbb{S}(s)w_T]^2\Big) ds. \end{aligned} \tag{4.21}$$

From (4.20) and (4.21) we find that

$$\limsup_{k\to\infty}|\mathbb{S}(t_{n_k})z_{n_k}|^2 \le e^{-\frac{\delta\lambda}{2}T}\rho^2 + |w|^2 - e^{-\frac{\delta\lambda}{2}T} \le |w|^2 + e^{-\frac{\delta\lambda}{2}T}\rho^2, \quad \forall T \in \mathbb{N}. \tag{4.22}$$

Let us take $T \to \infty$ in (4.22) to obtain

$$\limsup_{k\to\infty}|\mathbb{S}(t_{n_k})z_{n_k}|^2 \le |w|^2$$

and yields (4.11). This establishes that $\{\mathbb{S}(t_n)z_n\}_n$ is precompact in H and hence that $\{\mathbb{S}(t)\}_{t\ge 0}$ is asymptotically compact in H. □

5. Dimension of the attractor

In this section, we estimate the bounds for the Hausdorff as well as fractal dimensions of the global attractor $\mathcal{A}$ obtained in previous section. Due to this technical difficulty, we consider the cases $r = \overline{1,3}$ only in this section.

Let $z(t)$ be the unique weak solution of the Problem (3.1), we consider the linearized system

$$\begin{gathered} (\partial_t s(t), \tilde{s}) + a(s(t), \tilde{s}) + \sigma(s(t), \tilde{s}) + \mathbb{B}(s(t), z(t), \tilde{s}) + \mathbb{B}(z(t), s(t), \tilde{s}) \\ + \alpha(v, \tilde{v}) + \beta\langle\mathcal{C}_r'(u)v, \tilde{v}\rangle_{V_1', V_1} = 0, \quad \forall\tilde{s} = (\tilde{v}, \tilde{\psi}, \tilde{\zeta}) \in V, \text{ a.e. } t \in (0,T), \\ s(0) = s_0 \quad \text{in } H, \end{gathered} \tag{5.1}$$

where $\mathcal{C}_r'(\cdot)$ is defined in (2.13). As in the case of nonlinear Problem (3.1), one can show that there exists a unique solution $s = (v, \psi, \zeta) \in L^2(0,T;V) \cap L^\infty(0,T;H)$ with $v \in L^{r+1}(0,T;\tilde{L}^{r+1})$ and $\partial_t s \in L^{\frac{r+1}{r}}(0,T;V')$.

We define a map $\Lambda(t;z_0) : H \to H$ by setting $\Lambda(t;z_0)s_0 = s(t)$. In the next Lemma, we show that the map $\Lambda(t;z_0)$ is bounded and the semigroup $\{\mathbb{S}(t)\}_{t\ge 0}$ is uniformly differentiable on global attractor $\mathcal{A}$, i.e.,

$$\lim_{\varepsilon\to 0}\ \sup_{0<|\bar{z}_0 - z_0|<\varepsilon,\, \bar{z}_0, z_0\in\mathcal{A}} \frac{|\mathbb{S}(t)\bar{z}_0 - \mathbb{S}(t)z_0 - \Lambda(t;z_0)(\bar{z}_0 - z_0)|}{|\bar{z}_0 - z_0|} = 0. \tag{5.2}$$

Theorem 5.1. *Let $\bar{z}_0, z_0 \in H$. Then for $r = \overline{1,3}$, there exists a constant $M(|\bar{z}_0|, |z_0|)$ such that*

$$|\mathbb{S}(t)\bar{z}_0 - \mathbb{S}(t)z_0 - \Lambda(t;z_0)(\bar{z}_0 - z_0)| \le M|\bar{z}_0 - z_0|, \tag{5.3}$$

where the linear operator $\Lambda(t;z_0)$ for $t > 0$ is the solution operator of the problem (5.1). Or in other words, for every $t > 0$, the map $\mathbb{S}(t) : H \to H$ is Fréchet differentiable with respect to the initial data, and its Fréchet derivative $D_{z_0}(\mathbb{S}(t)z_0)s_0 = \Lambda(t;z_0)s_0$. Moreover, (5.2) is satisfied.

Proof. We denote $\bar{z}(t) = \mathbb{S}(t)\bar{z}_0$, $z(t) = \mathbb{S}(t)z_0$ and let $s(t)$ be the solution of (5.1) with $s_0 = \bar{z}_0 - z_0$. From (3.3), we easily find that

$$\int_0^t \|z(s)\|^2 ds \le \frac{2}{\delta}\Big(|z_0|^2 + \frac{2}{\delta}t\|\Phi\|^2_{V'}\Big), \quad \forall t \in [0,T]. \tag{5.4}$$

Writing $\omega(t) = \bar{z}(t) - z(t)$, $t \geq 0$ and using the same arguments as in proving the uniqueness of solution in Theorem 3.1, we deduce

$$\frac{d}{dt}|\omega(t)|^2 + \frac{\delta}{2}\|\omega(t)\|^2 \leq 2\delta^{-1}c_b^2|\omega(t)|^2\|z(s)\|^2$$

and

$$|\omega(t)|^2 \leq |\omega(0)|^2 \exp\Big(2\delta^{-1}c_b^2\int_0^t \|z(s)\|^2 ds\Big).$$

Thus, using (5.4), we obtain

$$|\omega(t)|^2 \leq |\omega(0)|^2 \exp\Big(4\delta^{-2}c_b^2\left(|z_0|^2 + 2\delta^{-1}t\|\Phi\|_{V'}^2\right)\Big), \quad \forall t \in [0,T]. \tag{5.5}$$

Now, from (5.4) and (5.5), we have

$$\begin{aligned}
&\frac{\delta}{2}\int_0^t \|\omega(s)\|^2 ds \\
&\leq |\omega(0)|^2 + 2\delta^{-1}c_b^2\int_0^t |\omega(s)|^2\|z(s)\|^2 ds \\
&\leq |\omega(0)|^2 + 2\delta^{-1}c_b^2\int_0^t \|z(s)\|^2|\omega(0)|^2 \exp\Big[4\delta^{-2}c_b^2\left(|z_0|^2 + 2\delta^{-1}s\|\Phi\|_{V'}^2\right)\Big]ds \\
&\leq |\omega(0)|^2\Big[1 + 4\delta^{-2}c_b^2\left(|z_0|^2 + 2\delta^{-1}t\|\Phi\|_{V'}^2\right)\exp\Big[4\delta^{-2}c_b^2\left(|z_0|^2 + 2\delta^{-1}t\|\Phi\|_{V'}^2\right)\Big] \\
&\leq |\omega(0)|^2 \exp\Big[8\delta^{-2}c_b^2\left(|z_0|^2 + 2\delta^{-1}t\|\Phi\|_{V'}^2\right)\Big].
\end{aligned} \tag{5.6}$$

Let $\eta(t) = (u_\eta(t), B_\eta(t), \theta_\eta(t))$ be defined by

$$\eta(t) = \bar{z}(t) - z(t) - s(t) = \omega(t) - s(t), \quad t \geq 0.$$

Evidently, for all $\tilde{\eta} = (u_{\tilde{\eta}}, B_{\tilde{\eta}}, \theta_{\tilde{\eta}}) \in V$, $\eta(t)$ satisfies

$$\begin{gathered}
(\partial_t\eta(t), \tilde{\eta}) + a(\eta(t), \tilde{\eta}) + \sigma(\eta(t), \tilde{\eta}) + \alpha(u_\eta(t), u_{\tilde{\eta}}) + \beta\langle \mathcal{C}_r(\bar{u}) - \mathcal{C}_r(u) - \mathcal{C}_r'(u)v, u_{\tilde{\eta}}\rangle \\
= -\mathbb{B}(\bar{z}(t), \bar{z}(t), \tilde{\eta}) + \mathbb{B}(z(t), z(t), \tilde{\eta}) + \mathbb{B}(s(t), z(t), \tilde{\eta}) + \mathbb{B}(z(t), s(t), \tilde{\eta}), \\
\text{a.e. } t \in (0,T), \\
\eta(0) = \mathbf{0} \quad \text{in } H.
\end{gathered}$$

It is easy to see that

$$\begin{aligned}
&-\mathbb{B}(\bar{z}(t), \bar{z}(t), \tilde{\eta}) + \mathbb{B}(z(t), z(t), \tilde{\eta}) + \mathbb{B}(s(t), z(t), \tilde{\eta}) + \mathbb{B}(z(t), s(t), \tilde{\eta}) \\
&= -\mathbb{B}(z(t), \eta(t), \tilde{\eta}) - \mathbb{B}(\eta(t), z(t), \tilde{\eta}) - \mathbb{B}(\omega(t), \omega(t), \tilde{\eta})
\end{aligned}$$

and consequently, by Lemma 2.1, for all $t > 0$,

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}|\eta(t)|^2 + \frac{\delta}{2}\|\eta(t)\|^2 + \alpha|u_\eta(t)|^2 &\leq \mathbb{B}(\eta(t), \eta(t), z(t)) - \mathbb{B}(\omega(t), \omega(t), \eta(t)) \\
&\quad - \beta\langle \mathcal{C}_r(\bar{u}) - \mathcal{C}_r(u) - \mathcal{C}_r'(u)v, u_\eta\rangle.
\end{aligned} \tag{5.7}$$

By using the Lemma 2.3 and the Young inequality, we have

$$\begin{aligned}
&\mathbb{B}(\eta(t), \eta(t), z(t)) - \mathbb{B}(\omega(t), \omega(t), \eta(t)) \\
&\leq c_b(|\eta(t)|\|\eta(t)\|\|z(t)\| + |\omega(t)|\|\omega(t)\|\|\eta(t)\|) \\
&\leq \frac{\delta}{6}\|\eta(t)\|^2 + 3\delta^{-1}c_b^2(|\eta(t)|^2\|z(t)\|^2 + |\omega(t)|^2\|\omega(t)\|^2).
\end{aligned}$$

To eatimate the term $-\beta\langle \mathcal{C}_r(\bar{u})-\mathcal{C}_r(u)-\mathcal{C}_r'(u)v,u_\eta\rangle$, we consider the cases $r=\overline{1,3}$ separately. For $r=1$, it can be easily seen that

$$-\beta\langle \mathcal{C}_r(\bar{u})-\mathcal{C}_r(u)-\mathcal{C}_r'(u)v,u_\eta\rangle=-\beta|u_\eta(t)|_1^2.$$

It should be noted that

$$\begin{aligned}
&\Big\langle \frac{u_1}{|u_1|}(u_1\cdot v)-\frac{u_2}{|u_2|}(u_2\cdot v),w\Big\rangle\\
&=\Big\langle \frac{u_1}{|u_1|}[(u_1-u_2)\cdot v],w\Big\rangle+\Big\langle \Big(\frac{u_1}{|u_1|}-\frac{u_2}{|u_2|}\Big)(u_2\cdot v),w\Big\rangle\\
&=\Big\langle \frac{u_1}{|u_1|}[(u_1-u_2)\cdot v],w\Big\rangle+\Big\langle \frac{u_1(|u_1|-|u_2|)+(u_1-u_2)|u_1|}{|u_1||u_2|}(u_2\cdot v),w\Big\rangle\\
&\le 2\langle |u_1-u_2||v|,|w|\rangle
\end{aligned}\tag{5.8}$$

for all $u_1\neq \mathbf{0}$, $u_2\neq\mathbf{0}$ and $v,w\in\tilde{L}^3$ (one can also obtain same estimates for $u_1=\mathbf{0}$ or $u_2=\mathbf{0}$). For $r=2$, by using Taylor's formula (see [16, Theorem 7.9.1]), Hölder's, Ladyzhenskaya's, Young's inequalities, (2.14) and (5.8), we have

$$\begin{aligned}
&-\beta\langle \mathcal{C}_r(\bar{u})-\mathcal{C}_r(u)-\mathcal{C}_r'(u)v,u_\eta\rangle\\
&=-\beta\Big\langle \int_0^1 \mathcal{C}_r'(\epsilon\bar{u}+(1-\epsilon)u)(\bar{u}-u)d\epsilon-\mathcal{C}_r'(u)v,u_\eta\Big\rangle\\
&=-\beta\langle \mathcal{C}_r'(u)u_\eta,u_\eta\rangle+\beta\Big\langle \int_0^1 \left[\mathcal{C}_r'(u)-\mathcal{C}_r'(\epsilon\bar{u}+(1-\epsilon)u)\right](\bar{u}-u)d\epsilon,u_\eta\Big\rangle\\
&=-\beta\langle \mathcal{C}_r'(u)u_\eta,u_\eta\rangle+\beta\int_0^1 \Big\langle \big[|\bar{u}|-|\epsilon\bar{u}+(1-\epsilon)u|\big]\cdot(\bar{u}-u),u_\eta\Big\rangle\,d\epsilon\\
&\quad+\beta\int_0^1\Big\langle \frac{\bar{u}}{|\bar{u}|}\left[\bar{u}\cdot(\bar{u}-u)\right]-\frac{\epsilon\bar{u}+(1-\epsilon)u}{|\epsilon\bar{u}+(1-\epsilon)u|}\left[(\epsilon\bar{u}+(1-\epsilon)u)\cdot(\bar{u}-u)\right],u_\eta\Big\rangle d\epsilon\\
&\le 3\beta\int_0^1(1-\epsilon)\langle|\bar{u}-u|^2,|u_\eta|\rangle d\epsilon\\
&\le \frac{3\beta}{2}\|\bar{u}-u\|_{\tilde{L}^4}^2|u_\eta|_1\\
&\le \alpha|u_\eta|_1^2+\frac{9\beta^2}{16\alpha}|\omega|^2\|\omega\|^2.
\end{aligned}$$

For $r=3$, once again by using (2.14), Taylor's formula (see [16, Theorem 7.9.1]), Hölder's, Ladyzhenskaya's, Young's, Poincaré's inequalities and (2.15), we have

$$\begin{aligned}
&-\beta\langle \mathcal{C}_r(\bar{u})-\mathcal{C}_r(u)-\mathcal{C}_r'(u)v,u_\eta\rangle\\
&=-\beta\Big\langle \mathcal{C}_r'(u)(\bar{u}-u)+\frac12\int_0^1 \mathcal{C}_r''(\epsilon\bar{u}+(1-\epsilon)u)d\epsilon(\bar{u}-u)\otimes(\bar{u}-u)-\mathcal{C}_r'(u)v,u_\eta\Big\rangle\\
&=-\beta\langle \mathcal{C}_r'(u)u_\eta,u_\eta\rangle+3\beta\int_0^1\langle\left[(\epsilon\bar{u}+(1-\epsilon)u)\cdot(\bar{u}-u)\right](\bar{u}-u),u_\eta\rangle\,d\epsilon\\
&=3\beta\int_0^1\|\epsilon\bar{u}+(1-\epsilon)u\|_{\tilde{L}^4}\|\bar{u}-u\|_{\tilde{L}^4}^2\|u_\eta\|_{\tilde{L}^4}d\epsilon\\
&\le 3(2^{3/4})\beta(\|\bar{u}\|_{\tilde{L}^4}+\|u\|_{\tilde{L}^4})|\omega|\|\omega\||u_\eta|_1^{1/2}\|u_\eta\|_1^{1/2}\\
&\le \frac{\delta}{12}\|\eta\|^2+\frac{3^{8/3}}{2}\delta^{-1/3}\beta^{4/3}(\|\bar{u}\|_{\tilde{L}^4}+\|u\|_{\tilde{L}^4})^{4/3}|\omega|^{4/3}\|\omega\|^{4/3}|\eta|^{2/3}
\end{aligned}$$

$$\leq \frac{\delta}{12}\|\eta\|^2 + \frac{\beta}{2}(\|\bar{u}\|_{\tilde{L}^4} + \|u\|_{\tilde{L}^4})^4|\eta|^2 + \delta^{-1/2}\beta^{3/2}3^{5/2}|\omega|^2\|\omega\|^2.$$

Collecting the estimates above and applying to (5.7), we obtain

$$\begin{aligned}
&\frac{d}{dt}|\eta(t)|^2 + \frac{\delta}{2}\|\eta(t)\|^2 \\
&\leq \begin{cases} 6\delta^{-1}c_b^2(|\eta(t)|^2\|z(t)\|^2 + |\omega(t)|^2\|\omega(t)\|^2), & \text{for } r=1, \\ 6\delta^{-1}c_b^2|\eta(t)|^2\|z(t)\|^2 + \left(6\delta^{-1}c_b^2 + \frac{9\beta^2}{8\alpha}\right)|\omega(t)|^2\|\omega(t)\|^2, & \text{for } r=2, \\ \left[6\delta^{-1}c_b^2\|z(t)\|^2 + \beta(\|\bar{u}(t)\|_{\tilde{L}^4}^4 + \|u(t)\|_{\tilde{L}^4}^4)\right]|\eta(t)|^2 & \\ + \left(6\delta^{-1}c_b^2 + 2\delta^{-1/2}\beta^{3/2}3^{5/2}\right)|\omega(t)|^2\|\omega(t)\|^2, & \text{for } r=3. \end{cases}
\end{aligned} \tag{5.9}$$

For $r=3$, we deduce from (3.3) that

$$\int_0^t \left(\|\bar{u}(s)\|_{\tilde{L}^4}^4 + \|u(s)\|_{\tilde{L}^4}^4\right) ds \leq \frac{1}{\beta}(|z_0|^2 + 2\delta^{-1}t\|\Phi\|_{V'}^2), \quad \forall t \geq 0. \tag{5.10}$$

By integrating (5.9) from 0 to t, and using that $\eta(0)=0$, we obtain

$$\begin{aligned}
|\eta(t)|^2 &\leq \left(6\delta^{-1}c_b^2 + 2\delta^{-1/2}\beta^{3/2}3^{5/2}\right)\int_0^t |\omega(s)|^2\|\omega(s)\|^2 ds \\
&\quad + \int_0^t \left[6\delta^{-1}c_b^2\|z(s)\|^2 + \beta(\|\bar{u}(s)\|_{\tilde{L}^4}^4 + \|u(s)\|_{\tilde{L}^4}^4)\right]|\eta(s)|^2 ds
\end{aligned}$$

for all $t \geq 0$, and consequently, by the Gronwall inequality and (5.4)-(5.6), (5.10),

$$\begin{aligned}
|\eta(t)|^2 &\leq \left(6\delta^{-1}c_b^2 + 2\delta^{-1/2}\beta^{3/2}3^{5/2}\right)\int_0^t |\omega(s)|^2\|\omega(s)\|^2 ds \\
&\quad \times \exp\Big(\int_0^t \left[6\delta^{-1}c_b^2\|z(s)\|^2 + \beta(\|\bar{u}(s)\|_{\tilde{L}^4}^4 + \|u(s)\|_{\tilde{L}^4}^4)\right] ds\Big) \\
&\leq \left(12\delta^{-2}c_b^2 + 4(\delta^{-1}\beta)^{3/2}3^{5/2}\right)|\omega(0)|^4 \\
&\quad \times \exp\left[\,\left(12\delta^{-2}c_b^2 + 12\delta^{-1}c_b^2 + 1\right)\left(|z_0|^2 + 2\delta^{-1}t\|\Phi\|_{V'}^2\right)\,\right].
\end{aligned}$$

Thus, by the definition of η, it is immediate that

$$\frac{|\bar{z}(t) - z(t) - s(t)|}{|\bar{z}_0 - z_0|} = \chi(t)|\bar{z}_0 - z_0|$$

where

$$\begin{aligned}
\chi(t) &= \left(12\delta^{-2}c_b^2 + 4(\delta^{-1}\beta)^{3/2}3^{5/2}\right)^{1/2} \\
&\quad \times \exp\Big[\left(6\delta^{-2}c_b^2 + 6\delta^{-1}c_b^2 + \frac{1}{2}\right)\left(|z_0|^2 + 2\delta^{-1}t\|\Phi\|_{V'}^2\right)\Big],
\end{aligned}$$

and hence the differentiability of the semigroup $\mathbb{S}(t)$ with respect to the initial data as well as (5.2) and (5.3) follows. The cases of $r = 1, 2$ can be proved in a similar way. □

In the next Theorem, we show that the global attractor obtained in Theorem 4.2 has finite Hausdorff and fractal dimensions. To estimate the dimension of the global attractor $\mathcal{A}$, we need the assumption

$$\mathbb{R}^2 \setminus \overline{\Omega} \text{ contains a semicone} \tag{5.11}$$

to ensure that we can use the generalized Lieb-Thirring inequality in the general case (see [23]).

Theorem 5.2. *Assume the conditions of Theorem 3.1 and conditions* (5.11) *hold. For* $r = \overline{1,3}$*, then the global attractor* $\mathcal{A}$ *in Theorem 4.2 has finite Hausdorff and fractal dimensions, which can be estimated as*

$$\dim_{\mathrm{Hau}}(\mathcal{A}) \le 1 + \frac{\mathcal{M}}{\mathcal{K}}, \tag{5.12}$$

$$\dim_{\mathrm{Fra}}(\mathcal{A}) \le 2\big(1 + \frac{2\mathcal{M}}{\mathcal{K}}\big), \tag{5.13}$$

where

$$\mathcal{M} := \frac{1}{12}\left(\lambda_1 R_e^{-1} + c_0 R_m^{-1} + \kappa\lambda_1\right),$$
$$\mathcal{K} := 6\mu R_e^{1/2}\max\left(R_e; \frac{2}{7}R_m; 2\kappa^{-1}\right)\left[4(R_e^{1/2} + R_m^{1/2})^2 + 4R_m^{3/2} + \kappa^{-3/2}\right]\|\Phi\|_{V'}^2.$$

Proof. First, we can rewrite (5.1) as

$$\begin{aligned}
(\partial_t s(t), \tilde{s}) &= \langle \mathcal{F}'(z)s, \tilde{s}\rangle \\
&= -\Big[a(s(t),\tilde{s}) + \sigma(s(t),\tilde{s}) + \mathbb{B}(s(t), z(t), \tilde{s}) + \mathbb{B}(z(t), s(t), \tilde{s}) \\
&\quad + \alpha(v,\tilde{v}) + \beta\langle \mathcal{C}_r'(u)v, \tilde{v}\rangle\Big], \quad \forall \tilde{s} = (\tilde{v}, \tilde{\psi}, \tilde{\zeta}) \in V, \text{ a.e. } t \in (0,T), \\
s(0) &= s_0 \quad \text{in } H.
\end{aligned} \tag{5.14}$$

Then we define the numbers q_m for $m \in \mathbb{N}$ by

$$q_m = \limsup_{t\to\infty} \sup_{z_0\in\mathcal{A}} \sup_{\xi_i\in H,\, |\xi_i|\le 1,\, i=1,\dots,m} \frac{1}{t}\int_0^t \mathrm{Tr}(\mathcal{F}'(\mathbb{S}(s)z_0)\circ Q_m(s))ds \tag{5.15}$$

where $Q_m(s) = Q_m(s; z_0, \xi_1, \dots, \xi_m)$ is the orthogonal projector of H onto the space spanned by

$$\{\Lambda(t; z_0)\xi_1, \dots, \Lambda(t; z_0)\xi_m\}$$

and the trace (denoted by Tr) of $\mathcal{F}'(\mathbb{S}(s)z_0)\circ Q_m(s)$ in (5.15) is defined at least a.e. in t. From [40, section V.3.4] (see Proposition V.2.1 and Theorem V.3.3), we infer that if $q_m < 0$, for some $m \in \mathbb{N}$, then the global attractor $\mathcal{A}$ has finite Hausdorff and fractal dimensions estimated respectively as

$$\dim_{\mathrm{Hau}}(\mathcal{A}) \le m, \tag{5.16}$$

$$\dim_{\mathrm{Fra}}(\mathcal{A}) \le m\Big(1 + \max_{1\le i\le m}\frac{(q_i)_+}{|q_m|}\Big). \tag{5.17}$$

To estimate the numbers q_m, let $z_0 \in \mathcal{A}$ and $\xi_1, \dots, \xi_m \in H$. Set $z(t) = \mathbb{S}(t)z_0$ and $\eta_i = \Lambda(t, z_0)\xi_i$, $t \ge 0$. Let $\Big\{\tilde{w}_i(t), \tilde{C}_i(t), \tilde{\phi}_i(t)\Big\}_{i=1,\dots,m}$, $t \ge 0$ be a basis for $\mathrm{span}\{\eta_1(t), \dots, \eta_m(t)\}$ such that $\{\tilde{w}_i(t)\}_{i=1,\dots,m}$ is orthonormal in H_1, $\{\tilde{C}_i(t)\}_{i=1,\dots,m}$ is orthonormal in H_2 and $\{\tilde{\phi}_i(t)\}_{i=1,\dots,m}$ is orthonormal in H_3. Set

$$\varphi_i = (w_i, C_i, \phi_i) = \Big(\frac{\tilde{w}_i}{\sqrt{3}}, \frac{\tilde{C}_i}{\sqrt{3S}}, \frac{\tilde{\phi}_i}{\sqrt{3\gamma}}\Big).$$

An easy computation shows that $\{\varphi_i\}_{i=1,\dots,m}$ is orthonormal in H. Since $\eta_i(t) \in V$ for a.e. $t \ge 0$, we can assume that $\varphi_i(t) \in V$ for a.e. $t \ge 0$ (by the Gram–Schmidt orthogonalization process). Then, from Lemma 2.1, (2.10), (2.14) and (5.14), we have

$$\begin{aligned}
&\mathrm{Tr}(\mathcal{F}'(\mathbb{S}(s)z_0)\circ Q_m(s))\\
&=\sum_{i=1}^{m}\langle \mathcal{F}'(z(s))\varphi_i,\varphi_i\rangle_{V',V}\\
&=-\sum_{i=1}^{m}\Big[a(\varphi_i,\varphi_i)+\sigma(\varphi_i,\varphi_i)+\mathbb{B}(\varphi_i,z,\varphi_i)+\alpha(w_i,w_i)+\beta\langle \mathcal{C}_r'(u)w_i,w_i\rangle\Big]\\
&\le \sum_{i=1}^{m}\Big[-\frac{1}{2}\Big(\frac{1}{R_e}\|w_i\|_1^2+\frac{S}{R_m}\|C_i\|_2^2+\gamma\kappa\|\phi_i\|_3^2\Big)+|\mathbb{B}(\varphi_i,z,\varphi_i)|\Big].
\end{aligned}\tag{5.18}$$

Now let

$$\rho(x)=\sum_{i=1}^{m}\Big(R_e^{-1/2}|w_i(x)|^2+SR_m^{-1/2}|C_i(x)|^2+\gamma\kappa^{1/2}|\phi_i(x)|^2\Big),$$

then by calculating similarly as in [3, Theorem 5.1], we find that

$$\begin{aligned}
\Big|\sum_{i=1}^{m}\big[b(w_i,u,w_i)-Sb(C_i,u,C_i)\big]\Big| &\le (R_e^{1/2}+R_m^{1/2})\|u\|_1\|\rho\|_{L^2},\\
\Big|\sum_{i=1}^{m}S\big[b(w_i,B,C_i)-b(C_i,B,w_i)\big]\Big| &\le S^{1/2}(R_eR_m)^{1/4}\|B\|_2\|\rho\|_{L^2}.
\end{aligned}\tag{5.19}$$

Applying Cauchy's and Young's inequalities, we obtain

$$\begin{aligned}
\gamma|w_i(x)\cdot\nabla\theta(x)||\phi_i(x)| &\le \gamma|\theta(x)||w_i(x)||\phi_i(x)|\\
&\le \frac{\gamma^{1/2}R_e^{1/4}}{2\kappa^{1/4}}|\nabla\theta(x)|\Big(\frac{1}{R_e^{1/2}}|w_i(x)|^2+\gamma\kappa^{1/2}|\phi_i(x)|^2\Big).
\end{aligned}$$

Integrating this expression in x, summing up in i from 1 up to m, and using the definition of ρ, we deduce that

$$\begin{aligned}
&\Big|\sum_{i=1}^{m}\gamma\bar{b}(w_i,\theta,\phi_i)\Big|\\
&=\gamma\Big|\sum_{i=1}^{m}\int_\Omega w_i(x)\cdot\nabla\theta(x)\phi_i(x)dx\Big|\\
&\le \frac{\gamma^{1/2}R_e^{1/4}}{2\kappa^{1/4}}\int_\Omega|\nabla\theta(x)|\sum_{i=1}^{m}\Big(\frac{1}{R_e^{1/2}}|w_i(x)|^2+\gamma\kappa^{1/2}|\phi_i(x)|^2\Big)dx\\
&=\frac{\gamma^{1/2}R_e^{1/4}}{2\kappa^{1/4}}\|\theta\|_3\|\rho\|_{L^2}.
\end{aligned}\tag{5.20}$$

Hence, from (5.19) and (5.20), we obtain

$$\begin{aligned}
&\big|\sum_{i=1}^m \mathbb{B}(\varphi_i, z, \varphi_i)\big| \\
&= \big|\sum_{i=1}^m \big[b(w_i, u, w_i) - Sb(C_i, B, w_i) + Sb(w_i, B, C_i) - Sb(C_i, u, C_i) \\
&\quad + \gamma\bar{b}(w_i, \theta, \phi_i)\big]\big| \\
&\le \|\rho\|_{L^2}\big[(R_e^{1/2} + R_m^{1/2})\|u\|_1 + S^{1/2}(R_eR_m)^{1/4}\|B\|_2 + \frac{\gamma^{1/2}R_e^{1/4}}{2\kappa^{1/4}}\|\theta\|_3\big].
\end{aligned} \tag{5.21}$$

From the definition of $\rho, \tilde{w}_i, \tilde{C}_i$ and $\tilde{\phi}_i$, we observe that

$$\rho(x) = \frac{1}{3}\sum_{i=1}^m \Big(R_e^{-1/2}|\tilde{w}_i(x)|^2 + R_m^{-1/2}|\tilde{C}_i(x)|^2 + \kappa^{1/2}|\tilde{\phi}_i(x)|^2\Big).$$

Then the generalized Lieb-Thirring inequality (see [23]) can be applied to the orthonormal finite families $\{\tilde{w}_i\}_i$, $\{\tilde{C}_i\}_i$ and $\{\tilde{\phi}_i\}_i$ (by condition (5.11). This guarantees the existence of a constant μ independent of the number of functions m (but depending on the shape of Ω) such that

$$\begin{aligned}
&\|\rho\|_{L^2}^2 \\
&\le \frac{1}{3}\Big(\frac{1}{R_e}\|\sum_{i=1}^m |\tilde{w}_i(x)|^2\|_{L^2}^2 + \frac{1}{R_m}\|\sum_{i=1}^m |\tilde{C}_i(x)|^2\|_{L^2}^2 + \kappa\|\sum_{i=1}^m |\tilde{\phi}_i(x)|^2\|_{L^2}^2\Big) \\
&\le \frac{\mu}{3}\sum_{i=1}^m \Big(\frac{1}{R_e}\|\tilde{w}_i\|_1^2 + \frac{1}{R_m}\|\tilde{C}_i\|_{\mathbb{H}^1}^2 + \kappa\|\tilde{\phi}_i\|_3^2\Big) \\
&\le \mu\sum_{i=1}^m \Big(\frac{1}{R_e}\|w_i\|_1^2 + \frac{S}{R_m}\|C_i\|_2^2 + \gamma\kappa\|\phi_i\|_3^2\Big).
\end{aligned} \tag{5.22}$$

Inserting (5.22) into (5.21) and using Young's inequality, we obtain

$$\begin{aligned}
&\big|\sum_{i=1}^m \mathbb{B}(\varphi_i, z, \varphi_i)\big| \\
&\le \mu\Big[(R_e^{1/2} + R_m^{1/2})\|u\|_1 + S^{1/2}(R_eR_m)^{1/4}\|B\|_2 + \frac{\gamma^{1/2}R_e^{1/4}}{2\kappa^{1/4}}\|\theta\|_3\Big]^2 + \frac{1}{4\mu}\|\rho\|_{L^2}^2 \\
&\le 3\mu\Big[(R_e^{1/2} + R_m^{1/2})^2\|u\|_1^2 + S(R_eR_m)^{1/2}\|B\|_2^2 + \frac{\gamma R_e^{1/2}}{4\kappa^{1/2}}\|\theta\|_3^2\Big] \\
&\quad + \frac{1}{4}\sum_{i=1}^m \Big(\frac{1}{R_e}\|w_i\|_1^2 + \frac{S}{R_m}\|C_i\|_2^2 + \gamma\kappa\|\phi_i\|_3^2\Big).
\end{aligned}$$

Applying this inequality to (5.18), we obtain

$$\begin{aligned}
&\mathrm{Tr}(\mathcal{F}'(\mathbb{S}(s)z_0) \circ Q_m(s)) \\
&\le 3\mu\Big[(R_e^{1/2} + R_m^{1/2})^2\|u\|_1^2 + S(R_eR_m)^{1/2}\|B\|_2^2 + \frac{\gamma R_e^{1/2}}{4\kappa^{1/2}}\|\theta\|_3^2\Big] \\
&\quad - \frac{1}{4}\sum_{i=1}^m \Big(\frac{1}{R_e}\|w_i\|_1^2 + \frac{S}{R_m}\|C_i\|_2^2 + \gamma\kappa\|\phi_i\|_3^2\Big).
\end{aligned} \tag{5.23}$$

Since $\{\varphi_i\}_{i=1,\dots,m}$ is orthonormal in H, we see that $|w_i|^2 = S|C_i|^2 = \gamma|\phi_i|^2 = 1/3$. Then, by (2.2), we deduce from (5.23) that

$$
\begin{aligned}
&\operatorname{Tr}(\mathcal{F}'(\mathbb{S}(s)z_0)\circ Q_m(s))\\
&\le 3\mu\Big[(R_e^{1/2}+R_m^{1/2})^2\|u\|_1^2 + S(R_eR_m)^{1/2}\|B\|_2^2 + \frac{\gamma R_e^{1/2}}{4\kappa^{1/2}}\|\theta\|_3^2\Big]\\
&\quad - \frac{1}{4}\sum_{i=1}^{m}\Big(\frac{\lambda_1}{R_e}|w_i|^2 + \frac{Sc_0}{R_m}|C_i|^2 + \gamma\kappa\lambda_1|\phi_i|^2\Big)\\
&= 3\mu\Big[(R_e^{1/2}+R_m^{1/2})^2\|u\|_1^2 + S(R_eR_m)^{1/2}\|B\|_2^2 + \frac{\gamma R_e^{1/2}}{4\kappa^{1/2}}\|\theta\|_3^2\Big]\\
&\quad - \frac{m}{12}\Big(\frac{\lambda_1}{R_e} + \frac{c_0}{R_m} + \kappa\lambda_1\Big).
\end{aligned}
$$

For all $m\in\mathbb{N}$, we have

$$
\begin{aligned}
q_m &\le 3\mu \limsup_{t\to\infty}\sup_{z_0\in\mathcal{A}}\frac{1}{t}\int_0^t \Big[(R_e^{1/2}+R_m^{1/2})^2\|u(s)\|_1^2 + S(R_eR_m)^{1/2}\|B(s)\|_2^2\\
&\quad + \frac{\gamma R_e^{1/2}}{4\kappa^{1/2}}\|\theta(s)\|_3^2\Big]ds - \frac{m}{12}\Big(\frac{\lambda_1}{R_e} + \frac{c_0}{R_m} + \kappa\lambda_1\Big).
\end{aligned}
\tag{5.24}
$$

From (1.1), we multiply the first, second and third equations by u, SB and $\gamma\theta$, respectively. Then integrating over Ω, using (2.9), (2.5) and (2.6), we obtain the energy estimates

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}|u|^2 + R_e^{-1}\|u\|_1^2 - Sb(B,B,u) + \alpha|u|^2 + \beta\|u\|_{\tilde{L}^{r+1}}^{r+1}\\
&\le \frac{R_e^{-1}}{4}\|u\|_1^2 + \frac{R_e}{\lambda_1^2}\|\theta\|_3^2 + \frac{R_e^{-1}}{8}\|u\|_1^2 + 2R_e\|f\|_{V_1'}^2\\
&\le \frac{3R_e^{-1}}{8}\|u\|_1^2 + \frac{\gamma\kappa}{4}\|\theta\|_3^2 + 2R_e\|f\|_{V_1'}^2,
\end{aligned}
$$

$$
\frac{1}{2}\frac{d}{dt}S|B|^2 + R_m^{-1}S\|B\|_2^2 - Sb(B,u,B) \le \frac{7R_m^{-1}S}{8}\|B\|_2^2 + \frac{2R_mS}{7}\|\Psi\|_{V_2'}^2,
$$

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt}\gamma|\theta|^2 + \kappa\gamma\|\theta\|_3^2 &\le \frac{\epsilon^2\gamma}{2\kappa}\|u\|_1^2 + \frac{\kappa\gamma}{2}\|\theta\|_3^2 + \frac{\kappa\gamma}{8}\|\theta\|_3^2 + \frac{2\gamma}{\kappa}\|h\|_{V_3'}^2\\
&\le \frac{5\kappa\gamma}{8}\|\theta\|_3^2 + \frac{R_e^{-1}}{2}\|u\|_1^2 + \frac{2\gamma}{\kappa}\|h\|_{V_3'}^2.
\end{aligned}
$$

Adding the inequalities above and using (2.8), we deduce that

$$
\begin{aligned}
&\frac{d}{dt}|z(t)|^2 + \frac{R_e^{-1}}{4}\|u(t)\|_1^2 + \frac{R_m^{-1}S}{4}\|B(t)\|_2^2 + \frac{\kappa\gamma}{4}\|\theta(t)\|_3^2\\
&\le 2R_e\|f\|_{V_1'}^2 + \frac{4R_mS}{7}\|\Psi\|_{V_2'}^2 + \frac{4\gamma}{\kappa}\|h\|_{V_3'}^2\\
&\le 2\max\left(R_e; \frac{2}{7}R_m; \frac{2}{\kappa}\right)\|\Phi\|_{V'}^2.
\end{aligned}
$$

It follows that

$$
\limsup_{t\to\infty}\sup_{z_0\in\mathcal{A}}\frac{1}{t}\int_0^t \|u(s)\|_1^2 ds \le 8R_e\max\Big(R_e; \frac{2}{7}R_m; \frac{2}{\kappa}\Big)\|\Phi\|_{V'}^2,
$$

$$\limsup_{t\to\infty}\sup_{z_0\in\mathcal{A}}\frac{1}{t}\int_0^t S\|B(s)\|_2^2ds \le 8R_m\max\Big(R_e;\frac{2}{7}R_m;\frac{2}{\kappa}\Big)\|\Phi\|_{V'}^2,$$
$$\limsup_{t\to\infty}\sup_{z_0\in\mathcal{A}}\frac{1}{t}\int_0^t \gamma\|\theta(s)\|_3^2ds \le 8\kappa^{-1}\max\Big(R_e;\frac{2}{7}R_m;\frac{2}{\kappa}\Big)\|\Phi\|_{V'}^2.$$

Therefore,

$$\begin{aligned}&\limsup_{t\to\infty}\sup_{z_0\in\mathcal{A}}\frac{1}{t}\int_0^t\Big[(R_e^{1/2}+R_m^{1/2})^2\|u(s)\|_1^2+S(R_eR_m)^{1/2}\|B(s)\|_2^2\\&+\frac{\gamma R_e^{1/2}}{4\kappa^{1/2}}\|\theta(s)\|_3^2\Big]ds\\&\le 2R_e^{1/2}\max\Big(R_e;\frac{2}{7}R_m;\frac{2}{\kappa}\Big)\Big[4(R_e^{1/2}+R_m^{1/2})^2+4R_m^{3/2}+\kappa^{-3/2}\Big]\|\Phi\|_{V'}^2.\end{aligned}\tag{5.25}$$

Applying (5.25) to (5.24), we obtain

$$q_m\le -m\mathcal{K}+\mathcal{M},\quad \forall m\in\mathbb{N}$$

where $\mathcal{K}$ and $\mathcal{M}$ are as in the statement of the theorem. If $m'\in\mathbb{N}$ is defined by

$$m'-1\le\frac{\mathcal{M}}{\mathcal{K}}<m'$$

then $q_{m'}<0$ and thus from (5.16), we find that

$$\dim_{\mathrm{Hau}}(\mathcal{A})\le m'\le 1+\frac{\mathcal{M}}{\mathcal{K}}$$

and obtain (5.12). Furthermore, if $m''\in\mathbb{N}$ is defined by

$$m''-1\le\frac{2\mathcal{M}}{\mathcal{K}}<m''$$

then using [40, Lemma VI.2.2], we have

$$q_{m''}<0 \text{ and } \frac{(q_i)_+}{|q_{m''}|}\le 1,\quad \text{for all } i=1,\dots,m''.$$

From (5.17), we obtain

$$\dim_{\mathrm{Fra}}(\mathcal{A})\le 2m''\le 2\Big(1+\frac{2\mathcal{M}}{\mathcal{K}}\Big)$$

and have (5.13), which completes the proof. □

6. Upper semicontinuity of global attractor

In this section, we verify the upper semicontinuity of global attractors for (1.1). Let $\{\Omega_m\}_{m=1}^\infty$ be an expanding sequence of simply connected bounded and smooth subdomains of Ω (for example, one can take $\Omega=\mathbb{R}\times(-L,L)$) such that $\cup_{m=1}^\infty\Omega_m=\Omega$. Throughout this section, we differentiate the H spaces defined in Ω and Ω_m as H_Ω and H_{Ω_m}, respectively, and similar modifications are made for other spaces also. Then, we consider

$$\begin{gathered}(\partial_t z_m(t),\tilde z)+a(z_m(t),\tilde z)+\sigma(z_m(t),\tilde z)+\mathbb{B}(z_m(t),z_m(t),\tilde z)\\+\alpha(u_m,\tilde u)+\beta\langle\mathcal{C}_r(u_m),\tilde u\rangle=\langle\Phi_m,\tilde z\rangle,\quad\forall\tilde z=(\tilde u,\tilde B,\tilde\theta)\in V_{\Omega_m},\ \text{a.e. } t\in(0,T),\\ z_m(0)=z_{0,m}\quad\text{in } H_{\Omega_m},\end{gathered}\tag{6.1}$$

where

$$\Phi_m(x) = \begin{cases} \Phi(x), & x \in \Omega_m, \\ \mathbf{0}, & x \in \Omega \setminus \Omega_m, \end{cases} \quad \text{and} \quad z_{0,m}(x) = \begin{cases} z_0(x), & x \in \Omega_m, \\ \mathbf{0}, & x \in \Omega \setminus \Omega_m. \end{cases}$$

We also assume that $\Phi \in H_\Omega$. Let $z_m \in L^\infty(0,T;H_{\Omega_m}) \cap L^2(0,T;V_{\Omega_m})$ be the unique solution of the system (6.1) with $\partial_t z_m \in L^2(0,T;V'_{\Omega_m})$ and hence in $C([0,T];H_{\Omega_m})$. Then, by the same arguments in Section 4, the system (6.1) possesses an attractor $\mathcal{A}_m \in H_{\Omega_m}$. To check whether the global attractor $\mathcal{A}$ and $\mathcal{A}_m$ of Problem (3.1) corresponding to Ω and Ω_m, respectively, have the upper semicontinuity when $m \to \infty$, we follow the results for the 2D Navier-Stokes equations in [44].

We now prove the following important lemma.

Lemma 6.1. *If $z_{0,m} \in \mathcal{A}_m$, $m = 1, 2, \dots$ then there exists $z_0 \in \mathcal{A}$ such that up to a subsequence,*

$$z_{0,m} \to z_0 \quad \textit{strongly in } H. \tag{6.2}$$

Proof. First, we show that for given $z_{0,m} \in \mathcal{A}_m$, $m = 1, 2, \dots$, there exists $z_0 \in \mathcal{A}$ such that up to a subsequence,

$$\mathbb{S}_m(\cdot)z_{0,m} \to \mathbb{S}(\cdot)z_0 \quad \text{weakly in } L^2(-T,T;V_\Omega), \tag{6.3}$$

$$\mathbb{S}_m(t)z_{0,m} \to \mathbb{S}(t)z_0 \quad \text{weakly in } H_\Omega \text{ for each } t \in \mathbb{R}. \tag{6.4}$$

Indeed, by using (4.8), we have that

$$|\mathbb{S}_m(t)z_{0,m}| \le \rho, \quad \text{for given } z_{0,m} \in \mathcal{A}_m \text{ and } t \in \mathbb{R}. \tag{6.5}$$

By the same estimate as (3.3), we deduce that for each $T > 0$, the sequence $\{\mathbb{S}_m(t)z_{0,m}\}_{m\in\mathbb{N}}$ is uniformly bounded in $L^\infty(-T,T,H_\Omega) \cap L^2(-T,T;V_\Omega)$ and the sequence $\{\partial_t\mathbb{S}_m(t)z_{0,m}\}_{m\in\mathbb{N}}$ is bounded in $L^2(-T,T;V'_\Omega)$. From these uniform bounds, we can extract a subsequence, which is denoted again by $\{z_{0,m}\}_{m\in\mathbb{N}}$ such that

$$\begin{gathered} \mathbb{S}_m(t)z_{0,m} \to z \text{ weak-star in } L^\infty(-T,T;H_\Omega), \\ \mathbb{S}_m(t)z_{0,m} \to z \text{ weakly in } L^2(-T,T;V_\Omega), \\ \partial_t\mathbb{S}_m(t)z_{0,m} \to \partial_t z \text{ weakly in } L^2(-T,T;V'_\Omega), \\ \mathbb{S}_m(t)u_{0,m} \to u \text{ weakly in } L^{r+1}(-T,T;\tilde{L}^{r+1}_\Omega). \end{gathered} \tag{6.6}$$

From the first convergence in (6.6), we see that

$$\mathbb{S}_m(t)z_{0,m} \to z \quad \text{weakly in } L^2(-T,T;H_\Omega).$$

By the same arguments as in [36, Lemma 2.1], one can show that $z(\cdot)$ is a weak solution to Problem (3.1) defined on $\mathbb{R}$ and $z \in C(\mathbb{R};H_\Omega)$. Hence, we have $z(t) = \mathbb{S}(t)z(0)$ and obtain (6.3). Let us now fix $t^* \in [-T,T]$. From (6.5), we have that the sequence $\{\mathbb{S}_m(t^*)z_{0,m}\}_{m\in\mathbb{N}}$ is bounded in H_Ω. Therefore there exists $\tilde{z} \in H_\Omega$, a subsequence $\{\mathbb{S}_m(t^*)z_{0,m}\}_{m\in\mathbb{N}}$ (still denoted by $\{\mathbb{S}_m(t^*)z_{0,m}\}_{m\in\mathbb{N}}$) such that

$$\mathbb{S}_m(t^*)z_{0,m} \to \tilde{z} \quad \text{weakly in } H_\Omega.$$

Thus $z(t)$ is a solution of Problem (3.1) with $z(t^*) = \tilde{z}$. Since $t^* \in [-T,T]$ is arbitrary, we obtain

$$\mathbb{S}_m(t)z_{0,m} \to \mathbb{S}(t)z(0) \quad \text{weakly in } H_\Omega, \text{ for each } t \in \mathbb{R}. \tag{6.7}$$

Next, we show that $z_0 \in \mathcal{A}$. By using the weakly lower-semicontinuity of norm, (6.5) and (6.7), we obtain

$$|\mathbb{S}(t)z(0)| = \liminf_{m\to\infty} |\mathbb{S}_m(t)z_{0,m}| \le \rho, \quad \text{for all } t \in \mathbb{R},$$

which implies that the solution $\mathbb{S}(t)z(0)$ defined on $\mathbb{R}$ and bounded. Hence,

$$\mathbb{S}(t)z(0) \in \mathcal{A} \quad \text{for all } t \in \mathbb{R}$$

and in particular $z(0) = z_0 \in \mathcal{A}$. Finally, by repeating the arguments in Lemma 4.4, we prove (6.2) as follows. From (6.4), we see that there exists $z_0 \in \mathcal{A}$ such that up to a subsequence,

$$z_{0,m} \to z_0 \quad \text{weakly in } H_\Omega,$$

and the weakly lower-semicontinuity property of the H_Ω norm gives

$$\liminf_{m\to\infty} |z_{0,m}| \ge |z_0|. \tag{6.8}$$

From (4.12), we also have that

$$\begin{aligned}|z_m(t)|^2 &= e^{-\frac{\delta\lambda}{2}(t-\tau)}|z_m(\tau)|^2\\ &\quad + 2\int_\tau^t e^{-\frac{\delta\lambda}{2}(t-s)}\Big[\langle\Phi, z(s)\rangle - (\alpha|u(s)|^2 + \beta\|u(s)\|^{r+1}_{\tilde{L}^{r+1}} + [z(s)]^2)\Big]ds,\end{aligned}$$

and the fact that $z_{0,m} = \mathbb{S}_m(T)\mathbb{S}_m(-T)z_{0,m}$ for each $T \in \mathbb{R}$, thus

$$\begin{aligned}&|z_{0,m}|^2\\ &= |\mathbb{S}_m(T)\mathbb{S}_m(-T)z_{0,m}|^2\\ &= e^{-\frac{\delta\lambda}{2}(T-T_0)}|\mathbb{S}_m(T_0)\mathbb{S}_m(-T)z_{0,m}|^2 + 2\int_{T_0}^T e^{-\frac{\delta\lambda}{2}(T-s)}\langle\Phi, \mathbb{S}_m(s)\mathbb{S}_m(-T)z_{0,m}\rangle ds\\ &\quad - 2\int_{T_0}^T e^{-\frac{\delta\lambda}{2}(T-s)}\Big[\alpha|S_{1m}(s)S_{1m}(-T)u_{0,m}|^2 + \beta\|S_{1m}(s)S_{1m}(-T)u_{0,m}\|^{r+1}_{\tilde{L}^{r+1}}\Big]ds\\ &\quad - 2\int_{T_0}^T e^{-\frac{\delta\lambda}{2}(T-s)}[\mathbb{S}_m(s)\mathbb{S}_m(-T)z_{0,m}]^2 ds.\end{aligned}$$

Since $z_{0,m} \in \mathcal{A}_m$, the solution $\mathbb{S}_m(t)z_{0,m}$ of the system (6.1) is bounded on $\mathbb{R}$ and $\mathbb{S}_m(t)z_{0,m} \in \mathcal{A}_m$ for all $t \in \mathbb{R}$. Hence, for each $T \ge 0$, we deduce that $\mathbb{S}_m(-T)z_{0,m} \in \mathcal{A}_m$. Thus, from (6.4), there exists $z_T \in \mathcal{A}$ such that, up to a subsequence,

$$\mathbb{S}_m(t)\mathbb{S}_m(-T)z_{0,m} \to z_T \quad \text{weakly in } H_\Omega, \text{ for all } t \in \mathbb{R}.$$

By a calculation similar to (4.12), we obtain

$$\limsup_{m\to\infty} |z_{0,m}|^2 \le |z_0|^2 + (\rho^2 - |\mathbb{S}(T^*)z_T|^2)e^{-\frac{\delta\lambda}{2}(T-T^*)} \le |z_0|^2 + \rho^2 e^{-\frac{\delta\lambda}{2}(T-T^*)} \tag{6.9}$$

for all $T > T^*$. Since H_Ω is a Hilbert space, passing T to ∞ in (6.9) and using (6.8) to obtain (6.2). The proof is complete. □

We now state and prove the main result in this section as follows.

Theorem 6.2. *Assume that $\Phi \in H$. Let $\mathcal{A}$ and $\mathcal{A}_m$ be the global attractors corresponding to the systems* (3.1) *and* (6.1)*, respectively. Then*

$$\lim_{m\to\infty} \operatorname{dist}_{H_\Omega}(\mathcal{A}_m, \mathcal{A}) = 0, \tag{6.10}$$

where $\operatorname{dist}_{H_\Omega}(\mathcal{A}_m,\mathcal{A}) = \sup_{z\in\mathcal{A}_m}\operatorname{dist}_{H_\Omega}(z,\mathcal{A})$ *is the Hausdorff semidistance of space* H_Ω.

Proof. We use contradiction to prove this Theorem. Let us assume that (6.10) does not hold. Then there exists a fixed $\varepsilon_0>0$ and a sequence $z_m\in\mathcal{A}_m$ such that

$$\operatorname{dist}_{H_\Omega}(z_m,\mathcal{A})\ge\varepsilon_0>0,\quad m=1,2,\dots. \tag{6.11}$$

While by Lemma 6.1, we find that there exists a subsequence $\{z_{m_k}\}_k\subset\{z_m\}_m$ such that

$$\lim_{k\to\infty}\operatorname{dist}_{H_\Omega}(z_{m_k},\mathcal{A})=0,$$

which is a contradiction to (6.11) and completes the proof. $\square$

7. Existence and exponential stability of a stationary solution

In this section, we study the existence, uniqueness and exponential stability of stationary solutions of problem (1.1) with the external force $\Phi\in\mathbb{L}^2(\Omega)\times\mathbb{L}^2(\Omega)\times L^2(\Omega)$. We first consider the following steady state system associated with the equation (1.1).

$$\begin{gathered}
\begin{aligned}
&-R_e^{-1}\Delta u+(u\cdot\nabla)u-S(B\cdot\nabla)B+\alpha u+\beta|u|^{r-1}u+\nabla\Big(p+\frac{S}{2}|B|^2\Big)\\
&=\theta e_2+f(x),\quad\text{in }\Omega,
\end{aligned}\\
R_m^{-1}\nabla^\perp(\operatorname{curl}B)+(u\cdot\nabla)B-(B\cdot\nabla)u=\Psi(x),\quad\text{in }\Omega,\\
-\kappa\Delta\theta+(u\cdot\nabla)\theta=u_2+h(x),\quad\text{in }\Omega,\\
\nabla\cdot u=\nabla\cdot B=0,\quad\text{in }\Omega,\\
u=\mathbf{0},\quad B\cdot n=0,\quad\operatorname{curl}B=0,\quad\theta=0\quad\text{on }\partial\Omega.
\end{gathered} \tag{7.1}$$

Taking the inner product of the first equation of (7.1) with $\tilde u\in V_1$, we obtain

$$R_e^{-1}((u,\tilde u))_1+b(u,u,\tilde u)-Sb(B,B,\tilde u)+\alpha(u,\tilde u)+\beta\langle\mathcal{C}_r(u),\tilde u\rangle-(\theta e_2,\tilde u)=\langle f,\tilde u\rangle_{V_1',V_1}.$$

We take the inner product of the second equation of (7.1) with $S\tilde B$ ($\tilde B\in V_2$) to obtain

$$SR_m^{-1}((B,\tilde B))_2+Sb(u,B,\tilde B)-Sb(B,u,\tilde B)=S\langle\Psi,\tilde B\rangle_{V_2',V_2}.$$

And taking the inner product of the third equation of (7.1) with $\gamma\tilde\theta\in V_3$, we obtain

$$\gamma\kappa((\theta,\tilde\theta))_3+\gamma\bar b(u,\theta,\tilde\theta)-\gamma(u_2,\tilde\theta)=\gamma\langle h,\tilde\theta\rangle_{V_3',V_3}.$$

Then we give the following weak formulation of problem (7.1).

Definition 7.1. A triple $(u^*,B^*,\theta^*):=z^*\in V$ with $u^*\in\tilde L^{r+1}$ is called a weak solution of the system (7.1) if

$$a(z^*,\tilde z)+\sigma(z^*,\tilde z)+\mathbb{B}(z^*,z^*,\tilde z)+\alpha(u^*,\tilde u)+\beta\langle\mathcal{C}_r(u^*),\tilde u\rangle=(\Phi,\tilde z),$$

for all $\tilde z=(\tilde u,\tilde B,\tilde\theta)\in V$.

Theorem 7.2. *Suppose that* $\frac{|\Phi|}{\delta^2\lambda}<\frac{1}{4c_b}$ *(where* δ*,* c_b *and* λ *are in Lemmas 2.1, 2.3 and 4.3, respectively). Then, there exists a unique weak stationary solution of system* (1.1).

Proof. Let $\{w_j = (u_{w_j}, B_{w_j}, \theta_{w_j})\}_j$ be a Hilbert basis of V such that $V_m = \operatorname{span}\{w_j\}_j$ is dense in $V_1 \cap \tilde{L}^{r+1} \times V_2 \times V_3$. For each integer $m \geq 1$, we find the approximate stationary solution in the form

$$z_m(t) = \sum_{j=1}^{m} \xi_{mj}(t) w_j$$

where

$$\begin{aligned} &a(z_m(t), \tilde{z}) + \sigma(z_m(t), \tilde{z}) + \mathbb{B}(z_m(t), z_m(t), \tilde{z}) + \alpha(u_m(t), \tilde{u}) \\ &+ \beta\langle \mathcal{C}_r(u_m(t)), \tilde{u}\rangle = (\Phi, \tilde{z}) \end{aligned} \tag{7.2}$$

for all $\tilde{z} \in V_m$. We apply Lemma 2.5 to prove the existence of z_m as follows. Let $R_m : V_m \to V_m$ be defined by

$$((R_m z, \tilde{z})) = a(z, \tilde{z}) + \sigma(z, \tilde{z}) + \mathbb{B}(z, z, \tilde{z}) + \alpha(u, \tilde{u}) + \beta\langle \mathcal{C}_r(u), \tilde{u}\rangle - (\Phi, \tilde{z})$$

for all $z, \tilde{z} \in V_m$. For all $z \in V_m$, by using (2.2), Lemma 2.1 and (2.10), we have

$$\begin{aligned} ((R_m z, z)) &= a(z, z) + \sigma(z, z) + \alpha|u|^2 + \beta\|u\|_{\tilde{L}^{r+1}}^{r+1} - (\Phi, z) \\ &\geq \frac{\delta}{2}\|z\|^2 + \alpha|u|^2 + \beta\|u\|_{\tilde{L}^{r+1}}^{r+1} - \frac{1}{\lambda^{1/2}}\|z\||\Phi| \\ &\geq \frac{\delta}{2}\|z\|^2 - \frac{1}{\lambda^{1/2}}\|z\||\Phi|. \end{aligned}$$

Thus, if we take $k = 2\delta^{-1}\lambda^{-1/2}|\Phi|$, then $((R_m z, z)) \geq 0$ for all $z \in V_m$ satisfying $\|z\| = k$. Thus, there exists a solution $z_m \in V_m$ with $R_m(z_m) = 0$. From (7.2), we replace $\tilde{z}$ with z_m to obtain

$$\|z_m\| \leq 2\delta^{-1}\lambda^{-1/2}|\Phi|, \tag{7.3}$$

hence we can extract a subsequence of z_m (still denoted by z_m) such that $z_m \to z^*$ weakly in V. Moreover, applying the Aubin-Lions lemma (see [30]), we can conclude that

$$|u|^{r-1}u \to |u^*|^{r-1}u^* \quad \text{weakly in } \tilde{L}^{\frac{r+1}{r}}.$$

Therefore, we have that z^* is a weak stationary solution to problem (1.1). Now let z_1^* and z_2^* be two stationary solutions to problem (1.1) and set $y^* = z_1^* - z_2^*$, then by using the same arguments as in proving the uniqueness of solution in Theorem 3.1 and (7.3), we deduce that

$$\frac{\delta}{2}\|y^*\|^2 \leq \mathbb{B}(z_2^*, z_2^*, y^*) - \mathbb{B}(z_1^*, z_1^*, y^*) \leq \frac{c_b}{\lambda^{1/2}}\|z_1^*\|\|y^*\|^2 \leq 2(\delta\lambda)^{-1}c_b|\Phi|\|y^*\|^2.$$

Thus, we obtain

$$\Big(\delta - \frac{4c_b}{\delta\lambda}|\Phi|\Big)\|y^*\|^2 \leq 0$$

and obtain the uniqueness of stationary solutions. $\square$

Theorem 7.3. *Assume that the assumptions of Theorem 7.2 hold. Then the unique stationary solution z^* of problem* (1.1) *is exponentially stable.*

Proof. Notice that we can write the solution $z(t)$ to problem (1.1) in the form $z(t) = z^* + y(t)$, by repeating some arguments as in proving the uniqueness of solution in Theorem 3.1 and (7.3), we deduce that

$$\frac{1}{2}\frac{d}{dt}|y(t)|^2 + \frac{\delta}{2}\|y(t)\|^2 \leq c_b|y(t)|\|z^*\|\|y(t)\| \leq \frac{2c_b}{\delta\lambda}|\Phi|\|y(t)\|^2.$$

Then

$$\frac{d}{dt}|y(t)|^2 + \Big(\delta - \frac{4c_b}{\delta\lambda}|\Phi|\Big)\|y(t)\|^2 \le 0 \quad \text{or} \quad \frac{d}{dt}|y(t)|^2 + \vartheta|y(t)|^2 \le 0$$

where $\vartheta := \lambda^{-1}\big(\delta - \frac{4c_b}{\delta\lambda}|\Phi|\big) > 0$ and this yields

$$|y(t)|^2 = |z(t) - z^*| \le |z(0) - z^*|e^{-\vartheta t}.$$

The proof is complete. □

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