Differential Equations and Computational Simulations III
J. Graef, R. Shivaji, B. Soni J. & Zhu (Editors)
Electronic Journal of Differential Equations, Conference 01, 1997, pp. 81–95.
ISSN: 1072-6691. URL: http://ejde.math.swt.edu or http://ejde.math.unt.edu
ftp 147.26.103.110 or 129.120.3.113 (login: ftp)

Quadratic Convergence of Approximate Solutions of Two-Point Boundary Value Problems with Impulse *

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Abstract

The method of quasilinearization, coupled with the method of upper and lower solutions, is applied to a boundary value problem for an ordinary differential equation with impulse that has a unique solution. The method generates sequences of approximate solutions which converge monotonically and quadratically to the unique solution. In this work, we allow nonlinear terms with respect to velocity; in particular, Nagumo conditions are employed.

1 Introduction

Let $0 = t_0 < t_1 < \ldots < t_m < t_{m+1} = 1$ be given. In this paper, we shall apply the method of quasilinearization to the two-point conjugate boundary value problem (BVP) with impulse,

$$x''(t) = f(t, x(t), x'(t)), \quad t_k < t < t_{k+1}, \quad k = 0, \ldots, m, \tag{1}$$

$$x(0) = a, \quad x(1) = b, \tag{2}$$

and for $k = 1, \ldots, m$,

$$\begin{aligned} \Delta x(t_k) &= u_k \\ \Delta x'(t_k) &= v_k(x(t_k), x'(t_k)), \end{aligned} \tag{3}$$

where $f : [0,1] \times \mathbb{R}^2 \to \mathbb{R}$ is continuous, $u_k \in \mathbb{R}$, $v_k : \mathbb{R}^2 \to \mathbb{R}$ is continuous, $k = 1, \ldots, m$. Define the impulse, $\Delta x(t_k) = x(t_k^+) - x(t_k^-)$, and by convention, let $x(t_k) = x(t_k^-)$, $k = 1, \ldots, m$. We shall employ the method of upper and lower solutions and the method of quasilinearization to obtain a bilateral iteration scheme in which the iterates converge quadratically to the unique solution of the BVP with impulse, (1), (2), (3).

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*1991 *Mathematics Subject Classifications:* 34A37, 34B15.
Key words and phrases: Quasilinearization, boundary value problem with impulse, quadratic convergence, Nagumo conditions.

Published November 12, 1998.

The method of quasilinearization is described by Bellman [4, 5], and has recently been generalized by Lakshmikantham, Leela and various co-authors to apply to a wide variety of problems. See, for example, [14, 15], and references therein. The method generates sequences of approximate solutions which converge monotonically and quadratically to the problem of interest. Recently, Vatsala, et. al., [16], [17, 18], have applied the method of quasilinearization to families of two-point BVPs related to (1), in the case that f is independent of x', and the boundary conditions are more general than (2).

More recently, Eloe and Zhang [9] extended the work of Vatsala, et. al. [16, 17, 18] to the BVP, (1), (2), in the case where f depends on x'. As pointed out in [9], Knobloch [13] and Jackson and Schrader [12] have obtained conditions such that there exists a sequence of solutions of (1) converging monotonically and in $C^1[0,1]$ to a solution of the BVP, (1), (2). Neither Knoblach [13] nor Jackson and Schrader [12] considered the rate of convergence.

Also, recently, Devi, Chandrakala and Vatsala [6] applied the method of quasilinearization to initial value problems for scalar ordinary differential equations with impulse. Doddaballapur and Eloe [7] have extended the work of Vatsala, et. al. [16, 17], [18] to the BVP with impulse, (1), (2), (3), in the case that f and each v_k are independent of x'. Thus, the primary contribution of this paper then is that we extend the work in [16], [17], [18], [9] and [7] to the BVP with impulse, (1), (2), (3), when f and each v_k depend on x'.

The paper is organized in the following manner. We shall obtain a preliminary result in Theorem 1 concerning the properties of upper and lower solutions of the BVP with impulse, (1), (2), (3). In Theorem 2, we shall obtain a fundamental existence of solutions result for the BVP with impulse, (1), (2), (3). The proof of this result employs the Schauder fixed point theorem. Due to the dependence on x', technical difficulties arise which require the assumption of Nagumo type conditions and extensions of the Kamke convergence theorem [10, 11]. In Theorem 3, we shall state a uniqueness of solutions result for the BVP with impulse, (1), (2), (3). We shall state and prove the main result of this paper in Theorem 4. The proof of Theorem 4 employs a clever manipulation of Theorems 1 and 2. The iterative details in the proof of Theorem 4 are completely analogous to those found in [7, 9, 16, 17, 9] once Theorems 1 and 2 are obtained. Hence, we consider these details to be standard and only highlight those details in the proof of Theorem 4 that are particular to the BVP with impulse, (1), (2), (3).

2 Results

We begin with the definition of an appropriate Banach space, B. Let $PC[0,1]$ denote the piecewise continuous functions on $[0,1]$ and let $PC^1[0,1]$ denote the functions, x, such that $x \in PC[0,1]$ and $x' \in PC[0,1]$. Define

$$B = \{x \in PC^1[0,1] : x^{(i)}|_{[t_k,\ t_{k+1}]} \in C^i[t_k, t_{k+1}],\ k = 0, \ldots, m, i = 0, 1\},$$

with $\|x\|_B = \max_{k=0,\dots,m} \|x\|_k$ and $\|x\|_k = \max_{i=0,1} \sup_{t_k \le t \le t_{k+1}} |x^{(i)}(t)|$. We shall say that $\alpha \in B$ is a lower solution of the BVP with impulse, (1), (2), (3), if

$$\alpha''(t) \ge f(t, \alpha(t), \alpha'(t)), \quad t_k < t < t_{k+1}, \quad k = 0, \dots, m,$$
$$\alpha(0) \le a, \quad \alpha(1) \le b,$$

and for $k = 1, \dots, m$,

$$\Delta\alpha(t_k) = u_k$$
$$\Delta\alpha'(t_k) \ge v_k(\alpha(t_k), \alpha'(t_k))\,.$$

We shall say that $\beta \in B$ is an upper solution of the BVP with impulse, (1), (2), (3), if

$$\beta''(t) \le f(t, \beta(t), \beta'(t)), \quad t_k < t < t_{k+1}, \quad k = 0, \dots, m,$$
$$\beta(0) \ge a, \quad \beta(1) \ge b\,,$$

and for $k = 1, \dots, m$,

$$\Delta\beta(t_k) = u_k$$
$$\Delta\beta'(t_k) \le v_k(\beta(t_k), \beta'(t_k))\,.$$

For the remainder of this paper, we shall assume that

$$f \in C([0,1] \times \mathbb{R}^2), \quad (\partial f/\partial x) = f_x \in C([0,1] \times \mathbb{R}^2), \tag{4}$$
$$f_x(t, x, y) > 0, \ (t, x, y) \in [0,1] \times \mathbb{R}^2, \tag{5}$$
$$v_k \in C^1(\mathbb{R}^2), \tag{6}$$

and for $k = 1, \dots, m$,

$$v_{kx}(x, y) > 0, \ (x, y) \in \mathbb{R}^2, \quad v_{ky}(x, y) > 0, \ (x, y) \in \mathbb{R}^2. \tag{7}$$

In order to obtain Theorem 2, we shall define an appropriate fixed point operator, T. For $x \in B$, define an operator T on x by

$$Tx(t) = p(t) + I(t, x) + \int_0^1 G(t, s) f(s, x(s), x'(s))\, ds, \tag{8}$$

where $p(t) = a(1-t) + bt$, $I(t, x) = \sum_{k=1}^m I_k(t, x)$. For $k = 1, \dots, m$, let

$$I_k(t, x) = \begin{cases} t(-u_k - (1 - t_k) v_k(x(t_k), x'(t_k))) & , 0 \le t \le t_k, \\ (1 - t)(u_k - t_k v_k(x(t_k), x'(t_k))) & , t_k \le t \le 1\,. \end{cases}$$

Let

$$G(t, s) = \begin{cases} t(s-1) & , 0 \le t < s \le 1, \\ s(t-1) & , 0 \le s < t \le 1, \end{cases}$$

denote the Green's function for the BVP, $x''(t) = 0$, $0 \le t \le 1$, $x(0) = 0$, $x(1) = 0$. Eloe and Henderson [8] have argued that x is a solution of the BVP with impulse, (1), (2), (3), if, and only if, $x \in B$ and $Tx = x$. Finally, we shall define a partial order on B as follows: for α, $\beta \in B$, we say that $\alpha \le \beta$ if, and only if,

$$\alpha|_{[t_k,t_{k+1}]}(t) \le \beta|_{[t_k,t_{k+1}]}(t), t_k \le t \le t_{k+1}, k = 0, \dots, m.$$

Theorem 1 *Assume (4), (5), (6), and (7) hold. Let α, β be lower and upper solutions of the BVP with impulse, (1), (2), (3), respectively. Then $\alpha \le \beta$.*

Proof. Set $w(t) = \alpha(t) - \beta(t)$ and note that w is continuous on $[0, 1]$ by (3). Assume, for the sake of contradiction, that w is positive on $[0, 1]$. Since $w(0) \le 0$, $w(1) \le 0$, w has a positive maximum at some $\tau \in (0, 1)$. Assume $\tau \in \cup_{k=0}^{m}(t_k, t_{k+1})$. Then $w''(\tau) \le 0$ and $\alpha'(\tau) = \beta'(\tau)$. However, employing that α and β are lower and upper solutions of the BVP with impulse, (1), (2), (3), respectively, and employing (5), it follows that

$$w''(\tau) = \alpha''(\tau) - \beta''(\tau) \ge f(\tau, \alpha(\tau), \alpha'(\tau)) - f(\tau, \beta(\tau), \beta'(\tau)) > 0.$$

This provides a contradiction and so, $\tau \notin \cup_{k=0}^{m}(t_k, t_{k+1})$. Now, assume that $\tau = t_k$ for some $k \in \{1, \dots, m\}$. By Taylor's theorem, $w'(t_k^-) \ge 0$ and $w'(t_k^+) \le 0$, or $\Delta w'(t_k) \le 0$ and

$$\alpha'(t_k^-) = \alpha'(t_k) \ge \beta'(t_k) = \beta'(t_k^-).$$

But

$$\Delta w'(t_k) = \Delta\alpha'(t_k) - \Delta\beta'(t_k) \ge v_k(\alpha(t_k), \alpha'(t_k)) - v_k(\beta(t_k), \beta'(t_k)) > 0$$

by (7). Thus, $\tau \notin \{t_1, \dots t_m\}$, and $w(t) \le 0$, $0 \le t \le 1$.

Theorem 2 *Assume $g \in C([0, 1] \times \mathbb{R}^2)$, $z_k \in C(\mathbb{R}^2)$, $k = 1, \dots, m$, and assume that each $z_k(x, y)$ is monotone increasing in y for fixed x. Assume that each solution of $x''(t) = g(t, x(t), x'(t))$ extends to $[0, 1]$, or becomes unbounded on its maximal interval of convergence. Let α, β be lower and upper solutions of the BVP,*

$$x''(t) = g(t, x(t), x'(t)), \quad t_k < t < t_{k+1}, \tag{9}$$

$$\begin{aligned} \Delta x(t_k) &= u_k \\ \Delta x'(t_k) &= z_k(x(t_k), x'(t_k)), \end{aligned} \tag{10}$$

with $k = 1, \dots, m$ and boundary conditions given by (2), respectively, such that

$$\alpha \le \beta.$$

Then, there exists a solution, x, of the BVP with impulse, (9), (2), (10), satisfying

$$\alpha \le x \le \beta.$$

Proof. Define

$$\hat{f}(t,x,y)=\begin{cases} g(t,\beta(t),y)+(x-\beta(t))/[1+(x-\beta(t))], & x>\beta(t),\\ g(t,x,y), & \alpha(t)\le x\le\beta(t),\\ g(t,\alpha(t),y)+(x-\alpha(t))/[1+|x-\alpha(t)|], & x<\alpha(t),\end{cases}$$

and for $k=1,\dots,m$, define

$$\hat{v}_k(x,y)=\begin{cases} z_k(\beta(t_k),y)+(x-\beta(t_k))/[1+(x-\beta(t_k))], & x>\beta(t_k),\\ z_k(x,y), & \alpha(t_k)\le x\le\beta(t_k),\\ z_k(\alpha(t_k),y)+(x-\alpha(t_k))/[1+|x-\alpha(t_k)|], & x<\alpha(t_k).\end{cases}$$

Let $N>0$ be such that $|\alpha'(t)|\le N$, $|\beta'(t)|\le N$, $t\in[t_k,t_{k+1}]$, $k=0,\dots,m$. For each positive integer, l, define

$$f_l(t,x,y)=\begin{cases} \hat{f}(t,x,N+l), & y>N+l,\\ \hat{f}(t,x,y), & |y|\le N+l,\\ \hat{f}(t,x,-(N+l)), & y<-(N+l),\end{cases}$$

and

$$v_{kl}(t,x,y)=\begin{cases} \hat{v}_k(x,N+l), & y>N+l,\\ \hat{v}_k(x,y), & |y|\le N+l,\\ \hat{v}_k(x,-(N+l)), & y<-(N+l).\end{cases}$$

Notice that f_l and each v_{kl} are bounded and continuous. With a standard application of the Schauder fixed point theorem to the operator T, defined by (8), one obtains a solution, $x_l\in B$, to the BVP with impulse, (1), (2), (3), with $f=f_l$ and each $v_k=v_{kl}$ bounded and continuous.

We now argue that each solution, x_l, satisfies $\alpha\le x_l\le\beta$. We shall show that $x_l\le\beta$. As in the proof of Theorem 1, assume for the sake of contradiction that $x_l-\beta$ has a positive maximum at τ. As in the proof of Theorem 1, $\tau\in(0,1)$. If $\tau\in\cup_{k=0}^{m}(t_k,t_{k+1})$, then $x_l''(\tau)\le\beta''(\tau)$, and $|x_l'(\tau)|=|\beta'(\tau)|\le N<N+l$. Thus,

$$(x_l-\beta)''(\tau)\ge(x_l-\beta)(\tau)/[1+(x-\beta)(\tau)]>0,$$

which is a contradiction. If $\tau=t_k$, for some $k\in\{1,\dots,m\}$, then $x_l'(t_k)\ge\beta'(t_k)$. Since each $z_k(x,y)$ is monotone increasing in y for fixed x, it follows that each $v_{kl}(x,y)$ is monotone increasing in y for fixed x. Moreover, note that $v_{kl}(\beta(t_k),\beta'(t_k))=z_k(\beta(t_k),\beta'(t_k))$. Thus,

$$\begin{aligned}\Delta(x_l-\beta)'(t_k) &\ge v_{kl}(\beta(t_k),x_l'(t_k))-v_{kl}(\beta(t_k),\beta'(t_k))\\ &\quad+(x_l-\beta)(t_k)/[1+(x_l-\beta(t_k))]\\ &\ge (x_l-\beta)(t_k)/[1+(x_l-\beta(t_k))]>0\end{aligned}$$

which is also a contradiction. Therefore, $x_l\le\beta$. To show that $\alpha\le x_l$ we follow a similar procedure.

For each l there exists $t_l\in[0,t_1]$ such that

$$t_1|x_{kl}^{'}(t_l)|=|x_{kl}(t_1)-a|\le\max\{|\beta(0)-\alpha(t_1)|,|\beta(t_1)-\alpha(0)|\}.$$

Thus, each of the sequences $\{x_{kl}(t_l)\}$ and $\{x'_{kl}(t_l)\}$ are bounded. One can now apply the Kamke convergence theorem (see [11]) for solutions of initial value problems and obtain a subsequence of $\{x_{kl}\}$ which converges to a solution of $x''(t) = \hat{f}(t, x(t), x'(t))$ on a maximal subinterval of $[0, t_1]$. Clearly, $\alpha(t) \leq x(t) \leq \beta(t)$ and solutions of $x''(t) = g(t, x(t), x'(t))$ extend to all of $[0, 1]$ or become unbounded; thus, $x''(t) = \hat{f}(t, x(t), x'(t))$ on $[0, t_1]$.

Now, apply the impulse defined by (10) at t_1. Apply the Kamke theorem to the subsequence that was extracted in the preceding paragraph. Because of (10) one can employ $t_1 = t_l$ for each l. Thus, one obtains a further subsequence which converges to a solution, x, of $x''(t) = \hat{f}(t, x(t), x'(t))$ on $(0, t_1) \cup (t_1, t_2)$ such that x satisfies (10) at t_1.

Continue inductively, first applying (10) at each t_j and then applying the Kamke convergence theorem on that subinterval (t_j, t_{j+1}). Finally, since $\alpha \leq x \leq \beta$, $\hat{f}(t, x(t), x'(t)) = f(t, x(t), x'(t))$ and the proof of Theorem 2 is complete.

Remark. For simplicity, we can assume that g satisfies a Nagumo condition in x' ([10], [11]). That is, assume that for each $M > 0$ there exists a positive continuous function, $h_M(s)$, defined on $[0, \infty)$ such that

$$|g(t, x, x')| \leq h_M(|x'|)$$

for all $(t, x, x') \in [0, 1] \times [-M, M] \times \mathbb{R}$ and such that

$$\int_0^\infty (s/h_M(s))ds = +\infty.$$

The assumption that g satisfies a Nagumo condition implies that each solution of the differential equation, $x''(t) = g(t, x(t), x'(t))$, either extends to $[0, 1]$ or becomes unbounded on its maximal interval of existence ([10], [11]). In our main result, Theorem 4, g will represent a modification of f. Thus, we shall assume in Theorem 4 that f satisfies a Nagumo condition in x'.

Theorem 3 *Assume that (4), (5), (6), and (7) hold. Then, solutions of the BVP with impulse, (1), (2), (3), are unique.*

Proof. The uniqueness of solutions result follows immediately from Theorem 1 and the observation that solutions are respectively upper and lower solutions.

Theorem 4 *Assume that (4), (5), (6), and (7) hold, and assume that*

$$(\partial^2/\partial x^2)f \in C([0, 1] \times \mathbb{R}^2), v_k'' \in C(\mathbb{R}^2), k = 1, \ldots, m\,.$$

Assume that f satisfies a Nagumo condition in x'. Assume that α_0 and β_0 are lower and upper solutions of the BVP with impulse, (1), (2), (3), respectively. Then there exist monotone sequences, $\{\alpha_n(t)\}$ and $\{\beta_n(t)\}$, which converge in B to the unique solution, $x(t)$, of the BVP with impulse, (1), (2), (3), and the convergence is quadratic.

Proof. Let $F(t,x):[0,1]\times\mathbb{R}\to\mathbb{R}$ be such that F, F_x, F_{xx} are continuous on $[0,1]\times\mathbb{R}$ and

$$F_{xx}(t,x)\geq 0, (t,x)\in[0,1]\times\mathbb{R}\,. \tag{11}$$

Set $\phi_1(t,x_1,x_2)=F(t,x_1)-f(t,x_1,x_2)$ on $[0,1]\times\mathbb{R}^2$. From (11) it follows that, if $x_1, y_1\in\mathbb{R}$, then $F(t,x_1)\geq F(t,y_1)+F_x(t,y_1)(x_1-y_1)$. In particular, for $x_1,y_1,x_2,y_2\in\mathbb{R}$,

$$f(t,x_1,x_2)\geq f(t,y_1,y_2)+F_x(t,y_1)(x_1-y_1)-\phi_1(t,x_1,x_2)+\phi_1(t,y_1,y_2). \tag{12}$$

For each $k=1,\ldots,m$, let $V_k(x):\mathbb{R}\to\mathbb{R}$ be such that V_k, V_k', V_k'' are continuous on $\mathbb{R}$ and

$$V_k''(x)\geq 0,\quad x\in\mathbb{R}\,. \tag{13}$$

Set $\phi_{2k}(x_1,x_2)=V_k(x_1)-v_k(x_1,x_2)$ on $\mathbb{R}^2$. From (13) it follows that, if $x_1,y_1\in\mathbb{R}$, then $V_k(x_1)\geq V_k(y_1)+V_k'(y_1)(x_1-y_1)$. In particular, for $x_1,y_1,x_2,y_2\in\mathbb{R}$,

$$v_k(x_1,x_2)\geq v_k(y_1,y_2)+V_k'(y_1)(x_1-y_1)-(\phi_{2k}(x_1,x_2)-\phi_{2k}(y_1,y_2)). \tag{14}$$

Define

$$\begin{aligned}
g(t,x_1,x_2;\alpha_0,\beta_0,\alpha_0') &= f(t,\alpha_0(t),\alpha_0'(t))+F_x(t,\beta_0(t))(x_1-\alpha_0(t))\\
&\quad -\phi_1(t,x_1,x_2)+\phi_1(t,\alpha_0(t),\alpha_0'(t))\,,\\
G(t,x_1,x_2;\beta_0,\beta_0') &= f(t,\beta_0(t),\beta_0'(t))+F_x(t,\beta_0(t))(x_1-\beta_0(t))\\
&\quad -\phi_1(t,x_1,x_2)+\phi_1(t,\beta_0(t),\beta_0'(t))\,,\\
h_k(x_1,x_2;\alpha_0,\beta_0,\alpha_0') &= v_k(\alpha_0(t_k),\alpha_0'(t_k))+V_k'(\beta_0(t_k))(x_1-\alpha_0(t_k))\\
&\quad -(\phi_{2k}(x_1,x_2)-\phi_{2k}(\alpha_0(t_k),\alpha_0'(t_k)))\,,\\
H_k(x_1,x_2;\beta_0,\beta_0') &= v_k(\beta_0(t_k),\beta_0'(t_k))+V_k'(\beta_0(t_k))(x_1-\beta_0(t_k))\\
&\quad -(\phi_{2k}(x_1,x_2)-\phi_{2k}(\beta_0(t_k),\beta_0'(t_k)))\,.
\end{aligned}$$

First consider the BVP with impulse,

$$x''(t)=g(t,x(t),x'(t);\alpha_0,\beta_0,\alpha_0'),\ t_k<t<t_{k+1},\ k=0,\ldots,m, \tag{15}$$

and for $k=1,\ldots,m$,

$$\begin{gathered}\Delta x(t_k)=u_k\\ \Delta x'(t_k)=h_k(x(t_k),x'(t_k);\alpha_0,\beta_0,\alpha_0')\,,\end{gathered} \tag{16}$$

with boundary conditions given by (2). Each h_k readily satisfies the hypotheses of Theorem 2. A limit comparison implies that g satisfies a Nagumo condition in x'.

We now show that α_0 and β_0 are lower and upper solutions, respectively, of the BVP with impulse, (15), (2), (16); thus, by Theorem 2, there exists a solution $\alpha_1(t)$ of the BVP with impulse, (15), (2), (16), satisfying

$$\alpha_0\leq\alpha_1\leq\beta_0.$$

To this end, note that for $t_k < t < t_{k+1}$, $k = 0, \ldots, m$,

$$\alpha_0''(t) \geq f(t, \alpha_0(t), \alpha_0'(t)) = g(t, \alpha_0(t), \alpha_0'(t); \alpha_0, \beta_0, \alpha_0'),$$

and, for $k = 1, \ldots, m$,

$$\Delta\alpha_0'(t_k) \geq v_k(\alpha_0(t_k), \alpha_0'(t_k)) = h_k(\alpha_0(t_k), \alpha_0'(t_k); \alpha_0, \beta_0, \alpha_0').$$

Moreover, from (12) and (14), it follows that for $t_k < t < t_{k+1}$, $k = 0, \ldots, m$,

$$\begin{aligned} \beta_0''(t) \leq f(t, \beta_0(t), \beta_0'(t)) &\leq f(t, \alpha_0(t), \alpha_0'(t)) - F_x(t, \beta_0(t))(\alpha_0(t) - \beta_0(t)) \\ &\quad + \phi_1(t, \alpha_0(t), \alpha_0'(t)) - \phi_1(t, \beta_0(t), \beta_0'(t)) \\ &= g(t, \beta_0(t), \beta_0'(t); \alpha_0, \beta_0, \alpha_0'), \end{aligned}$$

and for $k = 1, \ldots, m$,

$$\begin{aligned} \Delta\beta_0'(t_k) &\leq v_k(\beta_0(t_k), \beta_0'(t_k)) \\ &\leq v_k(\alpha_0(t), \alpha_0'(t)) - V_k'(\beta_0(t_k))(\alpha_0(t_k) - \beta_0(t_k)) \\ &\quad + (\phi_{2k}(\alpha_0(t_k), \alpha_0'(t)) - \phi_{2k}(\beta_0(t_k), \beta_0'(t_k)) \\ &= h_k(\beta_0(t_k), \beta_0'(t_k); \alpha_0, \beta_0, \beta_0'). \end{aligned}$$

Since α_0 and β_0 satisfy (2), α_0 and β_0 are lower and upper solutions, respectively, of the BVP with impulse, (15), (2), (16), and thus, by Theorem 2, there exists a solution $\alpha_1(t)$ of the BVP with impulse, (15), (2), (16), such that

$$\alpha_0 \leq \alpha_1 \leq \beta_0\,.$$

Now, consider the BVP with impulse,

$$\begin{aligned} x''(t) = G(t, x(t), x'(t); \beta_0, \beta_0'), \quad t_k < t < t_{k+1}, \quad k = 0, \ldots, m, \\ x(0) = a, \quad x(1) = b, \end{aligned} \tag{17}$$

and for $k = 1, \ldots, m$,

$$\begin{aligned} \Delta x(t_k) &= u_k \\ \Delta x'(t_k) &= H_k(x(t_k), x'(t_k); \alpha_0, \beta_0, \beta_0')\,. \end{aligned} \tag{18}$$

Again, G and each H_k satisfy the hypotheses of Theorem 2, and, again, (12) and (14) are employed to show that α_0 and β_0 are lower and upper solutions, respectively, of the BVP with impulse, (17), (2), (18); thus, there exists a solution $\beta_1(t)$ of the BVP with impulse, (17), (2), (18), such that

$$\alpha_0 \leq \beta_1 \leq \beta_0\,.$$

We now show that α_1 and β_1 are lower and upper solutions, respectively, of the BVP with impulse, (1), (2), (3). Thus, it will follow by Theorem 1 that

$$\alpha_0 \leq \alpha_1 \leq \beta_1 \leq \beta_0\,.$$

Employ (12) and (11) to see that for $t \in \cup_{k=0}^{m}(t_k, t_{k+1})$,

$$\begin{aligned} \alpha_1''(t) &= g(t, \alpha_1(t), \alpha_1'(t); \alpha_0, \beta_0, \alpha_0') \\ &= f(t, \alpha_0(t), \alpha_0'(t)) + F_x(t, \beta_0(t))(\alpha_1(t) - \alpha_0(t)) \\ &\quad -(\phi(t, \alpha_1(t), \alpha_1'(t)) - \phi(t, \alpha_0(t), \alpha_0'(t))) \\ &\geq f(t, \alpha_1(t), \alpha_1'(t)) + F_x(t, \alpha_1(t))(\alpha_0(t) - \alpha_1(t)) + \phi(t, \alpha_1(t), \alpha_1'(t)) \\ &\quad -\phi(t, \alpha_0(t), \alpha_0'(t)) + F_x(t, \beta_0(t))(\alpha_1(t) - \alpha_0(t)) \\ &\quad -(\phi(t, \alpha_1(t), \alpha_1'(t)) - \phi(t, \alpha_0(t), \alpha_0'(t))) \\ &= f(t, \alpha_1(t), \alpha_1'(t)) + (F_x(t, \beta_0(t)) - F_x(t, \alpha_1(t)))(\alpha_1(t) - \alpha_0(t)) \\ &\geq f(t, \alpha_1(t), \alpha_1'(t)). \end{aligned}$$

Similarly, for $k = 1, \ldots, m$, employ (14) and (13) to see that

$$\begin{aligned} \Delta\alpha_1'(t_k) &= h_k(\alpha_1(t_k), \alpha_1'(t_k)); \alpha_0, \beta_0, \alpha_0') \\ &= v_k(\alpha_0(t_k), \alpha_0'(t_k)) + V_k'(\beta_0(t_k))(\alpha_1(t_k) - \alpha_0(t_k)) \\ &\quad -(\phi_{2k}(\alpha_1(t_k), \alpha_1'(t_k)) - \phi_{2k}(\alpha_0(t_k), \alpha_0'(t_k))) \\ &\geq v_k(\alpha_1(t_k), \alpha_1'(t_k)) + V_k'(\alpha_1(t_k))(\alpha_0(t_k) - \alpha_1(t_k)) \\ &\quad +\phi_{2k}(\alpha_1(t_k), \alpha_1'(t_k)) - \phi_{2k}(\alpha_0(t_k), \alpha_0'(t_k)) \\ &\quad +V_k'(\beta_0(t_k))(\alpha_1(t_k) - \alpha_0(t_k)) - (\phi_{2k}(\alpha_1(t_k), \alpha_1'(t_k)) \\ &\quad -\phi_{2k}(\alpha_0(t_k), \alpha_0'(t_k))) \\ &= v_k(\alpha_1(t_k)) + (V_k'(\beta_0(t_k)) - V_k'(\alpha_1(t_k)))(\alpha_1(t_k) - \alpha_0(t_k)) \\ &\geq v_k(\alpha_1(t_k)). \end{aligned}$$

Similarly, it follows by (11)-(14) that for $t \in \cup_{k=0}^{m}(t_k, t_{k+1})$,

$$\beta_1''(t) \leq f(t, \beta_1(t), \beta_1'(t)),$$

and for $k \in \{1, \ldots, m\}$,

$$\Delta\beta_1'(t_k) \leq v_k(\beta_1(t_k), \beta_1'(t_k)).$$

In particular, α_1 and β_1 are lower and upper solutions, respectively, of the BVP with impulse, (1), (2), (3), and by Theorem 1,

$$\alpha_0 \leq \alpha_1 \leq \beta_1 \leq \beta_0.$$

Inductively, define sequences of functions $\{g_l\}$, $\{G_l\}$, $\{h_{kl}\}$, and $\{H_{kl}\}$ by

$$\begin{aligned} g_l(t, x_1, x_2) &= g(t, x_1, x_2; \alpha_l, \beta_l, \alpha_l') \\ &= f(t, \alpha_l(t), \alpha_l'(t)) + F_x(t, \beta_l(t))(x_1 - \alpha_l(t)) \\ &\quad -\phi_1(t, x_1, x_2) + \phi_1(t, \alpha_l(t), \alpha_l'(t)) \\ G_l(t, x_1, x_2) &= G(t, x_1, x_2; \beta_l, \beta_l') \\ &= f(t, \beta_l(t), \beta_l'(t)) + F_x(t, \beta_l(t))(x_1 - \beta_l(t)) \end{aligned}$$

$$
\begin{aligned}
& -\phi_1(t,x_1,x_2)+\phi_1(t,\beta_l(t),\beta_l'(t))\,,\\
h_{kl} &= h_k(x_1,x_2;\alpha_l,\beta_l,\alpha_l')\\
&= v_k(\alpha_l(t_k),\alpha_l'(t_k))+V_k'(\beta_l(t_k))(x_1-\alpha_l(t_k))\\
&\quad -(\phi_{2k}(x_1,x_2)-\phi_{2k}(\alpha_l(t_k)))\,,\\
H_{kl} &= H_k(x_1,x_2;\beta_l,\beta_l')\\
&= v_k(\beta_l(t_k),\beta_l'(t_k))+V_k'(\beta_l(t_k))(x_1-\beta_l(t_k))\\
&\quad -(\phi_{2k}(x_1,x_2)-\phi_{2k}(\beta_l(t_k),\beta_l'(t_k)))\,.
\end{aligned}
$$

Inductively, Theorem 2 implies there exists a solution $\alpha_{l+1}(t)$ of the BVP with impulse, (15), (2), (16), with $g=g_l$ and each $h_k=h_{kl}$ satisfying

$$\alpha_0\le\ldots\le\alpha_l\le\alpha_{l+1}\le\beta_l\le\ldots\le\beta_0\,.$$

Similarly, there exists a solution $\beta_{l+1}(t)$ of the BVP with impulse, (17), (2), (18), with $G=G_l$ and each $H_k=H_{kl}$ satisfying

$$\alpha_0\le\ldots\le\alpha_l\le\beta_{l+1}\le\beta_l\le\ldots\le\beta_0.$$

Finally, inductively, α_{l+1} and β_{l+1} are lower and upper solutions, respectively, of the BVP with impulse, (1), (2), (3), and by Theorem 1,

$$\alpha_0\le\ldots\le\alpha_l\le\alpha_{l+1}\le\beta_{l+1}\le\beta_l\le\ldots\le\beta_0.$$

We now show that each sequence $\{\alpha_l\}$ and $\{\beta_l\}$ converge in B to x, the unique solution of the BVP with impulse, (1), (2), (3). Recall

$$B=\{x\in PC^1[0,1]:x^{(i)}|_{[t_k,t_{k+1}]}\in C^i[t_k,t_{k+1}],k=0,\ldots,m,i=0,1\},$$

with $\|x\|_B=\max_{k=0,\ldots,m}\|x\|_k$ and $\|x\|_k=\max_{i=0,1}\sup_{t_k\le t\le t_{k+1}}|x^{(i)}(t)|$. The Kamke convergence theorem does not apply directly to either sequence, $\{\alpha_l\}$ or $\{\beta_l\}$ since neither g_l nor G_l converge uniformly on compact sets to f. To see this, note that

$$g_l(t,x_1,x_2)=f(t,x_1,x_2)+F_x(t,\beta_l(t))(x_1-\alpha_l(t))+F(t,\alpha_l(t))-F(t,x_1)$$

and

$$G_l(t,x_1,x_2)=f(t,x_1,x_2)+F_x(t,\beta_l(t))(x_1-\beta_l(t))+F(t,\beta_l(t))-F(t,x_1).$$

Define

$$\hat g_l(t,x_1,x_2)=f(t,x_1,x_2)+F_x(t,\beta_l(t))(\alpha_{l+1}-\alpha_l)(t)+F(t,\alpha_l(t))-F(t,\alpha_{l+1}(t))$$

and

$$\hat G_l(t,x_1,x_2)=f(t,x_1,x_2)+F_x(t,\beta_l(t))(\beta_{l+1}-\beta_l)(t)+F(t,\beta_l(t))-F(t,\beta_{l+1}(t)).$$

Theorem 3 applies to the BVP with impulse, (1), (2), (3), with $f=\hat g_l$ and each $v_k=h_{kl}$ and note that α_{k+1} is the unique solution. The Kamke convergence theorem now does apply and, with omitted details that are similar to those given in the proof of Theorem 2, $\{\alpha_l\}$ converges in B to x, the unique solution of the BVP with impulse, (1), (2), (3). Similarly, $\{\beta_l\}$ converges in B to x, the unique solution of the BVP with impulse, (1), (2), (3).

We now argue that the convergence is quadratic. Let $q_n(t) = \beta_n(t) - x(t)$ and $p_n(t) = x(t) - \alpha_n(t)$, where $x(t)$ denotes the unique solution of the BVP with impulse, (1), (2), (3). Set

$$e_n = \max\{\|q_n\|_B, \|p_n\|_B\}.$$

First, consider $q_{n+1}(t)$ and note that $q_{n+1} \geq 0$. For $t \in \cup_{k=0}^{m}(t_k, t_{k+1})$,

$$\begin{aligned} q''_{n+1}(t) &= F(t, \beta_n(t)) + F_x(t, \beta_n(t))(\beta_{n+1} - \beta_n)(t) \\ &\quad -\phi_1(t, \beta_{n+1}(t), \beta'_{n+1}(t)) - F(t, x(t)) + \phi_1(t, x(t), x'(t)) \\ &= F_x(t, c_1(t))q_n(t) - F_x(t, \beta_n(t))q_n(t) + F_x(t, \beta_n(t))q_{n+1}(t) \\ &\quad -\phi_{1x}(t, c_2(t), c_3(t))q_{n+1}(t) - \phi_{1x'}(t, c_2(t), c_3(t))q'_{n+1}(t)\,, \end{aligned}$$

where $x(t) \leq c_1(t) \leq \beta_n(t)$, $x(t) \leq c_2(t) \leq \beta_{n+1}(t)$, and $c_3(t)$ is between $x'(t)$ and $\beta'_{n+1}(t)$. Thus, there exists $c_1(t) \leq c_4(t) \leq \beta_n(t)$ such that

$$\begin{aligned} &q''_{n+1}(t) \\ &= F_{xx}(t, c_4(t))q_n(t)(c_1(t) - \beta_n(t)) \\ &\quad +(F_x(t, \beta_n(t)) - \phi_{1x}(t, c_2(t), c_3(t)))q_{n+1}(t) - \phi_{1x'}(t, c_2(t), c_3(t))q'_{n+1}(t) \\ &\geq -F_{xx}(t, c_4(t))q_n^2(t) + f_{x'}(t, c_2(t), c_3(t))q'_{n+1}(t)\,. \end{aligned}$$

Note that to obtain this inequality, we have employed the monotonicity of F_x in the second component. In particular, there exists $M > 0$, such that

$$q''_{n+1}(t) - f_{x'}(t, c_2(t), c_3(t))q'_{n+1}(t) \geq -Me_n^2, \tag{19}$$

where $M > \max_i \max_{(t,x)\in D_i} F_{xx}(t, x)$, and for $i = 0, \ldots m$,

$$D_i = \{(t, x) : t_i \leq t \leq t_{i+1}, \alpha_0(t) \leq x \leq \beta_0(t)\}\,.$$

Similarly, there exist appropriate c_4 and c_5 such that for $k = 1, \ldots, m$,

$$\Delta q'_{n+1}(t_k) - v_{ky}(c_4, c_5)q'_{n+1}(t_k) \geq -Me_n^2\,. \tag{20}$$

Let $m(t) = \exp\Big(-\int_0^t f_{x'}(s, c_2(s), c_3(s))ds\Big)$ denote the integrating factor associated with (19). Then

$$(q'_{n+1}(t)m(t))' \geq -Mm(t)e_n^2\,. \tag{21}$$

Thus, for $t_m \leq t \leq 1$,

$$q'_{n+1}(1)m(1) - q'_{n+1}(t)m(t) \geq -Me_n^2 \int_t^1 m(s)ds\,.$$

Since, $q'_{n+1}(1) \le 0$, it follows that

$$q'_{n+1}(t) \le Me_n^2 \int_t^1 m(s)ds/m(t)\,.$$

Since q_{n+1} converges to 0 in B, eventually $(s, c_2(s), c_3(s))$ belongs to

$$\hat{D} = \{(s, x_1, x_2) : t_m \le s \le 1, \alpha_0(s) \le x_1 \le \beta_0(s), x'(s) - 1 \le x_2 \le x'(s) + 1\}.$$

Thus, we can bound $m(t)$ away from both 0 and ∞ for n sufficiently large; in particular, there exists $N_1 > 0$ such that for $t_m \le t \le 1$ and n sufficiently large,

$$q'_{n+1}(t) \le N_1 e_n^2\,. \tag{22}$$

Apply (20) at t_m. Then

$$q'_{n+1}(t_m^+) - q'_{n+1}(t_m) - v_{my}(c_4, c_5)q'_{n+1}(t_m) \ge -Me_n^2\,.$$

Employ (7) and also bound v_{my} away from both 0 and ∞ to obtain some $\hat{M} > 0$ such that

$$q'_{n+1}(t_m^-) \ge -\hat{M}e_n^2\,. \tag{23}$$

Now, employ (21) and (23) to obtain (22) for $t_{m-1} \le t \le t_m$ for some $N_2 > 0$. Again, apply (20) to obtain a suitable (23) at t_{m-1}. Proceed inductively and obtain that there exists $N > 0$ such that for $t \in \cup_{k=0}^m [t_k, t_{k+1}]$ and n sufficiently large,

$$q'_{n+1}(t) \le Ne_n^2. \tag{24}$$

Recall that $q_{n+1}(t) \ge 0$, and that $q_{n+1} \in C[0,1]$. Integrate (24) from 0 to t; then for n sufficiently large,

$$0 \le q_{n+1} \le Ne_n^2\,. \tag{25}$$

Beginning again at (21), integrate from 0 to $t \le t_1$ to obtain

$$q'_{n+1}(t)m(t) - q'_{n+1}(0) \ge -Me_n^2 \int_0^t m(s)\,ds\,.$$

Since, $q'_{n+1}(0) \ge 0$, it follows that for $0 \le t \le t_1$, there exists $N_1 > 0$, such that

$$q'_{n+1}(t) \ge -Me_n^2 \int_0^t m(s)\,ds/m(t) \ge -N_1 e_n^2\,,$$

for n sufficiently large. This is analogous to (22). Proceed analogously, then, and choose N large enough such that for $t \in \cup_{k=0}^m [t_k, t_{k+1}]$ for n sufficiently large,

$$q'_{n+1}(t) \ge -Ne_n^2\,. \tag{26}$$

It now follows from (24), (25), and (26) that β_n converges to x quadratically in B.

The argument that $\{\alpha_n\}$ converges quadratically to x in B is similar and we provide some details.

$$\begin{aligned}
&p''_{n+1}(t)\\
&= F(t,x(t)) - \phi_1(t,x(t),x'(t))\\
&\quad -(F(t,\alpha_n(t)) + F_x(t,\beta_n(t))(\alpha_{n+1}-\alpha_n)(t) - \phi_1(t,\alpha_{n+1}(t),\alpha'_{n+1}(t)))\\
&= F_x(t,c_1(t))p_n(t) - F_x(t,\beta_n(t))p_n(t) + F_x(t,\beta_n(t))p_{n+1}(t)\\
&\quad -\phi_{1x}(t,c_2(t),c_3(t))p_{n+1}(t) - \phi_{1x'}(t,c_2(t),c_3(t))p'_{n+1}(t)\\
&= F_{xx}(t,c_4(t))p_n(t)(c_1(t)-\beta_n(t))\\
&\quad +(F_x(t,\beta_n(t)) - \phi_{1x}(t,c_2(t),c_3(t)))p_{n+1}(t) - \phi_{1x'}(t,c_2(t),c_3(t))p'_{n+1}(t)\\
&\geq -F_{xx}(t,c_4(t))p_n(t)(p_n(t)+q_n(t)) + f_{x'}(t,c_2(t),c_3(t))p'_{n+1}(t)\,.
\end{aligned}$$

In particular,

$$p''_{n+1}(t) - f_{x'}(t,c_2(t),c_3(t))p'_{n+1}(t) \geq -2Me_n^2$$

on an appropriate set and for sufficiently large n. A similar inequality is obtained with respect to the impulse and the details for quadratic convergence follow as above.

Corollary 5 *The sequence $\{\beta''_n(t) - f(t,\beta_n(t),\beta'_n(t))\}$ converges quadratically to 0 in B.*

Proof: There exist $\beta_n \geq c_2 \geq c_1 \geq \beta_{n+1}$ such that

$$\begin{aligned}
f(t,\beta_{n+1}(t),\beta'_{n+1}(t)) &\geq \beta''_{n+1}(t)\\
&= f(t,\beta_n(t),\beta'_n(t)) + F_x(t,\beta_n(t))(\beta_{n+1}(t)-\beta_n(t))\\
&\quad -(\phi_1(t,\beta_{n+1}(t),\beta'_{n+1}(t)) - \phi_1(t,\beta_n(t),\beta'_n(t)))\\
&= f(t,\beta_{n+1}(t),\beta'_{n+1}(t))\\
&\quad +F_{xx}(t,c_2(t))(\beta_{n+1}(t)-\beta_n(t))(\beta_n(t)-c_1(t))\,.
\end{aligned}$$

Thus,

$$\begin{aligned}
0 &\leq f(t,\beta_{n+1}(t),\beta'_{n+1}(t)) - \beta''_{n+1}(t)\\
&\leq F_{xx}(t,c_2(t))(\beta_{n+1}(t)-\beta_n(t))^2\\
&\leq F_{xx}(t,c_2(t))e_n^2\,.
\end{aligned}$$

Similar inequalities are obtained for the impulse. Quadratic convergence can also be obtained for the sequence

$$\{f(t,\alpha_n(t),\alpha'_n(t)) - \alpha''_n(t)\}.$$

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